A Personal Tour Through Symplectic Topology and Geometry

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Tour Goal

Describe the following 3 trails and how my work fits in them.

- Gromov trail: holomorphic curves and topology of symplectomorphism groups.
- Delzant-Guillemin trail: moment polytopes and Kähler geometry of toric orbifolds.
- Donaldson trail: Hamiltonian action of the group of symplectomorphisms on the space of compatible complex structures.

Holomorphic Curves

Definition

A parametrized holomorphic curve is a map from a closed Riemann surface to an almost complex manifold, $f : (\Sigma, j) \to (M, J)$, such that $df \circ j = J \circ df$. Its image $C = f(\Sigma)$ is an unparametrized holomorphic curve.

- $\mathcal{J}(M, \omega)$ - space of compatible almost complex structures on a symplectic manifold (M, ω), i.e. $J \in \Gamma(\mathrm{End}(TM))$ with $J^2 = -Id$, such that

$$g_J(\cdot, \cdot) \equiv \omega(\cdot, J\cdot) \text{ is a Riemannian metric on } M.$$

- $\mathcal{J}(M, \omega)$ is always non-empty, infinite-dimensional and contractible.

Simple Version of Gromov's Compactness Theorem (1985)

Gromov'85

$f_n : (\Sigma, j) \to (M, \omega, J_n)$ - holomorphic curves representing $0 \neq A \in H_2(M, \mathbb{Z})$. Assume that $J_n \to J_\infty$ in $\mathcal{J}(M, \omega)$.

Then, modulo $PSL(2, \mathbb{C})$-reparametrizations when $\Sigma = S^2$,

(i) either a subsequence of f_n converges to a holomorphic curve $f_\infty : (\Sigma, j) \to (M, J_\infty)$, also representing A;

(ii) or $A = B + A'$, where $B \neq 0$ can be represented by a J_∞-holomorphic sphere ("bubble") and $A' \neq 0$ can be represented by a J_∞-holomorphic curve.

Simple Corollary of Gromov's Compactness Theorem

Any compatible almost complex structure $J \in \mathcal{J}(M, \omega)$ satisfies the positivity condition

$$\omega(X, JX) > 0\,,\ \forall\, 0 \neq X \in TM\,.$$

Hence

$$[\omega](A) > 0$$

for any $0 \neq A \in H_2(M, \mathbb{Z})$ that can be represented by a holomorphic curve.

Corollary

If the homology class $A \in H_2(M, \mathbb{Z})$ cannot be written as $A = A_1 + A_2$, where $[\omega](A_i) > 0\,,\ i = 1, 2$, and either A_1 or A_2 is a spherical class, then (i) holds (i.e. no "bubbles").

Properties of Holomorphic Curves in Dimension 4 - Homology Controls Geometry

Positivity of Intersections implies that two distinct holomorphic curves C and C', representing homology classes $A = [C]$ and $A' = [C']$, satisfy:

- $A \cdot A' \geq 0$ (i.e. if $A \cdot A' < 0$ then $C = C'$).
- $A \cdot A' = 0$ iff C and C' are disjoint.
- $A \cdot A' = 1$ iff C and C' intersect transversally at 1 point.

The Adjunction Formula implies that a somewhere injective holomorphic curve $C = f(\Sigma) \subset M$, with $A = [C] \in H^2(M)$, is embedded iff

$$2 + A \cdot A - c_1(A) = 2 \cdot (\text{genus of } \Sigma)\,.$$

Holomorphic Spheres and the Symplectomorphism Group of $(S^2 \times S^2, \sigma \times \sigma)$

Proposition

For any $J \in \mathcal{J}(S^2 \times S^2, \sigma \times \sigma)$ and any point $p \in S^2 \times S^2$, there exist J-holomorphic spheres representing the classes $A = [S^2 \times \{p\}]$ and $F = [\{p\} \times S^2]$, both passing through the point p.

Proof.

The statement is true for a subset of $\mathcal{J}(S^2 \times S^2, \sigma \times \sigma)$ which is:

(i) non-empty (contains $J_0 = j_0 \times j_0$);

(ii) open (holomorphic curve equation is elliptic);

(iii) closed (Corollary of Gromov's Compactness).

□

Holomorphic Spheres and the Symplectomorphism Group of $(S^2 \times S^2, \sigma \times \sigma)$

Gromov'85

$$\mathrm{Diff}^{H_2}(S^2 \times S^2, \sigma \times \sigma) \sim SO(3) \times SO(3)\,.$$

Proof.

The natural action map

$$\beta : \mathrm{Diff}^{H_2}(S^2 \times S^2, \sigma \times \sigma)/SO(3) \times SO(3) \to \mathcal{J}\,,$$

given by $\beta([\varphi]) = \varphi_*(J_0)$, is an homotopy equivalence.

In fact, one can construct an homotopy inverse using the coordinate systems provided by holomorphic sphere foliations. □

Holomorphic Spheres and the Symplectomorphism Group of $(S^2 \times S^2, \omega_\lambda)$

Given $1 < \lambda \in \mathbb{R}$, let $M_\lambda \equiv (S^2 \times S^2, \omega_\lambda = \lambda\sigma \times \sigma)$, $\mathcal{J}_\lambda \equiv \mathcal{J}(M_\lambda)$ and $G_\lambda \equiv \text{Symp}(M_\lambda)$.

For each $k \in \mathbb{N}_0$, define $U_k \subset \mathcal{J}_\lambda$ by

$$J \in U_k \Leftrightarrow \exists\ J\text{-holomorphic sphere representing } A - kF.$$

Note: $[\omega_\lambda](A - kF) > 0 \Leftrightarrow k < \lambda \Leftrightarrow k \leq \ell$, where $\ell \in \mathbb{N}$ is such that $\ell < \lambda \leq \ell + 1$. Hence $U_k \neq \emptyset \Rightarrow 0 \leq k \leq \ell$.

In fact, when $0 \leq k \leq \ell$, there exists integrable $J_k \in U_k$ such that:

- $(S^2 \times S^2, J_k) \cong$ Hirzebruch surface H_{2k};
- $K_k \equiv \text{Isom}(S^2 \times S^2, g_{\lambda,k}(\cdot,\cdot) \equiv \omega_\lambda(\cdot, J_k\cdot))$ is $SO(3) \times SO(3)$ $(k = 0)$ or $SO(3) \times S^1$ $(k \geq 1)$.

Holomorphic Spheres and the Symplectomorphism Group of $(S^2 \times S^2, \omega_\lambda)$

A.'98, McDuff'00

$\mathcal{J}_\lambda$ is stratified as $\mathcal{J}_\lambda = U_0 \sqcup U_1 \sqcup \cdots \sqcup U_\ell$, where U_0 is open and dense, $\mathrm{codim}(U_k) = 4k - 2$ when $k \geq 1$, and $\overline{U_k} = U_k \sqcup U_{k+1} \sqcup \cdots \sqcup U_\ell$. Moreover,

$$G_\lambda / K_k \sim U_k \,, \ \forall\, k \geq 0 \,.$$

A.-McDuff'00

$$H^*(G_\lambda; \mathbb{Q}) = \Lambda(a, x, y) \otimes S(w_\ell) \,,$$

where $\Lambda(a, x, y)$ = exterior algebra over $\mathbb{Q}$ on generators a, x and y of degrees $\deg(a) = 1$, $\deg(x) = \deg(y) = 3$, and $S(w_\ell)$ = polynomial algebra over $\mathbb{Q}$ on the generator w_ℓ of degree 4ℓ.

Symplectic Toric Manifolds

Definition

A symplectic toric manifold is a connected symplectic manifold (M, ω) of dimension $2n$, equipped with an effective Hamiltonian $\mathbb{T}^n$-action.

The corresponding moment map will be denoted by $\mu : M \to \mathbb{R}^n$ and, for compact M, the corresponding moment polytope $\mu(M) = P \subset \mathbb{R}^n$ will be called its Delzant polytope.

Delzant polytopes are always convex (by the A-G-S Convexity Theorem), simple and integral, and these 3 properties can be used to define them.

Classification Theorems

Delzant'82

The Delzant polytope is a complete invariant of a compact symplectic toric manifold.

Lerman-Tolman'97

The rational Delzant polytope with a positive integer label attached to each of its facets is a complete invariant of a compact symplectic toric orbifold.

A facet of a polytope is a codimension-1 face.

Examples - Projective Spaces

$(\mathbb{P}^n, \omega_{\mathrm{FS}})$, with homogeneous coordinates $[z_0; z_1; \ldots; z_n]$.

$\mathbb{T}^n$-action on $\mathbb{P}^n$ given by

$$(y_1, \ldots, y_n) \cdot [z_0; z_1; \ldots; z_n] = [z_0; e^{-iy_1} z_1; \ldots; e^{-iy_n} z_n] \,,$$

is Hamiltonian, with moment map $\mu_{\mathrm{FS}} : \mathbb{P}^n \to \mathbb{R}^n$ given by

$$\mu_{\mathrm{FS}}[z_0; z_1; \ldots; z_n] = \frac{1}{\|z\|^2} (\|z_1\|^2, \ldots, \|z_n\|^2) \,.$$

$(\mathbb{P}^n, \omega_{\mathrm{FS}})$ is a compact symplectic toric manifold.

Delzant polytope $P = \mu_{\mathrm{FS}}(\mathbb{P}^n)$ is the standard simplex in $\mathbb{R}^n$.

Examples - Labeled Projective Spaces

$$(\mathbb{P}^n_{\vec{m}}, \omega_{\vec{m}}), \text{ where } \vec{m} = (m_1, \ldots, m_{n+1}) \in \mathbb{N}^{n+1},$$

is the symplectic toric orbifold associated with the standard simplex in $\mathbb{R}^n$, with labels $m_1, \ldots, m_{n+1} \in \mathbb{N}$ on its facets.

The better known weighted projective spaces,

$$\mathbb{CP}^n_{\vec{a}} \quad \text{with} \quad \vec{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{N}^{n+1},$$

are orbifold covering spaces of labeled projective spaces, i.e. there is an orbifold covering map $\mathbb{CP}^n_{\vec{a}} \to \mathbb{P}^n_{\vec{m}}$ of degree $(\prod a_k)^{n-1}$, where

$$m_r = \prod_{k=1, k\neq r}^{n+1} a_k, \; 1 \leq r \leq n+1.$$

Toric Compatible Complex Structures

Definition

A toric compatible complex structure on a symplectic toric manifold (M^{2n}, ω) is a

$$\mathbb{T}^n\text{-invariant } J \in \mathcal{I}(M, \omega) \subset \mathcal{J}(M, \omega)\,.$$

The space of all such will be denoted by

$$\mathcal{I}^{\mathbb{T}^n}(M, \omega) \subset \mathcal{J}^{\mathbb{T}^n}(M, \omega)\,.$$

- It follows from the classification theorems of Delzant and Lerman-Tolman that $\mathcal{I}^{\mathbb{T}^n}(M, \omega)$ is non-empty for any compact symplectic toric manifold or orbifold.

Description of Action-Angle Coordinates

$$\mathbb{R}^n \supset \mu(M) = P \supset \check{P} \equiv \text{interior of } P.$$

$$\check{M} \equiv \mu^{-1}(\check{P}) \equiv \{\text{points of } M \text{ where } \mathbb{T}^n\text{-action is free}\}$$

$$\cong \check{P} \times \mathbb{T}^n = \left\{(x,y) \, : \, x \in \check{P} \subset \mathbb{R}^n, \; y \in \mathbb{R}^n/2\pi\mathbb{Z}^n\right\}$$

such that $\omega|_{\check{M}} = dx \wedge dy \equiv$ standard symplectic form

Definition

$(x,y) \equiv$ symplectic/Darboux/action-angle coordinates.

- $\check{M}$ is an open dense subset of M.
- In these coordinates, the moment map $\mu : \check{M} \to \check{P}$ is given by $\mu(x,y) = x$.

Local Form of Toric Compatible Complex Structures

For any integrable $J \in \mathcal{I}^{\mathbb{T}^n}$ there exist action-angle coordinates (x, y) on $\check{M} \cong \check{P} \times \mathbb{T}^n$ such that

$$J = \begin{bmatrix} 0 & \vdots & -S^{-1} \\ \cdots & \cdots & \cdots \\ S & \vdots & 0 \end{bmatrix}$$

with

$$S = S(x) = \left(s_{ij}(x)\right) = \left(\frac{\partial^2 s}{\partial x_i \partial x_j}\right) > 0$$

for some real potential function $s : \check{P} \to \mathbb{R}$.

We will call s the symplectic potential of the complex structure J.

Examples - Compact Symplectic Toric Manifolds

- To any Delzant polytope $P \subset \mathbb{R}^n$, a construction of Delzant associates a canonical compact Kähler toric manifold $(M_P^{2n}, \omega_P, J_P)$ with moment map $\mu_P : M_P \to \mathbb{R}^n$ and moment polytope $\mu_P(M_P) = P$.

- Let F_i denote the i-th facet of the polytope. The affine defining function of F_i is the function

$$\begin{aligned} \ell_i : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto \ell_i(x) = \langle x, \nu_i \rangle - \lambda_i \,, \end{aligned}$$

where $\nu_i \in \mathbb{Z}^n$ is a primitive inward pointing normal to F_i and $\lambda_i \in \mathbb{R}$ is such that $\ell_i|_{F_i} \equiv 0$. Note that $\ell_i|_{\breve{P}} > 0$.

Examples - Compact Symplectic Toric Manifolds

- Lerman-Tolman extended Delzant's construction to rational and labeled Delzant polytopes, associating to each a compact Kähler toric orbifold $(M_P^{2n}, \omega_P, J_P)$.

Guillemin'94, A.'01

In appropriate action-angle coordinates (x, y), the canonical symplectic potential $s_P : \breve{P} \to \mathbb{R}$ for $J_P|_{\breve{P}}$ is given by

$$s_P(x) = \frac{1}{2} \sum_{i=1}^{d} m_i \, \ell_i(x) \log \ell_i(x) \,,$$

where d is the number of facets of P and $m_i \in \mathbb{N}$ is the label of the i-th facet of P, $i = 1, \ldots, d$.

Examples - Projective Space $(\mathbb{P}^n, \omega_{\mathrm{FS}}, \mu_{\mathrm{FS}})$

The Delzant polytope of $\mathbb{P}^n$ is the standard simplex, with defining affine functions

$$\ell_i(x) = x_i\,,\ i = 1, \ldots, n\,, \quad \text{and} \quad \ell_{n+1}(x) = 1 - r\,,$$

where $r = \sum_i x_i$. In this smooth case all labels $= 1$.

The canonical symplectic potential, $s_P : \breve{P} \to \mathbb{R}$, is then given by

$$s_P(x) = \frac{1}{2}\sum_{i=1}^{n} x_i \log x_i + \frac{1}{2}(1-r)\log(1-r)\,.$$

It defines the standard complex structure J_{FS} and Fubini-Study metric g_{FS} on $\mathbb{P}^n$.

Remark - Symplectic Potentials and Reduction

Calderbank-David-Gauduchon'02

Symplectic potentials restrict naturally under toric symplectic reduction.

- If (M', ω', P') is a toric symplectic reduction of (M, ω, P), then there is a natural affine inclusion $P' \subset P$ and any $J \in \mathcal{I}^{\mathbb{T}}(M, \omega)$ induces a reduced $J' \in \mathcal{I}^{\mathbb{T}'}(M', \omega')$. This theorem says that if $s : \breve{P} \to \mathbb{R}$ and $s' : \breve{P}' \to \mathbb{R}$ denote the corresponding symplectic potentials, then

$$s' = s|_{\breve{P}'} \, .$$

It easily implies, for example, Guillemin's formula for the canonical symplectic potential s_P.

Toric $\partial\bar{\partial}$-Lemma in Action-Angle Coordinates

A.'01

Let $J \in \mathcal{I}^{\mathbb{T}^n}(M_P, \omega_P)$. Then, in suitable action-angle coordinates (x, y) on $\check{M} \cong \check{P} \times \mathbb{T}^n$, J is given by a symplectic potential $s : \check{P} \to \mathbb{R}$ of the form

$$s(x) = s_P(x) + h(x)\,,$$

where $h \in C^\infty(P)$, $\text{Hess}_x(s) > 0$ in $\check{P}$ and $\det(\text{Hess}_x(s)) = (\delta(x) \prod_i \ell_i)^{-1}$, with $\delta \in C^\infty(P)$ and $\delta(x) > 0\,,\ \forall x \in P$.

Conversely, any such s is the symplectic potential of a $J \in \mathcal{I}^{\mathbb{T}^n}(\check{P} \times \mathbb{T}^n)$ that compactifies to a well defined $J \in \mathcal{I}^{\mathbb{T}^n}(M_P, \omega_P)$.

Extremal Kähler Metrics

Definition (Calabi'82)

Extremal Kähler metrics are critical points of the square of the L^2-norm of the scalar curvature, considered as a functional on the space of all Kähler metrics in a fixed Kähler cohomology class.

- Extremal Kähler metrics generalize constant scalar curvature Kähler metrics.
- The extremal Euler-Lagrange equation is equivalent to the gradient of the scalar curvature being an holomorphic vector field.
- From a symplectic point of view, "Kähler metrics in a fixed Kähler cohomology class" means compatible complex structures in a fixed diffeomorphism class.

Remarks on Toric Case

- Calabi proved that extremal Kähler metrics are always invariant under a maximal compact subgroup of the group of holomorphic diffeomorphisms of the underlying complex manifold.

- This implies that on a toric manifold extremal Kähler metrics are toric.

- On a fixed symplectic toric manifold, there is a unique diffeomorphism class of compatible toric complex structures.

Toric Kähler Metrics and Scalar Curvature

- Toric Kähler metric

$$g = \begin{bmatrix} (s_{ij}) & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & (s^{ij}) \end{bmatrix}$$

where $(s_{ij}) = \text{Hess}_x(s)$ for a symplectic potential $s : \check{P} \subset \mathbb{R}^n \to \mathbb{R}$.

- Formula for its scalar curvature [A.'98]:

$$Sc \equiv -\sum_{j,k} \frac{\partial}{\partial x_j}\left[s^{jk} \frac{\partial \log(\det \text{Hess}_x(s))}{\partial x_k} \right] = -\sum_{j,k} \frac{\partial^2 s^{jk}}{\partial x_j \partial x_k}$$

- Extremal $\Leftrightarrow$ Sc is affine function on $\check{P} \subset \mathbb{R}^n$.

Remarks

- Many interesting explicit constructions in this setting. For example, Calabi's first examples of extremal metrics with non-constant scalar curvature, on non-trivial holomorphic $\mathbb{P}^1$-bundles over $\mathbb{P}^n$, have simple explicit symplectic potentials [A.'98].

- (Donaldson'04,'06) Geometric and analytic study of the general nonlinear, fourth order PDE for a convex function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, given by

$$-\sum_{j,k} \frac{\partial^2 u^{jk}}{\partial x_j \partial x_k} = A \,,$$

  where A is some given function (e.g. Ω a polytope and A a suitable affine function).

Examples - Extremal Metrics on Labeled Projective Spaces

- (R. Bryant'01) Constructs extremal Kähler metrics on weighted projective spaces, as particular examples of Bochner-Kähler metrics.

- In our setting these can be written down explicitly, on the corresponding labeled projective spaces, in the following simple way.

- Let $P \subset \mathbb{R}^n$ be the standard simplex, with defining affine functions

$$\ell_i(x) = x_i\,,\ i = 1, \ldots, n\,, \quad \text{and} \quad \ell_{n+1}(x) = 1 - r\,,$$

where $r = \sum_i x_i$, and facet labels $m_1, \ldots, m_{n+1} \in \mathbb{N}$.

Examples - Extremal Metrics on Labeled Projective Spaces

Define

$$\ell_{\vec{m}}(x) = \sum_{i=1}^{n+1} m_i \ell_i(x) \,.$$

A.'01

The symplectic potential

$$s(x) = \frac{1}{2}\sum_{i=1}^{n+1} m_i \, \ell_i(x) \log \ell_i(x) - \frac{1}{2}\ell_{\vec{m}}(x) \log \ell_{\vec{m}}(x)$$

defines an extremal toric Kähler metric on the labeled projective space $(\mathbb{P}^n_{\vec{m}}, \omega_{\vec{m}}, \tau_{\vec{m}})$. If the integer labels m_i are not all equal, this metric has non-constant scalar curvature.

General Moment Map Framework (Atiyah-Bott-Kirwan-Donaldson)

(X, J, Ω) with Kähler action of a Lie group G, such that:

(i) holomorphic action of G on (X, J) extends to $G^{\mathbb{C}}$;

(ii) symplectic action of G on (X, Ω) admits moment map

$$\mu : X \to \mathcal{G}^* \,.$$

Then we have the following two general principles:

PI. $X^s/G^{\mathbb{C}} = \mu^{-1}(0)/G$, i.e. on each stable $G^{\mathbb{C}}$-orbit there is a point $p \in \mu^{-1}(0)$, unique up to the action of G.

PII. $\|\mu\|^2 : X \to \mathbb{R}$ is a G-invariant Morse-Bott function - critical manifolds compute $H^*_G(X) \equiv H^*(X \times_G EG)$.

Action of Symplectomorphism Groups on Compatible Complex Structures

- (M^{2n}, ω) - compact symplectic manifold, $H^1(M, \mathbb{R}) = 0$.
- $G \equiv \mathrm{Diff}(M, \omega)$ - infinite dim'l Lie group with Lie algebra

$$\mathcal{G}^* \supset \mathcal{G} = C_0^\infty(M) \equiv \left\{ f : M \to \mathbb{R} \; : \; \int_M f \, \frac{\omega^n}{n!} = 0 \right\} .$$

- $\mathcal{J}(M, \omega)$ - infinite dim'l Kähler manifold $=$ space of sections of $Sp(2n, \mathbb{R})/U(n)$-bundle over (M, ω).
- Natural Kähler action of G on $\mathcal{J}(M, \omega)$:

$$\phi \cdot J \equiv \phi_*(J) = d\phi \circ J \circ d\phi^{-1} \, , \; \forall \phi \in G \, , \; J \in \mathcal{J}(M, \omega) \, .$$

Action of Symplectomorphism Groups on Compatible Complex Structures

On $X = \mathcal{I}(M, \omega) \subset \mathcal{J}(M, \omega)$, an invariant Kähler submanifold, Donaldson'97 shows that:

(i) "$G^{\mathbb{C}}$"-orbits are diffeomorphism classes of compatible complex structures;

(ii) moment map $\mu : X \to C_0^\infty(M)$ is given by scalar curvature: $\mu(J) = Sc(J) -$ const.

The two general principles should then say that:

PI. Each diffeomorphism class of stable compatible complex structures has one with constant scalar curvature, unique up to $G = \mathrm{Diff}(M, \omega)$.

PII. Critical manifolds of $\|\mu\|^2 : X \to \mathbb{R}$ compute $H_G^*(X)$, i.e. extremal Kähler metrics compute $H^*_{\mathrm{Diff}(M,\omega)}(\mathcal{I}(M, \omega))$.

Remarks

- Framework not rigorous in infinite dimensions, but formal application can be useful as guide.
- Donaldson follows this guide to study constant scalar curvature Kähler metrics. In particular, shows how the use of action-angle coordinates on toric manifolds fits well in this framework.
  For example, special toric formula for scalar curvature is its natural expression as a moment map.
- We can also follow this guide to get a new approach to understanding the topology of the symplectomorphism groups of $S^2 \times S^2$.

Compatible Complex Structures and the Symplectomorphism Group of $(S^2 \times S^2, \omega_\lambda)$

Given $1 < \lambda \in \mathbb{R}$, let $M_\lambda \equiv (S^2 \times S^2, \omega_\lambda = \lambda\sigma \times \sigma)$, $X_\lambda \equiv \mathcal{I}(M_\lambda) \subset \mathcal{J}(M_\lambda)$ and $G_\lambda \equiv \text{Symp}(M_\lambda)$.

For each $k \in \mathbb{N}_0$, define $V_k \subset X_\lambda$ by

$$J \in V_k \Leftrightarrow \exists \text{ } J\text{-holomorphic sphere representing } A - kF.$$

Note: $[\omega_\lambda](A - kF) > 0 \Leftrightarrow k < \lambda \Leftrightarrow k \leq \ell$, where $\ell \in \mathbb{N}_0$ is such that $\ell < \lambda \leq \ell + 1$. Hence $V_k \neq \emptyset \Rightarrow 0 \leq k \leq \ell$.

In fact, when $0 \leq k \leq \ell$, there exists extremal $J_k \in V_k$, unique up to the action of G_λ such that:

- $(S^2 \times S^2, J_k) \cong$ Hirzebruch surface H_{2k};
- $K_k \equiv \text{Isom}(S^2 \times S^2, g_{\lambda,k}(\cdot,\cdot) \equiv \omega_\lambda(\cdot, J_k\cdot))$ is $SO(3) \times SO(3)$ $(k = 0)$ or $SO(3) \times S^1$ $(k \geq 1)$.

Compatible Complex Structures and the Symplectomorphism Group of $(S^2 \times S^2, \omega_\lambda)$

A.-Granja-Kitchloo'05 and'06

X_λ is stratified as $X_\lambda = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_\ell$, where:

- Each stratum V_k consists of a unique "$G_\lambda^{\mathbb{C}}$"-orbit and $G_\lambda / K_k \sim V_k$, $\forall\, k \geq 0$.
- For $1 \leq k \leq \ell$, each V_k has a tubular neighborhood with normal slice given by $H^1(H_{2k}, \Theta) \cong \mathbb{C}^{2k-1}$, where $\Theta =$ sheaf of holomorphic vector fields on H_{2k}.

  The isotropy normal representation of $K_k \cong SO(3) \times S^1$ on $\mathbb{C}^{2k-1}$ is the following: S^1 acts diagonally and $SO(3)$ acts irreducibly with highest weight $2(k-1)$.

Standard equivariant cohomology gives the following

Corollary

$$H^*_{G_\lambda}(X_\lambda;\mathbb{Z}) \cong H^*(BSO(3) \times BSO(3);\mathbb{Z}) \oplus \oplus_{k=1}^{\ell} \Sigma^{4k-2} H^*(BS^1 \times BSO(3);\mathbb{Z}),$$

where $\cong$ indicates a group isomorphism.

Comparing $X_\lambda = \mathcal{I}(M_\lambda)$ with $\mathcal{J}(M_\lambda)$ gives the following

Theorem

X_λ is weakly contractible.

In particular, this means that

$$H^*_{G_\lambda}(X_\lambda; \mathbb{Z}) \cong H^*(BG_\lambda; \mathbb{Z}),$$

which implies the following

Corollary

$$\begin{aligned} H^*(BG_\lambda; \mathbb{Z}) \cong & \, H^*(BSO(3) \times BSO(3); \mathbb{Z}) \oplus \\ & \oplus_{k=1}^{\ell} \Sigma^{4k-2} H^*(BS^1 \times BSO(3); \mathbb{Z}), \end{aligned}$$

where $\cong$ indicates a group isomorphism.

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Outlook

- Symplectomorphism groups and compatible complex structures of ruled surfaces (e.g. $\Sigma \times S^2$).
- Balanced Kähler metrics on toric orbifolds.
- Sasakian geometry of contact toric orbifolds in “action-angle” coordinates.

Local Form of Toric Compatible Complex Structures

Any $J \in \mathcal{J}^{\mathbb{T}^n}(\breve{M}, \omega|_{\breve{M}})$ can be written in action-angle coordinates (x,y) on $\breve{M} \cong \breve{P} \times \mathbb{T}^n$ as

$$J = \begin{bmatrix} S^{-1}R & \vdots & -S^{-1} \\ \cdots & \cdots & \cdots \\ RS^{-1}R + S & \vdots & -RS^{-1} \end{bmatrix}$$

where $R = R(x)$ and $S = S(x)$ are real symmetric $(n \times n)$ matrices, with S positive definite, i.e.

$$Z(x) \equiv R(x) + iS(x) \in \text{Siegel Upper Half Space}, \forall\, x \in \breve{P}.$$

Local Form of Toric Compatible Complex Structures

For integrable toric compatible complex structures we have that:

$$J \in \mathcal{I}^{\mathbb{T}^n} \subset \mathcal{J}^{\mathbb{T}^n} \Leftrightarrow \frac{\partial Z_{ij}}{\partial x_k} = \frac{\partial Z_{ik}}{\partial x_j}$$

$$\Leftrightarrow \exists\ f : \check{P} \to \mathbb{C}\,,\ f(x) = r(x) + is(x)\,,\ \text{such that}$$

$$Z_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 r}{\partial x_i \partial x_j} + i \frac{\partial^2 s}{\partial x_i \partial x_j} = R_{ij} + iS_{ij}$$

Local Form of Toric Compatible Complex Structures

Any real function

$$h : \check{P} \to \mathbb{R}$$

is the Hamiltonian of a 1-parameter family

$$\phi_t : \check{M} \to \check{M}$$

of $\mathbb{T}^n$-equivariant symplectomorphisms, given in action-angle coordinates (x, y) on $\check{M} \cong \check{P} \times \mathbb{T}^n$ by

$$\phi_t(x, y) = (x, y - t\frac{\partial h}{\partial x}) .$$

Local Form of Toric Compatible Complex Structures

The natural action of such a ϕ_t on $\mathcal{J}^{\mathbb{T}^n}$, given by

$$\phi_t \cdot J = (d\phi_t)^{-1} \circ J \circ (d\phi_t)\,,$$

corresponds in the Siegel Upper Half Space parametrization to

$$\phi_t \cdot (Z = R + iS) = (R + tH) + iS\,,$$

where

$$H = (h_{ij}) = \left(\frac{\partial^2 h}{\partial x_i \partial x_j}\right)\,.$$

Local Form of Toric Compatible Complex Structures

Hence, for any integrable $J \in \mathcal{I}^{\mathbb{T}^n}$ there exist action-angle coordinates (x, y) on $\breve{M} \cong \breve{P} \times \mathbb{T}^n$ such that $R \equiv 0$ in the Siegel Upper Half Space Parametrization, i.e. such that

$$J = \begin{bmatrix} 0 & \vdots & -S^{-1} \\ \cdots & \cdots & \cdots \\ S & \vdots & 0 \end{bmatrix}$$

with

$$S = S(x) = (s_{ij}(x)) = \left(\frac{\partial^2 s}{\partial x_i \partial x_j} \right) > 0$$

for some

real potential function $\quad s : \breve{P} \to \mathbb{R}\,.$