Symmetric periodic Reeb orbits on the sphere and contact homology of Lens spaces

(joint with Hui Liu and L. Macarini)

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(I) Introduction

Basic setup

- $(\mathbb{R}^{2n+2}, \omega)$, $\omega = \sum_i dq_i \wedge dp_i = d\lambda$
  where $\lambda = \frac{1}{2} \sum_i (q_i \, dp_i - p_i \, dq_i)$.

- Unit sphere $S^{2n+1} \subset \mathbb{R}^{2n+2}$ with its standard contact structure $\xi = \ker \lambda|_{S^{2n+1}}$

- Contact form $\alpha = f \lambda|_{S^{2n+1}}$, for some $f : S^{2n+1} \longrightarrow \mathbb{R}^+$, and Reeb vector field R_α, characterized by $L_{R_\alpha} d\alpha = 0$ and $\alpha(R_\alpha) = 1$.

- Study dynamics of Reeb flows on standard contact sphere (S^{2n+1}, ξ).

- In particular: periodic orbits.

Multiplicity Problem

- $\mathcal{P}_\alpha$ = set of simple periodic orbits of R_α
- $\boxed{\text{Conjecture: } \#\mathcal{P}_\alpha \geq n+1, \ \forall \alpha}$
- Note that irrational ellipsoids have precisely $n+1$ simple periodic orbits:
$$(R_\alpha)^t(z_0, \dots, z_n) = (e^{ia_0 t} z_0, \dots, e^{ia_n t} z_n)$$
with $a_0, \dots, a_n$ rationally independent.

General Results (no assumption on α)

- Rabinowitz '1978: $\#\mathcal{P}_\alpha \geq 1, \ \forall n$.
- Cristofaro Gardiner - Hutchings '2016, Ginzburg - Hein - Hryniewicz - Macarini '2015, Liu - Long '2016 (using a result of GHHM):
$$\#\mathcal{P}_\alpha \geq 2 \text{ for } n = 1.$$

Results assuming convexity

- Bijection between contact forms α on (S^{2n+1}, ξ) and starshaped hyper-surfaces $\Sigma_\alpha \subset \mathbb{R}^{2n+2}$:

$$\alpha = f\lambda\big|_{S^{2n+1}} \longleftrightarrow \Sigma_\alpha = \left\{\sqrt{f(x)}\cdot x \,:\, x \in S^{2n+1}\right\}.$$

- α is convex if Σ_α bounds a strictly convex subset.

- Ekeland-Hofer '1987: $\#\mathcal{P}_\alpha \geq 2$, $\forall$ convex α

- Long-Zhu '2002: $\#\mathcal{P}_\alpha \geq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$, $\forall$ convex α

- Wang '2016: $\#\mathcal{P}_\alpha \geq \left\lceil \frac{n+1}{2} \right\rceil + 1$, $\forall$ convex α

- The convexity condition is not a natural contact geometry condition: it is not invariant under contactomorphisms.

Dynamical Convexity

- Def. [Hofer - Wysocki - Zehnder]
  A contact form α on (S^{2n+1}, ξ) is dynamically convex if $\mu_{CZ}(\gamma) \geq n+2$ for every periodic R_α-orbit γ.

μ_{CZ} ≡ Conley - Zehnder index

- Convex $\Longrightarrow$ Dynamical Convex (DC)
  $\cancel{\Longleftarrow}$

[Chaidez - Edtmair, A.- Macarini, Ginzburg - Macarini, preprint in arXiv today by J. Dardennes - J. Gutt - - V. Ramos - J. Zhang]

- A.- Macarini '2017: $\# \mathcal{P}_\alpha \geq 2$, $\forall$ DC α.

- Ginzburg - Gurel '2020 and independently Duan - Liu '2017: $\# \mathcal{P}_\alpha \geq \left\lceil \frac{n+1}{2} \right\rceil + 1$, $\forall$ DC α.

(II) Our problem and results

- $S^{2n+1} \circlearrowleft \mathbb{Z}_p$-action, $p \in \mathbb{N}$, generated by

$$\psi(z_0, \dots, z_n) = \left(e^{\frac{2\pi i l_0}{p}} z_0, \dots, e^{\frac{2\pi i l_n}{p}} z_n\right),$$

$l_0, \dots, l_n \in \mathbb{Z}$ weights of the action.

- Will always assume that $\gcd(l_j, p) = 1$, $j = 0, \dots, n$, so that $\mathbb{Z}_p$-action is free and

$$L_p^{2n+1}(l_0, \dots, l_n) := S^{2n+1} / \mathbb{Z}_p .$$

- Given a $\mathbb{Z}_p$-inv. contact form α a R_α-periodic orbit γ is said to be symmetric if $\psi(\gamma(\mathbb{R})) = \gamma(\mathbb{R})$, and

$\mathcal{P}_{\alpha,s}$ = set of simple symmetric R_α-periodic orbits.

Clearly, $\# \mathcal{P}_\alpha \geq \# \mathcal{P}_{\alpha,s}$.

- $\boxed{\text{Conjecture}: \# \mathcal{P}_{\alpha,s} \geq n+1.}$

- Note that irrational ellipsoids in $\mathbb{R}^{2n+2}$ are invariant under these $\mathbb{Z}_p$-actions (for any p and weights) and have precisely $n+1$ simple periodic orbits which are all symmetric.

Previous Results

- Girardi '1984 : $p=2$, $\# \mathcal{P}_{\alpha,s} \geq 1$, $\forall \alpha$.

- Bähni '2021 : p even, $\# \mathcal{P}_{\alpha,s} \geq 1$, $\forall \alpha$.

- Zhang '2013 : $\# \mathcal{P}_{\alpha,s} \geq 2$, $\forall$ convex α.

- Liu-Zhang '2022 : $p=2$, $\# \mathcal{P}_{\alpha,s} \geq 2$, $\forall$ DC α.

Our results [A.-Liu-Macarini '2022]

$\# \mathcal{P}_{\alpha,s} \geq 1, \forall \alpha,$ and $\# \mathcal{P}_{\alpha,s} \geq 2, \forall$ DC α.

(III) Main ingredients in the proofs

- β = contact form on $L_p^{2n+1}(l_0, \dots, l_n)$ induced by symmetric α on S^{2n+1}.

  Then, simple symmetric closed R_α-orbits

  $$\Big\updownarrow 1{:}1$$

  $$\boxed{\text{simple closed } R_\beta\text{-orbits which are generators of } \pi_1(L_p^{2n+1}(l_0,\dots,l_n))}$$

- Symplectic cohomology of orbifold filling [F. Gironella - Z. Zhou '2021]

$$\begin{tikzcd} H^*_{CR}(\mathbb{C}^{n+1}/\mathbb{Z}_p;\mathbb{Q}) \arrow[rr] \arrow[dr, "{[1]}"'] & & \underset{\substack{\| \\ 0}}{SH^*(\mathbb{C}^{n+1}/\mathbb{Z}_p)} \arrow[dl] \\ & SH^*_+(\mathbb{C}^{n+1}/\mathbb{Z}_p) & \end{tikzcd}$$

$$\Rightarrow SH^*_+(\mathbb{C}^{n+1}/\mathbb{Z}_p) \simeq H^{*+1}_{CR}(\mathbb{C}^{n+1}/\mathbb{Z}_p;\mathbb{Q}) \simeq$$

$$\simeq \bigoplus_{k=0}^{p-1} \mathbb{Q}\left[-2\sum_{i=0}^{n}\left\{\frac{k l_i}{p}\right\}\right], \quad \{x\} = x - \lfloor x \rfloor$$

$$\Rightarrow \# P_{\alpha,s} \geq 1, \forall \alpha$$

- Equivariant symplectic cohomology of symplectization [Bourgeois-Oancea, 2010's]

$$\begin{tikzcd}
SH^{*+2}(L_p^{2n+1}) \arrow[rr, "{[-1]}"] & & ESH^{*}(L_p^{2n+1}) \arrow[dl, "D"] \\
& ESH^{*+2}(L_p^{2n+1}) \arrow[ul] &
\end{tikzcd}$$

D = shift operator

- Claim 1: $SH^*(L_p^{2n+1}) \simeq SH_+^*(\mathbb{C}^{n+1}/\mathbb{Z}_p)$.

$\left(\cong H_{CR}^{*+1}(\mathbb{C}^{n+1}/\mathbb{Z}_p ; \mathbb{Q}) \right)$

$\Rightarrow D : ESH^{*,a}(L_p^{2n+1}) \longrightarrow ESH^{*+2,a}(L_p^{2n+1})$

is $\cong$, $\forall * \geq 2n$, $a \in \pi_1(L_p^{2n+1})$.

- Claim 2:

$$ESH^{*,a}(L_p^{2n+1}) \cong \begin{cases} \mathbb{Q}, & * = k_a + 2k,\ k \in \mathbb{N}_0 \\ 0, & \text{otherwise} \end{cases}$$

$$k_a = \min\{k \in \mathbb{Q} : ESH^{k,a}(L_p^{2n+1}) \neq 0\} \in \mathbb{Q}$$

- If R_β had only one simple closed orbit $\bar{\gamma}$ with homotopy class a, then Claim 1, Claim 2 and Lusternik-Schnirelmann theory [Ginzburg - Gurel '2020] would imply

$$\frac{1}{\Delta(\bar{\gamma})} \geq P/2 \,,$$

while dynamical convexity implies

$$\frac{1}{\Delta(\bar{\gamma})} < P/2 \,,$$

where

$$\Delta(\bar{\gamma}) := \lim_{j \to +\infty} \frac{\mu_{CZ}(\bar{\gamma}^N)}{N} = \text{mean index.}$$

(IV) Contact homology of Lens spaces

- $H^{2j}(L_p^{2n+1}(l_0,\dots,l_n);\mathbb{Z}) \cong \mathbb{Z}_p$ and

$$\boxed{c_j(\xi_{std}) = \sigma_j \mod p, \quad j=1,\dots,n,}$$

σ_j = j-th elementary symmetric polynomial of $l_0,\dots,l_n$.

- m = smallest positive integer s.t. $m\cdot c_1(\xi)=0$ (note: $m \mid p$)

- $(R_\alpha)^t(z_0,\dots,z_n) = (e^{ia_0t}z_0,\dots,e^{ia_nt}z_n)$,

with $a_0,\dots,a_n$ rationally independent and $a_0\sim 1$, $0<a_1,\dots,a_n \ll 1$, induces Reeb flow on $L_p^{2n+1}(l_0,\dots,l_n)$ with $n+1$ distinct simple periodic orbits $\gamma_0,\gamma_1,\dots,\gamma_n$, such that:

(i) they are all generators of $\pi_1(L_p^{2n+1})$.

(ii) $\mu_{CZ}(\gamma_j^N)$, $j=1,\dots,n$, arbitrarily large for all $N\in\mathbb{N}$.

(iii) $\mu_{CZ}(\gamma_0^N)$ "=" $2\left(\sum_{j=0}^{n}\left\lfloor\frac{-N\ell_j}{p}\right\rfloor + \frac{N}{m}\right) + n$,

$\boxed{K_a = \mu_{CZ}(\gamma_0^a), \quad a = 1, \dots, p,}$ and

$$\boxed{\mu_{CZ}(\gamma_0^{N+p}) = \mu_{CZ}(\gamma^N) + 2}$$ up to

an arbitrarily large $N \in \mathbb{N}$.

Note: definition of μ_{CZ} follows Seidel '2007 and McLean '2016.

- <u>Examples</u>

① $L_p^{2n+1}(1, \dots, 1) \Rightarrow K_a = \frac{2a(n+1)}{p} - n$,

$a = 1, \dots p$ (in particular $K_a \neq K_b$, $a \neq b$).

② $L_p^{2n+1}(1, -1, \dots, 1, -1)$, $p > 2$, n odd

$\Rightarrow K_a = 1$, $a = 1, \dots, p-1$.

- See A.-Macarini-Moreira, Adv. Math. 2023, for contact homology of toric mflds.