On the number of periodic orbits of non-degenerate lacunary contact forms on prequantizations

(joint with L. Macarini)

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Plan:

(I) Motivating example

(II) Main result and applications

(III) Main ingredients in the proofs

(IV) Some natural questions

(I) Motivating example

Diagonal convex Reeb flows on $S^{2n+1} \subset \mathbb{C}^{n+1}$

$$S^{2n+1} := \left\{ z \in \mathbb{C}^{n+1} : \|z\|^2 = \sum_{j=1}^{n+1} |z_j|^2 = 1 \right\}$$

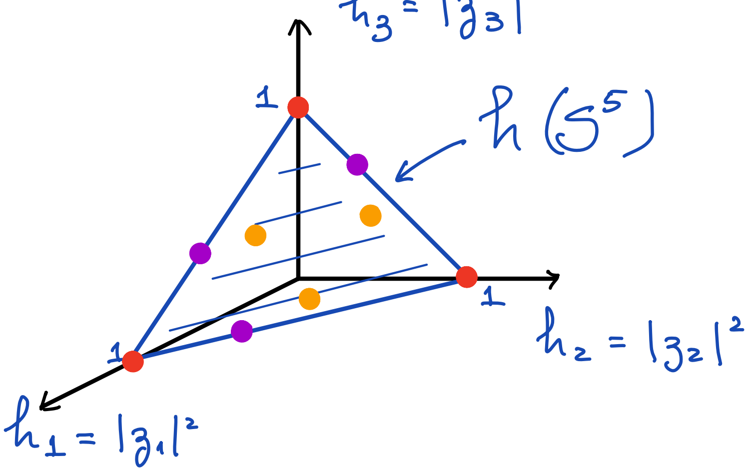
$$R_t : S^{2n+1} \longrightarrow S^{2n+1}, \quad t \in \mathbb{R}$$

$$\boxed{R_t(z_1, \dots, z_{n+1}) = \left(e^{ia_1 t} z_1, \dots, e^{ia_{n+1} t} z_{n+1}\right)}$$

for some given constants $a_1, \dots, a_{n+1} \in \mathbb{R}^+$

Proposition: Any diagonal convex Reeb flow on S^{2n+1} has either $\underline{n+1}$ or $\underline{\infty\text{-many}}$ distinct simple periodic orbits.

Proof:



$h^{-1}(\bullet) = S^1 \times S^1 \times S^1 = \mathbb{T}^3$

$h^{-1}(\bullet) = S^1 \times S^1 = \mathbb{T}^2$

$h^{-1}(\bullet) = S^1 = \mathbb{T}^1$

Conley-Zehnder index of $\bullet$-orbits

Measure of "flow twisting" around $\bullet$-orbit.

For $\mathbb{Q}$-independent $a_1, \dots, a_{n+1}$ and $\bullet$-orbit γ_k that passes at the point $z_k = 1$ & $z_j = 0$, $j \neq k$, we have that (up to a dim'l shift)

$$\boxed{\mu_{CZ}(\gamma_k^N) = 2 \sum_{j=1}^{n+1} \left\lfloor N \frac{a_j}{a_k} \right\rfloor}$$

Proposition: For any $\mathbb{Q}$-independent $a_1, \dots, a_{n+1}$ and even $2m$, $m \in \mathbb{N}$, there exist unique $k \in \{1, \dots, n+1\}$ and $N \in \mathbb{N}$ such that

$$\boxed{\mu_{CZ}(\gamma_k^N) = 2m}.$$

Proof: "exercise" (cf. A.-Macarini-Moreira '23). ∎

Remarks

[eigenvalues of lin. Poincaré map] [all CZ-indices of same parity]

① $n+1$ orbits $\Rightarrow$ nondeg $(\lambda \neq 1)$, elliptic $(|\lambda| = 1)$, lacunary

(2) $a_1 \sim 1,\ a_2, \dots, a_{n+1} \sim \varepsilon \implies$ 1 periodic orbit can generate everything up to an arbitrarily high degree

(3) S^{2n+1} has Reeb flow with all orbits periodic of same period ($a_1 = \cdots = a_{n+1} = 1$)

$$\Downarrow$$

S^{2n+1} is the prequantization of an integral symplectic manifold

$$\begin{array}{ccc} S^1 & \hookrightarrow & (S^{2n+1}, \xi_{std}) \\ & & \downarrow \\ & & (\mathbb{C}P^n, \omega_0) \end{array}, \quad \omega_0(\mathbb{C}P^1) = 1$$

(4)

$$H_*(\mathbb{C}P^n; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{if } * = \text{even and } 0 \le * \le 2n \\ 0, & \text{otherwise} \end{cases}$$

$$\implies \dim H_*(\mathbb{C}P^n; \mathbb{Q}) = n+1$$

(II) Main result and applications

- (M^{2n+1}, ξ) = closed (co-oriented) contact mfld, i.e. $\xi = \mathrm{Ker}(\alpha)$, α = 1-form s.t. $\alpha \wedge (d\alpha)^n \neq 0$. ($\alpha$ contact form $\Rightarrow f\alpha$ contact form, $\forall f > 0$)

- R_α = Reeb vector field ($R_\alpha \lrcorner d\alpha \equiv 0$, $\alpha(R_\alpha) \equiv 1$)

- Examples: starshaped hypersurfaces in $\mathbb{R}^{2(n+1)}$, geodesic flows on unit cosphere bundles S^*N and

- prequantizations of integral symp. mflds:

$$S^1 \hookrightarrow (M^{2n+1}, \xi) \longrightarrow (B^{2n}, \omega)$$

- [Ginzburg - Gürel - Macarini] $\omega|_{\pi_2(B)} = 0$ + minor (technical) assumptions on M $\Rightarrow$ ∞-many closed R_α orbits, $\forall \alpha$.

- $c_B := \inf\{k \in \mathbb{N} : \exists S \in \pi_2(B) \text{ w/ } \langle c_1(TB), S \rangle = k\}$ = minimal Chern number of B

- α lacunary $\equiv$ same parity $\mu_{CZ}(\gamma)$, $\forall$ contractible closed R_α-orbit γ.

- α index-admissible $\equiv$ $\mu_{CZ}(\gamma) > 3-n$, $\forall$ contractible closed R_α-orbit γ.

- Main Theorem Suppose $\omega|_{\pi_2(B)} \neq 0$, $c_B > n/2$ and $H_k(B;\mathbb{Q}) = 0$, $\forall$ odd k.
  Suppose α non-degenerate and lacunary.
  Suppose one of the following:
  (F) M has strong symp. filling W s.t. $c_1(TW)|_{\pi_2(W)} = 0$ and $\pi_1(M) \hookrightarrow \pi_1(W)$.
  (NF) $c_1(\xi)|_{\pi_2(M)} = 0$, $c_1(TB)|_{\pi_2(B)} = \lambda\, \omega|_{\pi_2(B)}$ for some $\lambda > 0$, and α index-admissible.
  Then α has precisely r_B contractible closed orbits, where $r_B = \dim H_*(B;\mathbb{Q})$.

Corollary 1 $(M^{2n+1}, \xi) = (S^{2n+1}, \xi_{std})$ or S^*N, N = CROSS. Any non-deg. lacunary α has precisely r_B simple closed orbits, r_B = table below. [Uses finite π_1.]

Prequantization	$r_B = \dim H_*(B;\mathbb{Q})$	c_B
S^{2n+1}	$n+1$	$n+1$
S^*S^2 or $S^*\mathbb{R}P^2$	2	2
S^*S^m or $S^*\mathbb{R}P^m$, $m>2$ even	m	$m-1$
S^*S^m or $S^*\mathbb{R}P^m$, m odd	$m+1$	$m-1$
$S^*\mathbb{C}P^m$	$m(m+1)$	m
$S^*\mathbb{H}P^m$	$2m(m+1)$	$2m+1$
S^*CaP^2	24	11

Remark 1: Replacing "lacunary" in Main Thm. by a much weaker index condition, Ginzburg-Gürel-Macarini '2018 proved that there are at least r_B contractible closed orbits.

Remark 2: All known (to us) examples of Reeb flows with finitely many closed orbits are non-degenerate and lacunary.

Remark 3: Cor. 1 improves results due to Duan-Liu-Ren '2022, Duan-Xie '2023.

Remark 4: The Motivating Example and the Katok-Ziller Finsler metrics fully illustrate Cor. 1.

Remark 5: Main Theorem also applies to the prequantizations of $Gr_{\mathbb{C}}(2;m)$, $Gr_{\mathbb{C}}(3;6)$, $Gr_{\mathbb{C}}(3;7)$, monotone $\mathbb{C}P^m \times \mathbb{C}P^m$ and monotone $\mathbb{C}P^m \times Gr^+_{\mathbb{R}}(2;m+3)$.

Remark 6: Proof of Main Theorem also applies to $L_p^{2n+1}(\ell_1,\dots,\ell_{n+1})$ and $S^*L_p^{2m-1}(\ell_1,\dots,\ell_m)$.

Recall: $L_p^{2m-1}(\ell_1,\dots,\ell_m) = S^{2m-1}/\mathbb{Z}_p$,
where $\mathbb{Z}_p$-action on S^{2m-1} is gen. by
$\psi(z_1,\dots,z_m) = (e^{\frac{2\pi i \ell_1}{p}} z_1, \dots, e^{\frac{2\pi i \ell_m}{p}} z_m)$
with all weights $\ell_1,\dots,\ell_m$ coprime with $p \in \mathbb{N}$.

Note: $L_p^{2n+1}(\ell_1,\dots,\ell_{n+1})$ and $S^*L_p^{2m-1}(\ell_1,\dots,\ell_m)$ are prequantizations of integral symp. orbifolds (not manifolds in general).

Corollary 2:

Every non-deg. lacunary contact form on $L_p^{2n+1}(\ell_1, \ldots, \ell_{n+1})$ has precisely $n+1$ closed Reeb orbits.

Every non-deg. lacunary contact form on $S^* L_p^{2m-1}(\ell_1, \ldots, \ell_m)$ has precisely $2m$ closed Reeb orbits.

III Main ingredients in the proofs

- Equivariant Symplectic Homology $ESH_*(M,\xi)$

$\equiv$ equivariant Morse homology for the Action Functional $\mathcal{A}(\gamma) = \int_\gamma \alpha$.

Corresponding chain complex is generated by

critical pts of $\mathcal{A}$ $\equiv$ periodic Reeb orbits

and graded by Conley-Zehnder index μ_{CZ}.

For prequantizations of Main Theorem:

$$\boxed{ESH_*^{\circ}(M,\xi) \cong \bigoplus_{m\in\mathbb{N}} H_{*-2mc_B+n}(B;\mathbb{Q})}$$

$$\begin{array}{ccccccccc}
 & & 2c_B & & 4c_B & & \cdots & & \\
\hline
H_0(B) & \cdots & H_n(B) & \cdots & H_{2n}(B) & & & & \\
 & & & & H_0(B) & \cdots & H_n(B) & \cdots & H_{2n}(B) \\
 & & & & & & & & \ddots
\end{array}$$

ESH_*° ≡ equivariant symp. homology of the contractible periodic orbits.

- Two Key Points

α nondeg. + lacunary $\Rightarrow$ $\partial \equiv 0$ in ESH

$$c_B > n/2 \Rightarrow \begin{cases} \min\{k\in\mathbb{Z} : ESH_k \neq 0\} \geq 1 \\ \text{shift } 2c_B > n \end{cases}$$

quite restrictive and likely just technical, but plays a crucial role in our proof

- Common Index Jump Theorem or Index Recurrence Theorem, of Y. Long and collaborators.

- Key point in proof of Cor. 2 (Lens spaces)

$$ESH_*^{\circ}\left(L_p^{2n+1}(\ell_1,\dots,\ell_{n+1})\right) \cong ESH_*^{\circ}(S^{2n+1})$$

$$ESH_*^{\circ}\left(S^* L_p^{2m-1}(\ell_1,\dots,\ell_m)\right) \cong ESH_*^{\circ}\left(S^* S^{2m-1}\right)$$

IV Some natural questions

- $(M,\xi) \xrightarrow{\text{preq.}} (B,\omega) \overset{?}{\Longrightarrow}$ any contact form α has either r_B or ∞-many periodic orbits

[YES on S^3 by the recent work of Cristofaro-Gardiner, Hryniewicz, Hutchings and Liu (arXiv: 2310.07636), seems to be currently out of reach in higher dimensions]

- Contact form with finitely many closed orbits ("Reeb pseudo rotation")

$$\overset{?}{\Downarrow}?$$

all closed orbits are non-degenerate and elliptic (in particular, the contact form is lacunary)

[positive answer to this question + Main Theorem

$$\Downarrow$$

(partial) positive answer to previous question]

- "Reeb pseudo-rotation" $\overset{?}{\Longrightarrow}$ prequant. of an orbifold, i.e. has Reeb flow with all orbits periodic