Dynamical Convexity and Elliptic Orbits for Reeb Flows

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Workshop on Conservative Dynamics and Symplectic Geometry

IMPA, Rio de Janeiro, September 2–6, 2013

Introduction: basic setup

- $(M^{2n-1}, \xi = \ker \alpha)$ contact manifold, that is, $\alpha \wedge (d\alpha)^{n-1}$ is a volume form.
- Examples: starshaped hypersurfaces in $\mathbb{R}^{2n}$ and the unit sphere bundle of a closed Riemannian manifold.
- Let R_α be the Reeb vector field uniquely characterized by $\alpha(R_\alpha) = 1$ and $i_{R_\alpha} d\alpha = 0$.
- If M is the energy level of a Hamiltonian then the corresponding Hamiltonian flow is a reparametrization of the Reeb flow.

Introduction: existence of elliptic periodic orbits

- Problem: existence of elliptic periodic orbits for the Reeb flow.
- A periodic orbit is called elliptic if every eigenvalue of its linearized Poincaré map is in the unit circle.
- The existence of elliptic orbits has several dynamical consequences: under generic conditions it implies the presence of KAM tori, transversal homoclinic connections and positivity of the topological entropy.
- Classical conjecture: every convex hypersurface in $\mathbb{R}^{2n}$ carries an elliptic periodic orbit.
- Unfortunately, with this degree of generality, this is far from being known.

Introduction: goal

Theorem (Dell'Antonio-D'Onofrio-Ekeland'1995)

If $M \subset \mathbb{R}^{2n}$ is convex and invariant by the antipodal map then it carries an elliptic closed orbit.

Our goal: understand and generalize this result using contact homology.

Conley-Zehnder Index

- Suppose, for simplicity, that $\xi = \ker \alpha$ has a global trivialization $\Phi : \xi \to M \times \mathbb{R}^{2n-2}$.
- Then the linearized Reeb flow along a periodic orbit γ defines a path $\Gamma : [0, T] \to Sp(2n-2)$ starting at the identity.
- Let $\mathcal{M} \subset Sp(2n-2)$ be the Maslov cycle, that is, the subset of symplectic linear maps A such that $\det(A - Id) = 0$.
- This is a stratified submanifold of codimension one.
- Then one can associate to Γ an intersection number with $\mathcal{M}$, called the Conley-Zehnder index of γ and denoted by $\mu_{CZ}(\gamma)$.

Contact Homology

- Contact homology is a Morse homology for the action functional $A_\alpha(\gamma) = \int_\gamma \alpha$. The chain complex is generated by the periodic orbits of R_α graded by the Conley-Zehnder index and the differential counts rigid finite energy pseudo-holomorphic cylinders in the symplectization.

- The chain complex depends on the contact form α but contact homology is an invariant of the contact structure ξ.

Dynamical Convexity: S^3

- **Definition. (Hofer-Wysocki-Zehnder'1998)** A contact form α on S^3 is dynamically convex if every periodic orbit γ of R_α satisfies $\mu_{CZ}(\gamma) \geq 3$.
- **Theorem. (HWZ)** The contact form induced on a convex hypersurface in $\mathbb{R}^4$ is dynamically convex.
- Note that, in contrast to convexity, dynamical convexity is invariant by contactomorphisms.
- HWZ proved that the Reeb flow of a dynamically convex contact form admits global sections given by the pages of an open book decomposition (it works only in dim 3!).

Dynamical Convexity: S^{2n-1}

- The proof of HWZ shows that every periodic orbit γ on a convex hypersurface in $\mathbb{R}^{2n}$ satisfies $\mu_{CZ}(\gamma) \geq n + 1$.

- It turns out that the term $n + 1$ has an important meaning: it corresponds to the lowest CZ-degree with non-trivial contact homology. Indeed, a computation shows that

$$HC_*(S^{2n-1}) \cong \begin{cases} \mathbb{Q} & \text{if } * = n + 2k + 1 \text{ and } k \in \mathbb{N}_0 \\ 0 & \text{otherwise.} \end{cases}$$

Dynamical Convexity: general contact manifolds

Definition.
Let a be a free homotopy class in M and define

$$k_- = \inf\{k \in \mathbb{Z}; HC_k^a(M) \neq 0\},\ k_+ = \sup\{k \in \mathbb{Z}; HC_k^a(M) \neq 0\}.$$

A contact form α is positively (resp. negatively) a-dynamically convex if k_- is an integer and $\mu_{CZ}(\gamma) \geq k_-$ (resp. k_+ is an integer and $\mu_{CZ}(\gamma) \leq k_+$) for every periodic orbit γ of R_α with free homotopy class a.

Main Result: preliminaries

- A contact manifold (M, ξ) is called Boothby-Wang if it supports a contact form β whose Reeb flow generates a free circle action.
- Example: spheres.
- Given a Boothby-Wang contact manifold $(M, \xi = \ker \beta)$, an arbitrary contact form α and a finite subgroup $G \subset S^1$, we say that α is G-invariant if $(\varphi_\beta^{t_0})^* \alpha = \alpha$, where φ_β^t is the flow of R_β and $t_0 \in S^1$ is a generator of G.

Main Result: statement

Theorem. (A.-Macarini'2013)

Let $(M, \xi = \ker \beta)$ be a Boothby-Wang contact manifold and G a non-trivial finite subgroup of S^1. Let a be the free homotopy class of the (simple) closed orbits of R_β and assume that one of the following two conditions holds:

1. M/S^1 admits a Morse function such that every critical point has even Morse index;
2. $a^j \neq 0$ for every $j \in \mathbb{N}$.

Then every G-invariant positively (resp. negatively) a-dynamically convex contact form α supporting ξ has an elliptic closed orbit γ with free homotopy class a. Moreover, $\mu_{CZ}(\gamma) = k_-$ (resp. $\mu_{CZ}(\gamma) = k_+$).

Applications and Examples: geodesic flows

Corollary.
Let g be a Riemannian metric on S^2 with sectional curvature K satisfying $1/4 \leq K \leq 1$. Then g carries an elliptic closed geodesic γ. Moreover, γ is contractible in SS^2 and satisfies $\mu_{CZ}(\gamma) = 3$.

In fact, Harris-Paternain proved that if $1/4 < K \leq 1$ then the geodesic flow on $SS^2 \simeq \mathbb{R}P^3$ lifts to a (positively) dynamically convex contact form on S^3. An easy perturbation argument implies the result above where the pinching condition is not strict.

Applications and Examples: geodesic flows

- Contreras-Oliveira proved that C^2**-densely** a Riemannian metric on S^2 has an elliptic closed geodesic. They used the global sections constructed by HWZ.

- Ballmann-Thorbergsson-Ziller proved the previous corollary using different methods.

Applications and Examples: magnetic flows

- Let (N, g) be a Riemannian manifold with a closed 2-form Ω.
- Let ω_0 be the pullback of the canonical symplectic form of T^*N to TN via g and consider the symplectic form $\omega = \omega_0 + \pi^*\Omega$, where $\pi : TN \to N$ is the projection.
- The Hamiltonian flow of $H(x, v) = \frac{1}{2}g(v, v)$ is the magnetic flow associated to the pair (g, Ω).

Applications and Examples: magnetic flows

- G. Benedetti proved in his thesis that if N is a closed orientable surface of genus $\mathfrak{g} \neq 1$ and Ω is a symplectic form then there exists $c > 0$ such that $H^{-1}(k)$ is of contact type for every $k < c$.
- He also proved that if $\mathfrak{g} = 0$ then the lift to S^3 is positively dynamically convex and one can prove, using his thesis and some contact homology computations, that if $\mathfrak{g} > 1$ then there is a $|\chi(N)|$-covering $\widetilde{M} \to H^{-1}(k)$ such that the lift of the magnetic flow to $\widetilde{M}$ is negatively dynamically convex. Moreover, $\widetilde{M}$ is Boothby-Wang.

Applications and Examples: magnetic flows

Corollary.

Let (N, g) be a closed orientable Riemannian surface of genus $\mathfrak{g} \neq 1$ and Ω a symplectic magnetic field on N. Then the magnetic flow has an elliptic closed orbit γ on every sufficiently small energy level. Moreover, γ is freely homotopic to a $|\chi(N)|$-covering of the fiber of SN and satisfies $\mu_{CZ}(\gamma) = 3$ if $\mathfrak{g} = 0$ and $\mu_{CZ}(\gamma) = 2\chi(N) + 1$ otherwise.

Applications and Examples: toric contact manifolds

- Toric contact manifolds can be defined as contact manifolds of dimension $2n-1$ equipped with an effective Hamiltonian action of a torus of dimension n.
- A good toric contact manifold has the property that its symplectization can be obtained by symplectic reduction of $\mathbb{C}^d$, where d is the number of facets of the corresponding convex polyhedral cone, by the action of a subtorus $K \subset \mathbb{T}^d$, with the action of $\mathbb{T}^d$ given by the standard linear one.
- The sphere S^{2n-1} is an example of a good toric contact manifold and its symplectization is obtained from $\mathbb{C}^n$ with K being trivial (that is, there is no reduction at all; the symplectization of S^{2n-1} can be identified with $\mathbb{C}^n \setminus \{0\}$).

Applications and Examples: toric contact manifolds

- Consequently, given a contact form α on a good toric contact manifold *M* we can always find a Hamiltonian $H_\alpha : \mathbb{C}^d \to \mathbb{R}$ invariant by *K* such that the reduced Hamiltonian flow of *H* is the Reeb flow of α.
- Notice that H_α is not unique.
- We say that a contact form α on *M* is convex if such H_α can be chosen convex.
- Clearly, in the case of the sphere this holds if and only if the corresponding hypersurface in $\mathbb{C}^n$ is convex.

Theorem. (A.-Macarini'2013)

A convex contact form on a good toric simply-connected contact manifold is positively dynamically convex.