Remarks on Lagrangian Intersections in Toric Manifolds

Miguel Abreu
(joint with Leonardo Macarini - UFRJ, arXiv:1105.0640)

Center for Mathematical Analysis, Geometry and Dynamical Systems
Instituto Superior Técnico, Lisbon, Portugal

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Introduction

- Lagrangian intersection questions play a fundamental role in symplectic topology.
- Two examples:
  - Arnold Conjectures.
  - Fukaya Category and Mirror Symmetry.
- Main tool: Floer (co)homology.
- Technical difficulties: cf. two-volume work of Fukaya-Oh-Ohta-Ono (FOOO), 2009 $\sim$ 800 pages.

Introduction

- Toric manifolds provide examples that are both interesting and tractable.
- FOOO:
  - four preprints in arXiv since 2008 $\sim$ 450 pages,
  - survey (2010) $\sim$ 60 pages.
- Woodward (2010) $\sim$ 70 pages.
- Simplest toric example: $(S^2, \sigma_{\text{area}})$ where the equator is a non-displaceable Lagrangian submanifold and all other S^1-orbits are displaceable.

Toric Symplectic Manifolds

- A toric symplectic manifold is a connected symplectic $2n$-manifold (M, ω), equipped with an effective Hamiltonian action of the n-torus: $\mathbb{T}^n \hookrightarrow \mathrm{Ham}(M^{2n}, \omega)$.
- The corresponding moment map, unique up to an additive constant, will be denoted by $\mu : M \to \mathbb{R}^n$.
- Atiyah-Guillemin-Sternberg'82: the image of the moment map is the convex polytope given by the convex hull of the images of the fixed points of the action. This will be called the moment polytope and denoted by $P := \mu(M) \subset \mathbb{R}^n$.
- Delzant'82: the moment polytope is a complete invariant of a compact toric symplectic manifold.

Moment Polytopes

The moment polytope $P \subset \mathbb{R}^n$ of a toric symplectic manifold (M^{2n}, ω) can be defined by

$$x \in P \Leftrightarrow \ell_i(x) := \langle x, \nu_i \rangle + a_i \geq 0\,,\ i = 1, \ldots, d,$$

where

- d is the number of facets of P,
- each vector $\nu_i \in \mathbb{Z}^n$ is the primitive integral interior normal to the facet F_i of P,
- the a_i's are real numbers that determine $[\omega] \in H^2(M; \mathbb{R})$.

(M, ω) is monotone, i.e. $[\omega] = \lambda(2\pi c_1(\omega)) \in H^2(M; \mathbb{R})$ with $\lambda \in \mathbb{R}^+$, iff $P \subset \mathbb{R}^n$ can be defined as above with

$$a_1 = \cdots = a_d = \lambda\,.$$

Such a P will be called a monotone polytope and note that, in this case, $0 \in P$.

Lagrangian Intersections in Toric Manifolds

- Torus Fibers: $x \in \check{P} :=$ interior$(P) \Rightarrow T_x := \mu^{-1}(x) \cong \mathbb{T}^n \equiv$ Lagrangian submanifold of (M,ω). When P is monotone T_0 is called the centered, special or monotone torus fiber.
- Real Part: $\tau : M \to M\,,\ \tau^2 = \text{id}\,,\ \tau^*\omega = -\omega \Rightarrow R := M^\tau \equiv$ Lagrangian submanifold of (M,ω).
- Explicit description in action-angle coordinates:

$$\check{M} \equiv \mu^{-1}(\check{P}) \equiv \{\text{points of } M \text{ where } \mathbb{T}^n\text{-action is free}\}$$
$$\cong \check{P} \times \mathbb{T}^n = (x_1, \ldots, x_n) \times (e^{i\theta_1}, \ldots, e^{i\theta_n}) \text{ such that}$$
$$\omega|_{\check{M}} = dx \wedge d\theta\,,\ \mu(x, e^{i\theta}) = x \text{ and } \tau(x, e^{i\theta}) = (x, e^{-i\theta})$$

$$\Rightarrow R \cap \check{M} = (x, \pm 1) \quad \text{and} \quad \sharp(R \pitchfork T_x) = 2^n\,,\ \forall x \in \check{P}\,.$$

- Question: does there exist $\phi \in \text{Ham}(M,\omega)$ such that

$$\phi(T_x) \cap T_x = \emptyset \quad \text{or} \quad \phi(T_x) \cap R = \emptyset \quad \text{or} \quad \phi(R) \cap R = \emptyset \ ?$$

Example

$(\mathbb{C}\mathrm{P}^n, \omega_{FS})$ is a monotone toric symplectic manifold whose moment polytope $P \subset \mathbb{R}^n$ is the simplex given by

$$P = \{(x_1, \dots, x_n) \;:\; x_j + 1 \geq 0\,,\; j = 1, \dots, n;\; -\sum_{j=1}^{n} x_j + 1 \geq 0\}.$$

Moreover

$$R = \mathbb{R}\mathrm{P}^n \quad \text{and} \quad T_0 = \text{ Clifford torus.}$$

Biran-Entov-Polterovich and Cho, 2003:

$$\psi(T_0) \cap T_0 \neq \emptyset \quad \text{and} \quad \sharp(\psi(T_0) \pitchfork T_0) \geq 2^n\,,\; \forall\, \psi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^n, \omega_{FS})\,.$$

Entov-Polterovich'07, Biran-Cornea'08 and Tamarkin'09:

$$\psi(T_0^n) \cap \mathbb{R}\mathrm{P}^n \neq \emptyset\,,\; \forall\, \psi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^n, \omega_{FS})\,.$$

Alston'09:

$$\sharp(\psi(T_0^{2n-1}) \pitchfork \mathbb{R}\mathrm{P}^{2n-1}) \geq 2^n\,,\; \forall\, \psi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^{2n-1}, \omega_{FS})\,.$$

Symplectic Reduction Remark

- $(\tilde{M}, \tilde{\omega})$: $\tilde{\mathbb{T}}$-manifold of dimension $2N$, with moment map $\tilde{\mu} : \tilde{M} \to (\mathbb{R}^N)^*$.
- $K \subset \tilde{\mathbb{T}}$: subtorus of dimension N-n, determined by inclusion of Lie algebras $\iota : \mathbb{R}^{N-n} \to \mathbb{R}^N$.
- Induced action of K on $\tilde{M}$ has moment map $\tilde{\mu}_K = \iota^* \circ \tilde{\mu} : \tilde{M} \to (\mathbb{R}^{N-n})^*$.
- $c \in \tilde{\mu}_K(\tilde{M}) \subset (\mathbb{R}^{N-n})^*$ regular value and consider level set $Z := \tilde{\mu}_K^{-1}(c) \subset \tilde{M}$.
- $(M := Z/K, \omega)$ is $\mathbb{T} := \tilde{\mathbb{T}}/K$-manifold of dimension $2n$, with moment map $\mu : M \to P \subset (\mathbb{R}^n)^* \cong \ker(\iota^*)$ characterized by

$$\begin{array}{ccc} \tilde{M} \supset Z & \xrightarrow{\tilde{\mu}} & \tilde{P} \subset (\mathbb{R}^N)^* \\ \pi \downarrow & & \uparrow \\ M & \xrightarrow{\mu} & P \subset (\mathbb{R}^n)^* \end{array}$$

Symplectic Reduction Remark (cont.)

- $T_x := \mu^{-1}(x)\,,\ x \in \mathrm{int}(P) \subset \mathrm{int}(\tilde{P}) \ \Rightarrow\ \pi^{-1}(T_x) = \tilde{T}_x$.
- $\tilde{\tau}(Z) = Z$, $Z^{\tilde{\tau}} = Z \cap \tilde{R}$ and $\tilde{\tau}$ induces $\tau : M \to M$ via $\pi \circ \tilde{\tau} = \tau \circ \pi$.
- For $p \in R := M^{\tau}$ a simple counting argument shows that $\sharp(\pi^{-1}(p) \cap \tilde{R}) = 2^{N-n}$.

Lemma

Given $\psi \in \mathrm{Ham}(M, \omega)$ *there is* $\tilde{\psi} \in \mathrm{Ham}(\tilde{M}, \tilde{\omega})$ *such that* $\tilde{\psi}(Z) = Z$ *and* $\pi(\tilde{\psi}(\tilde{p})) = \psi(\pi(\tilde{p}))\,,\ \forall\, \tilde{p} \in Z$*.*

Symplectic Reduction Remark (cont.)

Proposition

$\psi, \tilde{\psi}$ as in Lemma and $x \in \text{int}(P) \subset \text{int}(\tilde{P})$. Then

(i) $\sharp(\psi(T_x) \cap R) = r \Rightarrow \sharp(\tilde{\psi}(\tilde{T}_x) \cap \tilde{R}) = r2^{N-n}$.

(ii) $\psi(T_x) \cap T_x = \emptyset \Rightarrow \tilde{\psi}(\tilde{T}_x) \cap \tilde{T}_x = \emptyset$.

Moreover

(iii) $\sharp(\psi(T_x) \pitchfork T_x) = r \Rightarrow \exists\, \tilde{\varphi} \in \text{Ham}(\tilde{M}, \tilde{\omega})$:
$\sharp(\tilde{\varphi}(\tilde{T}_x) \pitchfork \tilde{T}_x) = r2^{N-n}$.

Remark

Dusa McDuff's probes, so far the only displaceability tool for torus fibers, can be thought of as a particular case of (ii).

Symplectic Reduction Remark (cont.)

Corollary

(i) *If* $\sharp(\tilde{\psi}(\tilde{T}_x) \cap \tilde{R}) \geq m$ *for any* $\tilde{\psi} \in \mathrm{Ham}(\tilde{M}, \tilde{\omega})$ *then*

$$\sharp(\psi(T_x) \cap R) \geq \frac{m}{2^{N-n}}\,, \ \forall\, \psi \in \mathrm{Ham}(M, \omega)\,.$$

(ii) $\tilde{T}_x \subset (\tilde{M}, \tilde{\omega})$ *non-displ.* $\Rightarrow$ $T_x \subset (M, \omega)$ *non-displ.*

(iii) *If* $\sharp(\tilde{\psi}(\tilde{T}_x) \pitchfork \tilde{T}_x) \geq m$ *for any* $\tilde{\psi} \in \mathrm{Ham}(\tilde{M}, \tilde{\omega})$ *then*

$$\sharp(\psi(T_x) \pitchfork T_x) \geq \frac{m}{2^{N-n}}\,, \ \forall\, \psi \in \mathrm{Ham}(M, \omega)\,.$$

Remark

Tamarkin uses similar idea in his work on microlocal analysis of sheaves and Lagrangian intersections. Also present in the work of Borman on reduction properties of quasi-morphisms and quasi-states.

Applications - 1

- Alston: $\sharp(\psi(T_0^{2n-1}) \pitchfork \mathbb{R}\mathrm{P}^{2n-1}) \geq 2^n$, $\forall \psi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^{2n-1})$.
- $\mathbb{C}\mathrm{P}^{2n}$ obtained as centered reduction of $\mathbb{C}\mathrm{P}^{2n+1}$

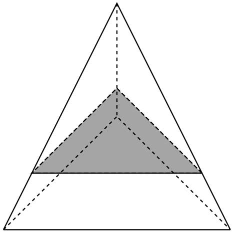


Figure: $\mathbb{C}\mathrm{P}^2$ as reduction of $\mathbb{C}\mathrm{P}^3$.

$$\Rightarrow \sharp(\psi(T_0^{2n}) \pitchfork \mathbb{R}\mathrm{P}^{2n}) \geq \frac{2^{n+1}}{2} = 2^n \quad \text{for any } \psi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^{2n}).$$

Applications - 2

Toric (M^{2n}, ω), moment polytope $P \subset \mathbb{R}^n$, normals $\nu_1, \ldots, \nu_d \in \mathbb{Z}^n$.

- $\sum \nu_i = 0 \Rightarrow M$ is symplectic reduction of $\mathbb{C}\mathrm{P}^{d-1}$.
- (M^{2n}, ω) monotone $\Rightarrow$ reduction is centered.

Proposition

Under these two conditions, T_0 and the pair (R, T_0) are non-displaceable. Moreover,

$$\sharp(\psi(T_0) \pitchfork T_0) \geq 2^n \, , \; \forall \, \psi \in \mathrm{Ham}(M, \omega) \, .$$

Remark

Can remove zero-sum condition by using weighted projective spaces (with work of Woodward and Cho-Poddar), obtaining general result due to Entov-Polterovich, Cho and FOOO.

Cartesian Product Remarks

- $L_1, L_2 \subset (M, \omega)$ Lagrangian submanifolds.

  Non-trivial intersection properties of the pair $(L_1 \times L_1, L_2 \times L_2)$ in $(M \times M, \omega \times \omega)$ give rise to non-trivial intersection properties of the pair (L_1, L_2) in (M, ω).

  The extension of Alston's result on $\mathbb{C}\mathrm{P}^{2n-1}$ to $\mathbb{C}\mathrm{P}^{2n}$ can also be proved using this simple observation (A.-Macarini).

- If T_i or the pair (T_i, R_i) have non-zero Floer homology or are non-displaceable in (M_i, ω_i), $i = 1, 2$, then the same should be true for the corresponding $T_1 \times T_2$ and $(T_1 \times T_2, R_1 \times R_2)$ in $(M_1 \times M_2, \omega_1 \times \omega_2)$.

  OK for Lagrangian torus fibers in toric symplectic manifolds (e.g. using Woodward's set-up).

Applications - 3

Proposition

(M^{2n}, ω) monotone toric manifold with symmetric polytope $P \subset \mathbb{R}^n$, i.e. if $\nu \in \mathbb{Z}^n$ normal to a facet of P then $-\nu$ is also the normal to a facet of P. Then $\sharp(\psi(T_0) \pitchfork R) \geq 2^n$ for any $\psi \in \mathrm{Ham}(M)$ and this bound is optimal.

Proof.

P symmetric implies that M can be obtained as reduction of a product of d copies of $\mathbb{C}\mathrm{P}^1$, $2d =$ number of facets of P.
M monotone implies that $\mathbb{C}\mathrm{P}^1$'s have the same area and reduction is centered. Hence, we get from previous remarks

$$\sharp(\psi(T_0) \pitchfork R) \geq \frac{2^d}{2^{d-n}} = 2^n \,, \ \forall \psi \in \mathrm{Ham}(M, \omega)\,.$$

Since $\sharp(T_0 \pitchfork R) = 2^n$ the bound is indeed optimal. $\square$

Applications - 3 (cont.)

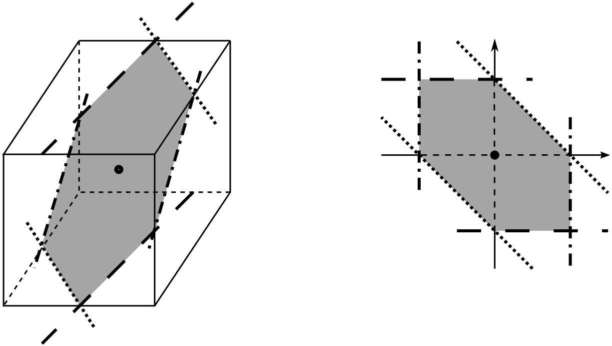
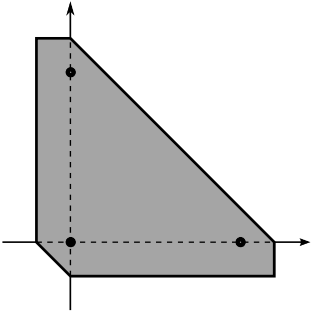
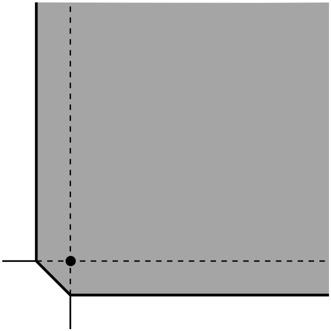


Figure: $\mathbb{C}P^2 \sharp 3 \overline{\mathbb{C}P}^2$ as reduction of $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$.

One More Ingredient

In $\mathcal{O}(-1) \to \mathbb{C}\mathrm{P}^1$, the special torus T_0 sitting over the origin in the polygon on the right side of the figure is non-displaceable. This follows from result of Cho (left side of figure) and was also directly proved by Woodward.





Application - 4: $M = \mathbb{C}P^2 \sharp \overline{\mathbb{C}P}^2$ (blow-up of $\mathbb{C}P^2$)

Two non-displaceable fibers when exceptional divisor is small.

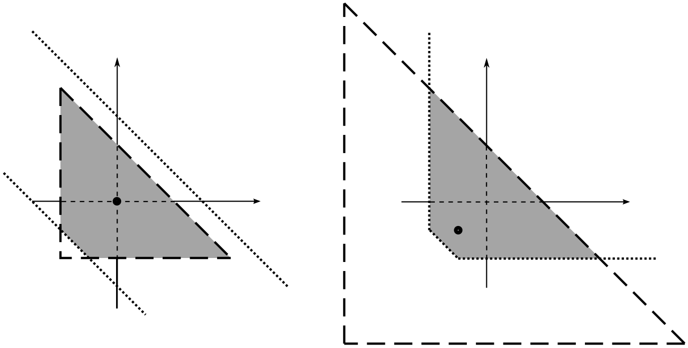


Figure: reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1$ (left) and of $\mathcal{O}(-1) \times \mathbb{C}P^2$ (right).

Application - 4: $M = \mathbb{C}P^2 \sharp \overline{\mathbb{C}P}^2$ (blow-up of $\mathbb{C}P^2$)

One non-displaceable fiber when exceptional divisor is big.

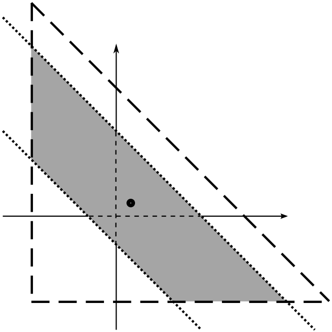


Figure: reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1$, now with $\mathbb{C}P^2$ “bigger" than $\mathbb{C}P^1$.

Application - 5: $M = \mathbb{C}\mathrm{P}^2 \sharp 2\overline{\mathbb{C}\mathrm{P}}^2$

FOOO: with blow-ups of different sizes, one small and the other big, get closed segment of non-displaceable fibers.

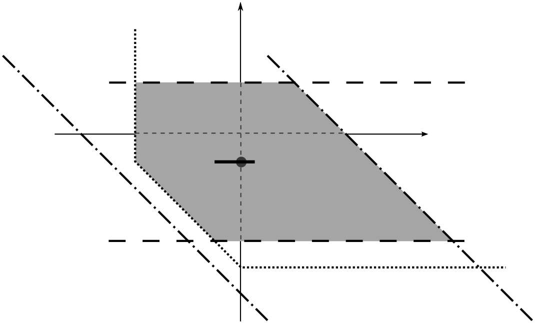


Figure: reduction of $\mathcal{O}(-1) \times \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$.

One More Ingredient

Woodward and Cho-Poddar: the special torus fiber of a weighted projective space is non-displaceable.

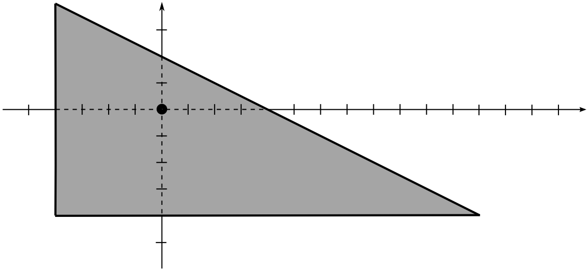


Figure: $\mathbb{C}P(1,1,2)$.

Application - 6: Hirzebruch surfaces

FOOO: any Hirzebruch surface has at least one non-displaceable fiber.

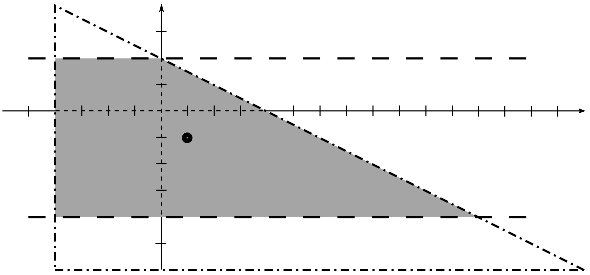


Figure: Hirzebruch surface H_2 as reduction of $\mathbb{C}P(1,1,2) \times \mathbb{C}P^1$.

Remark: Hirzebruch surfaces include non-Fano examples.

Application - 7: $M = H_2 \sharp \overline{\mathbb{C}\mathrm{P}}^2$ (non-Fano)

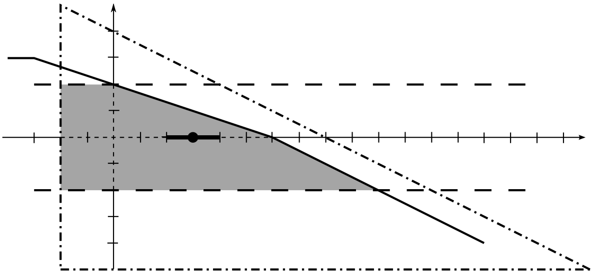


Figure: reduction of $\mathbb{C}\mathrm{P}(1,1,2) \times \mathbb{C}\mathrm{P}^1 \times \mathcal{O}(-1)$.

McDuff: this gives rise, under a repeated wedge construction, to a monotone toric 12-manifold with a segment of fibers that have vanishing Floer homology and cannot be displaced by probes. Still unknown if these are, in fact, (non-)displaceable.

Beyond Probes (A. – Borman – McDuff)

Observation 1: odd Hirzebruch surfaces have segment of fibers that cannot be displaced by probes.

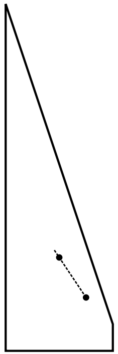


Figure: Hirzebruch surface H_3.

Beyond Probes (A. – Borman – McDuff)

Observation 2: use Karshon's S^1-equiv. symplectomorphisms $H_{\text{odd}} \to H_1 = \mathbb{C}P^2 \sharp \overline{\mathbb{C}P}^2$ to prove that fibers with $HF = 0$ are, in fact, displaceable.

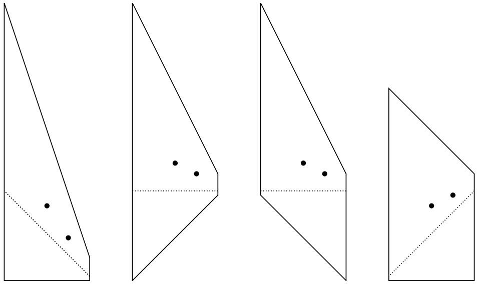


Figure: H_3 - left dot is unique non-displaceable fiber.

In progress: generalizations of this example to "probe" further the set of displaceable fibers on toric symplectic manifolds.