Periodic Orbits of Reeb Flows on Odd Dimensional Spheres

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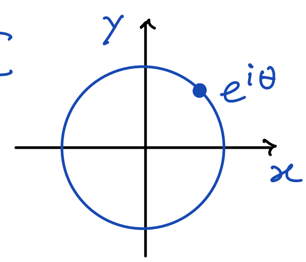
(I) Periodic Orbits of Linear Flows on Tori

- $\mathbb{T}^n = \underbrace{S^1 \times \dots \times S^1}_{n \text{ times}}$ with coordinates $(\theta_1, \dots, \theta_n)$

$\mathbb{R}^2 \cong \mathbb{C}$



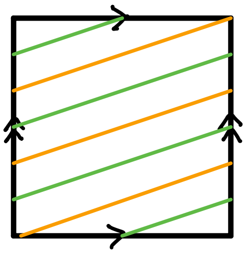
Linear flows on $\mathbb{T}^n$:

$F_t : \mathbb{T}^n \longrightarrow \mathbb{T}^n$, $t \in \mathbb{R}$

$$\boxed{F_t(\theta_1, \dots, \theta_n) = (\theta_1 + a_1 t, \dots, \theta_n + a_n t)}$$

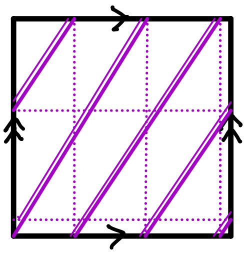
for some given constants $a_1, \dots, a_n \in \mathbb{R}$

- <u>Example</u> $(n=2)$



rational slope $m = \frac{a_2}{a_1}$ $\left(= \frac{1}{3}\right)$

∞-many periodic orbits



irrational slope $m = \frac{a_2}{a_1}$ $\left(= \sqrt{2}\right)$

no periodic orbits

<u>Exercise</u>: state and prove a proposition regarding the number of periodic orbits of a linear flow on $\mathbb{T}^n$.

(II) Periodic Orbits of Diagonal Linear Convex Flows on $S^{2n+1} \subset \mathbb{R}^{2n+2}$

$$\mathbb{R}^{2n+2} = \mathbb{C}^{n+1} \supset S^{2n+1} := \left\{ z \in \mathbb{C}^{n+1} : \|z\|^2 = \sum_{j=1}^{n+1} |z_j|^2 = 1 \right\}$$

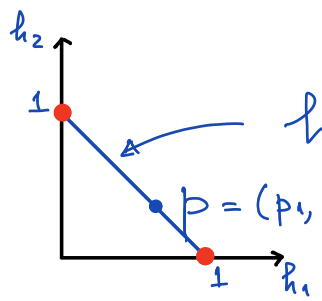
$$R_t : S^{2n+1} \longrightarrow S^{2n+1}, \quad t \in \mathbb{R}$$

$$\boxed{R_t(z_1, \ldots, z_{n+1}) = \left(e^{ia_1 t} z_1, \ldots, e^{ia_{n+1} t} z_{n+1}\right)}$$

for some given constants $a_1, \ldots, a_{n+1} \in \mathbb{R}^+$

- <u>Example</u> $(n=1)$ $\quad S^3 \subset \mathbb{C}^2 = \mathbb{R}^4$

$$h : \mathbb{C}^2 \longrightarrow \mathbb{R}^2$$
$$(z_1, z_2) \longmapsto h(z_1, z_2) = (|z_1|^2, |z_2|^2)$$



$$\boxed{h \circ R_t = h}$$

$$h^{-1}(p) = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 = p_1 \text{ and } |z_2|^2 = p_2 \}$$
$$= S^1 \times S^1 = \mathbb{T}^2, \quad \text{whenever } p_1, p_2 > 0.$$

$R_t \big|_{h^{-1}(p)}$ is a linear flow on $\mathbb{T}^2$.

Hence, as before, it has either

(i) <u>∞-many</u> periodic orbits $\left(a_2/a_1 \in \mathbb{Q} \right)$

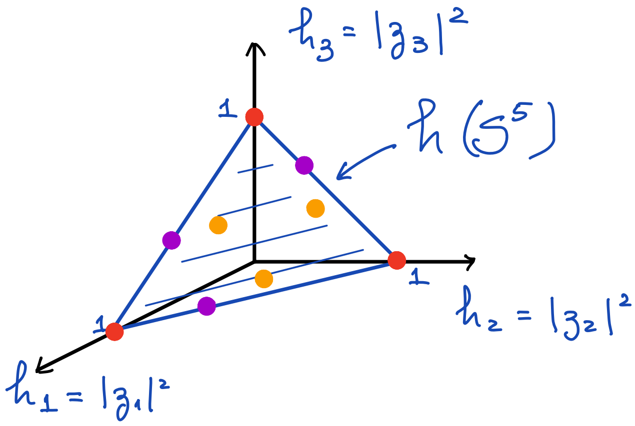
(ii) <u>no</u> periodic orbits $\left(a_2/a_1 \in \mathbb{R} \setminus \mathbb{Q} \right)$

<u>But</u>, note that

$$h^{-1}(\bullet = (1,0)) = S^1 \quad \text{and} \quad h^{-1}(\bullet = (0,1)) = S^1$$

are always <u>2 distinct</u> periodic orbits.

- Example $(n=2)$ $S^5 \subset \mathbb{C}^3 = \mathbb{R}^6$



$h^{-1}(\bullet) = S^1 \times S^1 \times S^1 = \mathbb{T}^3$

$h^{-1}(\bullet) = S^1 \times S^1 = \mathbb{T}^2$

$h^{-1}(\bullet) = S^1 = \mathbb{T}^1$

- Proposition: *Any diagonal linear convex flow on S^{2n+1} has either $n+1$ or ∞-many distinct periodic orbits.*

- Conley-Zehnder index of $\bullet$-orbits

Measure of "flow twisting" around $\bullet$-orbit.

For $\mathbb{Q}$-independent $a_1, \ldots, a_{n+1}$ and $\bullet$-orbit γ_k that passes at the point $z_k = 1$ & $z_j = 0$, $j \neq k$, we have that (up to a dim'l shift)

$$\boxed{\mu_{CZ}(\gamma_k^N) = 2 \sum_{j=1}^{n+1} \left\lfloor N \frac{a_j}{a_k} \right\rfloor}$$

- <u>Example</u> $(n=1)$ $\quad S^3 \subset \mathbb{C}^2 = \mathbb{R}^4$

(i) $a_1 = 1$, $\quad a_2 = 1+\varepsilon$

$$\mu_{CZ}(\gamma_1^N) = 2\left(\lfloor N\rfloor + \lfloor N(1+\varepsilon)\rfloor\right) = 4N \quad \text{for } N < 1/\varepsilon$$

$$\mu_{CZ}(\gamma_2^N) = 2\left(\left\lfloor N\tfrac{1}{1+\varepsilon}\right\rfloor + \lfloor N\rfloor\right) = 4N-2 \quad \text{for } N-1 < 1/\varepsilon$$

Suppose $10 < \frac{1}{\varepsilon} < 11$. Then

$$\mu_{CZ}(\gamma_1^9) = 36, \quad \mu_{CZ}(\gamma_1^{10}) = 40, \quad \mu_{CZ}(\gamma_1^{11}) = 46$$

$$\mu_{CZ}(\gamma_2^{10}) = 38, \quad \mu_{CZ}(\gamma_2^{11}) = 42, \quad \mu_{CZ}(\gamma_2^{12}) = 44$$

(ii) $a_1 = 1$, $\quad a_2 = \varepsilon$

$$\mu_{CZ}(\gamma_1^N) = 2\left(\lfloor N\rfloor + \lfloor N\varepsilon\rfloor\right) = 2N \quad \text{for } N < 1/\varepsilon$$

$$\mu_{CZ}(\gamma_2^N) = 2\left(\left\lfloor N\tfrac{1}{\varepsilon}\right\rfloor + \lfloor N\rfloor\right) = 2N + 2\left\lfloor \tfrac{N}{\varepsilon}\right\rfloor$$

Suppose $10 < \frac{1}{\varepsilon} < 11$. Then

$$\mu_{CZ}(\gamma_1^{10}) = 20, \quad \mu_{CZ}(\gamma_1^{11}) = 24, \quad \mu_{CZ}(\gamma_2^{1}) = 22$$

- <u>Proposition</u>: For any $\mathbb{Q}$-independent $a_1, \dots, a_{n+1}$ and even $2m$, $m \in \mathbb{N}$, there exist unique $k \in \{1, \dots, n+1\}$ and $N \in \mathbb{N}$ such that

$$\boxed{\mu_{CZ}(\gamma_k^N) = 2m}.$$

III Reeb flows on S^{2n+1}

- $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}, \quad (x,y) \longmapsto H(x,y)$

$\rightsquigarrow$ $$\boxed{\dot{x} = -\frac{\partial H}{\partial y}, \quad \dot{y} = \frac{\partial H}{\partial x}}$$ Hamilton's equations

$\rightsquigarrow$ solutions $(x(t), y(t))$ are such that

$$\frac{d}{dt}\big(H(x(t),y(t))\big) = \frac{\partial H}{\partial x} \cdot \dot{x}(t) + \frac{\partial H}{\partial y} \cdot \dot{y}(t)$$
$$= -\frac{\partial H}{\partial x} \cdot \frac{\partial H}{\partial y} + \frac{\partial H}{\partial y} \frac{\partial H}{\partial x} = 0$$

$\rightsquigarrow$ Hamiltonian flow

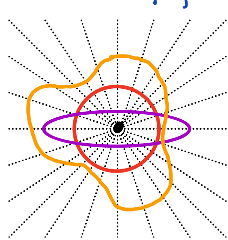
- Examples

1. $$H(x,y) = \frac{1}{2} \sum_{j=1}^{n+1} a_j \left(x_j^2 + y_j^2\right), \quad a_j > 0$$

$\rightsquigarrow$ solutions $z(t) = \left(e^{i a_1 t} z_1, \dots, e^{i a_{n+1} t} z_{n+1}\right)$

$\rightsquigarrow$ diagonal linear convex flow

2. $f : S^{2n+1} \hookrightarrow \mathbb{R}^{2n+2}$ s.t. $\Sigma_f := f(S^{2n+1})$ bounds a star-shapped domain w.r.t. $0 \in \mathbb{R}^{2n+2}$



Let $H_f : \mathbb{R}^{2n+2} \longrightarrow \mathbb{R}$ be the homogeneous of degree 2 function s.t. $H_f^{-1}(1) = \Sigma_f$.

$\rightsquigarrow$ corresponding Hamiltonian flow on $S^{2n+1} = f^{-1}(\Sigma_f)$ $\equiv$ Reeb flow

If Σ_f is the ellipsoid $\frac{1}{2}\sum_j a_j |z_j|^2 = 1$ we get the previous example.

- **Conjecture**: any Reeb flow on S^{2n+1} has at least $n+1$ geometrically distinct periodic orbits.

Status:

- $n=1$: proved by Cristofaro-Gardiner and Hutchings (2016, in a more general setting), and independently by Ginzburg, Hein, Hryniewicz and Macarini (2015) and Liu and Long (2016).

- $n>1$: widely open without extra assumptions.

(IV) Some Results

- P. Rabinowitz (1978): any Reeb flow on S^{2n+1} has at least one periodic orbit.
- Y. Long, C. Zhu (2002):
  (i) any convex Reeb flow on S^{2n+1} has at least $\lfloor \frac{n+1}{2} \rfloor + 1$ geometrically distinct periodic orbits.
  (ii) if also non-degenerate then it has at least $n+1$ geom. distinct periodic orbits.
- Dynamical Convexity (Hofer, Wysocki, Zehnder (1998))
  (i) J. Gutt, J. Kang (2016): any dynamical convex and non-degenerate Reeb flow on S^{2n+1} has at least $n+1$ geom. distinct periodic orbits.
  (ii) A.-L. Macarini (2017): any dynamical convex Reeb flow on S^{2n+1} has at least 2 geom. distinct periodic orbits.
  [Ekeland - Hofer (1987): same result for convex Reeb flows.]
- Reeb flows on S^*S^{n+1} (e.g. geodesic flows)
  A.-Macarini (2017): any dynamical convex and non-degenerate Reeb flow on S^*S^{n+1} has at least $2\left(\lfloor \frac{n}{2} \rfloor + 1\right)$ geometrically distinct periodic orbits.

[Katok - Ziller (1983): dyn. convex and non-deg. Finsler metric examples with exactly $2(\lfloor n/2 \rfloor + 1)$ closed geodesics]

V Main Tools

1 Variational Principle

- Action Functional: $\mathcal{A} : C^\infty(S^1, \mathbb{R}^{2n}) \longrightarrow \mathbb{R}$

$$[\gamma(e^{it}) = (x(t), y(t))] \longmapsto \mathcal{A}(\gamma) = \int_0^{2\pi} \Big(\sum_{j=1}^{n} x_j(t) \cdot \dot{y}_j(t) \Big)\, dt$$

- Any critical point γ_0 of $\mathcal{A}$, subject to the constraint $\int_0^{2\pi} H_f(\gamma_0(t))\, dt = 2\pi$ with nonzero Lag. multiplier λ, is such that $\gamma_0(e^{it/\lambda})$ is a $2\pi\lambda$ periodic orbit of Reeb flow on Σ_f.

- Bad variational problem: $\mathcal{A}$ is not bounded from below or above, critical points have ∞ index and coindex, etc.

- Rabinowitz (1978): first to overcome some of these problems.

2 Floer Homology $\equiv$ "Morse homology" of $\mathcal{A}$ graded by μ_{CZ}

- Gromov (1985): pseudo-holomorphic curves

  Floer (1988): Floer homology

  Hofer (1993) and Eliashberg-Givental-Hofer (2000): contact homology, Symplectic Field Theory (SFT).

- Theorem: For any non-degenerate Reeb flow on S^{2n+1} and any even $2m$, $m \in \mathbb{N}$, there exists at least one periodic orbit γ_m such that $\mu_{CZ}(\gamma_m) = 2m$. ["Morse Inequalities"]

Note: this only guarantees 1 geom. distinct periodic orbit, since apriori we could have a single γ_1 such that $\mu_{CZ}(\gamma_1^m) = 2m$.

③ Index Theory

- R. Bott (1956): index theory in the context of geodesic flows, which are examples of Reeb flows on the unit cosphere bundle S^*M of a manifold M.
- Y. Long (1990's, ...): cf. book with title "Index Theory for Symplectic Paths with Applications" (2002).

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(VI) Reeb flows on toric contact manifolds

A.- Macarini (2012, 2020), A.- Macarini - Liu (preprint 2022), A. - Macarini - Moreira (2022, preprint 2022):

use symplectic/contact reduction and convex polytopes as additional tools to compute contact homology.

Most recent application:

Theorem. [A.-Macarini-H. Liu, arXiv: 2211.16470]

Let $M = L_p^{2n+1}(\ell_0, \dots, \ell_n)$ and $a \in \pi_1(M)$ a generator. Then any dynamically convex Reeb flow on M has at least $\underline{\underline{2}}$ simple periodic orbits with homotopy class a.

[Generalizes Ekeland-Hofer (1987, $p=1$, convex), D. Zhang (2013, convex), A.-Macarini (2017, $p=1$), H. Liu-L. Zhang (2022, $p=2$).]