ON THE MEAN EULER CHARACTERISTIC OF GORENSTEIN TORIC CONTACT MANIFOLDS

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Abstract. We prove that the mean Euler characteristic of a Gorenstein toric contact manifold, i.e. a good toric contact manifold with zero first Chern class, is equal to half the normalized volume of the corresponding toric diagram and give some applications. A particularly interesting one, obtained using a result of Batyrev and Dais, is the following: twice the mean Euler characteristic of a Gorenstein toric contact manifold is equal to the Euler characteristic of any crepant toric symplectic filling, i.e. any toric symplectic filling with zero first Chern class.

1. Introduction

Good toric contact manifolds are the odd dimensional analogues of closed toric symplectic manifolds. As proved by Lerman in [12], they can be classified by the associated moment cones, in the same way that Delzant's theorem classifies closed toric symplectic manifolds by the associated moment polytopes.

In [1] we showed that any good toric contact manifold admits even non-degenerate toric contact forms, i.e. non-degenerate toric contact forms with all contractible closed orbits of its Reeb flow having even contact homology degree. As we will see in Proposition 2.16, this is also true for non-contractible closed Reeb orbits of such contact forms. The corresponding cylindrical contact homology, isomorphic to the chain complex associated to any such contact form, is a well-defined invariant that can be combinatorially computed from the associated good moment cone.

The contact homology degree of a non-degenerate closed Reeb orbit is determined by an appropriate dimensional shift of the Conley-Zehnder index. For it to be a well defined integer one needs vanishing of the first Chern class of the contact structure. Following [19], for such contact manifolds (M, ξ) and when their cylindrical contact homology $HC_*(M, \xi)$ is well defined, one defines the mean Euler characteristic as

$$\chi(M, \xi) := \lim_{N \to \infty} \frac{1}{N} \sum_{j=-N}^{N} (-1)^j \dim HC_j(M, \xi) \, ,$$

Date: November 1, 2016.

2010 *Mathematics Subject Classification.* 53D42 (primary), 53D20, 53D35 (secondary).

Key words and phrases. toric contact manifolds; toric symplectic cones; contact homology; mean index; mean Euler characteristic; volume of simplicial polytopes; Gorenstein toric isolated singularities; crepant toric resolutions.

MA was partially funded by FCT/Portugal through UID/MAT/04459/2013 and project EXCL/MAT-GEO/0222/2012, and by the Institut Mittag-Leffler, Sweden, and CNPq/Brazil through visiting grants. LM was partially supported by CNPq/Brazil and by FCT/Portugal through a visiting grant. The present work is part of the authors activities within BREUDS, a research partnership between European and Brazilian research groups in dynamical systems, supported by an FP7 International Research Staff Exchange Scheme (IRSES) grant of the European Union.

provided that the limit exists.

Definition 1.1. *A* Gorenstein toric contact manifold *is a good closed toric contact manifold with zero first Chern class.*

Remark 1.2. *The name Gorenstein is motivated by the fact that, in Algebraic Geometry, the corresponding toric cones are also called Gorenstein toric isolated singularities, the singular point being at the apex of the cone. Hence, good toric contact manifolds with zero first Chern class are links of Gorenstein toric isolated singularities.*

Remark 1.3. *As mentioned above, any Gorenstein toric contact manifold (M, ξ) has nonzero cylindrical contact homology only in even degrees. Moreover, as proved in* [2, Proposition 7.5], *the* $\inf\{j \in \mathbb{Z} \mid HC_j(M, \xi) \neq 0\}$ *is finite. Hence, for any Gorenstein toric contact manifold (M, ξ) we have that*

$$\chi(M, \xi) := \lim_{N\to\infty} \frac{1}{2N} \sum_{j=0}^{N} \dim HC_{2j}(M, \xi)\,.$$

Moreover, since any such (M, ξ) admits contact forms with finitely many simple closed Reeb orbits (cf. Section 4), this limit always exists (cf. [9]).

1.1. **Main Result.** Any $(2n + 1)$-dimensional good toric contact manifold is completely determined by a good cone C in the dual of the Lie algebra of the $(n + 1)$-dimensional acting torus (see Definition 2.1 and Theorem 2.2). For a Gorenstein toric contact manifold, there exists a $\mathbb{Z}$-basis of that torus such that the defining normals of $C \subset \mathbb{R}^{n+1}$ are of the form (see Corollary 2.4)

$$\nu_j = (v_j, 1)\,,\ v_j \in \mathbb{Z}^n\,,\ j = 1, \dots, d\,.$$

The integral simplicial convex polytope $D = \mathrm{conv}(v_1, \dots, v_d) \subset \mathbb{R}^n$ is called a toric diagram (see Definition 2.5) and to any such diagram there corresponds a unique Gorenstein toric contact manifold (M_D, ξ_D) of dimension $2n + 1$ (see Theorem 2.7).

Any vector $\nu = (v, 1) \in \mathbb{R}^{n+1}$, with v in the interior of D, determines a suitably normalized toric contact form α_ν and corresponding toric Reeb vector R_ν (see Subsection 2.4). Moreover, each facet F_ℓ of D determines a particular simple closed R_ν-orbit γ_ℓ. Denote by $v_{\ell_1}, \dots, v_{\ell_n} \in \{v_1, \dots, v_d\}$ the vertices of F_ℓ. Note that any facet of D is an $(n-1)$-dimensional integral simplex in $\mathbb{R}^n$, whose only integral points are its n vertices.

Our main result is the following formula for the mean index

$$\Delta(\gamma_\ell) := \lim_{N\to\infty} \frac{\mu_{\mathrm{RS}}(\gamma_\ell^N)}{N}\,,$$

where μ_{RS} denotes the Robbin-Salamon index (see Section 3 for a discussion about this index and the trivialization of the contact structure).

Theorem 1.4.

$$\frac{1}{\Delta(\gamma_\ell)} = \frac{n!\,\mathrm{vol}(S_{v,\ell})}{2}\,,$$

where $S_{v,\ell} \subset \mathbb{R}^n$ denotes the convex hull of $\{v, v_{\ell_!}, \dots, v_{\ell_n}\}$.

Since $S_{v,1} \dots, S_{v,m}$, where m is the number of facets of D, give a (simplicial) subdivision of D (see Figure 1), we have that

$$\sum_{\ell=1}^{m} \frac{1}{\Delta(\gamma_\ell)} = \frac{n!\,\mathrm{vol}(D)}{2}\,.$$

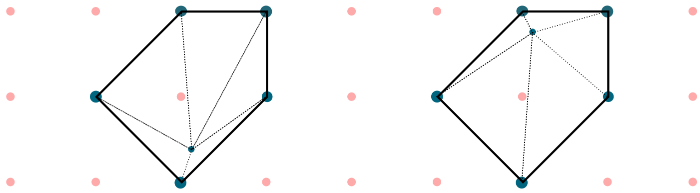

FIGURE 1. Two toric Reeb vectors and corresponding (simplicial) subdivisions of a toric diagram.

Now, take $v = (r_1, \ldots, r_n)$ with r_j's irrational and $\mathbb{Q}$-independent. Notice that this is a generic condition. Then α_ν is non-degenerate and $\gamma_1, \ldots, \gamma_\ell$ are its only simple closed Reeb orbits (see Section 4). By applying the resonance relation

$$\sum_{\ell=1}^{m} \frac{1}{\Delta(\gamma_\ell)} = \chi(M_D, \xi_D)$$

proved by Ginzburg and Kerman in [8], one gets the following very geometric formula for the mean Euler characteristic of a Gorenstein toric contact manifold.

Theorem 1.5. *Let $D \subset \mathbb{R}^n$ be a toric diagram and (M_D, ξ_D) its corresponding Gorenstein toric contact manifold. Then*

$$\chi(M_D, \xi_D) = \frac{n! \operatorname{vol}(D)}{2}\,.$$

1.2. **Applications.** Using well-known facts about integral polytopes and their volumes, one gets the following immediate applications of Theorem 1.5.

Corollary 1.6. *The mean Euler characteristic of a Gorenstein toric contact manifold is a half-integer.*

Proof. The normalized volume of an integral polytope $P \subset \mathbb{R}^n$, i.e. $n! \operatorname{vol}(P)$, is always an integer. □

Corollary 1.7. *In any given odd dimension and for any fixed upper bound, there are only finitely many Gorenstein toric contact manifolds with bounded mean Euler characteristic.*

Proof. In any given dimension there are only finitely many integral polytopes with a fixed volume upper bound. □

Remark 1.8. *These corollaries reflect the very special nature of toric contact structures. For example, one might want to compare them with the following result of Kwon and van Koert [11]: given any rational number x, there is a Stein fillable contact structure ξ_x on the 5-sphere whose mean Euler characteristic equals x.*

Batyrev-Dais show in [3, Corollary 4.6] that the Euler characteristic of any crepant (i.e. with zero first Chern class) toric smooth resolution of a Gorenstein toric isolated singularity is equal to the normalized volume of the corresponding toric diagram. Since any crepant toric symplectic filling of a Gorenstein toric contact manifold gives rise to a crepant toric smooth resolution of the corresponding Gorenstein toric isolated singularity, we have the following very interesting application of Theorem 1.5. It provides a relation between invariants of the contact structure and the topology of the corresponding toric symplectic filling.

Theorem 1.9. *Twice the mean Euler characteristic of a Gorenstein toric contact manifold is equal to the Euler characteristic of any crepant toric symplectic filling.*

In Section 5 we illustrate this theorem with some families of examples of crepant toric symplectic fillings of Gorenstein toric contact manifolds. Note that in odd dimensions higher than 5 there are Gorenstein toric contact manifolds with no crepant toric symplectic filling. In fact, as we show at the end of Section 5, the real projective spaces $(\mathbb{RP}^{4n+3}, \xi_{\rm std})$, $n \in \mathbb{N}$, are examples of that.

Remark 1.10. *As far as we know, it is unknown whether or not $(\mathbb{RP}^{4n+3}, \xi_{\rm std})$, $n \in \mathbb{N}$, have (necessarily non-toric) symplectic fillings with zero first Chern class.*

Remark 1.11. *Theorem 1.9 is not true if the filling is not toric. Indeed, as discussed in Example 2.12, there exists a family of toric contact structure ξ_p on $S^2 \times S^3$, with $p \in \mathbb{N}$, such that ξ_1 can be identified with the standard contact structure on the unit cosphere bundle of S^3 (see Section 6). The mean Euler characteristic of ξ_1 is equal to one, but clearly the Euler characteristic of the filling given by the unit disk bundle in T^*S^3 vanishes. As in Remark 1.8, this reflects the rigid nature of toric structures.*

Cho-Futaki-Ono show in [5] that there exists an infinite family of inequivalent toric Sasaki-Einstein metrics on $\#_k S^2 \times S^3$ for each $k \in \mathbb{N}$. The invariant they use to prove their inequivalence is precisely the volume of the corresponding toric diagrams. It then follows from Theorem 1.5 that the underlying Gorenstein toric contact structures are also distinct *as contact structures*. Hence we have the following result.

Corollary 1.12. *For each $k \in \mathbb{N}$, there are infinitely many non-equivalent contact structures on $\#_k S^2 \times S^3$ in the unique homotopy class determined by the vanishing of the first Chern class. These contact structures are toric and can be distinguished by their mean Euler characteristic.*

Remark 1.13. *In the case $k = 1$, i.e. $S^2 \times S^3$, these contact structures were first considered in [7], underlying new Sasaki-Einstein metrics, and proved to have non-isomorphic cylindrical contact homology in [1].*

In Section 6 we give one explicit description for toric diagrams associated to these Gorenstein toric contact structures, having also in mind the following natural question.

Question 1.14. *For each $k \in \mathbb{N}$, what is the minimal mean Euler characteristic of a Gorenstein toric contact structure on $\#_k S^2 \times S^3$?*

Lerman showed in [13] that the second homotopy group of any good toric contact manifold of dimension $2n + 1$ is a free abelian group of rank equal to $d - n - 1$, where d is the number of facets of the corresponding good moment cone $C \subset \mathbb{R}^{n+1}$. It is then natural to reformulate and generalize Question 1.14 in the following purely combinatorial terms.

Question 1.15. *Given $d, n \in \mathbb{N}$ with $d \geq n + 1$, what is the minimal volume of a toric diagram $D \subset \mathbb{R}^n$ with d vertices?*

When $n = 2$ and $3 \leq d \leq 16$ the answer is known (see [4], where you can also find the answer for a few higher values of d). Moreover, in this dimension it is possible to obtain a general bound using

1) Pick's formula for the volume of a lattice polygon in $\mathbb{R}^2$ with g interior lattice points and b lattice points on the boundary:

$$\mathrm{vol} = g + \frac{b}{2} - 1\,;$$

2) Coleman's conjecture, proved in [10] (see also [4]), stating that for such a lattice d-gon one has

$$b \leq 2g + 10 - d\,.$$

Since $d = b$ for toric diagrams $D \subset \mathbb{R}^2$, we get the inequality

$$\mathrm{vol}(D) \geq \frac{3(d-4)}{2}\,.$$

This is far from optimal, e.g. gives 18 as the lower bound for the volume of a 16-gon toric diagram while its minimal volume is known to be 59 (within the family of 16-gons described in Section 6 the minimal volume is 63). In any case, it can be combined with Theorem 1.5 to give a (very) partial answer to Question 1.14.

Corollary 1.16. *Let (M,ξ) be a Gorenstein toric 5-manifold. Then*

$$\chi(M,\xi) \geq \frac{3}{2}(\mathrm{rank}(\pi_2(M)) - 1)\,.$$

In particular,

$$\chi(\#_k S^2 \times S^3, \xi) \geq \frac{3}{2}(k-1)\,.$$

Acknowledgements. We are grafeful to Leonor Godinho for pointing our attention to [4] and for helping us identify real projectives spaces $(\mathbb{RP}^{4n+3}, \xi_{\mathrm{std}})$, $n \in \mathbb{N}$, as examples of Gorenstein toric contact manifolds with no crepant toric symplectic filling (see the end of Section 5). During the 2015-16 academic year, Miguel Abreu benefited from the hospitality of the following institutions in addition to his own: Institut Mittag-Leffler (Sweden), Instituto de Matemática Pura e Aplicada (Brazil) and Universidade Federal do Rio de Janeiro (Brazil).

2. Gorenstein toric contact manifolds

In this section we provide the necessary information on Gorenstein toric contact manifolds. For further details we refer the interested reader to [12] and [1].

2.1. Toric symplectic cones.

Via symplectization, there is a 1-1 correspondence between co-oriented contact manifolds and symplectic cones, i.e. triples (W,ω,X) where (W,ω) is a connected symplectic manifold and X is a vector field, the Liouville vector field, generating a proper $\mathbb{R}$-action $\rho_t : W \to W$, $t \in \mathbb{R}$, such that $\rho_t^*(\omega) = e^t\omega$. A closed symplectic cone is a symplectic cone (W,ω,X) for which the corresponding contact manifold $M = W/\mathbb{R}$ is closed.

A toric contact manifold is a contact manifold of dimension $2n+1$ equipped with an effective Hamiltonian action of the standard torus of dimension $n+1$: $\mathbb{T}^{n+1} = \mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1}$. Also via symplectization, toric contact manifolds are in 1-1 correspondence with toric symplectic cones, i.e. symplectic cones (W,ω,X) of dimension $2(n+1)$ equipped with an effective X-preserving Hamiltonian $\mathbb{T}^{n+1}$-action, with moment map $\mu : W \to \mathbb{R}^{n+1}$ such that $\mu(\rho_t(w)) = e^t\mu(w)$, for all $w \in W$ and $t \in \mathbb{R}$. Its moment cone is defined to be $C := \mu(W) \cup \{0\} \subset \mathbb{R}^{n+1}$.

A toric contact manifold is *good* if its toric symplectic cone has a moment cone with the following properties.

Definition 2.1. *A cone $C \subset \mathbb{R}^{n+1}$ is* good *if it is strictly convex and there exists a minimal set of primitive vectors $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$, with $d \geq n+1$, such that*

(i) $C = \bigcap_{j=1}^d \{x \in \mathbb{R}^{n+1} \mid \ell_j(x) := \langle x, \nu_j \rangle \geq 0\}$.
(ii) *Any codimension-k face of C, $1 \leq k \leq n$, is the intersection of exactly k facets whose set of normals can be completed to an integral basis of $\mathbb{Z}^{n+1}$.*

The primitive vectors $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$ are called the defining normals of the good cone $C \subset \mathbb{R}^{n+1}$.

The analogue for good toric contact manifolds of Delzant's classification theorem for closed toric symplectic manifolds is the following result (see [12]).

Theorem 2.2. *For each good cone $C \subset \mathbb{R}^{n+1}$ there exists a unique closed toric symplectic cone $(W_C, \omega_C, X_C, \mu_C)$ with moment cone C.*

The existence part of this theorem follows from an explicit symplectic reduction of the standard euclidean symplectic cone $(\mathbb{R}^{2d} \setminus \{0\}, \omega_{\rm st}, X_{\rm st})$, where d is the number of defining normals of the good cone $C \subset \mathbb{R}^{n+1}$, with respect to the action of a subgroup $K \subset \mathbb{T}^d$ induced by the standard action of $\mathbb{T}^d$ on $\mathbb{R}^{2d} \setminus \{0\} \cong \mathbb{C}^d \setminus \{0\}$. More precisely,

$$K := \left\{ [y] \in \mathbb{T}^d \mid \sum_{j=1}^d y_j \nu_j \in 2\pi \mathbb{Z}^{n+1} \right\} , \tag{1}$$

where $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$ are the defining normals of C.

One source for examples of good toric contact manifolds is the prequantization construction over integral closed toric symplectic manifolds, i.e. (M, ξ) with M given by the S^1-bundle over (B, ω) with Chern class $[\omega]/2\pi$ and ξ being the horizontal distribution of a connection with curvature ω. The corresponding good cones have the form

$$C := \{z(x,1) \in \mathbb{R}^n \times \mathbb{R} \mid x \in P\,,\ z \geq 0\} \subset \mathbb{R}^{n+1}$$

where $P \subset \mathbb{R}^n$ is a Delzant polytope with vertices in the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Note that if

$$P = \bigcap_{j=1}^d \{x \in \mathbb{R}^n \mid \langle x, v_j \rangle + \lambda_j \geq 0\}\,,$$

with integral $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}$ and primitive $v_1, \ldots, v_d \in \mathbb{Z}^n$, then the defining normals of $C \subset \mathbb{R}^{n+1}$ are

$$\nu_j = (v_j, \lambda_j)\,,\ j = 1, \ldots, d\,.$$

2.2. **First Chern class and toric diagrams.** The Chern classes of a co-oriented contact manifold can be canonically identified with the Chern classes of the tangent bundle of the associated symplectic cone. The vanishing of the first Chern class for good toric symplectic cones can be characterized in several equivalent ways. Proposition 2.16 in [1] does it in terms of the group K defined by (1), more precisely in terms of the characters that define its representation in $\mathbb{C}^d$. The following proposition gives a characterization in terms of the moment cone that will be more useful for our purposes and is commonly used in toric Algebraic Geometry (see, e.g., §4 of [3]).

Proposition 2.3. *Let (W_C, ω_C, X_C) be a good toric symplectic cone. Let $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$ be the defining normals of the corresponding moment cone $C \in \mathbb{R}^{n+1}$. Then $c_1(TW_C) = 0$ if and only if there exists $\nu^* \in (\mathbb{Z}^{n+1})^*$ such that*

$$\nu^*(\nu_j) = 1\,,\ \forall\ j = 1, \ldots, d\,.$$

By an appropriate change of basis of the torus $\mathbb{T}^{n+1}$, i.e. an appropriate $SL(n+1,\mathbb{Z})$ transformation of $\mathbb{R}^{n+1}$, this implies the following.

Corollary 2.4. *Let (W_C, ω_C, X_C) be a good toric symplectic cone with $c_1(TW_C) = 0$. Then there exists an integral basis of $\mathbb{T}^{n+1}$ for which the defining normals $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$ of the corresponding moment cone $C \subset \mathbb{R}^{n+1}$ are of the form*

$$\nu_j = (v_j, 1)\,,\ v_j \in \mathbb{Z}^n\,,\ j = 1, \ldots, d\,.$$

The next definition and theorem are then the natural analogues for Gorenstein toric contact manifolds of Definition 2.1 and Theorem 2.2.

Definition 2.5. *A* toric diagram *$D \subset \mathbb{R}^n$ is an integral simplicial polytope with all of its facets $\mathrm{Aff}(n,\mathbb{Z})$-equivalent to $\mathrm{conv}(e_1, \ldots, e_n)$, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.*

Remark 2.6. *The group $\mathrm{Aff}(n,\mathbb{Z})$ of integral affine transformations of $\mathbb{R}^n$ can be naturally identified with the elements of $SL(n+1,\mathbb{Z})$ that preserve the hyperplane $\{(v,1) \mid v \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$.*

Theorem 2.7. *For each toric diagram $D \subset \mathbb{R}^n$ there exists a unique Gorenstein toric contact manifold (M_D, ξ_D) of dimension $2n+1$.*

Here are some examples of toric diagrams and corresponding Gorenstein toric contact manifolds.

Example 2.8. *The toric diagram $D = \mathrm{conv}(0, p) \subset \mathbb{R}$, with $p \in \mathbb{N}$, gives $(M_D, \xi_D) \cong (L(p, p-1), \xi_{\mathrm{std}})$, i.e. a 3-dimensional lens space with contact structure induced from the standard one on $S^3 = L(1, 0)$. Figure 2 a) shows the toric diagram of $L(3, 2)$.*

Example 2.9. *The toric diagram $D = \mathrm{conv}(e_1, \ldots, e_n, \mathbf{0}) \subset \mathbb{R}^n$, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$ and $\mathbf{0} \in \mathbb{R}^n$ is the origin, gives $(M_D, \xi_D) \cong (S^{2n+1}, \xi_{\mathrm{std}})$, i.e. the standard contact structure on the $(2n+1)$-sphere. In fact, $(S^{2n+1}, \xi_{\mathrm{std}})$ is the prequantization of $(\mathbb{P}^n, \omega_{\mathrm{FS}})$ with Delzant polytope*

$$P = \bigcap_{j=1}^{n+1} \{x \in \mathbb{R}^n \mid \langle x, v_j \rangle + \lambda_j \geq 0\}\,,$$

where $v_j = e_j$, $\lambda_j = 0$, $j = 1, \ldots, n$, and $v_{n+1} = -(e_1 + \cdots + e_n)$, $\lambda_{n+1} = 1$, and so the corresponding good cone $C \subset \mathbb{R}^{n+1}$ has defining normals

$$(e_j, 0)\,,\ j = 1, \ldots, n\,,\ \text{and}\ (-(e_1 + \cdots + e_n), 1)$$

which are $SL(n+1,\mathbb{Z})$-equivalent to

$$(e_j, 1)\,,\ j = 1, \ldots, n\,,\ \text{and}\ (\mathbf{0}, 1)$$

(since both sets of normals form a $\mathbb{Z}$-basis of $\mathbb{Z}^{n+1}$). Figure 2 b) shows the toric diagram of $(S^5, \xi_{\mathrm{std}})$.

Example 2.10. *The toric diagram* $D = \mathrm{conv}(e_1, \ldots, e_n, -(e_1 + \cdots + e_n)) \subset \mathbb{R}^n$ *gives* $(M_D, \xi_D) =$ *prequantization of* $(\mathbb{P}^n, \omega = (n+1)\omega_{\mathrm{FS}} = 2\pi c_1(\mathbb{P}^n))$*, i.e. a* $\mathbb{Z}_{n+1}$*-quotient of* $(S^{2n+1}, \xi_{\mathrm{std}})$*. Figure 2 c) shows the toric diagram of* $(S^5/\mathbb{Z}_3, \xi_{\mathrm{std}})$*.*

Example 2.11. *Any monotone Delzant polytope* $P \subset \mathbb{R}^n$ *with primitive normal vectors* $v_1, \ldots, v_d \in \mathbb{Z}^n \subset \mathbb{R}^n$ *determines a toric diagram* $D := \mathrm{conv}(v_1, \ldots, v_d) \subset \mathbb{R}^n$*. The corresponding Gorenstein toric contact manifold* (M_D, ξ_D) *is the prequantization of the monotone toric symplectic manifold* (B_P, ω_P) *determined by* P*, with* $[\omega_P] = 2\pi c_1(M_P)$*.*

Example 2.12. *The toric diagram* $D = \mathrm{conv}((0,0),(1,0),(0,1),(p,p)) \subset \mathbb{R}^2$*, with* $p \in \mathbb{N}$*, gives* $(M_D, \xi_D) \cong (S^2 \times S^3, \xi_p)$*, i.e. a family of contact structures on* $S^2 \times S^3$*. This is the family of Remark 1.13. Figure 2 d) shows the toric diagram of* $(S^2 \times S^3, \xi_3)$*.*

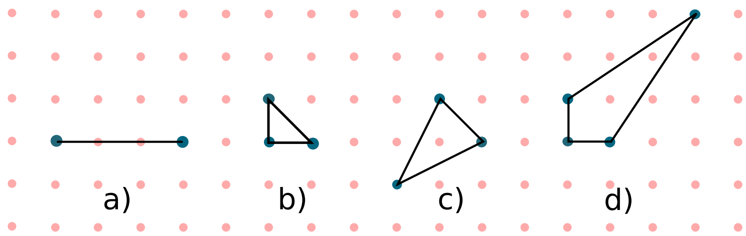

FIGURE 2. Examples of toric diagrams.

2.3. **Fundamental group.** As shown by Examples 2.8 and 2.10, a Gorenstein toric contact manifold (M_D, ξ_D) can have nontrivial fundamental group. In fact, it follows from a result of Lerman [13] that if $D = \mathrm{conv}(v_1, \ldots, v_d)$ then the fundamental group of M_D is the finite abelian group

$$\mathbb{Z}^{n+1}/\mathcal{N}\,,$$

where $\mathcal{N}$ denotes the sublattice of $\mathbb{Z}^{n+1}$ generated by $\{\nu_1 = (v_1, 1), \ldots, \nu_d = (v_d, 1)\}$ Hence, for simply connected Gorenstein toric contact manifolds we have that the $\mathbb{Z}$-span of the set of defining normals $\{\nu_1, \ldots, \nu_d\}$ is the full integer lattice $\mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1}$.

2.4. **Normalized toric Reeb vectors.** Let (W, ω, X) be a good toric symplectic cone of dimension $2(n+1)$, with corresponding closed toric contact manifold (M, ξ). Denote by $\mathcal{X}_X(W, \omega)$ the set of X-preserving symplectic vector fields on W and by $\mathcal{X}(M, \xi)$ the corresponding set of contact vector fields on M. The $\mathbb{T}^{n+1}$-action associates to every vector $\nu \in \mathbb{R}^{n+1}$ a vector field $R_\nu \in \mathcal{X}_X(W, \omega) \cong \mathcal{X}(M, \xi)$. We will say that a contact form $\alpha_\nu \in \Omega^1(M, \xi)$ is *toric* if its Reeb vector field R_{α_ν} satisfies

$$R_{\alpha_\nu} = R_\nu \quad \text{for some } \nu \in \mathbb{R}^{n+1}.$$

In this case we will say that $\nu \in \mathbb{R}^{n+1}$ is a *Reeb vector* and that R_ν is a *toric Reeb vector field.* The following proposition characterizes which $\nu \in \mathbb{R}^{n+1}$ are Reeb vectors of a toric contact form on (M, ξ).

Proposition 2.13 ([14] or [1, Proposition 2.19])**.** *Let* $\nu_1, \ldots, \nu_d \in \mathbb{R}^{n+1}$ *be the defining normals of the good moment cone* $C \in \mathbb{R}^{n+1}$ *associated with* (W, ω, X) *and* (M, ξ)*. The vector* *field $R_\nu \in \mathcal{X}_X(W,\omega) \cong \mathcal{X}(M,\xi)$ is the Reeb vector field of a toric contact form $\alpha_\nu \in \Omega^1(M,\xi)$ if and only if*

$$\nu = \sum_{j=1}^{d} a_j \nu_j \quad \text{with } a_j \in \mathbb{R}^+ \text{ for all } j = 1, \ldots, d.$$

This motivates the following definition and corollary for toric Reeb vectors on Gorenstein toric contact manifolds.

Definition 2.14. *A* normalized *toric Reeb vector is a toric Reeb vector $\nu \in \mathbb{R}^{n+1}$ of the form*

$$\nu = \sum_{j=1}^{d} a_j \nu_j \quad \text{with } a_j \in \mathbb{R}^+ \text{ for all } j = 1, \ldots, d, \text{ and } \quad \sum_{j=1}^{d} a_j = 1\,.$$

Corollary 2.15. *The interior of a toric diagram $D \subset \mathbb{R}^n$ parametrizes the set of normalized toric Reeb vectors on the Gorenstein toric contact manifold (M_D, ξ_D).*

Proof. If $D = \mathrm{conv}(v_1, \ldots, v_d)$ then $\nu_j = (v_j, 1)$, $j = 1, \ldots, d$, and any normalized toric Reeb vector is of the form

$$\nu = (v, 1) \quad \text{with} \quad v = \sum_{j=1}^{d} a_j v_j\,,\ a_j \in \mathbb{R}^+\,,\ j = 1, \ldots, d\,,\ \text{and} \ \sum_{j=1}^{d} a_j = 1\,.$$

□

2.5. **Parity of contact homology degree of closed toric Reeb orbits.** Let (W_C, ω_C, X_C) be a good toric symplectic cone, not necessarily Gorenstein, with $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$ being the defining normals of the corresponding good moment cone $C \in \mathbb{R}^{n+1}$. Denote by (M_C, ξ_C) the corresponding closed toric contact manifold and consider a toric Reeb vector field $R_\nu \in \mathcal{X}(M_C, \xi_C)$ determined by the toric Reeb vector

$$\nu = \sum_{j=1}^{d} a_j \nu_j\,,\ a_j \in \mathbb{R}^+\,,\ j = 1, \ldots, d\,.$$

The toric Reeb flow of R_ν on (M_C, ξ_C) has at least m simple closed orbits $\gamma_1, \ldots, \gamma_m$, corresponding to the m edges $E_1, \ldots, E_m$ of the cone C, i.e. one simple closed toric R_ν-orbit for each S^1-orbit of the $\mathbb{T}^{n+1}$-action on (M_C, ξ_C). When R_ν is non-degenerate, i.e. when $\nu \in \mathbb{R}^{n+1} \equiv \mathrm{Lie}\,(\mathbb{T}^{n+1})$ generates a dense 1-parameter subgroup of $\mathbb{T}^{n+1}$, then $\gamma_1, \ldots, \gamma_m$ are its only simple closed orbits.

Proposition 2.16. *When R_ν is non-degenerate the contact homology degree of γ_j^N, for any $j = 1, \ldots, m$, and any iterate $N \in \mathbb{N}$, is even.*

Proof. The parity of the contact homology degree is independent of the trivialization of the contact structure used to define it. In fact, the contact homology degree of a non-degenerate closed Reeb orbit is given by the Conley-Zehnder index of its linearization plus $n-2$, and different trivializations can only change the Conley-Zehnder index by twice the value of the Maslov index of an identity based loop in the symplectic linear group $Sp(2n, \mathbb{R})$.

Hence, it is enough to show that the contact homology degree is even for some trivialization. We will prove that by showing that any γ_j, $j = 1, \ldots, m$, has a $\mathbb{T}^{n+1}$-invariant neighborhood that is $\mathbb{T}^{n+1}$-equivariantly contactomorphic to a $\mathbb{T}^{n+1}$-invariant neighborhood of a simple closed orbit of a non-degenerate toric Reeb flow on the standard sphere $(S^{2n+1}, \xi_{\mathrm{std}})$. The result follows since the parity of the contact homology degree of any closed orbit of any such Reeb flow on the standard sphere is even. In fact, any such Reeb flow is just the Hamiltonian flow on a irrational ellipsoid in $\mathbb{R}^{2(n+1)}$.

Let us denote by E an arbitrary edge of the good moment cone $C \in \mathbb{R}^{n+1}$. Applying an appropriate $SL(n+1,\mathbb{Z})$ change of basis we may assume that the normals to the n facets of C that contain E are the first n vectors of the canonical basis of $\mathbb{R}^{n+1}$, which we denote by $\{e_1, \ldots, e_{n+1}\}$, and that E is generated by e_{n+1}. Write the Reeb vector ν as

$$\nu = \sum_{j=1}^{n} b_j e_j + r e_{n+1}\,,\ b_1, \ldots, b_n, r \in \mathbb{R}\,.$$

Strict convexity of C and Proposition 2.13 imply that $r > 0$. We can then pick

$$m_j \in \mathbb{Z} \quad \text{such that} \quad r_j := b_j - m_j r > 0\,,\ j = 1, \ldots, n\,,$$

and write the Reeb vector ν as

$$\nu = \sum_{j=1}^{n} (r_j + m_j r) e_j + r e_{n+1}\,,\ r_1, \ldots, r_n, r \in \mathbb{R}^+\,,\ m_1, \ldots, m_n \in \mathbb{Z}\,.$$

Using again Proposition 2.13, this implies that ν can be seen as a Reeb vector on the good moment cone with $n+1$ facets defined by the following set of normals:

$$\left\{ e_1\,,\ \ldots\,,\ e_n\,,\ \sum_{j=1}^{n} m_j e_j + e_{n+1} \right\}.$$

Since this set of vectors forms an integral basis of $\mathbb{Z}^{n+1}$, the corresponding toric contact manifold is indeed the sphere $(S^{2n+1}, \xi_{\rm std})$. □

Remark 2.17. *Although the above argument is a very simple and direct way to prove Proposition 2.16, it does not give any further information regarding the actual value of the contact homology degree, which is a well defined integer when the first Chern class of the contact structure vanishes. See* [1, Section 5] *for a way to compute that integer value for contractible non-degenerate closed toric Reeb orbits in Gorenstein toric contact manifolds, which we will use in Section 4.*

3. Index of closed orbits and trivialization of the contact structure

In this section we will discuss the trivialization of the contact structure that we use to define the index of closed orbits. While this trivialization is standard for *contractible* closed orbits, it depends on some choices when we consider not contractible periodic orbits. However, as will be explained below, it turns out that for Gorenstein toric contact manifolds there is a natural way to get this trivialization, which will be very handy for the computation of the mean index in the proof of Theorem 1.4.

Let (M^{2n+1}, ξ) be a contact manifold, α a contact form supporting ξ and J an almost complex structure on ξ compatible with $d\alpha|_\xi$. It is well known that the first Chern class $c_1(\xi) \in H^2(M;\mathbb{Z})$ of the complex vector bundle (ξ, J) does not depend on the choices of α and J. The same holds for the top complex exterior power $\Lambda^n_{\mathbb{C}}\xi$ up to complex bundle isomorphism.

Suppose that $c_1(\xi) = 0$ so that $\Lambda^n_{\mathbb{C}}\xi$ is a trivial line bundle. Choose a trivialization $\tau : \Lambda^n_{\mathbb{C}}\xi \to M \times \mathbb{C}$ which corresponds to a choice of a non-vanishing section $\mathfrak{s}$ of $\Lambda^n_{\mathbb{C}}\xi$. The choice of this trivialization furnishes a unique way to symplectically trivialize the contact structure along periodic orbits of α up to homotopy. As a matter of fact, given a periodic orbit $\gamma : S^1 \to M$ of α, let $\Phi : \gamma^*\xi \to S^1 \times \mathbb{C}^n$ be a trivialization of ξ over γ as a Hermitian vector bundle such that its highest complex exterior power coincides with τ. This condition fixes the homotopy class of Φ: given any other such trivialization Ψ we have, for every $t \in S^1$, that $\Phi_t \circ \Psi_t^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ has complex determinant equal to one and therefore the Maslov index of the symplectic path $t \mapsto \Phi_t \circ \Psi_t^{-1}$ vanishes, where $\Phi_t := \pi_2 \circ \Phi|_{\gamma^*\xi(t)}$ and $\Psi_t := \pi_2 \circ \Psi|_{\gamma^*\xi(t)}$ with $\pi_2 : S^1 \times \mathbb{C}^n \to \mathbb{C}^n$ being the projection onto the second factor; cf. [6, 15]. Notice that this trivialization is *closed under iterations*, that is, the trivialization induced on γ^N coincides, up to homotopy, with the N-th iterate of the trivialization over γ.

Thus, the choice of a non-vanishing section of $\Lambda^n_{\mathbb{C}}\xi$ furnishes a way to trivialize the contact structure over *any* closed orbit γ of α. However, if γ is *contractible* there is a canonical way to trivialize $\gamma^*\xi$ unique up to homotopy. More precisely, consider a capping disk σ of γ, that is, a smooth map $\sigma : D^2 \to M$, where D^2 is the two-dimensional disk, such that $\sigma|_{\partial D^2} = \gamma$. Choose a trivialization of $\sigma^*\xi$ and let $\Phi : \gamma^*\xi \to S^1 \times \mathbb{R}^{2n}$ be its restriction to the boundary. Since D^2 is contractible, the homotopy class of Φ does not depend on the choice of the trivialization of $\sigma^*\xi$. Moreover, the condition $c_1(\xi) = 0$ ensures that the homotopy class of Φ does not depend on the choice of σ as well.

The trivializations induced by a section $\mathfrak{s}$ of $\Lambda^n_{\mathbb{C}}\xi$ and a capping disk σ coincide up to homotopy. Indeed, consider a trivialization Φ of $\sigma^*\xi$ as a Hermitian vector bundle and let $\mathfrak{s}^\Phi_\sigma$ be the section of $\sigma^*\Lambda^n_{\mathbb{C}}\xi$ induced by the top complex exterior power of Φ. Since D^2 is contractible, $\mathfrak{s}^\Phi_\sigma$ is homotopic to $\sigma^*\mathfrak{s}$ and therefore the corresponding trivializations of $\gamma^*\xi$ are homotopic.

Given a non-vanishing section $\mathfrak{s}$ of $\Lambda^n_{\mathbb{C}}\xi$, one can define the Robbin-Salamon index $\mu_{\mathrm{RS}}(\gamma; \mathfrak{s})$ of any closed orbit γ in the usual way. More precisely, by the previous discussion $\mathfrak{s}$ induces a unique up to homotopy trivialization $\Phi : \gamma^*\xi \to S^1 \times \mathbb{R}^{2n}$. Using this trivialization, the linearized Reeb flow gives the symplectic path

$$\Gamma(t) = \Phi_t \circ d\phi^t_\alpha(\gamma(0))|_\xi \circ \Phi_0^{-1},$$

where ϕ^t_α is the Reeb flow of α. Then the Robbin-Salamon index $\mu_{\mathrm{RS}}(\gamma; \mathfrak{s})$ is defined as the Robbin-Salamon index of Γ [17]. It turns out that if $H^1(M; \mathbb{Q}) = 0$ then this index does not depend on the choice of $\mathfrak{s}$ since every two such sections are homotopic; see [15, Lemma 4.3].

The Robbin-Salamon index coincides with the Conley-Zehnder index if γ is non-degenerate and the mean index

$$\Delta(\gamma; \mathfrak{s}) := \lim_{N\to\infty} \frac{\mu_{\mathrm{RS}}(\gamma^N; \mathfrak{s})}{N}$$

varies continuously with respect to the C^2-topology in the following sense: if α_j is a sequence of contact forms converging to α in the C^2-topology and γ_j is a sequence of periodic orbits of α_j converging to γ then $\Delta(\gamma_j; \mathfrak{s}) \xrightarrow{j\to\infty} \Delta(\gamma; \mathfrak{s})$ [18].

Now, let (M^{2n+1}, ξ) be a Gorenstein toric contact manifold and α a toric contact form supporting ξ. As explained in the previous section, the symplectization W of M can be obtained by symplectic reduction of $\mathbb{C}^d$ by the action of a subtorus $K \subset \mathbb{T}^d$, where d is the number of vertices of the corresponding toric diagram. Given a contractible closed orbit γ of α, it is possible to construct a Hamiltonian $H : \mathbb{C}^d \to \mathbb{R}$ whose Hamiltonian flow is unitary linear and has a closed orbit $\hat{\gamma}$ that lifts γ and satisfies

$$\mu_{\mathrm{RS}}(\hat{\gamma}) = \mu_{\mathrm{RS}}(\gamma),$$

where the index in the left hand side can be computed using the canonical (constant) trivialization in $\mathbb{C}^d$ and the index in the right hand side is given by a trivialization using a capping disk; see [1, Lemma 3.4]. This fact was used in [1] to compute the cylindrical contact homology of M for *contractible* closed orbits.

Choose a non-vanishing section $\mathfrak{s}$ of $\Lambda^n_{\mathbb{C}}\xi$. By the discussion above, $\mathfrak{s}$ furnishes a unique way, up to homotopy, to trivialize $\gamma^*\xi$ for any closed orbit γ of α. Since $\pi_1(M)$ is finite (see Subsection 2.3) we have that $H^1(M;\mathbb{Q}) = 0$ and therefore $\mu_{\mathrm{RS}}(\gamma;\mathfrak{s})$ does not depend on the choice of $\mathfrak{s}$.

In this way, we are able to define the cylindrical contact homology of M for *every* closed orbit (contractible or not) and the computation of the index of contractible orbits reduces to the computation of the index of a lifted periodic orbit of a suitable linear unitary Hamiltonian flow on $\mathbb{C}^d$ (this lift exists if and only if the closed orbit is contractible). Using the homogeneity of the mean index (i.e. $\Delta(\gamma^N) = N\Delta(\gamma)$ for every N) and the fact that $\pi_1(M)$ is finite, we can use this lift to compute the mean index of any closed orbit (because it is enough to compute the mean index of some contractible iterate of γ) and it will be crucial in the proof of Theorem 1.4 presented in the next section. (It is important here that, as noticed before, the trivialization induced by a section is closed under iterations and this property is essential to achieve the homogeneity of the mean index.)

4. Proof of Theorem 1.4

Given a toric diagram $D = \mathrm{conv}(v_1, \ldots, v_d) \subset \mathbb{R}^n$ and corresponding Gorenstein toric contact manifold (M_D, ξ_D), the proof of Theorem 1.4 consists of the following simple steps:

1) Use the method developed in [1, Section 5] to compute the mean index of any closed simple Reeb orbit of any normalized non-degenerate toric Reeb vector field.
2) Compute the normalized volume of any simplicial pyramid with a normalized toric Reeb vector as vertex and a facet of D as base.
3) Check that the values obtained in 1) and 2) are the same when the orbit of 1) corresponds to the facet of 2).
4) Use the continuity of the mean index to conclude the result for possibly degenerate toric contact forms.

Consider a toric Reeb vector field $R_\nu \in \mathcal{X}(M_D, \xi_D)$ determined by the normalized toric Reeb vector

$$\nu = (v, 1) \quad \text{with} \quad v = \sum_{j=1}^d a_j v_j\,,\ a_j \in \mathbb{R}^+\,,\ j = 1, \ldots, d\,, \text{ and } \sum_{j=1}^d a_j = 1\,.$$

By a small abuse of notation, we will also write

$$R_\nu = \sum_{j=1}^d a_j \nu_j\,,$$

where $\nu_j = (v_j, 1)$, $j = 1, \ldots, n$, are the defining normals of the associated good moment cone $C \subset \mathbb{R}^n$. Making a small perturbation of ν if necessary, we can assume that

the 1-parameter subgroup generated by R_ν is dense in $\mathbb{T}^{n+1}$,

which means that if $v = (r_1, \ldots, r_n)$ then the r_j's are irrational and $\mathbb{Q}$-independent. This is equivalent to the corresponding toric contact form being non-degenerate. In fact, as already pointed out in Subsection 2.5, the toric Reeb flow of R_ν on (M_D, ξ_D) has exactly m simple closed orbits $\gamma_1, \ldots, \gamma_m$, all non-degenerate, corresponding to the m edges $E_1, \ldots, E_m$ of the cone C, i.e. one non-degenerate closed simple toric R_ν-orbit for each S^1-orbit of the $\mathbb{T}^{n+1}$-action on (M_D, ξ_D). Equivalently, there is

one non-degenerate closed simple toric R_ν-orbit for each facet of the toric diagram D.

Let F denote one of the facets of D, necessarily a simplex, and assume without loss of generality that its vertices are $v_1, \ldots, v_n \in \mathbb{Z}^n$. Let $\eta \in \mathbb{Z}^n$ be such that

$$\{\nu_1 = (v_1, 1), \ldots, \nu_n = (v_n, 1), (\eta, 1)\} \text{ is a } \mathbb{Z}\text{-basis of } \mathbb{Z}^{n+1}.$$

Then R_ν can be uniquely written as

$$R_\nu = \sum_{j=1}^{n} b_j \nu_j + b(\eta, 1)\,, \text{ with } b_1, \ldots, b_n \in \mathbb{R} \text{ and } b = 1 - \sum_{j=1}^{n} b_j \neq 0\,.$$

Let $k \in \mathbb{N}$ be the order in the fundamental group of M_D of the non-degenerate closed simple toric R_ν-orbit γ determined by F (that is, k is the smallest positive integer such that γ^k is contractible). Then the Conley-Zehnder index of γ^{kN}, for any $N \in \mathbb{N}$, is given by (see [1, Section 5])

$$\mu_{\mathrm{CZ}}(\gamma^{kN}) = 2\left(\sum_{j=1}^{n} \left\lfloor kN\frac{b_j}{|b|}\right\rfloor + kN\frac{b}{|b|}\right) + n\,.$$

Here we are taking a trivialization of ξ_D induced by a non-vanishing section of $\Lambda^n_{\mathbb{C}}\xi_D$ as discussed in Section 3. Hence, for the mean index of γ^k we have that

$$\Delta(\gamma^k) = \lim_{N\to\infty} \frac{\mu_{\mathrm{CZ}}(\gamma^{kN})}{N} = 2\left(\sum_{j=1}^{n} k\frac{b_j}{|b|} + k\frac{b}{|b|}\right) = \frac{2k}{|b|}\,,$$

which implies that

$$\Delta(\gamma) = \frac{2}{|b|},$$

where we are using the homogeneity of the mean index (see Section 3).

Let us now prove that

$$\frac{1}{\Delta(\gamma)} = \frac{n!\,\mathrm{vol}(S_v)}{2}\,, \text{ i.e. } n!\,\mathrm{vol}(S_v) = |b|\,,$$

where $S_v = \mathrm{conv}(v, v_1, \ldots, v_n)$. Using $\mathrm{Aff}(n, \mathbb{Z})$-invariance, we may assume without loss of generality that

$$\eta = 0 \text{ and } v_1 = e_1, \ldots, v_n = e_n\,, \text{ where } \{e_1, \ldots, e_n\} \text{ is the standard basis of } \mathbb{R}^n.$$

In this case we have that $v = (b_1, \ldots, b_n)$ and so

$$n!\,\mathrm{vol}(S_v) = |\det(I_n - A_v)|\,,$$

where I_n is the $(n \times n)$ identity matrix and A_v is the $(n \times n)$ matrix with all of its colums given by $v \in \mathbb{R}^n$. A simple determinant computation then gives

$$|\det(I_n - A_v)| = |1 - \sum_{j=1}^{n} b_j| = |b|\,.$$

Finally, using the continuity of the mean index (see Section 3) we easily conclude the proof of the Theorem when R_ν is degenerate.

5. Examples of crepant toric symplectic fillings

In this section we illustrate Theorem 1.9 with some families of examples of crepant toric symplectic fillings of Gorenstein toric contact manifolds and show that the real projective spaces $(\mathbb{RP}^{4n+3}, \xi_{\rm std})$, $n \in \mathbb{N}$, are examples of Gorenstein toric contact manifolds with no crepant toric symplectic filling. All these correspond to examples of isolated toric Gorenstein singularities well known in Algebraic Geometry. To check that the Euler characteristic of a filling is indeed equal to the normalized volume of the toric diagram, recall the following well known fact: the Euler characteristic of a toric symplectic manifold is equal to the number of vertices of the corresponding moment map image.

The first family of examples is given by the 3-dimensional lens spaces $(L(p,p-1), \xi_{\rm std}), p \in \mathbb{N}$ (cf. Example 2.8). The corresponding toric diagrams are $D = {\rm conv}(0,p) \subset \mathbb{R}$ with normalized volume equal to p and the moment map image of the corresponding symplectizations is a good cone in $\mathbb{R}^2$ with defining normals $(0,1)$ and $(p,1)$. The moment map image of a crepant toric symplectic filling is a convex polyhedral set in $\mathbb{R}^2$ with primitive interior normals to its edges given by $(0,1)$, $(1,1)$, $\ldots$, $(p-1,1)$, $(p,1)$. See Figure 3 for the case $p = 3$. The Euler characteristic of this fiiling is indeed equal to p.

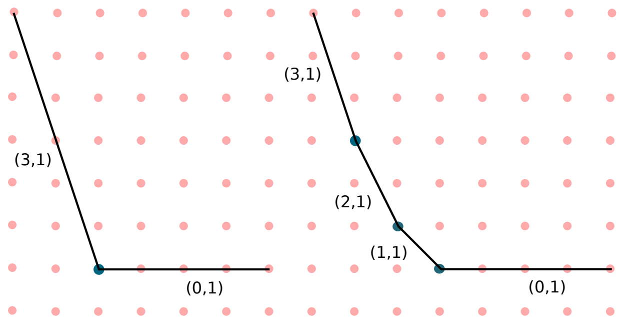

Figure 3. Moment map images of the symplectization of $(L(3,2), \xi_{\rm std})$ on the left and of its crepant toric symplectic filling on the right. Each label indicates the primitive interior normal to the corresponding edge.

The second family of examples is given by the prequantization of monotone toric symplectic manifolds, as described in Example 2.11. Here a crepant toric symplectic filling is obtained by blowing-up the Gorenstein toric isolated singularity. In the moment map image, this corresponds to cutting the good cone at level one with a horizontal facet. See Figure 4 for the case where the basis of the prequantization is $(\mathbb{P}^2, \omega = 3\omega_{\rm FS} = 2\pi c_1(\mathbb{P}^2))$ (cf. Example 2.10). In all these prequantization examples, one can use an elementary simplicial subdivision argument to show that the normalized volume of the toric diagram is equal to the number of its facets. That is also the number of vertices of the moment polytope of the basis, which is the Euler characteristic of the basis and of the crepant toric symplectic filling.

The third and final family of examples is given by the Gorenstein toric contact structures of Example 2.12, i.e. $(S^2 \times S^3, \xi_p)$, $p \in \mathbb{N}$. The corresponding toric diagrams are $D = {\rm conv}((0,0),(1,0),(0,1),(p,p)) \subset \mathbb{R}^2$ with normalized volume equal to $2p$ and the moment map image of the corresponding symplectizations is a good cone in $\mathbb{R}^3$ with defining normals $(0,0,1),(1,0,1),(0,1,1)$ and $(p,p,1)$. The moment map image of a crepant toric symplectic filling is a convex polyhedral set in $\mathbb{R}^3$ with primitive interior normals to its edges given by

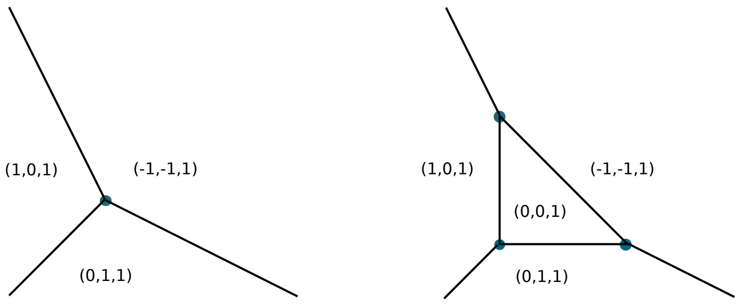

FIGURE 4. Moment map images of the symplectization of $(S^5/\mathbb{Z}_3, \xi_{\text{std}})$ on the left and of its crepant toric symplectic filling on the right. The viewer is on the inside of the moment map images and each label indicates the primitive interior normal to the corresponding facet.

$(0,0,1), (1,0,1), (0,1,1), (1,1,1), \ldots, (p-1,p-1,1), (p,p,1)$. See Figure 5 for the case $p=3$. The Euler characteristic of this filling is indeed equal to $2p$.

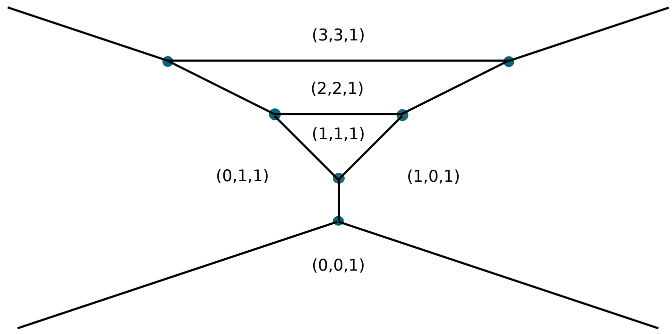

FIGURE 5. Moment map image of a crepant toric symplectic filling of $(S^2 \times S^3, \xi_3)$. The viewer is on the inside of the moment map image and each label indicates the primitive interior normal to the corresponding facet.

In Algebraic Geometry, the crepant toric resolutions of isolated toric Gorenstein singularities are obtained via regular simplicial subdivisions of the corresponding toric diagrams. The above examples correspond to obvious regular subdivisions of their toric diagrams. Figure 6 illustrates that correspondence for the crepant toric symplectic fillings of Figures 4 and 5. Note that the interior integral points determine the defining normals to the facets that are added to the moment cone for the construction of the crepant toric symplectic filling.

It is well known and easy to see that every toric diagram in $\mathbb{R}$ and $\mathbb{R}^2$ has a regular simplicial subdivision. The associated crepant toric resolution of the corresponding isolated toric Gorenstein singularity can then be used to show that all Gorenstein toric contact manifolds of dimensions 3 and 5 have crepant toric symplectic fillings. That is no longer the case in higher dimensions. In fact, real projective spaces $(\mathbb{RP}^{4n+3}, \xi_{\text{std}})$, $n \in \mathbb{N}$, already provide examples of Gorenstein toric contact manifolds with no crepant toric symplectic filling. Although this

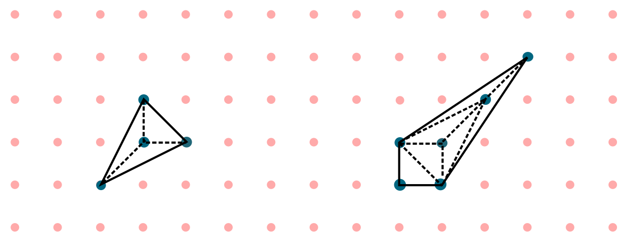

FIGURE 6. Regular simplicial subdivisions of the toric diagrams of $(S^5/\mathbb{Z}_3, \xi_{\rm std})$ (left) and $(S^2 \times S^3, \xi_3)$ (right), corresponding to the crepant toric symplectic fillings of Figures 4 and 5.

is just a consequence of the fact that the origin in $(\mathbb{C}^{2(n+1)}/\pm 1)$, $n \in \mathbb{N}$, is an example of an isolated Gorenstein singularity with no crepant resolution, let us give an argument in the spirit of this paper, i.e. using toric diagrams.

Similarly to Example 2.9, $(\mathbb{RP}^{2n+1}, \xi_{\rm std})$ can be described as the prequantization of $(\mathbb{P}^n, 2\omega_{\rm FS})$. The corresponding good moment cone $C \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ has defining normals

$$(e_j, 0)\,,\ j = 1, \ldots, n \quad \text{and} \quad (-(e_1 + \cdots + e_n), 2)\,,$$

where $\{e_1, \ldots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$. Using Proposition 2.3 one can show that

$$c_1(\mathbb{RP}^{2n+1}, \xi_{\rm std}) = 0 \Leftrightarrow n \text{ odd}.$$

Hence, any $(\mathbb{RP}^{4n+3}, \xi_{\rm std})$, $n \in \mathbb{N}_0$, is a Gorenstein toric contact manifold. In fact, in these dimensions the above set of normals is $SL(2(n+1), \mathbb{Z})$-equivalent to

$$(\mathbf{0}, 1)\,,\ (e_j, 0, 1)\,,\ j = 1, \ldots, 2n\,,\ \text{and } (e_1 + \cdots + e_{2n}, 2, 1)\,,$$

where $\mathbf{0} \in \mathbb{R}^{2n+1}$ and $\{e_1, \ldots, e_{2n}\}$ denotes the canonical basis of $\mathbb{R}^{2n}$.

When $n = 0$ this is just $(\mathbb{RP}^3, \xi_{\rm std})$ seen as the lens space $(L(2,1), \xi_{\rm std})$ with toric diagram $D = \text{conv}(0, 2) \subset \mathbb{R}$. As we have seen, it has a crepant toric symplectic filling. The corresponding crepant resolution is obtained from the regular subdivision of its toric diagram that comes from having an interior integral point: $D = \text{conv}(0,2) = \text{conv}(0,1) \cup \text{conv}(1,2)$.

In higher dimensions, i.e. when $n > 0$, the toric diagram

$$D = \text{conv}(\mathbf{0}, (e_1, 0), \ldots, (e_{2n}, 0), (e_1 + \cdots + e_{2n}, 2)) \subset \mathbb{R}^{2n} \times \mathbb{R} = \mathbb{R}^{2n+1}$$

has no interior integral points. Hence, the corresponding isolated toric Gorenstein singularity has no crepant toric resolution, which implies that $(\mathbb{RP}^{4n+3}, \xi_{\rm std})$, $n \in \mathbb{N}$, has no crepant toric symplectic filling. In other words, there are no normals one could use to define aditional facets to the moment cone for the construction of a toric symplectic filling that would be both smooth and with zero first Chern class.

6. Simply-connected Gorenstein toric contact structures in dimension 5

In this section we give one explicit description for the families of contact structures that one needs to construct in order to prove Corollary 1.12. First let us recall that the diffeomorphism type of a simply-connected Gorenstein toric 5-manifold is completely determined by the number of vertices of its toric diagram.

Proposition 6.1 ([16, Theorem 5.5]). *Let (M, ξ) be a simply-connected Gorenstein toric contact 5-manifold determined by a toric diagram $D \subset \mathbb{R}^2$ with d vertices. Then M is diffeomorphic to S^5 when $d = 3$ and to*

$$\#_{d-3} S^2 \times S^3 \quad \textit{when} \quad d > 3\,.$$

A good moment cone for a toric contact structure on S^5 has 3 normals that form a $\mathbb{Z}$-basis of $\mathbb{Z}^3$. Hence it is $SL(3,\mathbb{Z})$ equivalent to the good moment cone with normals

$$(0,0,1)\,,\ (1,0,1) \quad \text{and} \quad (0,1,1)\,,$$

and gives the standard contact structure on S^5. Its toric diagram is the unimodular simplex with vertices $(0,0)$, $(1,0)$ and $(0,1)$, shown in Figure 2 b). Its normalized volume is 1 and we recover the well known fact that

$$\chi(S^5, \xi_{\mathrm{st}}) = \frac{1}{2}\,.$$

A minimal volume toric diagram with 4 vertices is the square with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. It determines the standard contact structure on $S^2 \times S^3$, as the unit cosphere bundle of S^3, with mean Euler characteristic equal to 1. The family of 4-gons with vertices

$$(0,0)\,,\ (1,0)\,,\ (0,1) \quad \text{and} \quad (p,p)\,,\ p \in \mathbb{N}\,,$$

gives a family of inequivalent contact structures ξ_p on $S^2 \times S^3$ with

$$\chi(S^2 \times S^3, \xi_p) = p\,.$$

See Example 2.12 and Figure 2 d).

A minimal volume toric diagram with 5 vertices is the 5-gon with vertices $(-1,0)$, $(0,-1)$, $(1,0)$, $(0,1)$ and $(1,1)$. It determines a contact structure on $\#_2 S^2 \times S^3$ with mean Euler characteristic equal to $5/2$. The family of 5-gons with vertices

$$(0,-1)\,,\ (-1,0)\,,\ (1,0)\,,\ (0,1) \quad \text{and} \quad (p,p)\,,\ p \in \mathbb{N}\,,$$

gives a family of inequivalent contact structures ξ_p on $\#_2 S^2 \times S^3$ with

$$\chi(\#_2 S^2 \times S^3, \xi_p) = \frac{3}{2} + p\,.$$

A minimal volume toric diagram with 6 vertices is the 6-gon with vertices $(-1,-1)$, $(0,-1)$, $(-1,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ (see Figure 7 left). It determines a contact structure on $\#_3 S^2 \times S^3$ with mean Euler characteristic equal to 3. The family of 6-gons with vertices

$$(-1,-1)\,,\ (0,-1)\,,\ (-1,0)\,,\ (1,0)\,,\ (0,1) \quad \text{and} \quad (p,p)\,,\ p \in \mathbb{N}\,,$$

gives a family of inequivalent contact structures ξ_p on $\#_3 S^2 \times S^3$ with

$$\chi(\#_3 S^2 \times S^3, \xi_p) = 2 + p\,.$$

See Figure 7 right for this toric diagram with $p = 4$. Note that, as indicated in Figure 7, clipping off the vertex $(-1,-1)$ in this family of 6-gons gives the family of 5-gons considered above.

A minimal volume toric diagram with 8 vertices is the 8-gon with vertices $(0,0)$, $(1,0)$, $(0,3)$, $(1,3)$, $(-1,1)$, $(-1,2)$, $(2,1)$ and $(2,2)$ (see Figure 8 left). It determines a contact structure on $\#_5 S^2 \times S^3$ with mean Euler characteristic equal to 7. The family of 8-gons with vertices

$$(0,0)\,,\ (1,0)\,,\ (0,3)\,,\ (1,3)\,,\ (-1,1)\,,\ (-1,2)\,,\ (1+p,1) \quad \text{and} \quad (1+p,2)\,,\ p \in \mathbb{N}\,,$$

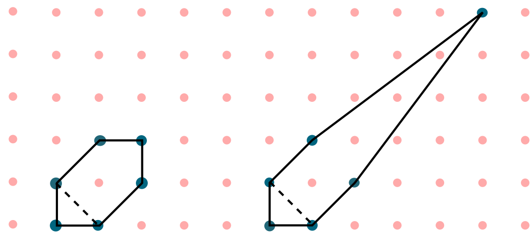

FIGURE 7. Examples of toric diagrams with 6 vertices. The one on the left has minimal volume. The dashed segment indicates that clipping off the bottom left vertex gives examples of toric diagrams with 5 vertices.

gives a family of inequivalent contact structures ξ_p on $\#_5 S^2 \times S^3$ with

$$\chi(\#_5 S^2 \times S^3, \xi_p) = 5 + 2p\,.$$

See Figure 8 right for this toric diagram with $p = 4$. Note that, as indicated in Figure 8, clipping off the vertex $(-1, 1)$ in this family of 8-gons gives a family of 7-gons that determine a family of inequivalent contact structures ξ_p on $\#_4 S^2 \times S^3$. The 7-gon on the left of Figure 8 also has minimal volume.

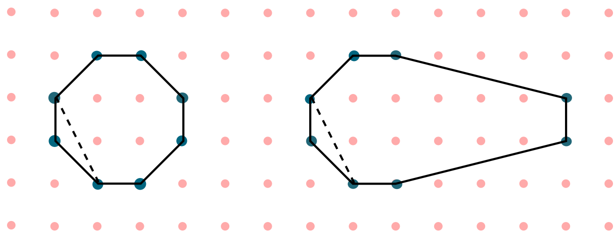

FIGURE 8. Examples of toric diagrams with 8 vertices. The one on the left has minimal volume. The dashed segment indicates how to obtain examples of toric diagrams with 7 vertices by clipping off one vertex.

One possibility to systematically increase the number of vertices is the following (see Figure 9). For each $k \in \mathbb{N}$ consider the family of $(4k + 4)$-gons, parametrized by $p \in \mathbb{N}$, with vertices

$$(0,0)\,,\ (1,0)\,,\ (0,2k+1)\,,\ (1,2k+1)\,,$$
$$\left(-\frac{(k-j)(k+j+1)}{2}, k-j\right)\,,\ \left(-\frac{(k-j)(k+j+1)}{2}, k+j+1\right)\,,$$
$$\left(\frac{(k-j)(k+j+1)}{2}+p, k-j\right)\,,\ \left(\frac{(k-j)(k+j+1)}{2}+p, k+j+1\right)\,,\ j = 0,\ldots,k-1\,.$$

It gives a family of inequivalent contact structures ξ_p on $\#_{4k+1} S^2 \times S^3$ with

$$\chi(\#_{4k+1} S^2 \times S^3, \xi_p) = 2k + 1 + \frac{2k(k+1)(2k+1)}{3} + (p-1)2k\,.$$

Clipping for example the vertex $(0, 0)$ in this family, one gets a family of $(4k + 3)$-gons that determine a family of inequivalent contact structures ξ_p on $\#_{4k} S^2 \times S^3$ with

$$\chi(\#_{4k} S^2 \times S^3, \xi_p) = 2k + \frac{1}{2} + \frac{2k(k+1)(2k+1)}{3} + (p-1)2k\,.$$

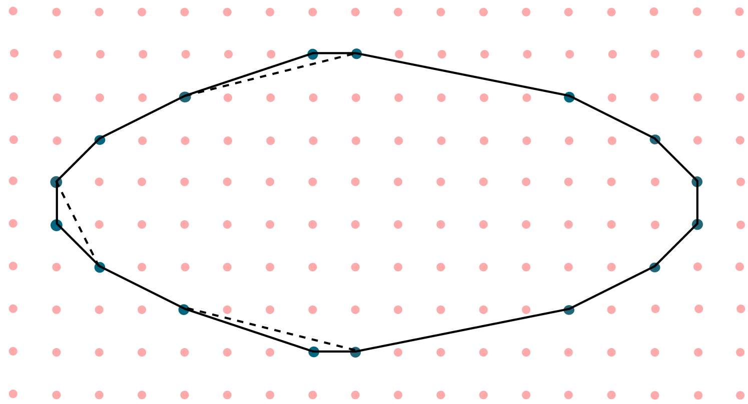

FIGURE 9. Example of a toric diagram with 16 vertices, corresponding to $k = p = 3$. The dashed segments indicate how to obtain examples of toric diagrams with 15, 14 and 13 vertices by clipping off one, two or three vertices.

For each $k > 1$ we can clip one or two more vertices in this family of $(4k + 4)$-gons, for example the vertices with coordinates $(0, 2k + 1)$ and $(-k(k + 1)/2, k)$, as shown in Figure 9 when $k = p = 3$. This gives families of $(4k+2)$-gons and $(4k+1)$-gons that determine families of inequivalent contact structures ξ_p on $\#_{4k-1}S^2 \times S^3$ and $\#_{4k-2}S^2 \times S^3$ with

$$\chi(\#_{4k-1}S^2 \times S^3, \xi_p) = 2k + \frac{2k(k + 1)(2k + 1)}{3} + (p - 1)2k$$

and

$$\chi(\#_{4k-2}S^2 \times S^3, \xi_p) = 2k - \frac{1}{2} + \frac{2k(k + 1)(2k + 1)}{3} + (p - 1)2k\,.$$

When $k = p = 1$ the corresponding 7-gon and 8-gon are the ones in Figure 8 left and have minimal volume. That is no longer the case when $k > 1$.

For the record, a minimal volume toric diagram with 9 vertices is the 9-gon with vertices $(-2, -3)$, $(-3, -2)$, $(-1, -3)$, $(-3, -1)$, $(0, -2)$, $(-2, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ (see Figure 10 left). It determines a contact structure on $\#_6S^2 \times S^3$ with mean Euler characteristic equal to $21/2$. The family of 9-gons with vertices

$$(-2, -3),\ (-3, -2),\ (-1, -3),\ (-3, -1),\ (0, -2),\ (-2, 0),\ (p, p-1),\ (p-1, p),\ (p, p),\ p \in \mathbb{N},$$

gives a family of inequivalent contact structures ξ_p on $\#_6S^2 \times S^3$ with

$$\chi(\#_6S^2 \times S^3, \xi_p) = \frac{9}{2} + 3(1 + p)\,.$$

See Figure 10 right for this toric diagram with $p = 3$

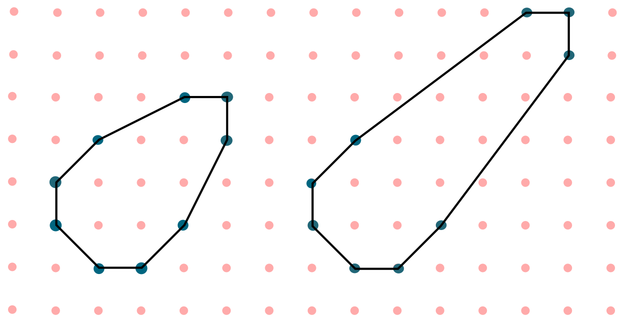

FIGURE 10. Examples of toric diagrams with 9 vertices. The one on the left has minimal volume.

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