Topology of Symplectomorphism Groups of $S^2 \times S^2$

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final version

December 4, 1996

0 Introduction

In dimension 4, due to non-existence of adequate tools, very little is known about the topology of groups of diffeomorphisms. For example, it is unknown if the group of compactly supported diffeomorphisms of $\mathbb{R}^4$ is connected.

The situation is much better if one wants to study groups of symplectomorphisms. This is due to the existence of powerful tools, going by the name of "pseudo-holomorphic curve techniques" and introduced in symplectic geometry by M.Gromov in his seminal paper of 1985 [5]. Gromov proved in that paper, among several other remarkable results, the contractibility of the group of compactly supported symplectomorphisms of $\mathbb{R}^4$ with its standard symplectic form $dx^1 \wedge dy^1 + dx^2 \wedge dy^2$.

Gromov also studied the following example. Let M_λ be the symplectic manifold $(S^2 \times S^2, \omega_\lambda = (1+\lambda)\sigma_0 \oplus \sigma_0)$, where $0 \leq \lambda \in \mathbb{R}$ and σ_0 is a standard area form on S^2 with total area equal to 1. Denote by G_λ the group of symplectomorphisms of M_λ that act as the identity on $H_2(S^2 \times S^2; \mathbb{Z})$. Gromov proved in [5] that G_0 is homotopy equivalent to its subgroup of standard isometries $SO(3) \times SO(3)$. He also showed why that would not be true for G_λ with $\lambda > 0$, and in 1987 D.McDuff [10] constructed explicitly an element of infinite order in $H_1(G_\lambda), \lambda > 0$.

In this paper we will give a more detailed description of G_λ, for $0 < \lambda \leq 1$. In particular we will prove the following:

*On leave from Instituto Superior Técnico, Lisbon, Portugal.

Partially supported by JNICT - Programas Ciência and Praxis, Fundação Luso-Americana para o Desenvolvimento and a Sloan Doctoral Dissertation Fellowship.

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Theorem 0.1 *For $0 < \lambda \leq 1$ we have that*

$$H^*(G_\lambda/(SO(3) \times SO(3)); \mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } * = 4k, 4k+1, k \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

As an algebra $H^*(G_\lambda/(SO(3) \times SO(3)); \mathbb{R})$ is isomorphic to the tensor product of an exterior algebra with one generator of degree 1 and a polynomial algebra with one generator of degree 4.

Remark 0.2 The algebraic techniques we use in §4 to compute $H^*(G_\lambda/(SO(3) \times SO(3)); \mathbb{R})$ can also be used to compute the cohomology with any other field coefficients, although they fall short of giving the integral cohomology due to the presence of torsion. For example, the application of the Leray-Hirsch Theorem to the fibrations described in §4.1 cannot be made with $\mathbb{Z}$ coefficients since the cohomology of the fibers is not free.

We will however prove, see the Remark after Proposition 4.1, that

$$H_1(G_\lambda/(SO(3) \times SO(3)); \mathbb{Z}) \cong \mathbb{Z} .$$

Since the fundamental group of a topological group is always abelian (I would like to thank Dusa McDuff for pointing this out) and $SO(3) \times SO(3)$ is connected we can conclude that

$$\pi_1(G_\lambda/(SO(3) \times SO(3))) \cong H_1(G_\lambda/(SO(3) \times SO(3)); \mathbb{Z}) \cong \mathbb{Z} .$$

The main steps in the proof of this theorem, which also determine the organization of the paper, are as follows.

Following Gromov, one looks at the contractible space $\mathcal{J}_\lambda$ of almost complex structures on $S^2 \times S^2$ compatible with ω_λ. Basic facts from the theory of pseudo-holomorphic curves, which we review in §1.1, determine that for an open dense subset of $J \in \mathcal{J}_\lambda$ the homology classes $[S_1] = [S^2 \times \{pt\}]$ and $[S_2] = [\{pt\} \times S^2]$, in $H_2(S^2 \times S^2; \mathbb{Z})$, are both represented by J-holomorphic spheres. We will denote by $\mathcal{J}_\lambda^g$, g for good, this subset of $\mathcal{J}_\lambda$, and by $\mathcal{J}_\lambda^b$, b for bad, its complement. In §1.2 we will see that $\mathcal{J}_0^g = \mathcal{J}_0$, while for $0 < \lambda \leq 1$ the space of bad structures $\mathcal{J}_\lambda^b$ is a non-empty, closed, co-oriented, codimension 2 submanifold of $\mathcal{J}_\lambda$, consisting of all $J \in \mathcal{J}_\lambda$ for which the homology class of the antidiagonal $\overline{D} = \{(s, -s) \in S^2 \times S^2\}$ is represented by a J-holomorphic sphere.

Gromov's theorem says that $G_0/(SO(3) \times SO(3))$ is homotopy equivalent to the contractible space $\mathcal{J}_0^g = \mathcal{J}_0$ (see §1.3 for the analogue of this theorem in the context of compactly supported symplectomorphisms of $\mathbb{R}^4$). It turns out that, for $\lambda > 0$, we still have $G_\lambda/(SO(3) \times SO(3))$ weakly homotopy equivalent to $\mathcal{J}_\lambda^g$. This was stated by Gromov in [5] and is proved in §2. When $0 < \lambda \leq 1$ one can extend the arguments of §2 to show that $\mathcal{J}_\lambda^b$ is weakly homotopy equivalent to $G_\lambda/(SO(3) \times S^1)$, where the $SO(3)$ factor is the diagonal in $SO(3) \times SO(3)$ and the S^1 factor corresponds to the element of infinite order in $H_1(G_\lambda)$ constructed by McDuff in [10]. This is explained in §3.

In §4 we use the above geometric data, together with some standard algebraic topology techniques, to compute $H^*(G_\lambda/(SO(3)\times SO(3));\mathbb{R})$.

Throughout the paper we assume that all manifolds, maps and structures are C^∞-smooth. This means that the infinite dimensional spaces we will be dealing with, like $\mathcal{J}_\lambda$, are not Banach manifolds but only Fréchet manifolds. This causes some problems when the use of the inverse mapping theorem is required. The statement "$\mathcal{J}_\lambda^b$ is a codimension 2, co-oriented submanifold of $\mathcal{J}_\lambda$", which is needed for the first algebraic computations in §4, falls in this category and so requires some additional justification. To avoid a break in the reasoning leading to the proof of Theorem 0.1, we deal with this technical issue in the Appendix. There we show, using the set-up and some results of [14], Chapter 3, that if we work in the C^l-category, for some $l \geq 1$, the above statement is true. Hence the first computations of §4, leading to Proposition 4.1, are correct in this setting. Since the algebraic topological invariants of the relevant spaces are independent of whether we define them in the C^l- or C^∞-category, Proposition 4.1 is true as stated.

If one tries to study the topology of G_λ for $\lambda > 1$ along the same lines, one faces the difficulty that $\mathcal{J}_\lambda^b$ is no longer a nice closed codimension 2 submanifold of $\mathcal{J}_\lambda$. Instead it has the structure of a stratified space. Each strata has codimension $4k-2$ in $\mathcal{J}_\lambda$ and corresponds to $J \in \mathcal{J}_\lambda$ for which $[S_1]-k[S_2]$ is represented by a J-holomorphic sphere, where $1 \leq k \leq m$ and $m-1 < \lambda \leq m$. It is clear from the arguments in §3 that each strata can be described geometrically, up to weak homotopy equivalence, as a suitable quotient of G_λ. It is however unclear how the way they fit together to give $\mathcal{J}_\lambda^b$ determines the topology of the complement $\mathcal{J}_\lambda^g \cong G_\lambda/(SO(3)\times SO(3))$.

The techniques used in this paper can give information on the topology of symplectomorphism groups for other symplectic 4-manifolds. Results in this direction for $\mathbb{C}P^2\sharp\overline{\mathbb{C}P}^2$, the nontrivial S^2-bundle over S^2, will appear elsewhere.

Acknowledgements: The results of this paper are part of my doctoral thesis, written at Stanford University under the supervision of Y.Eliashberg. I would like to thank him for all his guidance, advice and support.

I also want to thank V.Ginzburg for a very useful sugestion on how to compute the ring structure of $H^*(G_\lambda/(SO(3)\times SO(3));\mathbb{R})$, G.Carlsson for patiently answering all my questions on issues related to spectral sequences, D.McDuff and M.Symington for useful corrections on preliminary versions of this paper, and the referee for pointing out a gap in the Appendix and sugesting [7] as a way to fill it in.

1 J-holomorphic Spheres and Gromov's Theorem

In this section we review some basic facts about J-holomorphic curves, how these facts determine the structure of J-holomorphic spheres in M_λ and, as an example of how J-curves can be used, we recall Gromov's proof of the contractibility of the group of compactly supported symplectomorphisms of $\mathbb{R}^4$.

All manifolds, maps and structures will be assumed to be C^∞-smooth (see the remarks in the Introduction).

1.1 Basic facts on J-holomorphic curves

The following is based on the presentation of D.McDuff in [12].

Let (M, ω) be a symplectic 4-manifold and denote by $\mathcal{J}_\omega$ the space of compatible almost complex structures, i.e. automorphisms $J : TM \to TM$ of the tangent bundle such that $J^2 = -Id$ and $\omega(Jv, Jw) = \omega(v, w)$ for all $v, w \in TM$. This space is non-empty and contractible (see [13]).

A (parametrized) J-holomorphic curve in M is a map u from a Riemann surface (Σ, j) to M which satisfies the generalized Cauchy-Riemann equation:

$$\overline{\partial}_J u = \frac{1}{2}(du \circ j - J \circ du) = 0. \tag{1}$$

We will always assume that u is not a multiple cover, i.e. u is **somewhere injective** in the sense that there is a point $z \in \Sigma$ such that $du_z \neq 0$ and $u^{-1}(u(z)) = \{z\}$. In this case we say that $S = u(\Sigma)$ is an unparametrized J-curve and denote by $[S]$ the homology class in $H_2(M;\mathbb{Z})$ represented by S.

Given $A \in H_2(M;\mathbb{Z})$ let $\mathcal{M}(A, J)$ be the space of all rational J-holomorphic curves $u : S^2 \to M$ such that $A = [u(S^2)]$. The Mobius group $G = PSL(2, \mathbb{C})$ of holomorphic diffeomorphisms of S^2 acts freely on $\mathcal{M}(A, J)$ by reparametrization and the quotient $\mathcal{M}(A, J)/G$ is the moduli space of unparametrized (rational) A-curves.

Here are some key facts about J-holomorphic curves in symplectic 4-manifolds we will need.

Fact 1.1 (ω-positivity)

If $A \in H_2(M;\mathbb{Z})$ can be represented by a J-holomorphic curve, for some $J \in \mathcal{J}_\omega$, then $\omega(A) > 0$. Although it follows trivially from the compatibility condition between ω and J, this will be a key fact for us.

Fact 1.2 (Fredholm property and regularity) (see [14])

The universal moduli space $\mathcal{M}(A, \mathcal{J}_\omega) = \cup_{J \in \mathcal{J}_\omega} \mathcal{M}(A, J)$ is a Fréchet manifold and the projection operator $\pi_A : \mathcal{M}(A, \mathcal{J}_\omega) \to \mathcal{J}_\omega$ is Fredholm of index $2(c_1(A) + 2)$, where $c_1 \in H^2(M;\mathbb{Z})$ is the first Chern class of (M, J) (independent of $J \in \mathcal{J}_\omega$). An almost complex structure $J \in \mathcal{J}_\omega$ is said to be **regular** for the class $A \in H_2(M;\mathbb{Z})$ if it is a regular value for the projection π_A. In this case, the sets $\mathcal{M}(A,J)$ and $\mathcal{M}(A,J)/G$ are smooth manifolds of dimensions $2(c_1(A)+2)$ and $2(c_1(A)-1)$ respectively. The set of all such regular J is a subset of second category in $\mathcal{J}_\omega$, usually denoted by $\mathcal{J}_\omega^{reg}(A)$.

If $J \in \mathcal{J}_\omega$ is integrable and S is an embedded J-holomorphic sphere with self-intersection number $S \cdot S = p \geq -1$, then J is regular for the class $[S]$ (see [14]).

Fact 1.3 (Compactness and Evaluation Maps) (see [5] and [14])

Gromov's Compactness Theorem asserts that if a sequence $J_i \in \mathcal{J}_\omega$ converges C^∞ to $J \in \mathcal{J}_\omega$ and if S_i are (unparametrized) J_i-holomorphic curves of bounded area $\omega(S_i)$, then there is a subsequence of the S_i which converges weakly to a J-holomorphic curve or cusp-curve S'. A **cusp-curve** is a connected union of (possibly multiply-covered) curves. Thus the area $\omega(S')$ of the limit is the limit of the areas of the converging subsequence. In particular, if all the S_i lie in the same class A, the limit S' does too. Therefore, if it consists of several components $S_1 \cup \ldots \cup S_k$ their homology classes $A_i = [S_i]$ provide a decomposition of A with $\omega(A_i) > 0$.

We say that the homology class $A \in H_2(M;\mathbb{Z})$ is J-**simple** if A does not split as a sum of classes $A_1 + \ldots + A_k, k \geq 2$, all of which can be represented by J-holomorphic spheres, and **simple** if it is J-simple for all $J \in \mathcal{J}_\omega$. It follows that if A is a simple class then the moduli space $\mathcal{M}(A,J)/G$ of unparametrized J-holomorphic A-spheres is compact for all $J \in \mathcal{J}_\omega$. In this case we can consider the compact manifold $\mathcal{M}(A,J) \times_G S^2$, where $\phi \in G$ acts by $\phi \cdot (u,z) = (u \circ \phi^{-1}, \phi(z))$, and the **evaluation map** $e_{A,J} : \mathcal{M}(A,J) \times_G S^2 \to M$ given by $e_{A,J}(u,z) = u(z)$. We then have that if J_1 and J_2 are two almost complex structures from $\mathcal{J}_\omega^{reg}$, the evaluation maps e_{A,J_1} and e_{A,J_2} are compactly bordant.

Fact 1.4 (Positivity of Intersections) (see [11])

Two distinct closed J-curves S and S' in an almost complex 4-manifold (M,J) have only a finite number of intersection points. Each such point x contributes a number $k_x \geq 1$ to the algebraic intersection number $[S] \cdot [S']$. Moreover, $k_x = 1$ if and only if the curves S and S' intersect transversally at x.

Thus, $[S] \cdot [S'] = 0$ if and only if S and S' are disjoint, and $[S] \cdot [S'] = 1$ if and only if S and S' meet exactly once transversally and at a point which is non-singular on both curves.

Fact 1.5 (Adjunction Formula) (see [11])

If S is the image of a J-holomorphic map $u : \Sigma \to M$, define the **virtual genus** of S as the number

$$g(S) = 1 + \frac{1}{2}([S] \cdot [S] - c_1([S])). \tag{2}$$

Then, if u is somewhere injective, $g(S)$ is an integer. Moreover $g(S) \geq g = \text{genus}(\Sigma)$ with equality if and only if S is embedded.

Taking $\Sigma = S^2$, we have that if $c_1([S]) - [S]\cdot[S] = 2$ then u is a smooth embedding.

1.2 Structure of J-holomorphic spheres in M_λ: the spaces $\mathcal{J}_\lambda^g$ and $\mathcal{J}_\lambda^b$

Let M_λ denote the symplectic manifold $(S^2 \times S^2, \omega_\lambda)$ where $\omega_\lambda = (1+\lambda)\sigma_0 \oplus \sigma_0$, $0 \leq \lambda \in \mathbb{R}$, and $\int_{S^2}\sigma_0 = 1$. Let $\mathcal{J}_\lambda$ be the corresponding contractible space of compatible almost complex structures. Let $[S_1]$ (resp. $[S_2]$) be the homology class of $S^2 \times \{pt\}$ (resp. $\{pt\} \times S^2$), and denote by $[\overline{D}] = [S_1] - [S_2]$ the homology class of the antidiagonal $\overline{D} = \{(s,-s) \in S^2 \times S^2\}$. For each $J \in \mathcal{J}_\lambda$ we want to know when each of the above classes is represented by a J-holomorphic embedded 2-sphere.

Let $J_0 = j_0 \oplus j_0 \in \mathcal{J}_\lambda$ be a standard split compatible complex structure. For J_0, the class $[S_1]$ (resp. $[S_2]$) is represented by a 2-parameter family of holomorphic spheres given by $S^2 \times \{s\}$ (resp. $\{s\} \times S^2$) for any $s \in S^2$. Moreover $J_0 \in \mathcal{J}_\lambda^{reg}([S_i])$ since $[S_i]\cdot[S_i] = 0 \geq -1, i = 1,2$ (see Fact 1.2).

Definition 1.6 *We say that a compatible almost complex structure $J \in \mathcal{J}_\lambda$ is* **good** *if it has the same structure of J-holomorphic spheres as J_0, i.e. there exist two transversal foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of $S^2 \times S^2$ by embedded J-holomorphic spheres representing the homology classes $[S_1]$ and $[S_2]$ respectively. If $J \in \mathcal{J}_\lambda$ is not good we will call it* **bad**.

Denote by $\mathcal{J}_\lambda^g$ ($\mathcal{J}_\lambda^b$) the space of all good (bad) $J \in \mathcal{J}_\lambda$, so that $\mathcal{J}_\lambda = \mathcal{J}_\lambda^g \cup \mathcal{J}_\lambda^b$.

When $\lambda = 0$, the fact that ω_0 is integral and $\omega_0([S_1]) = \omega_0([S_2]) = 1$ implies, using ω-positivity, that $[S_1]$ and $[S_2]$ are simple and we can then prove that $\mathcal{J}_0^g = \mathcal{J}_0$ (see Theorem 1.8 below). However, if $\lambda > 0$ the fact that $\omega_\lambda([\overline{D}]) = 1 + \lambda - 1 = \lambda > 0$ means that $[S_1]$ can split as $[S_1] = [\overline{D}] + [S_2]$ and so it is not necessarily simple for all $J \in \mathcal{J}_\lambda$. The following lemma characterizes the homology classes in $H_2(S^2 \times S^2; \mathbb{Z})$ that can be represented by pseudo-holomorphic spheres.

Lemma 1.7 *If the class $A = a[S_1] + b[S_2] \in H_2(S^2 \times S^2; \mathbb{Z})$ can be represented by a somewhere injective J-holomorphic sphere for some $J \in \mathcal{J}_\lambda$, then either*

(a) $a, b \geq 2$*; or*

(b) $a = 1$ *and* $b > -(1+\lambda)$*; or*

(c) $b = 1$ *and* $a \geq 0$.

Proof: By the adjunction formula we must have

$$1 + \frac{1}{2}(A\cdot A - c_1(A)) \geq 0 \Rightarrow 1 + \frac{1}{2}(2ab - 2a - 2b) \geq 0 \Rightarrow (a-1)(b-1) \geq 0.$$

By ω-positivity we must also have

$$(1+\lambda)a + b > 0.$$

These two inequalities imply the result. QED

As a corollary of this lemma we have that $[S_2]$ is simple for any $\lambda \geq 0$ and if $0 < \lambda \leq 1$ the only possible splitting for $[S_1]$ is $[S_1] = [\overline{D}] + [S_2]$.

We are now ready to state and prove the main theorem of this subsection.

Theorem 1.8 *If $\lambda = 0$ we have that all compatible complex structures are good, i.e. $\mathcal{J}_0^g = \mathcal{J}_0$. If $0 < \lambda \leq 1$, the space of bad structures $\mathcal{J}_\lambda^b$ is a non-empty, closed, codimension 2 submanifold of $\mathcal{J}_\lambda$ consisting of all $J \in \mathcal{J}_\lambda$ for which the class $[\overline{D}]$ is represented by a unique embedded J-holomorphic sphere. For each such J we still have the existence of a foliation $\mathcal{F}_2$ of $S^2 \times S^2$ by embedded J-holomorphic spheres representing the homology class $[S_2]$.*

Proof: Let $\widetilde{\mathcal{J}}_\lambda^b$ denote the space of $J \in \mathcal{J}_\lambda$ for which the class $[\overline{D}]$ is represented by a J-holomorphic sphere. Note that, by positivity of intersections, this sphere is necessarily unique and, by the adjunction formula, embedded. Also, since $[S_1] \cdot [\overline{D}] = -1$, we have that for any $J \in \widetilde{\mathcal{J}}_\lambda^b$ the class $[S_1]$ is not represented by a J-holomorphic curve, and so $\widetilde{\mathcal{J}}_\lambda^b \subset \mathcal{J}_\lambda^b$.

If $\lambda = 0$ we have $\widetilde{\mathcal{J}}_0^b = \emptyset$ because $\omega_0([\overline{D}]) = 0$. If $\lambda > 0$, the standard antidiagonal $[\overline{D}] = \{(s,-s) \in S^2 \times S^2\}$ is an embedded symplectic sphere and we can easily construct a $J \in \mathcal{J}_\lambda$ such that it becomes J-holomorphic, proving that $\widetilde{\mathcal{J}}_\lambda^b \neq \emptyset$. Consider the universal moduli space $\mathcal{M}([\overline{D}], \mathcal{J}_\lambda)$ and the projection operator $\pi_{[\overline{D}]} : \mathcal{M}([\overline{D}], \mathcal{J}_\lambda) \to \mathcal{J}_\lambda$ (see Fact 1.2). The space $\widetilde{\mathcal{J}}_\lambda^b$ is exactly the image of $\pi_{[\overline{D}]}$. The Fredholm index in this case is 4. Given any point $p = (u, J) \in \mathcal{M}([\overline{D}], \mathcal{J}_\lambda)$, we have that $\pi_{[\overline{D}]}^{-1}(J)$ consists exactly of reparametrizations of u by elements of $G = PSL(2, \mathbb{C})$ due to uniqueness of the unparametrized J-sphere. Hence the dimension of the kernel of $(d\pi_{[\overline{D}]})_p$ is 6 which implies that the dimension of the cokernel is 2 for all $p \in \mathcal{M}([\overline{D}], \mathcal{J}_\lambda)$. We then have that the map $\widetilde{\mathcal{J}}_\lambda^b = \mathcal{M}([\overline{D}], \mathcal{J}_\lambda)/G \to \mathcal{J}_\lambda$ induced by $\pi_{[\overline{D}]}$ is just the inclusion $\widetilde{\mathcal{J}}_\lambda^b \hookrightarrow \mathcal{J}_\lambda$, realizing $\widetilde{\mathcal{J}}_\lambda^b$ as a codimension 2 submanifold of $\mathcal{J}_\lambda$ (see the remarks in the Introduction and the Appendix for further explanation).

If $0 < \lambda \leq 1$ we have from the previous lemma that $[\overline{D}]$ is simple and so we can apply Gromov's compactness theorem to conclude that $\widetilde{\mathcal{J}}_\lambda^b$ is closed inside $\mathcal{J}_\lambda$.

We now prove that $\widetilde{\mathcal{J}}_\lambda^b = \mathcal{J}_\lambda^b$ by showing that $\widetilde{\mathcal{J}}_\lambda^g = \mathcal{J}_\lambda \backslash \widetilde{\mathcal{J}}_\lambda^b$ is the same as $\mathcal{J}_\lambda^g$. Let A be equal to either $[S_1]$ or $[S_2]$. We have that $\widetilde{\mathcal{J}}_\lambda^g$ is open and connected, and A is J-simple for all $J \in \widetilde{\mathcal{J}}_\lambda^g$. This implies, using Facts 1.2 and 1.3, that the moduli spaces $\mathcal{M}(A, J)/G$, $J \in \mathcal{J}_\lambda^{reg} \cap \widetilde{\mathcal{J}}_\lambda^g = \mathcal{J}_\lambda^{reg}$, are compact, have dimension 2 (since $c_1(A) = 2$) and the evaluation maps $e_{A,J} : \mathcal{M}(A, J) \times_G S^2 \to S^2 \times S^2$, as maps between compact 4-manifolds, have a well defined degree that is independent of $J \in \mathcal{J}_\lambda^{reg}$. Because $A \cdot A = 0$, positivity of intersections implies that this degree is at most 1. On the other hand, for $J_0 \in \mathcal{J}_\lambda^{reg}$ the degree is 1, and so the degree is 1 for every $J \in \mathcal{J}_\lambda^{reg}$. This means that $\mathcal{J}_\lambda^{reg} \subset \mathcal{J}_\lambda^g$.

Given any point $p \in S^2 \times S^2$ and $J \in \widetilde{\mathcal{J}}_\lambda^g$, let $J_i \in \mathcal{J}_\lambda^{reg}$ be a sequence converging C^∞ to J and S_i the corresponding sequence of J_i-holomorphic A-spheres containing p. Because A is $\widetilde{\mathcal{J}}_\lambda^g$-simple, compactness gives us a subsequence of S_i converging to a J-holomorphic A-sphere S'. Since $p \in S_i$ for all $i \in \mathbb{N}$, we have that $p \in S'$. Hence, for any $J \in \widetilde{\mathcal{J}}_\lambda^g$ and any $p \in S^2 \times S^2$ there is a unique J-holomorphic A-sphere containing p. This proves that $\widetilde{\mathcal{J}}_\lambda^g \subset \mathcal{J}_\lambda^g$, and since we already knew that $\widetilde{\mathcal{J}}_\lambda^g \supset \mathcal{J}_\lambda^g$, we conclude that $\widetilde{\mathcal{J}}_\lambda^g = \mathcal{J}_\lambda^g$ as desired.

The same argument applied just to the class $[S_2]$, and using the fact that it is simple, gives the existence of the foliation $\mathcal{F}_2$ for any $J \in \mathcal{J}_\lambda$. QED

1.3 Gromov's theorem

Let D^2 denote the unit disc in $\mathbb{R}^2$ and let $\omega_\lambda = (1 + \lambda)dx^1 \wedge dy^1 + dx^2 \wedge dy^2$ be a split symplectic form on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$. The following was proved by Gromov in [5].

Theorem 1.9 (Gromov) *If σ is a symplectic form on $D^2 \times D^2$ which equals ω_λ near the boundary, the space $\mathcal{G}_\sigma$ of all diffeomorphisms ψ_σ of $D^2 \times D^2$, which equal the identity near the boundary and are such that $\psi_\sigma^*(\omega_\lambda) = \sigma$, is non-empty and contractible.*

Proof: Let $\mathcal{J}_\sigma$ be the contractible space of all almost complex structures on $D^2 \times D^2$ that are compatible with σ and equal to the standard split complex structure J_0 near the boundary. We prove that the space $\mathcal{G}_\sigma$ of ψ_σ defined above is homotopy equivalent to $\mathcal{J}_\sigma$.

Given $J \in \mathcal{J}_\sigma$ we claim that there are two families $\mathcal{F}_1$ and $\mathcal{F}_2$ of embedded J-holomorphic discs in $D^2 \times D^2$, one with boundaries $\partial D^2 \times \{y\}, y \in D^2$, and one with boundaries $\{x\} \times \partial D^2, x \in D^2$. To see this, collapse $D^2 \times D^2$ to $S^2 \times S^2$ by making the obvious identifications along the boundary. Note that the two spheres in $S^2 \times S^2$ coming from $D^2 \times \partial D^2$ and $\partial D^2 \times D^2$ are holomorphic for the induced J. Use the arguments in the proof of Theorem 1.8 to show that this induced J is good for σ. Finally, get the desired families of holomorphic discs in $D^2 \times D^2$ from the corresponding holomorphic foliations of $S^2 \times S^2$ by J-spheres. We can then define a diffeomorphism $\phi_J : D^2 \times D^2 \to D^2 \times D^2$ to be the unique map which is the identity on the boundary and takes the discs in the families $\mathcal{F}_1$ and $\mathcal{F}_2$ to the discs $D^2 \times \{y\}$ and $\{x\} \times D^2$. It is easy to check that $\phi_J^*(\omega_\lambda)$ is J-positive ($\phi_J^*(\omega_\lambda)(X, JX) > 0$, for all $X \in T(D^2 \times D^2)$) and so the forms $\tau_t = t\sigma + (1 - t)\phi_J^*(\omega_\lambda)$, $t \in [0, 1]$, are all non-degenerate. Using

Moser's Theorem [16] we get an isotopy from ϕ_J to a symplectomorphism $\psi_{\sigma,J}$. Since, by construction, ϕ_J is the identity on the boundary of $D^2 \times D^2$ and near $\partial D^2 \times \partial D^2$ (recall that J is standard near the boundary), the isotopy τ_t is constant when restricted to the boundary and near $\partial D^2 \times \partial D^2$. Therefore we can assume that $\psi_{\sigma,J}$ is the identity in this region. A final adjustment, which can be made in a canonical way, allows us to assume that $\psi_{\sigma,J}$ is the identity near the boundary of $D^2 \times D^2$.

We have thus constructed a map $\beta : \mathcal{J}_\sigma \to \mathcal{G}_\sigma$. To prove that β is a homotopy equivalence we construct a homotopy inverse by sending $\psi_\sigma \in \mathcal{G}_\sigma$ to $(\psi_\sigma^{-1})_*(J_0)$. Observe that $\beta((\psi_\sigma^{-1})_*(J_0)) = \psi_\sigma$, while the composition the other way around is homotopic to the identity because $\mathcal{J}_\sigma$ is contractible. QED

Remark 1.10 In the above theorem, two choices of ψ_σ differ by an element of the group of compactly supported ω_λ-preserving diffeomorphisms of $Int(D^2 \times D^2)$, and so we conclude as a corollary that this group is contractible.

2 Geometric Description of $\mathcal{J}_\lambda^g$

Recall that G_λ denotes the group of symplectomorphisms of M_λ acting as the identity on $H_2(S^2 \times S^2; \mathbb{Z})$. The following was proved for $\lambda = 0$ and stated for $\lambda > 0$ by Gromov in [5]:

Theorem 2.1 *$G_\lambda/(SO(3) \times SO(3))$ is weakly homotopy equivalent to $\mathcal{J}_\lambda^g$.*

Proof: Let $\star = (s_0, s_0) \in S^2 \times S^2$ be a fixed base point and denote by $G_{\lambda,\star}$ the subgroup of G_λ fixing $\star$. Let $SO(2)$ be the subgroup of $SO(3)$ fixing s_0. We clearly have $G_\lambda/(SO(3) \times SO(3)) \cong G_{\lambda,\star}/(SO(2) \times SO(2))$ and we prove that the later is homotopy equivalent to $\mathcal{J}_\lambda^g$.

Let $\mathcal{S}_\lambda^g$ denote the space of embedded symplectic 2-spheres in M_λ containing $\star$ and representing the homology class $[S_1]$. Associated to any $J \in \mathcal{J}_\lambda^g$ we have a well defined element of $\mathcal{S}_\lambda^g$ given by the unique J-sphere in $\mathcal{F}_1$ that goes through $\star$. Conversely, given any element in $\mathcal{S}_\lambda^g$ we can construct a $J \in \mathcal{J}_\lambda^g$ that makes it J-holomorphic. Moreover the space of all such J is contractible and so we conclude that $\mathcal{J}_\lambda^g$ is weakly homotopy equivalent to $\mathcal{S}_\lambda^g$.

Lemma 2.2 *$G_{\lambda,\star}$ acts transitively on $\mathcal{S}_\lambda^g$.*

Proof: Let S_1 (resp. S_2) denote the standard $S^2 \times \{s_0\}$ (resp. $\{s_0\} \times S^2$). Given any $S_1' \in \mathcal{S}_\lambda^g$, pick a $J \in \mathcal{J}_\lambda^g$ that makes it J-holomorphic, and denote by S_2'' the unique J-holomorphic sphere in $\mathcal{F}_2$ that goes through $\star$. We have that S_1' and S_2'' intersect transversally at $\star$, although not necessarily in an ω_λ-orthogonal way. By using, for example, Lemma 2.3 of [4] we can construct a symplectic isotopy h_t, supported in a neighborhood of $\star$ and fixing it, such that $h_0 = id$ and $S_2' = h_1(S_2'')$ intersects S_1' at $\star$ in an ω_λ-orthogonal way.

Using the symplectic neighborhood theorem we can then construct a diffeomorphism $\alpha : S^2 \times S^2 \to S^2 \times S^2$ such that $\alpha(S_i) = S_i', i = 1, 2$, and α is a symplectomorphism from a neighborhood U of $S_1 \cup S_2$ onto a neighborhood $U' = \alpha(U)$ of $S_1' \cup S_2'$.

The complement of $S_1 \cup S_2$ inside M_λ is clearly symplectomorphic to $D^2 \times D^2 \subset \mathbb{R}^4$ with standard split form $\omega_\lambda = (1+\lambda)dx^1 \wedge dy^1 + dx^2 \wedge dy^2$. The form $\alpha^*(\omega_\lambda)$, considered on $D^2 \times D^2$, is symplectic and agrees with ω_λ near the boundary. Hence, by Theorem 1.9, there is a diffeomorphism β of $D^2 \times D^2$, which equals the identity near the boundary and such that $\beta^*(\alpha^*(\omega_\lambda)) = \omega_\lambda$.

By taking $\varphi = \alpha \circ \beta$ we have a symplectomorphism of M_λ such that $\varphi(S_1) = S_1'$, proving the lemma. QED

Let $H_{\lambda,\star}$ be the subgroup of $G_{\lambda,\star}$ consisting of symplectomorphisms ψ preserving S_1, i.e. $\psi(S_1) = S_1$. From the previous lemma we have that $G_{\lambda,\star}/H_{\lambda,\star} \cong \mathcal{S}_\lambda^g$.

The obvious map $G_{\lambda,\star}/(SO(2) \times SO(2)) \to G_{\lambda,\star}/H_{\lambda,\star}$ is a fibration, with fiber over the identity given by $H_{\lambda,\star}/(SO(2) \times SO(2))$.

Lemma 2.3 *$H_{\lambda,\star}/(SO(2) \times SO(2))$ is contractible.*

Proof: This is just an extension of the proof of Theorem 1.9. We prove that $H_{\lambda,\star}/(SO(2) \times SO(2))$ is homotopy equivalent to the contractible space $\mathcal{J}_{\lambda,\star}^g$ of compatible almost complex structures for which $S_1 = S^2 \times \{s_0\}$ is a holomorphic curve. Note that $\mathcal{J}_{\lambda,\star}^g \subset \mathcal{J}_\lambda^g$.

Given $J \in \mathcal{J}_{\lambda,\star}^g$, let S_2' be the unique J-sphere in $\mathcal{F}_2$ that passes through $\star$. We construct holomorphic diffeomorphisms $a_1 : (S_1, J|_{S_1}, \star) \to (S_1, J_0|_{S_1}, \star)$ and $a_2 : (S_2', J|_{S_2'}, \star) \to (S_2, J_0|_{S_2}, \star)$ by a procedure which is canonical modulo rotations of the target fixing $\star$ and gives an isometry when possible. For example, by identifying S_1 and S_2 with the standard $S^2 \subset \mathbb{R}^3$, one can normalize the choice of a conformal map by requiring the center of mass of the induced metric on S^2 to be the origin. With the help of the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, the pair (a_1, a_2) defines a diffeomorphism $\phi_J : S^2 \times S^2 \to S^2 \times S^2$. We again have that the form $\phi_J^*(\omega_\lambda)$ is J-positive and so the family of forms $\tau_t = t\sigma + (1-t)\phi_J^*(\omega_\lambda)$ is non-degenerate for all $t \in [0, 1]$. Applying Moser's Theorem we get an isotopy from ϕ_J to a symplectomorphism ψ_J and, because the family $\tau_t|_{S_1}$ is also non-degenerate (since $\phi_J(S_1) = S_1$), this isotopy can be chosen so that we also have $\psi_J(S_1) = S_1$ and $\psi_J(\star) = \star$.

We have thus constructed a map $\beta : \mathcal{J}_{\lambda,\star}^g \to H_{\lambda,\star}/(SO(2) \times SO(2))$. To prove that β is a homotopy equivalence we construct, as in Theorem 1.9, a homotopy inverse by sending $[\psi] \in H_{\lambda,\star}/(SO(2) \times SO(2))$ to $(\psi^{-1})_*(J_0)$. We have that $\beta((\psi^{-1})_*(J_0)) = \psi \bmod(SO(2) \times SO(2))$, while the composition the other way around is homotopic to the identity because $\mathcal{J}_{\lambda,\star}^g$ is contractible. QED

This completes the proof of the theorem. QED

3 Geometric Description of $\mathcal{J}_\lambda^b$

In this section we identify the space $\mathcal{J}_\lambda^b$ with the quotient of G_λ by a particular subgroup. This will be described in the first subsection, where we construct M_λ as a symplectic reduction of $\mathbb{C}^4$ with a particular torus action. For general expositions on symplectic group actions and reduction see, for example, [1] and [13]. The second subsection is devoted to the proof of the description theorem.

3.1 Symplectic reduction construction of M_λ

Consider the unitary action of $\mathbb{T}^2$ on $\mathbb{C}^4$ given by

$$(s,t)\cdot(z_1,z_2,z_3,z_4) = (s^2tz_1, tz_2, sz_3, sz_4),\ \ |s| = |t| = 1, \tag{3}$$

and let $\mu : \mathbb{C}^4 \to \mathbb{R}^2$ be the corresponding moment map:

$$\mu(z_1,z_2,z_3,z_4) = \frac{1}{2}(2|z_1|^2 + |z_3|^2 + |z_4|^2, |z_1|^2 + |z_2|^2). \tag{4}$$

If $\lambda > 0$ one checks easily that $\xi = (2+\lambda, 1) \in \mathbb{R}^2$ is a regular value of μ and so we can consider the symplectic reduction $\mu^{-1}(2+\lambda, 1)/\mathbb{T}^2$ which we denote by R_λ.

The subgroup $SU(2) \times S^1 \subset U(4)$ given by matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ with } A = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}, \alpha\overline{\alpha} = 1, \text{ and } D \in SU(2),$$

acts on $\mathbb{C}^4$ by matrix multiplication. This action preserves μ and commutes with the $\mathbb{T}^2$ action defined above. Since the intersection of these two subgroups of $U(4)$ is $\{\pm 1\} \times \{\pm 1\}$, we get an effective symplectic action of $SO(3) \times S^1 \cong SU(2)/\{\pm 1\} \times S^1/\{\pm 1\}$ on R_λ.

Note that the $SO(3)$-action on R_λ has generic orbits diffeomorphic to $SO(3)$ and exactly two exceptional 2-dimensional orbits given by

$$E = \{ \text{ reduction of } \{z_1 = 0\} \subset \mathbb{C}^4\}$$

and

$$\overline{E} = \{ \text{ reduction of } \{z_2 = 0\} \subset \mathbb{C}^4\}\ ,$$

which are also the fixed point sets of the S^1 action. The symplectic form on R_λ evaluates to $2+\lambda$ on E and to λ on $\overline{E}$. The same picture appears for M_λ with the diagonal action of $SO(3)$, if we replace E by the diagonal $D = \{(s,s) : s \in S^2\}$ and $\overline{E}$ by the antidiagonal $\overline{D} = \{(s,-s) : s \in S^2\}$. It then follows from the classification of $SO(3)$-symplectic 4-manifolds by P.Iglesias (see [8]) that M_λ is $SO(3)$-equivariantly symplectomorphic to R_λ. By pulling back, via the equivariant symplectomorphism, the action of S^1 on R_λ we have proved the following:

Proposition 3.1 *If $\lambda > 0$, M_λ has a symplectic S^1-action that commutes with the diagonal action of $SO(3)$ and has the diagonal D and the antidiagonal $\overline{D}$ as its fixed point sets.*

Remark 3.2 This S^1-action corresponds to the element of infinite order in $H_1(G_\lambda)$ constructed by D.McDuff in [10].

3.2 Description theorem

The following is the main theorem of this section, giving us a geometric description of $\mathcal{J}^b_\lambda$:

Theorem 3.3 *If $0 < \lambda \leq 1$, the space $\mathcal{J}^b_\lambda$ is weakly homotopy equivalent to $G_\lambda/(SO(3) \times S^1)$, where the $SO(3)$ factor acts diagonally on $S^2 \times S^2$ and the S^1 factor was described in Proposition 3.1.*

Remark 3.4 If $\lambda > 1$ this theorem and its proof are still valid, provided we replace $\mathcal{J}^b_\lambda$ by the space of $J \in \mathcal{J}_\lambda$ for which the class $[\overline{D}]$ is represented by a (necessarily unique) embedded J-holomorphic sphere.

Proof: We follow the main steps in the proof of Theorem 2.1.

Let $\mathcal{S}^b_\lambda$ denote the space of embedded symplectic 2-spheres in M_λ representing the homology class $[\overline{D}]$. Associated to any $J \in \mathcal{J}^b_\lambda$ we have a well defined element of $\mathcal{S}^b_\lambda$ given by the unique J-sphere representing $[\overline{D}]$. Conversely, given any element in $\mathcal{S}^b_\lambda$ the space of $J \in \mathcal{J}^b_\lambda$ that make it J-holomorphic is non-empty and contractible. It follows that $\mathcal{J}^b_\lambda$ is weakly homotopy equivalent to $\mathcal{S}^b_\lambda$.

Lemma 3.5 *G_λ acts transitively on $\mathcal{S}^b_\lambda$.*

Before starting the proof of this lemma it is useful to have yet another description of M_λ, this time as a Hirzebruch Surface.

Consider $\mathbb{C}P^1 \times \mathbb{C}P^2$ with Kähler form $\sigma_\lambda = \lambda u + v$, where u and v are the standard integral Kähler forms on $\mathbb{C}P^1$ and $\mathbb{C}P^2$ respectively, and let W_2 be the complex submanifold defined in homogeneous coordinates by

$$W_2 = \{([a,b],[x,y,z]) \in \mathbb{C}P^1 \times \mathbb{C}P^2 : a^2y - b^2x = 0\}.$$

The projection $\pi : \mathbb{C}P^1 \times \mathbb{C}P^2 \to \mathbb{C}P^1$ restricted to W_2 gives it the structure of a $\mathbb{C}P^1$ bundle over $\mathbb{C}P^1$, which is diffeomorphic (but not complex isomorphic) to $S^2 \times S^2$ (see [2], Chapter 2).

The symplectic manifold $(W_2, \sigma_\lambda|_{W_2})$ is symplectomorphic to M_λ (see [10] or the construction below) and we have that:

- the antidiagonal $\overline{D}$ in M_λ corresponds to the zero section in W_2:

$$\overline{D} = \{([a,b],[0,0,1])\} \subset W_2;$$

- the diagonal D in M_λ corresponds to the section at infinity in W_2:

$$D = \{([a,b],[a^2,b^2,0])\} \subset W_2;$$

- a representative for the class $[S_2]$ in M_λ corresponds to a fiber in W_2:

$$F = \{([1,0],[a,0,b])\} \subset W_2, \;\; [F]=[S_2].$$

We abuse notation slightly and denote $\sigma_\lambda|_{W_2}$ also by ω_λ. Note that in W_2 it can be easily checked that F and $\overline{D}$ intersect in an ω_λ-orthogonal way.

We can now give the proof of the lemma.

Proof: Given any $\overline{D}' \in \mathcal{S}^b_\lambda$, pick a $J \in \mathcal{J}^b_\lambda$ that makes it J-holomorphic, and denote by F'' some J-holomorphic sphere representing the homology class $[S_2]$. By again using Lemma 2.3 of [4], there is a symplectic sphere F' which is isotopic to F'' and intersects $\overline{D}'$ in an ω_λ- orthogonal way.

Using the symplectic neighborhood theorem we can now construct a diffeomorphism $\alpha : W_2 \to S^2 \times S^2$ such that $\alpha(\overline{D}) = \overline{D}'$, $\alpha(F) = F'$ and α is a symplectomorphism from a neighborhood U of $\overline{D} \cup F$ onto a neighborhood U' of $\overline{D}' \cup F'$.

In order to finish the proof of this lemma we need to check that Theorem 1.9 can be applied to the complement of $\overline{D} \cup F$ in W_2. From the above description of W_2 we have that

$$W_2 - (\overline{D} \cup F) = \{([a,1],[a^2,1,z])\} \subset \mathbb{C}P^1 \times \mathbb{C}P^2$$

and the map from $\mathbb{C}^2$ to $W_2 - (\overline{D} \cup F)$ given by $(a,z) \mapsto ([a,1],[a^2,1,z])$ identifies these two manifolds. The pull-back of $\sigma_\lambda|_{W_2-(\overline{D}\cup F)}$ can be easily computed and one gets:

$$\begin{aligned} \omega_\lambda &= \frac{i}{2\pi}[\lambda \partial\overline{\partial} \log(1+\|a\|^2) + \partial\overline{\partial}\log(1+\|a\|^4+\|z\|^2)] \\ &= \frac{i}{2\pi}\partial\overline{\partial}\log[(1+\|a\|^2)^\lambda(1+\|a\|^4+\|z\|^2)]. \end{aligned} \tag{5}$$

Let $f : \mathbb{C}^2 \to \mathbb{R}$ be the function

$$f(a,z) = \log[(1+\|a\|^2)^\lambda(1+\|a\|^4+\|z\|^2)].$$

Then $\omega_\lambda = \frac{i}{2\pi}\partial\overline{\partial} f$ and a simple calculation shows that the gradient vector field X_f of f, with respect to the Riemannian metric induced by ω_λ and the standard complex structure on $\mathbb{C}^2$, is conformally symplectic: $L_{X_f}\omega_\lambda = \omega_\lambda$. The function

f has only one critical point at the origin, strictly increases along rays from the origin and its level sets bound star shaped domains and exhaust $\mathbb{C}^2$. For our purposes f behaves like the standard potential $\log(1 + \|a\|^2 + \|z\|^2)$.

If we are given some other symplectic form on $\mathbb{C}^2$, say $\alpha^*(\omega_\lambda)$, that agrees with ω_λ at infinity (i.e. outside some big compact subset), we can use the backward flow of X_f together with rescaling to make it agree with ω_λ outside some arbitrarily small neighborhood of the origin, where we can apply Theorem 1.9 to show that it is equivalent to ω_λ. Hence, there exists a compactly supported diffeomorphism $\beta : W_2 - (\overline{D} \cup F) \to W_2 - (\overline{D} \cup F)$ with $\beta^*(\alpha^*(\omega_\lambda)) = \omega_\lambda$, which means that $(\alpha \circ \beta) : W_2 \to S^2 \times S^2$ is a symplectomorphism sending $\overline{D}$ to $\overline{D}'$.

This completes the proof of the lemma.

QED

Let H_λ be the subgroup of G_λ consisting of symplectomorphisms ψ preserving $\overline{D}$, i.e. $\psi(\overline{D}) = \overline{D}$. From the previous lemma we have that $G_\lambda / H_\lambda \cong \mathcal{S}^b_\lambda$.

The obvious map $G_\lambda/(SO(3) \times S^1) \to G_\lambda / H_\lambda$ is a fibration, with fiber over the identity given by $H_\lambda/(SO(3) \times S^1)$.

Lemma 3.6 *$H_\lambda/(SO(3) \times S^1)$ is weakly contractible.*

Proof: We start by showing that H_λ deformation retracts to its subgroup consisting of elements ψ such that $\psi|_{\overline{D}} \in SO(3)$. Recall that the group of orientation preserving diffeomorphisms of S^2 strongly deformation retracts to $SO(3)$, by S.Smale's theorem (see [17]). Together with Moser's theorem (see [16]) and the fact that in dimension 2 the space of symplectic (or area) forms is convex, this implies that the symplectomorphism group of S^2, denote it by $Diff_{\sigma_0}(S^2)$, deformation retracts to $SO(3)$. Fix a deformation retraction and given $\varphi \in Diff_{\sigma_0}(S^2)$ let $\varphi_t, 0 \le t \le 1$, denote the path of elements in $Diff_{\sigma_0}(S^2)$ such that $\varphi \circ \varphi_t$ realizes the retraction of φ to an element of $SO(3)$ (φ_0 =identity and $\varphi \circ \varphi_1 \in SO(3)$). Given any $\psi \in H_\lambda$ denote by φ its restriction to $\overline{D}$, which is an element of $Diff_{\lambda\sigma_0}(S^2)$. The associated path $\varphi_t \in Diff_{\lambda\sigma_0}(S^2)$, $0 \le t \le 1$, can be lifted to a path $\varphi_t^{(2)} \in H_\lambda$ simply by letting $\varphi_t^{(2)} = \varphi_t \times \varphi_t : S^2 \times S^2 \to S^2 \times S^2$. Then the composition $\psi \circ \varphi_t^{(2)}$ gives a retraction of ψ to the element $\psi \circ \varphi_1^{(2)}$ of H_λ, whose restriction to $\overline{D}$ is in $SO(3)$ as required.

The above argument shows that $H_\lambda/SO(3)$ is homotopy equivalent to $H_{\lambda,\overline{D}} = \{\psi \in H_\lambda : \psi|_{\overline{D}} = \text{ identity}\}$. The symplectic neighbourhood theorem and the local contractibility of the group of symplectomorphisms (see [13], Theorems 3.29 and 10.1) imply that the behaviour of any $\varphi \in H_{\lambda,\overline{D}}$ in a neighbourhood of $\overline{D}$ is determined, up to a canonical symplectic isotopy, by the symplectic linear map

$$(d\psi) : N_{\overline{D}} \to N_{\overline{D}}$$

induced by its differential on the symplectic normal bundle to $\overline{D}$, where $N_{\overline{D}}$ is identified with the ω_λ-orthogonal complement to $T\overline{D}$.

The space of symplectic bundle maps of $N_{\overline{D}}$ is homotopy equivalent to the space of unitary ones, with respect to the unitary structure induced by a compatible complex structure. This space, $Map(S^2, S^1)$, has an evaluation map

$$ev : Map(S^2, S^1) \to S^1$$

given by $ev(f) = f(s_0)$, for some fixed $s_0 \in S^2$. Since $\pi_1(S^2) = 0$ we have that ev is a homotopy equivalence. Note that this S^1 is naturally identified with the S^1 factor referred to in the statement of the lemma.

We conclude that the space of symplectic bundle maps of $N_{\overline{D}}$ is homotopy equivalent to S^1, and from the above discussion it follows that $H_\lambda/(SO(3)\times S^1)$ is homotopy equivalent to

$$H^0_{\lambda,\overline{D}} = \{\psi \in H_\lambda : \psi \equiv \text{ identity in a neighbourhood of } \overline{D}\}.$$

Let F be an embedded symplectic 2-sphere, representing the homology class $[S_2]$, and intersecting $\overline{D}$ positively at a unique point $\star = (s_0, -s_0)$ in an ω_λ-orthogonal way, and denote by $\mathcal{S}^0_{\lambda,F}$ the space of all such symplectic 2-spheres that agree with F in a neighbourhood of $\star$. As in the proof of the previous lemma, we can show that $H^0_{\lambda,\overline{D}}$ acts transitively on $\mathcal{S}^0_{\lambda,F}$. Moreover, it follows from the symplectic isotopy extension theorem (see [13], Theorem 3.18), that the map

$$\begin{array}{ccc} H^0_{\lambda,\overline{D}} & \to & \mathcal{S}^0_{\lambda,F} \\ \psi & \mapsto & \psi(F) \end{array}$$

is a fibration. The fiber over F can be shown, by arguments already used in the proof of this lemma, to be homotopy equivalent to

$$H^0_{\lambda,\overline{D}\cup F} = \{\psi \in H_\lambda : \psi \equiv \text{ identity in a neighbourhood of } \overline{D}\cup F\}.$$

It follows from the fact that Theorem 1.9 can be applied to the complement of $\overline{D}\cup F$ in $S^2\times S^2$ (see the proof of the previous lemma) that $H^0_{\lambda,\overline{D}\cup F}$ is contractible, and so we conclude that $H^0_{\lambda,\overline{D}}$ is homotopy equivalent to $\mathcal{S}^0_{\lambda,F}$.

We finally prove that $\mathcal{S}^0_{\lambda,F}$ is weakly homotopy equivalent to the contractible space $\mathcal{J}_{\lambda,\overline{D}}$ of compatible almost complex structures J for which $\overline{D}$ is a J-holomorphic sphere. Given $F' \in \mathcal{S}^0_{\lambda,F}$ we can easily construct, using the fact that $\overline{D}$ and F' intersect in an ω_λ- orthogonal way at the single point $\star$, a $J \in \mathcal{J}_{\lambda,\overline{D}}$ for which F' is J-holomorphic. Moreover the space of all such J is contractible. On the other hand, given $J \in \mathcal{J}_{\lambda,\overline{D}}$ let F'' be the unique J-holomorphic sphere representing the class $[S_2]$ and going through $\star$. F'' does not necessarily agree with F in a neighbourhood of $\star$, but we can use again Lemma 2.3 of [4] to construct a canonical symplectic isotopy h_t, supported in a neighbourhood U of $\star$, such that $F' = h_1(F'')$ does agree with F in some $U' \subset U$. Here canonical means the construction can be made to depend continuously on any finite number of real parameters that run on some compact set. This shows that $\mathcal{S}^0_{\lambda,F}$ is weakly contractible, completing the proof of the lemma. QED

This completes the proof of the theorem. QED

4 Algebraic Computations

This section is devoted to the algebraic computations necessary to finish the proof of Theorem 0.1. Unless noted otherwise, we assume real coefficients throughout.

4.1 The vector space $H^*(G_\lambda/(SO(3)\times SO(3)))$

Denote by X the contractible space $\mathcal{J}_\lambda$ of compatible almost complex structures, by A the space $\mathcal{J}^b_\lambda \cong G_\lambda/(SO(3)\times S^1)$ and by $X-A$ the space $\mathcal{J}^g_\lambda \cong G_\lambda/(SO(3)\times SO(3))$. Because A has codimension 2 in X we know that $X-A$ is connected. This means that G_λ is connected, which in turn implies that A is connected. Hence

$$H^0(X-A) \cong \mathbb{R} \cong H^0(A)\ . \qquad (6)$$

A version of Alexander-Pontrjagin duality proved by J.Eells in [3] can be applied to the pair (X,A) (see the remarks in the Introduction and the Appendix). Using singular cohomology with arbitrary (closed) supports, this gives the following exact sequence:

$$\begin{array}{ccccccccc} \cdots \leftarrow & H^{p+1}(X) & \leftarrow & H^{p-1}(A) & \leftarrow & H^p(X-A) & \leftarrow & H^p(X) & \leftarrow \cdots \\ & 0 & & & \cong & & & 0 & \qquad (p\geq 1), \end{array}$$

proving the following

Proposition 4.1 *For $p \geq 1$, $H^p(X-A) \cong H^{p-1}(A)$.*

Remark 4.2 Since $H^0(A) \cong \mathbb{R}$, this already implies that $H^1(X-A) \cong \mathbb{R}$. Moreover, because this proposition is still true if we use cohomology with any other coefficients and using the Universal Coefficient Theorem, we have that for any abelian group G:

$$G \cong H^0(A;G) \cong H^1(X-A;G) \cong Hom(H_1(X-A;\mathbb{Z}),G)\ .$$

This implies that $H_1(X-A;\mathbb{Z}) \cong \mathbb{Z}$.

We now explore the geometric descriptions of $\mathcal{J}_\lambda^g$ and $\mathcal{J}_\lambda^b$. To avoid confusion we denote by $SO_d(3)$ the diagonal in $SO(3) \times SO(3)$. Consider the following principal fibrations:

$$\begin{array}{ccccc} SO(3) \times SO(3) & \xrightarrow{i_1} & G_\lambda & \xleftarrow{i_2} & SO_d(3) \times S^1 \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ (X - A) \cong G_\lambda/(SO(3) \times SO(3)) & & & & G_\lambda/(SO_d(3) \times S^1) \cong A \end{array}$$

The associated Leray-Serre cohomology spectral sequences give two different ways of computing the cohomology of G_λ, which of course have to yield the same result. This, together with the above proposition, will enable us to prove the first part of Theorem 0.1. Note that the system of local coefficients in these principal fibrations is trivial since both structure groups are connected.

In order to understand the above fibrations we start by analysing the inclusions $i_k,\ k = 1, 2$. As was already remarked, it is proved by D.McDuff in [10] that the inclusion of the S^1-factor in G_λ is injective in homology. The following proposition implies the analogous result for the inclusion of $SO(3) \times SO(3)$ in G_λ.

Proposition 4.3 *Let $Diff_0(S^2 \times S^2)$ denote the group of diffeomorphisms of $S^2 \times S^2$ that act as the identity on $H_2(S^2 \times S^2; \mathbb{Z})$. The inclusion*

$$i : SO(3) \times SO(3) \to Diff_0(S^2 \times S^2)$$

is injective in homology.

Proof: Given $\varphi \in Diff_0(S^2 \times S^2)$ we define a self map of S^2, denoted by $\tilde{\varphi}_1$, via the composite

$$\tilde{\varphi}_1 : S^2 \xrightarrow{\iota_1} S^2 \times S^2 \xrightarrow{\varphi} S^2 \times S^2 \xrightarrow{\pi_1} S^2 \ ,$$

where ι_1, resp. π_1, denote inclusion as, resp. projection onto, the first S^2-factor of $S^2 \times S^2$. Because φ acts as the identity on $H_2(S^2 \times S^2; \mathbb{Z})$, $\tilde{\varphi}_1$ is an orientation preserving self homotopy equivalence of S^2. Denote the space of all such by $Map_1(S^2)$. Defining $\tilde{\varphi}_2$ in an analogous way using the second S^2-factor of $S^2 \times S^2$, we have thus constructed a map

$$F : Diff_0(S^2 \times S^2) \to Map_1(S^2) \times Map_1(S^2)$$

given by

$$\varphi \mapsto \tilde{\varphi}_1 \times \tilde{\varphi}_2 \ .$$

It is clear from the construction that F restricted to $SO(3) \times SO(3)$ is just the inclusion

$$SO(3) \times SO(3) \hookrightarrow Map_1(S^2) \times Map_1(S^2) \ .$$

Although $Map_1(S^2)$ is not homotopy equivalent to $SO(3)$, the above inclusion is injective in homology (see [6] for these two facts). This implies the result.

QED

Since we are working over the reals it follows from the above that the maps in cohomology

$$i_1^* : H^*(G_\lambda) \to H^*(SO(3) \times SO(3))$$

and

$$i_2^* : H^*(G_\lambda) \to H^*(SO_d(3) \times S^1) ,$$

induced by the inclusions i_1 and i_2, are surjective. Using the Leray-Hirsch Theorem (see [15]) this implies that the spectral sequences of both fibrations collapse at the E_2 term, giving the following two vector space isomorphisms for $H^*(G_\lambda)$:

$$H^*(G_\lambda) \cong H^*(X - A) \otimes H^*(SO(3) \times SO(3)) \tag{7}$$

and

$$H^*(G_\lambda) \cong H^*(A) \otimes H^*(SO_d(3) \times S^1) . \tag{8}$$

One can now easily check, using (6), (7), (8) and Proposition 4.1, that

$$H^2(X - A) = 0 = H^3(X - A)$$

and

$$H^{p+4}(X - A) \cong H^p(X - A), \ p \geq 0 .$$

Hence we conclude that

$$H^*(X - A) \cong \begin{cases} \mathbb{R} & \text{if } * = 4k, 4k+1, k \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

as desired.

4.2 The ring structure

As V.Ginzburg pointed out to me, to compute the ring structure on the cohomology of $G_\lambda/(SO(3) \times SO(3))$ it is enough to show that

$$H^*(G_\lambda) \cong H^*(G_\lambda/(SO(3) \times SO(3))) \otimes H^*(SO(3) \times SO(3)) \tag{9}$$

as graded algebras. The reason is that since G_λ is a group, in particular an H-space, we have that $H^*(G_\lambda)$ is a Hopf algebra. The Leray Structure Theorem for Hopf algebras over a field of characteristic 0 then says that $H^*(G_\lambda)$ is the tensor product of exterior algebras with generators of odd degree and polynomial algebras with generators of even degree (see [18] for relevant definitions and a proof of this structure theorem). The isomorphism (9) will then imply that $H^*(G_\lambda/(SO(3) \times SO(3)))$ is the tensor product of an exterior algebra with one generator of degree 1 and a polynomial algebra with one generator of degree 4, as stated in Theorem 0.1.

We know from the previous subsection that the Leray-Serre cohomology spectral sequence for the fibration

$$\begin{array}{ccc} SO(3)\times SO(3) & \xrightarrow{i_1} & G_\lambda \\ & & \Big\downarrow{\scriptstyle p_1} \\ & & G_\lambda/(SO(3)\times SO(3)) \end{array}$$

collapses with

$$E_\infty^{*,*} \cong E_2^{*,*} \cong H^*(G_\lambda/(SO(3)\times SO(3))) \otimes H^*(SO(3)\times SO(3))$$

as bigraded algebras.

Although proving (7), a vector space isomorphism, this does not directly prove (9), a graded algebra isomorphism. What this says is that the bigraded algebra $E_0^{*,*}(H^*(G_\lambda), F)$, associated to $H^*(G_\lambda)$ with filtration F coming from the above fibration, is isomorphic to $H^*(G_\lambda/(SO(3)\times SO(3)))\otimes H^*(SO(3)\times SO(3))$ (see [15]).

The last step in the proof of (9) is now the following. $H^*(G_\lambda)$ has a subalgebra $p^*(H^*(G_\lambda/(SO(3)\times SO(3)))) \cong H^*(G_\lambda/(SO(3)\times SO(3)))$. From Proposition 4.3 we know that

$$i^* : H^*(G_\lambda) \to H^*(SO(3)\times SO(3))$$

is surjective. Choose $\alpha, \beta \in H^3(G_\lambda)$ so that $i^*(\alpha)$ and $i^*(\beta)$ generate the ring $H^*(SO(3)\times SO(3))$. The subalgebra of $H^*(G_\lambda)$ generated by α and β is isomorphic to $H^*(SO(3)\times SO(3))$ since the only relations in this algebra hold universally on any cohomology algebra (skew-symmetry of the cup product of elements of odd degree). Composing these inclusions of $H^*(G_\lambda/(SO(3)\times SO(3)))$ and $H^*(SO(3)\times SO(3))$ as subalgebras of $H^*(G_\lambda)$ with cup product multiplication in $H^*(G_\lambda)$ we get a map

$$\nu : H^*(G_\lambda/(SO(3)\times SO(3))) \otimes H^*(SO(3)\times SO(3)) \to H^*(G_\lambda)\ .$$

Because $H^*(G_\lambda)$ is (graded) commutative we have that ν is an algebra homomorphism. Moreover, by construction, ν is compatible with filtrations (the obvious one on $H^*(G_\lambda/(SO(3)\times SO(3)))\otimes H^*(SO(3)\times SO(3))$ and F on $H^*(G_\lambda)$). As was already remarked, the degeneration of the spectral sequence at the E_2 term implies that the associated bigraded version of ν is an algebra isomorphism, which in turn shows that ν itself is an algebra isomorphism, concluding the proof of (9) and hence of Theorem 0.1.

Appendix

The purpose of this Appendix is to justify the statement "$\mathcal{J}_\lambda^b$ is a codimension 2, co-oriented submanifold of $\mathcal{J}_\lambda$", which was needed in the beginning of Section 4. We use the set-up and some results of [14], Chapter 3.

Fix an integer $l \geq 1$. Let $\mathcal{J}_\lambda^l$ be the smooth Banach manifold of C^l-almost complex structures on $M = S^2 \times S^2$ that are compatible with ω_λ. Its tangent space $T_J\mathcal{J}_\lambda^l$ at J consists of C^l-sections Y of the bundle $End(TM, J, \omega_\lambda)$ whose fiber at $p \in M$ is the space of linear maps $Y : T_pM \to T_pM$ such that

$$YJ + JY = 0\ ,\ \omega_\lambda(Yv, w) + \omega_\lambda(v, Yw) = 0.$$

This space of C^l-sections is a Banach space and gives rise to a local model for the space $\mathcal{J}_\lambda^l$ via the map $Y \mapsto J\exp(-JY)$. Let $\mathcal{X}^{l+1,p}$, $p > 2$, be the Banach manifold of $W^{l+1,p}$-maps $u : S^2 \to M$, with $[u(S^2)] = [\overline{D}]$. This is the completion of the space of smooth maps with respect to the Sobolev $W^{l+1,p}$-norm given by the sum of the L^p-norms of all derivatives of u up to order $l+1$. Its tangent space at $u \in \mathcal{X}^{l+1,p}$ is $W^{l+1,p}(u^*TM)$, the space of $W^{l+1,p}$ sections of the bundle u^*TM over S^2. Recall that, since $p > 2$, any $u \in \mathcal{X}^{l+1,p}$ is of class C^l by the Sobolev estimates.

Over $\mathcal{X}^{l+1,p} \times \mathcal{J}_\lambda^l$ there is an infinite dimensional vector bundle $\mathcal{E}^{l,p}$ whose fiber at (u, J) is the space

$$\mathcal{E}^{l,p}_{(u,J)} = W^{l,p}(\wedge^{0,1}T^*S^2 \otimes_J u^*TM)$$

of complex anti-linear 1-forms on S^2, of class $W^{l,p}$, with values in the pullback tangent bundle u^*TM. This bundle has a natural section

$$\mathcal{F} : \mathcal{X}^{l+1,p} \times \mathcal{J}_\lambda^l \longrightarrow \mathcal{E}^{l,p}$$

given by

$$\mathcal{F}(u, J) = \overline{\partial}_J(u) = \frac{1}{2}(du \circ j_0 - J \circ du)\ ,$$

where j_0 is the standard complex structure on S^2. Its zero-set is the universal moduli space $\mathcal{M}^l([\overline{D}], \mathcal{J}_\lambda^l)$. In [14], Chapter 3, it is proved that the differential

$$D\mathcal{F}(u, J) : W^{l+1,p}(u^*TM) \times C^l(End(TM, J, \omega_\lambda)) \to W^{l,p}(\wedge^{0,1}T^*S^2 \otimes_J u^*TM)$$

at a zero (u, J) is surjective, and so it follows from the implicit function theorem that the space $\mathcal{M}^l([\overline{D}], \mathcal{J}_\lambda^l)$ is a smooth Banach manifold. This differential $D\mathcal{F}$ is given by the formula

$$D\mathcal{F}(u, J)(\xi, Y) = D_u\xi + \frac{1}{2}Y(u) \circ du \circ j_0$$

for $\xi \in W^{l+1,p}(u^*TM)$ and $Y \in C^l(End(TM, J, \omega_\lambda))$, where D_u is a 0th-order perturbation (coming from the possible non-integrability of J) of the usual Dolbeault $\overline{\partial}$-operator, and so is Fredholm.

Consider now the projection

$$\pi : \mathcal{M}^l([\overline{D}], \mathcal{J}^l_\lambda) \longrightarrow \mathcal{J}^l_\lambda,$$

whose image is exactly the space $\mathcal{J}^{l,b}_\lambda$ of "bad" almost complex structures of class C^l. The tangent space

$$T_{(u,J)}\mathcal{M}^l([\overline{D}], \mathcal{J}^l_\lambda)$$

consists of all pairs (ξ, Y) such that

$$D_u\xi + \frac{1}{2}Y(u) \circ du \circ j_0 = 0$$

and the derivative

$$d\pi(u,J) : T_{(u,J)}\mathcal{M}^l([\overline{D}], \mathcal{J}^l_\lambda) \longrightarrow T_J\mathcal{J}^l_\lambda$$

is just the projection $(\xi, Y) \mapsto Y$. Hence the kernel of $d\pi(u,J)$ is isomorphic to the kernel of D_u whose dimension we will now determine.

Using the metric induced by J and ω_λ on M, and the fact that u is a C^l-embedding (see Fact 1.5 in subsection 1.1), we have the isomorphisms of complex vector bundles

$$u^*TM \cong TM|_{u(S^2)} \cong T_u \oplus N_u \ ,$$

where $T_u = Im(du)$ and $N_u = (T_u)^\perp$ are complex line sub-bundles of $TM|_{u(S^2)}$ because u is J-holomorphic and J is orthogonal with respect to the metric. Given $\xi \in W^{l+1,p}(u^*TM)$ we decompose it accordingly as $\xi = \xi^t + \xi^n$, where $\xi^t \in W^{l+1,p}(T_u)$ and $\xi^n \in W^{l+1,p}(N_u)$. An easy computation, using the explicit formula for D_u given in Chapter 3 of [14], gives the decomposition

$$D_u\xi = D^n_u\xi^n + H^t(\xi^n) + D^t_u\xi^t \ , \tag{10}$$

where $D^t_u : W^{l+1,p}(T_u) \to W^{l,p}(\wedge^{0,1}T^*S^2 \otimes_J T_u)$ is a Dolbeault $\overline{\partial}$-operator on T_u, $D^n_u : W^{l+1,p}(N_u) \to W^{l,p}(\wedge^{0,1}T^*S^2 \otimes_J N_u)$ is a 0th-order perturbation of a Dolbeault $\overline{\partial}$-operator on N_u and $H^t : W^{l+1,p}(N_u) \to W^{l,p}(\wedge^{0,1}T^*S^2 \otimes_J T_u)$ is a 0th-order linear map related to the second fundamental form of $u(S^2) \subset M$. The point to note is that if $\xi = \xi^t \in W^{l+1,p}(T_u)$ then $D_u\xi^t \in W^{l+1,p}(\wedge^{0,1}T^*S^2 \otimes_J T_u)$, i.e. $D_u\xi^t$ has no normal component. This implies that if $\xi = \xi^n + \xi^t \in Ker(D_u)$, then $\xi^n \in Ker(D^n_u)$. The main result proved in [7] implies that D^n_u is injective, because the degree $c_1(N_u) = [\overline{D}] \cdot [\overline{D}] = -2$ is negative. When J is integrable, and consequently D^n_u is equal to a $\overline{\partial}$-operator, this is clear since there can be no nonzero holomorphic sections of a holomorphic line bundle with negative degree. In [7] this result is extended to "generalized $\overline{\partial}$-operators" on complex line bundles over Riemann surfaces, i.e. operators such as D^n_u.

We have thus concluded that if $\xi = \xi^n + \xi^t \in Ker(D_u)$ then $\xi^n = 0$, and from (10) this means that $Ker(D_u) \cong Ker(D^t_u)$. The kernel of D^t_u is naturally identified with the vector space of holomorphic vector fields on S^2, which is the tangent space of the reparametrization group $G = PSL(2, \mathbb{C})$ of real dimension 6.

The image of $d\pi(u, J)$ consists of all Y such that $Y(u) \circ du \circ j_0 \in Im(D_u)$. This is a closed subspace of $T_J \mathcal{J}^l_\lambda$ and, since $D\mathcal{F}(u, J)$ is onto, it has the same (finite) codimension as the image of D_u. Indeed we have a natural isomorphism

$$C^l(End(TM, J, \omega_\lambda))/Im(d\pi(u, J)) \longrightarrow W^{l,p}(\wedge^{0,1} T^* S^2 \otimes_J u^* TM)/Im(D_u)$$

given by

$$Y \longmapsto \frac{1}{2} Y(u) \circ du \circ j_0 \ .$$

It follows that $d\pi(u, J)$ is a Fredholm operator and has the same index as D_u, which in this case is 4. This means that the map between smooth Banach manifolds

$$\mathcal{J}^{l,b}_\lambda = \mathcal{M}^l([\overline{D}], \mathcal{J}^l_\lambda)/G \longrightarrow \mathcal{J}^l_\lambda$$

induced by π is a codimension 2 inclusion. Then an application of the inverse mapping theorem (see [9], Chapter 2) shows that $\mathcal{J}^{l,b}_\lambda$ is a codimension 2 submanifold of $\mathcal{J}^l_\lambda$.

To prove that $\mathcal{J}^{l,b}_\lambda$ is co-oriented in $\mathcal{J}^l_\lambda$ it is enough to show, because of the above identification, that $Coker(D_u)$ is oriented. For that we again follow [14], Chapter 3. The determinant line of a complex linear Fredholm operator carries a natural orientation. Due to the possible non-integrability of J, D_u is not in general a complex linear operator, but it does lie in a component of the space of Fredholm operators which contains the complex linear operator $\overline{\partial}$. Therefore its determinant line

$$det(D_u) = \wedge^{max}(Ker(D_u)) \otimes \wedge^{max}(Coker(D_u)) \tag{11}$$

carries a natural orientation. Since $Ker(D_u)$ has a natural orientation as the tangent space of $G = PSL(2, \mathbb{C})$, we have that (11) determines an orientation for $Coker(D_u)$.

This justifies the use of Alexander-Pontrjagin duality in the beginning of Section 4, provided we work with the spaces $\mathcal{J}^{l,g}_\lambda$ and $\mathcal{J}^{l,b}_\lambda$. As was already remarked in the Introduction, the algebraic topological invariants of these spaces are the same as the ones of $\mathcal{J}^g_\lambda$ and $\mathcal{J}^b_\lambda$ and so Proposition 4.1 is correct as stated.

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