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Kähler geometry of toric manifolds in symplectic coordinates

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Abstract. A theorem of Delzant states that any symplectic manifold (M, ω) of dimension $2n$, equipped with an effective Hamiltonian action of the standard n-torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$, is a smooth projective toric variety completely determined (as a Hamiltonian $\mathbb{T}^n$-space) by the image of the moment map $\phi : M \to \mathbb{R}^n$, a convex polytope $P = \phi(M) \subset \mathbb{R}^n$. In this paper we show, using symplectic (action-angle) coordinates on $P \times \mathbb{T}^n$, how all ω-compatible toric complex structures on M can be effectively parametrized by smooth functions on P. We also discuss some topics suited for application of this symplectic coordinates approach to Kähler toric geometry, namely: explicit construction of extremal Kähler metrics, spectral properties of toric manifolds and combinatorics of polytopes.

1 Introduction

Kähler geometry can be thought of as a "compatible" intersection of complex and symplectic geometries. Indeed, the triple (M^{2n}, J, ω), with $2n$ the **real** dimension of M, is a **Kähler manifold** if

(i) (M^{2n}, J) is a **complex manifold**, i.e. the automorphism $J : TM \to TM$, $J^2 = -I$, is an integrable complex structure;
(ii) (M^{2n}, ω) is a **symplectic manifold**, i.e. the 2-form ω is closed and non-degenerate;
(iii) J and ω are **compatible** in the sense that the bilinear form $\omega(\cdot, J\cdot)$ is a **Riemannian metric**, i.e. symmetric and positive definite.

There are two natural types of local coordinates on a Kähler manifold: complex (holomorphic) and symplectic (Darboux) coordinates. In almost all differential geometric questions on Kähler manifolds, like local Riemannian Kähler geometry or

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existence of Kähler-Einstein metrics, the complex point of view plays the dominant role. There are several reasons for that, one being that in complex coordinates all compatible symplectic forms can be effectively parametrized using functions. In fact, if one fixes a complex structure J_0 and local holomorphic coordinates $(z_1, \ldots, z_n)$, any compatible symplectic form ω is locally of the form

$$\omega = 2i\partial\overline{\partial}f, \ f \equiv \text{local smooth real function.} \tag{1.1}$$

Moreover, if one is given a fixed compatible symplectic form ω_0 on M, defining a cohomology class $\Omega \in H^2(M, \mathbb{R})$, any other compatible symplectic form ω_1 in the same class Ω is globally of the form

$$\omega_1 = \omega_0 + 2i\partial\overline{\partial}f, \ f \in C^\infty(M). \tag{1.2}$$

Because the family $\omega_t = \omega_0 + t(\omega_1 - \omega_0), \ t \in [0,1]$, is an isotopy of symplectic forms in the same cohomology class Ω, Moser's theorem [19] gives a family of diffeomorphisms $\varphi_t : M \to M, \ t \in [0,1]$, such that $\varphi_t^*(\omega_t) = \omega_0$. Hence φ_1 is a Kähler isomorphism between (M, J_0, ω_1) and (M, J_1, ω_0), where $J_1 = (\varphi_1)_*^{-1} \circ J_0 \circ (\varphi_1)_*$. This means that *fixing the complex structure J_0 and varying the compatible symplectic form ω in a fixed cohomology class is equivalent to fixing the symplectic form ω_0 and varying the compatible complex structure J in a fixed diffeomorphism class.* This simple fact makes it reasonable, at least in principle, to try to do Kähler geometry starting from a symplectic coordinate chart, i.e. Darboux coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on which ω_0 has the standard form

$$\omega_0 = \sum_j dx_j \wedge dy_j \ .$$

Unfortunately this does not take us very far in general, since one does not know of any effective way of parametrizing (locally or globally) compatible complex structures.

Nevertheless, the main purpose of this paper is to illustrate how

symplectic coordinates can be useful in Kähler geometry,

more precisely in Kähler toric geometry. As is explained in §2.1, a Kähler toric manifold M can be described either as a compactification of a complex torus $\mathbb{T}^n_{\mathbb{C}} = \mathbb{C}^n / 2\pi i \mathbb{Z}^n$ (complex point of view), or as a compactification of $P^\circ \times \mathbb{T}^n$ (symplectic point of view), where $P^\circ \subset \mathbb{R}^n$ is the interior of a certain convex polytope (moment polytope), $\mathbb{T}^n$ is a real torus $\mathbb{R}^n / 2\pi\mathbb{Z}^n$, and the symplectic form is the standard $\omega_0 = \sum_i dx_i \wedge dy_i$ (the x's and y's being linear coordinates on $P^\circ \subset \mathbb{R}^n$ and $\mathbb{T}^n$, respectively). In §2.2, which in part is based on Guillemin's paper [13], we will see how to change from one point of view to the other and back again by an appropriate Legendre transform and its dual, and also how all ω_0-compatible toric complex structures can be effectively parametrized using smooth functions on P (more precisely, through their "Hessian"). Then, in §2.3, we present Guillemin's explicit formula for a "canonical" ω_0-compatible toric complex structure J_0 on M, expressed only in terms of combinatorial data on P. The analogue of (1.2) for ω_0-compatible toric complex structures is presented in §2.4 and proved in the Appendix. Finally, in section 3, we discuss some applications and open problems related to this symplectic approach to Kähler toric geometry. Specifically, we talk about:

- explicit constructions of extremal Kähler metrics in §3.1;
- spectral properties of toric manifolds in §3.2;

- relation between Kähler toric geometry and combinatorics of polytopes in §3.3.

These topics, chosen according to the author's current knowledge and previous work, are obviously not exhaustive. We believe that other differential-geometric questions can be successfully approached through this "symplectic coordinates" point of view.

My interest in this subject originated from Guillemin's paper [13]. The reading of recent papers by Donaldson ([8] and [9]) and Hitchin ([15] and [16]), where this "duality" between changes in complex structure and changes in symplectic structure is also present (in the context, respectively, of symplectic/complex quotients of infinite dimensional spaces and the Strominger-Yau-Zaslow approach to mirror symmetry), contributed to the clarification in my mind of some of the ideas presented in this paper.

2 Kähler toric manifolds

In this section we present a description of toric manifolds and their invariant Kähler metrics. The point of view is that of a symplectic geometer trying to translate into its own natural context and language some well-known facts from Kähler geometry. Good references for most of what will be presented in this section are Guillemin's paper [13], book [12] and some of the bibliography listed there.

2.1 Definition and "canonical" examples.

Definition 2.1 A **Kähler toric manifold** is a closed connected $2n$-dimensional Kähler manifold (M^{2n}, ω, J) equipped with an effective Hamiltonian holomorphic action

$$\tau : \mathbb{T}^n \to \mathrm{Diff}(M^{2n}, \omega, J)$$

of the standard (real) n-torus $\mathbb{T}^n$.

A few comments are in order regarding this definition. The category where it fits is, of course, Kähler geometry and not just complex geometry or symplectic geometry. As we recall below, almost all closed smooth complex toric varieties and all closed symplectic toric manifolds are Kähler. However, in the complex or symplectic categories, the particular form of a Kähler metric determined by a compatible symplectic form or complex structure is secondary and not part of the original data, although the fact that one exists is important and often used. In this paper, in particular Definition 2.1, the object of study is the Kähler metric involving both the complex and symplectic structures. Two Kähler toric manifolds are isomorphic if they are equivariantly Kähler isomorphic, which implies in particular that they have to be isometric as Riemannian manifolds. For example, while in the purely complex or symplectic (with fixed volume) worlds there is only one toric manifold of dimension 2, the standard sphere in $\mathbb{R}^3$, in the Kähler world different S^1-invariant Riemannian metrics on S^2 determine different Kähler toric manifolds of dimension 2 (see §3.2). Related to this, see also Remark 2.4.

We now recall the usual complex and symplectic definitions of a toric manifold, not only for comparing with Definition 2.1, but also to introduce some of the purely complex and symplectic ingredients we will need.

From the complex geometry point of view, see [6] or [11], a toric variety is a normal [1] variety X that contains a complex torus $\mathbb{T}^n_{\mathbb{C}} = \mathbb{C}^n/2\pi i\mathbb{Z}^n$ as a dense open subset, together with an holomorphic action

$$\mathbb{T}^n_{\mathbb{C}} \times X \to X$$

of $\mathbb{T}^n_{\mathbb{C}}$ on X that extends the natural action of $\mathbb{T}^n_{\mathbb{C}}$ on itself. Hence, a toric variety is a suitable compactification of the complex torus $\mathbb{T}^n_{\mathbb{C}}$. Each particular compactification is determined by a combinatorial gadget called a "fan". Here we will not discuss the details of this complex point of view, although we will make some use of the holomorphic coordinate description of the open dense $\mathbb{T}^n_{\mathbb{C}}$ orbit as $\mathbb{C}^n/2\pi i\mathbb{Z}^n$ (which, as will be seen in the Appendix, is always present on a Kähler toric manifold).

Not all closed smooth complex toric varieties are projective (example on p.71 of [11]), but they are all dominated birationally by a smooth projective toric variety (exercise on p.72 of [11]). Any smooth projective toric variety is Kähler and an example of a Kähler toric manifold (the action of the real torus

$$\mathbb{T}^n = i\mathbb{R}^n/2\pi i\mathbb{Z}^n \subset \mathbb{C}^n/2\pi i\mathbb{Z}^n = \mathbb{T}^n_{\mathbb{C}}$$

is Hamiltonian with respect to the Kähler form). Moreover the converse is also true, i.e. any Kähler toric manifold is equivariantly biholomorphic (but not necessarily Kähler isomorphic) to a smooth projective toric variety (see Remark 2.4 and Proposition A.1 in the Appendix).

From the symplectic geometry point of view, a symplectic toric manifold is a symplectic manifold (M^{2n}, ω) equipped with an effective Hamiltonian action

$$\tau : \mathbb{T}^n \to \mathrm{Diff}(M^{2n}, \omega)$$

of the standard (real) n-torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. Associated to (M, ω, τ) there is a **moment map** $\phi : M \to \mathbb{R}^n$, whose image $P = \phi(M) \subset \mathbb{R}^n$ is a convex polytope. Denoting by P° the interior of P, we have that the set $M^\circ = \phi^{-1}(P^\circ)$ is an open dense subset of M consisting of the points where the $\mathbb{T}^n$-action is free. M° is symplectomorphic to $P^\circ \times \mathbb{T}^n \subset \mathbb{R}^n \times \mathbb{T}^n$ (with its standard linear symplectic structure induced from $\mathbb{R}^{2n}$). Hence, a symplectic toric manifold can be thought of as a suitable compactification of an n-parameter family of Lagrangian tori $\mathbb{T}^n$, the compactification being determined by the combinatorics of a convex polytope $P \subset \mathbb{R}^n$.

Not every convex polytope in $\mathbb{R}^n$ is the moment polytope of some triple (M, ω, τ). The following definition characterizes the ones that are.

Definition 2.2 A convex polytope P in $\mathbb{R}^n$ is **Delzant** if:

(1): there are n edges meeting at each vertex p;
(2): the edges meeting at the vertex p are rational, i.e. each edge is of the form $p + tv_i$, $0 \le t \le \infty$, where $v_i \in \mathbb{Z}^n$;
(3): the $v_1, \ldots, v_n$ in (2) can be chosen to be a basis of $\mathbb{Z}^n$.

In [7] Delzant associates to every Delzant polytope $P \subset \mathbb{R}^n$ a closed connected symplectic manifold (M_P, ω_P) of dimension $2n$, together with a Hamiltonian $\mathbb{T}^n$-action

$$\tau_P : \mathbb{T}^n \to \mathrm{Diff}(M_P, \omega_P)$$

[1] A variety X is **normal** if for any point $p \in X$ the local ring of germs of holomorphic functions $\mathcal{O}(X, p)$ is integrally closed. In particular, any smooth variety X is normal.

with moment map $\phi_P : M_P \to \mathbb{R}^n$, such that the image $\phi_P(M)$ is precisely P. Moreover, he shows that this is a bijective correspondence. More precisely, he proves the following:

Theorem 2.3 *Let (M, ω, τ) be a compact, connected, $2n$-dimensional Hamiltonian $\mathbb{T}^n$-space, on which the action of $\mathbb{T}^n$ is effective with moment map $\phi : M \to \mathbb{R}^n$. Then the image P of ϕ is a Delzant polytope, and (M, ω, τ) is isomorphic as a Hamiltonian $\mathbb{T}^n$-space to (M_P, ω_P, τ_P).*

In Delzant's construction (M_P, ω_P, τ_P) is obtained as the symplectic quotient of the standard symplectic linear space $\mathbb{R}^{2d}$ with respect to the linear action of a $(d-n)$-dimensional subtorus N of the standard torus $\mathbb{T}^d$. Here $d =$ number of facets (or $(n-1)$-dimensional faces) of the polytope $P \subset \mathbb{R}^n$, and N is determined by its combinatorics, more precisely by the integral normals to its facets. The action τ_P is the residual action of $\mathbb{T}^d/N \cong \mathbb{T}^n$ on the symplectic quotient.

One has on $\mathbb{R}^{2d} \cong \mathbb{C}^d$ a standard linear complex structure compatible with the standard symplectic form. This complex structure is invariant under the linear actions of N and $\mathbb{T}^d$, hence descends to a compatible $\mathbb{T}^n$-invariant complex structure on the symplectic quotient. This means that (M_P, ω_P, τ_P) (or, because of the above theorem, any effective Hamiltonian $\mathbb{T}^n$-space with the same moment polytope) is equipped with a "canonical" $\mathbb{T}^n$-invariant complex structure J_P compatible with the symplectic form ω_P. In other words, for each Delzant polytope $P \subset \mathbb{R}^n$, the quadruple $(M_P, \omega_P, J_P, \tau_P)$ is an example of a Kähler toric manifold. These are the **"canonical" examples** of the title of this subsection. Delzant's construction also shows that the $\mathbb{T}^n$-action can be complexified to a holomorphic $\mathbb{T}^n_{\mathbb{C}}$-action on (M_P, J_P), giving M_P the structure of a smooth projective toric variety. Note however that the "canonical" toric Kähler metric $\omega_P(\cdot, J_P \cdot)$ is not projective in general, i.e. it need not be induced from the standard Fubini-Study one on complex projective space.

Remark 2.4 For any Kähler toric manifold (M, ω, J, τ), with moment polytope $P \subset \mathbb{R}^n$, we have that:

(i) (M, ω, τ) is equivariantly symplectomorphic to the "canonical" (M_P, ω_P, τ_P) (by Theorem 2.3);

(ii) (M, J, τ) is equivariantly biholomorphic to the "canonical" (M_P, J_P, τ_P) (see Proposition A.1 in the Appendix).

Hence, the polytope P completely determines the symplectic and complex structures of (M, ω, J, τ), but only if we consider them separately. The toric Kähler metric $\omega(\cdot, J \cdot)$, obtained by combining the two, is **not** determined by P.

2.2 Kähler metrics in symplectic coordinates: set-up. Here we explain the set-up that will be used to describe invariant Kähler metrics on a toric manifold in symplectic coordinates. We first recall the usual set-up in complex coordinates, then present the one in symplectic coordinates and finally justify it by showing how the moment map (or Legendre transform) is the explicit translator from complex to symplectic.

Let $(M^{2n}, \omega, J, \tau)$, $\tau : \mathbb{T}^n \to \mathrm{Diff}(M, \omega, J)$, be a Kähler toric manifold. We again denote by M° the open dense subset of M defined by

$$M^\circ = \{p \in M : \mathbb{T}^n\text{-action is free at } p\}\,.$$

We can describe M° in complex (holomorphic) coordinates as

$$M^\circ \cong \mathbb{C}^n/2\pi i\mathbb{Z}^n = \mathbb{R}^n \times i\mathbb{T}^n = \{u+iv : u \in \mathbb{R}^n,\ v \in \mathbb{R}^n/\mathbb{Z}^n\}$$

(see the proof of Proposition A.1 in the Appendix). In this $z = u+iv$ coordinates, the $\mathbb{T}^n$-action is given by

$$t\cdot(u+iv) = u + i(v+t),\ t \in \mathbb{T}^n,$$

and the complex structure J, which is just multiplication by i, is given by the standard

$$J = \begin{bmatrix} 0 & \vdots & -I \\ \cdots & \cdots & \cdots \\ I & \vdots & 0 \end{bmatrix}$$

where I denotes the $(n\times n)$ identity matrix. The symplectic Kähler form ω is given by a potential $f \in C^\infty(M^\circ)$ through the usual relation $\omega = 2i\partial\overline{\partial}f$. Since ω is invariant by the $\mathbb{T}^n$-action, the potential f depends only on the u coordinates: $f = f(u) \in C^\infty(\mathbb{R}^n)$. Hence, the matrix that represents the skew-symmetric bilinear form ω in this (u,v) coordinates is of the form

$$\begin{bmatrix} 0 & \vdots & F \\ \cdots & \cdots & \cdots \\ -F & \vdots & 0 \end{bmatrix}$$

with $F = \mathrm{Hess}_u(f) \equiv$ Hessian of f in the u coordinates:

$$F = [f_{jk}]_{j,k=1}^{n,n}\ ,\ f_{jk} = \frac{\partial^2 f}{\partial u_j \partial u_k},\ 1 \le j,k \le n\ .$$

Note that the Riemannian Kähler metric $\omega(\cdot, J\cdot)$ is then given in matrix form by

$$\begin{bmatrix} F & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & F \end{bmatrix} \tag{2.1}$$

Not every function $f \in C^\infty(\mathbb{R}^n)$ is the Kähler potential of a symplectic form ω. Because (2.1) defines a Riemannian metric, we have that $F = \mathrm{Hess}_u(f)$ is positive definite. This means that f is strictly convex. Moreover, the fact that ω and the metric compactify smoothly to give a Kähler form and metric on M, puts restrictions on the behaviour of f at infinity. However we will not have to worry with what those restrictions are since, as we will see in § 2.3 and § 2.4, they become simple and explicit in the symplectic set-up that we are interested in and will work with.

In symplectic (or action-angle) coordinates, M° can be described as

$$M^\circ \cong P^\circ \times \mathbb{T}^n = \{(x,y) : x \in P^\circ \subset \mathbb{R}^n, y \in \mathbb{R}^n/\mathbb{Z}^n\}$$

where P° is the interior of a Delzant polytope $P \subset \mathbb{R}^n$. In these (x,y) coordinates, the $\mathbb{T}^n$-action is given by

$$t\cdot(x,y) = (x, y+t),\ t \in \mathbb{T}^n,$$

and the symplectic form ω is the standard $\omega = dx \wedge dy = \sum_j dx_j \wedge dy_j$, which in matrix form is

$$\begin{bmatrix} 0 & \vdots & I \\ \cdots & \cdots & \cdots \\ -I & \vdots & 0 \end{bmatrix}$$

The interesting part now is how one describes the complex Kähler structure J. It turns out, and we will see below why, that J is given by a "potential" $g = g(x) \in C^\infty(P^\circ)$ through the relation $J =$ "Hessian" of g. More precisely, the matrix that represents the complex structure J in these (x, y) coordinates is of the form

$$J = \begin{bmatrix} 0 & \vdots & -G^{-1} \\ \cdots & \cdots & \cdots \\ G & \vdots & 0 \end{bmatrix} \tag{2.2}$$

with $G = \mathrm{Hess}_x(g) \equiv$ Hessian of g in the x coordinates:

$$G = [g_{jk}]_{j,k=1}^{n,n}\ ,\ g_{jk} = \frac{\partial^2 g}{\partial x_j \partial x_k},\ 1 \le j, k \le n\ .$$

Note that the Riemannian Kähler metric $\omega(\cdot, J\cdot)$ is now given in matrix form by

$$\begin{bmatrix} G & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & G^{-1} \end{bmatrix} \tag{2.3}$$

Let us address the question of how to change from one set of coordinates to the other and vice-versa. In complex coordinates, the moment map ϕ of the $\mathbb{T}^n$-action with respect to ω is given by the Legendre transform

$$\phi(u, v) = \frac{\partial f}{\partial u}\ .$$

The restriction $\phi(u, 0),\ u \in \mathbb{R}^n$, is a diffeomorphism of $\mathbb{R}^n$ onto P°. Hence the map

$$x = \frac{\partial f}{\partial u},\ y = v, \tag{2.4}$$

gives us a diffeomorphism from $\mathbb{R}^n \times \mathbb{T}^n$ to $P^\circ \times \mathbb{T}^n$. Moreover, the symplectic form transforms under this change of coordinates from $\omega = 2i\partial\overline{\partial} f$ to the standard $\omega = dx \wedge dy$.

The complex structure J, standard in the (u, v) coordinates, clearly becomes of the form (2.2) where G at the point $x = \frac{\partial f}{\partial u}$ is equal to the inverse of F at the point u. We can now argue in two different ways to show that G is of the form $\mathrm{Hess}_x(g)$ for some $g \in C^\infty(P^\circ)$. One is to quote some general facts about the Legendre transform (see pp.118-126 of [12]). The other is to note that any almost complex structure of the form (2.2), with $G = [g_{jk}]$ a symmetric matrix, is integrable iff

$$g_{jk,l} \equiv \frac{\partial g_{jk}}{\partial x_l} = \frac{\partial g_{jl}}{\partial x_k} \equiv g_{jl,k},\ 1 \le j, k, l \le n,$$

and this is clearly equivalent to $G = \mathrm{Hess}_x(g)$ for some function $g \in C^\infty(P^\circ)$. Since G at the point $x = \frac{\partial f}{\partial u}$ is equal to the inverse of F at the point u, we also have that the map

$$u = \frac{\partial g}{\partial x},\ v = y, \tag{2.5}$$

is (up to a constant) the inverse of (2.4). In other words, f and g are (up to a linear factor) Legendre dual to each other:

$$f(u) + g(x) = \sum_j \frac{\partial f}{\partial u_j} \cdot \frac{\partial g}{\partial x_j},\ \text{at } x = \frac{\partial f}{\partial u} \text{ or } u = \frac{\partial g}{\partial x}. \tag{2.6}$$

2.3 Kähler metrics in symplectic coordinates: "canonical" examples. As was already mentioned in §2.1, a construction of Delzant [7] associates to every Delzant polytope $P \subset \mathbb{R}^n$ a "canonical" Kähler toric manifold $(M_P, \omega_P, J_P, \tau_P)$. In [13] Guillemin gives an explicit formula for the Kähler metric $\omega_P(\cdot, J_P\,\cdot)$ in terms of combinatorial data on P. The purpose of this subsection is to present that formula in the set-up just explained.

A Delzant polytope P can be described by a set of inequalities of the form $\langle x, \mu_r \rangle \geq \lambda_r,\ r = 1, \dots, d$, each μ_r being a primitive element of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ and inward-pointing normal to the r-th $(n-1)$-dimensional face of P. Consider the affine functions $\ell_r : \mathbb{R}^n \to \mathbb{R},\ r = 1, \dots, d$, defined by

$$\ell_r(x) = \langle x, \mu_r \rangle - \lambda_r\ . \tag{2.7}$$

Then $x \in P^\circ$ if and only if $\ell_r(x) > 0$ for all r, and hence the function $g_P : P^\circ \to \mathbb{R}$ defined by

$$g_P(x) = \frac{1}{2} \sum_{r=1}^d \ell_r(x) \log \ell_r(x) \tag{2.8}$$

is smooth on P°.

Theorem 2.5 (Guillemin,[13]**)** *The "canonical" compatible toric complex structure J_P on (M_P, ω_P) is given in the (x, y) symplectic coordinates of $M_P^\circ \cong P^\circ \times \mathbb{T}^n$ by*

$$J_P = \begin{bmatrix} 0 & \vdots & -G_P^{-1} \\ \cdots & \cdots & \cdots \\ G_P & \vdots & 0 \end{bmatrix}$$

with $G_P = \mathrm{Hess}_x(g_P)$, $g_P \equiv$ smooth function on P° defined by (2.8).

Note that, as it should be, g_P is singular on the boundary of P° (and so are G_P and J_P). A consequence of Guillemin's theorem is that the type of singular behaviour that g_P has on ∂P° is the right one to allow for the smooth compactification of J_P (and also the associated Riemannian Kähler metric $\omega_P(\cdot, J_P\,\cdot)$) to the whole M_P.

Example 2.6 (2-sphere) Let $S^2 \subset \mathbb{R}^3$ be the standard sphere of radius 1, with symplectic form given by the standard area form with total area 4π, and S^1-action given by rotation around an axis. The moment map $\phi : S^2 \to \mathbb{R}$ is just projection to this axis of rotation, and so the moment polytope is given by $P = [-1, 1]$. This is determined by the two affine functions

$$\ell_{-1}(x) = 1 + x \text{ and } \ell_1(x) = 1 - x\ ,$$

which means that the function $g_P : P^\circ = (-1,1) \to \mathbb{R}$ has the form

$$g_P(x) = \frac{1}{2}\left[(1+x)\log(1+x) + (1-x)\log(1-x)\right] \ .$$

Computing two derivatives on gets that

$$G_P = \left[\frac{1}{1-x^2}\right] \ ,$$

and so, on $P^\circ \times S^1 = (-1,1)\times S^1$ with (x,y) coordinates, we have that

$$\omega_P = dx\wedge dy\,;\; J_P = \begin{bmatrix} 0 & -(1-x^2) \\ \frac{1}{1-x^2} & 0 \end{bmatrix}\,;\; \omega_P(\cdot, J_P\,\cdot) = \begin{bmatrix} \frac{1}{1-x^2} & 0 \\ 0 & (1-x^2) \end{bmatrix}.$$

As we will see in §3.1, and one can easily check, these correspond to the standard area form, complex structure and round metric on S^2.

Example 2.7 (Projective plane) Let $P \subset \mathbb{R}^2$ be the Delzant polytope defined by

$$\ell_1(x_1,x_2) = 1 + x_1\ ,\ \ell_2(x_1,x_2) = 1 + x_2 \text{ and } \ell_3(x_1,x_2) = 1 - x_1 - x_2\ ,$$

an isosceles right triangle that is well-known to correspond to $\mathbb{CP}^2$ (see e.g. Chapter 1 of [12]). We then have that

$$\begin{aligned} g_P(x_1,x_2) \;&=\; \frac{1}{2}\left[(1+x_1)\log(1+x_1) + (1+x_2)\log(1+x_2) + \right.\\ &\qquad \left.(1-x_1-x_2)\log(1-x_1-x_2)\right] \end{aligned}$$

and

$$G_P(x_1,x_2) = \mathrm{Hess}_x(g_P) = \frac{1}{2(1-x_1-x_2)}\begin{bmatrix} \frac{2-x_2}{1+x_1} & 1 \\ 1 & \frac{2-x_1}{1+x_2} \end{bmatrix} \ .$$

One can check, either by tracing down Delzant's construction or using the curvature computations of §3.1, that this corresponds to the standard Fubini-Study Kähler metric on $\mathbb{CP}^2$ with area 6π on $\mathbb{CP}^1$.

2.4 Kähler metrics in symplectic coordinates: general description. Our goal in this subsection is to describe all possible compatible toric complex structures J on the symplectic toric manifold (M_P, ω_P, τ_P) associated to a Delzant polytope $P \subset \mathbb{R}^n$. If possible, we would like this description to be similar to the one for compatible symplectic forms given by (1.2) .

In light of the set-up of §2.2, in particular (2.2), any such J is determined in the (x,y) symplectic coordinates of $M^\circ \cong P^\circ \times \mathbb{T}^n$ by a "potential" $g = g(x) \in C^\infty(P^\circ)$. Because of the form (2.3) of the Riemannian Kähler metric $\omega_P(\cdot, J\cdot)$, we have that $G = \mathrm{Hess}_x(g)$ is positive definite and so g has to be strictly convex on P°.

To understand what the behaviour of g has to be near the boundary of P°, so that ω_P and J compactify smoothly on M_P, it is useful to look more carefully at the standard g_P, given by (2.8), and the corresponding G_P. Explicit calculations show easily that, although G_P is singular on the boundary of P°, G_P^{-1} is smooth on the whole P and its determinant has the form

$$\det(G_P^{-1}) = \delta_P(x)\cdot \prod_{r=1}^{d} \ell_r(x)\ ,$$

where the ℓ_r's are defined by (2.7) and δ_P is a smooth and strictly positive function on the whole P. The geometric interpretation of this formula is clear. As one reaches the r-th $(n-1)$-dimensional face of P, the positive definite matrix G_P^{-1} acquires a kernel that is generated by the normal μ_r. Because of the nondegeneracy properties of a Delzant polytope (see Definition 2.2), this means for example that at any vertex p of P we have $G_P^{-1} \equiv$ zero matrix.

This type of singular behaviour of g_P, and degenerate behaviour of G_P^{-1}, has to be present in any g and G^{-1} that correspond to honest smooth symplectic and complex structures on the whole M_P. The following theorem, which describes all possible compatible toric J's in a way very similar to (1.2), is just a consequence of that. A detailed proof is given in the Appendix.

Theorem 2.8 *Let (M_P, ω_P, τ_P) be the toric symplectic manifold associated to a Delzant polytope $P \subset \mathbb{R}^n$, and J any compatible toric complex structure. Then J is determined, using (2.2), by a "potential" $g \in C^\infty(P^\circ)$ of the form*

$$g = g_P + h \ , \tag{2.9}$$

where g_P is given by (2.8), h is smooth on the whole P, and the matrix $G = \mathrm{Hess}_x(g)$ is positive definite on P° and has determinant of the form

$$\det(G) = \left[\delta(x) \cdot \prod_{r=1}^{d} \ell_r(x)\right]^{-1} ,$$

with δ being a smooth and strictly positive function on the whole P.

Conversely, any such g determines a compatible toric complex structure J on (M_P, ω_P), which in the (x, y) symplectic coordinates of $M_P^\circ \cong P^\circ \times \mathbb{T}^n$ has the form (2.2).

3 Applications and some interesting problems

In this section we discuss some applications of the symplectic approach to Kähler geometry of toric manifolds, presented in §2, and also state some open problems where we think this approach should be useful. These applications and problems, chosen solely according to the author's current knowledge and previous work, deal with the following differential-geometric issues and tools: scalar curvature (§3.1), spectrum of the Laplacian (§3.2) and differential forms (§3.3).

3.1 Explicit construction of extremal Kähler metrics. In [4] and [5], Calabi introduced the notion of **extremal** Kähler metrics. These are defined, for a fixed closed complex manifold (M, J_0), as critical points of the square of the L^2-norm of the scalar curvature, considered as a functional on the space of all symplectic Kähler forms ω in a fixed Kähler class $\Omega \in H^2(M, \mathbb{R})$. The extremal Euler-Lagrange equation is equivalent to the gradient of the scalar curvature being an holomorphic vector field (see [4]), and so these metrics generalize constant scalar curvature Kähler metrics. Moreover, Calabi showed in [5] that extremal Kähler metrics are always invariant under a maximal compact subgroup of the group of holomorphic transformations of (M, J_0). Hence, on a complex toric manifold, extremal Kähler metrics are automatically toric Kähler metrics, and one should be able to write them down using the general description given by Theorem 2.8.

An attempt in this direction was made in [1], and we now briefly summarize what is obtained there. A Riemannian Kähler metric, given in holomorphic coordinates (u, v) by (2.1), has scalar curvature S given by

$$S = -\frac{1}{2}\sum_{j,k} f^{jk}\,\frac{\partial^2 \log(\det F)}{\partial u_j \partial u_k}\,, \tag{3.1}$$

where the f^{jk}, $1 \le j, k \le n$, denote the entries of the inverse of the matrix $F = \mathrm{Hess}_u(f)$ (see §2.2). Using (2.4) this can be written in symplectic (x, y) coordinates as

$$S = -\frac{1}{2}\sum_{j,k} \frac{\partial}{\partial x_j}\left(g^{jk}\,\frac{\partial \log(\det G)}{\partial x_k}\right)\,, \tag{3.2}$$

which after some algebraic manipulations becomes the more compact

$$S = -\frac{1}{2}\sum_{j,k} \frac{\partial^2 g^{jk}}{\partial x_j \partial x_k}\,, \tag{3.3}$$

where now the g^{jk}, $1 \le j, k \le n$, are the entries of the inverse of the matrix $G = \mathrm{Hess}_x(g)$. The Euler-Lagrange equation defining an extremal Kähler metric can be shown to be equivalent to

$$\frac{\partial S}{\partial x_j} \equiv \text{ constant, } j = 1, \ldots, n, \tag{3.4}$$

i.e. *the metric is extremal if and only if its scalar curvature S is an affine function of x.*

Example 3.1 For the "canonical" toric metric on S^2, presented in Example 2.6, we have that

$$G_P^{-1} = \left[1 - x^2\right]$$

and so

$$S_P(x) = -\frac{1}{2}(1 - x^2)'' \equiv 1.$$

For the "canonical" toric metric on $\mathbb{CP}^2$, presented in Example 2.7, we have that

$$G_P^{-1}(x_1, x_2) = \frac{2}{3}\begin{bmatrix} (2 - x_1)(1 + x_1) & -(1 + x_1)(1 + x_2) \\ -(1 + x_1)(1 + x_2) & (2 - x_2)(1 + x_2) \end{bmatrix}$$

and so

$$S_P(x_1, x_2) \equiv -\frac{1}{3}\left[-2 - 1 - 1 - 2\right] = 2\ .$$

In both these cases, the fact that the metrics have constant scalar curvature makes them (trivial) examples of extremal Kähler metrics and can be used to show that they correspond to standard metrics on S^2 and $\mathbb{CP}^2$.

In [4], Calabi constructed families of extremal Kähler metrics of non-constant scalar curvature. In [1], Calabi's simplest family (on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$) is written down very simply and explicitly in symplectic coordinates, providing an example of the effectiveness of the parametrization given by Theorem 2.8. The following exercise describes a particular metric in that family.

Exercise 3.2 (Blow-up of $\mathbb{CP}^2$) Let $P \subset \mathbb{R}^2$ be the Delzant polytope defined by

$$\ell_1(x_1,x_2) = 1 + x_1\,,\ \ell_2(x_1,x_2) = 1 + x_2\,,\ \ell_3(x_1,x_2) = 1 - x_1 - x_2$$

and

$$\ell_{-3}(x_1,x_2) = 1 + x_1 + x_2\ .$$

(a) Check that the manifold M_P, corresponding to P under Delzant's construction, is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ (the blow-up of $\mathbb{CP}^2$ at one point). Hint: read Chapter 1 of [12].

(b) Show that the "canonical" toric Kähler metric defined by the "potential"

$$g_P(x) = \frac{1}{2}\left(\sum_{r=1}^{3} \ell_r(x)\log \ell_r(x) + \ell_{-3}(x)\log \ell_{-3}(x)\right)$$

is not extremal.

(c) Show that the toric Kähler metric defined by the "potential"

$$g(x_1,x_2) = g_P(x_1,x_2) + \frac{1}{2}\, h(x_1+x_2)\,,$$

with h a function of one variable satisfying

$$h''(t) = \frac{2}{t^2+11t+21} - \frac{1}{t+2}\ ,$$

is extremal.

The toric manifolds $\mathbb{CP}^2 \# \, 2\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2 \# \, 3\overline{\mathbb{CP}}^2$ already give rise to interesting open questions. $\mathbb{CP}^2 \# \, 3\overline{\mathbb{CP}}^2$ is probably the simpler and more appealing of the two. When its Kähler class is 2π times its first Chern class, the corresponding Delzant polytope $P \subset \mathbb{R}^2$ is the hexagon defined by

$$\ell_{\pm 1}(x_1,x_2) = 1 \pm x_1\,,\ \ell_{\pm 2}(x_1,x_2) = 1 \pm x_2 \text{ and } \ell_{\pm 3}(x_1,x_2) = 1 \mp (x_1+x_2)\ .$$

A simple but somewhat tedious computation shows that the "canonical" toric Kähler metric defined by the "potential"

$$g_P(x) = \frac{1}{2}\left(\sum_{r=-3,\, r\neq 0}^{3} \ell_r(x)\log \ell_r(x)\right)$$

is not extremal. It is known, due to an analytic existence result first proved by Siu [20], that a Kähler-Einstein (hence extremal) metric exists and one should be able to

explicitly write it down on P.

Regarding $\mathbb{CP}^2 \# \, 2\overline{\mathbb{CP}}^2$, it is known that it does not have any Kähler metric of constant scalar curvature, but nothing prevents it from having extremal Kähler metrics. To the author's knowledge these have not been constructed or even proved to exist, and

this question should also be analysed in the set-up presented here.

This symplectic approach to the problem of finding extremal Kähler metrics on toric manifolds can be considered as an explicit particular example of a much more general set-up proposed by Donaldson in [8] and [9]. The part of that set-up more relevant for our discussion here is the following. Consider any compact symplectic manifold (M,ω), and suppose for simplicity that $H^1(M,\mathbb{R}) = 0$. Let

$\mathcal{J}$ be the space of all complex structures on M that are compatible with ω, and assume that this space is nonempty. $\mathcal{J}$ can be naturally endowed with the structure of an infinite-dimensional Kähler manifold. Let $\mathcal{G}$ denote the identity component of the group of symplectomorphisms of (M, ω), with Lie algebra naturally identified with the space C_0^∞ of smooth functions on M with integral 0. $\mathcal{G}$ acts naturally on $\mathcal{J}$, and Donaldson shows in [8] that an equivariant moment map for this action is given by

$$\begin{array}{ccl} \mathcal{J} & \longrightarrow & (C_0^\infty)^* \\ J & \longmapsto & S_J \equiv \text{scalar curvature of } \omega(\cdot, J\cdot)\,, \end{array}$$

under the pairing

$$(S, H) = \int_M SH\frac{\omega^n}{n!}\ ,\ \forall H \in C_0^\infty\ .$$

Hence, critical points of the norm square of the moment map are critical points of the square of the L^2-norm of the scalar curvature, considered now as a functional on the space $\mathcal{J}$ of all compatible complex structures. In general this space contains several different diffeomorphism classes of complex structures, and so the critical condition in this context could be more restrictive than the extremal condition of Calabi.

When the compact symplectic manifold (M^{2n}, ω) is toric, with action $\tau : \mathbb{T}^n \to \mathcal{G}$ and moment map $\phi : M \to P \subset \mathbb{R}^n$, we can restrict the above considerations to the space $\mathcal{J}^{inv}$ of invariant compatible complex structures and to the group $\mathcal{G}^{inv}$ of equivariant symplectomorphisms, with Lie algebra naturally identified with the space $(C_0^\infty)^{inv}$ of invariant functions on M (i.e. smooth functions on P) with integral 0. The moment map of the natural action of $\mathcal{G}^{inv}$ on $\mathcal{J}^{inv}$ is again given by the scalar curvature. Since by Proposition A.1 all complex structures in $\mathcal{J}^{inv}$ are in the same diffeomorphism class, we do have that critical points of the norm square of the moment map are in one to one correspondence with extremal Kähler metrics. Moreover, (3.4) gives a symplectic formulation for the complex extremal condition saying that the gradient of the scalar curvature is an holomorphic vector field:

> an invariant compatible complex structure $J \in \mathcal{J}^{inv}$ is extremal if and only if the scalar curvature S_J of the metric $\omega(\cdot, J\cdot)$ is a constant plus a linear combination of the components $\phi_1, \ldots, \phi_n$ of the moment map ϕ.

3.2 Spectral properties of toric manifolds. One can define on any Riemannian manifold M the Laplace operator

$$\Delta : C^\infty(M) \to C^\infty(M)\ .$$

If M is closed, then Δ is a self-adjoint elliptic operator on $L_1^2(M) \equiv$ completion of $C^\infty(M)$ with respect to the norm

$$\|\psi\|^2 = \int_M \psi^2 + \int_M \|\nabla\psi\|^2\ .$$

It follows from the spectral theory of self-adjoint operators that Δ has discrete eigenvalues

$$0 = \lambda_0 < \lambda_1 \le \cdots \le \lambda_j \le \cdots\ ,\ \lambda_j \overset{j\to\infty}{\longrightarrow} \infty\,,$$

and the corresponding eigenfunctions $\{\psi_j\}_{j=0}^\infty$, satisfying

$$\Delta\psi_j = \lambda_j\psi_j,\ \psi_j \in C^\infty(M) \cap L_1^2(M)\,,$$

can be chosen so that they form an orthonormal basis of $L^2_1(M)$.

When one tries to give estimates for the eigenvalues, the Min-Max principle plays a fundamental role. It can be formulated as follows. Define $H_j \subset C^\infty(M),\ j = 1, \dots,$ by

$$H_j = \left\{ \psi \in C^\infty(M) : \psi \neq 0 \text{ and } \int_M \psi \cdot \psi_k = 0,\ k = 0, \dots, j-1 \right\} .$$

Then we have that

$$\lambda_j = \inf_{\psi \in H_j} \frac{\displaystyle\int_M \|\nabla \psi\|^2}{\displaystyle\int_M \psi^2} \,,\ j = 1, \dots \ . \tag{3.5}$$

The relation between the spectrum of Δ and geometric properties of M has a long history, an important chapter of which is Mark Kac's celebrated paper [17] on the question:

can one hear the shape of a drum?

When a group acts by isometries on M, the Laplace operator can be restricted to the subspace of invariant functions and one can consider its spectral theory there. The corresponding eigenvalues will be called here **invariant eigenvalues**, and one might ask how much of the geometry of M can be recovered from them? Our purpose in this subsection is to show how asking this question for Kähler toric manifolds quickly leads to interesting problems.

The Laplace operator Δ of a toric Kähler metric given in holomorphic coordinates (u, v) by (2.1), has the form

$$(\Delta\psi)(u) = -\frac{1}{\det F} \sum_{j,k=1}^n \frac{\partial}{\partial u_j} \left((\det F) f^{jk} \frac{\partial \psi}{\partial u_k} \right)$$

for any $\mathbb{T}^n$-invariant smooth function $\psi = \psi(u)$. Using (2.4) this can be written in symplectic (x, y) coordinates as

$$(\Delta\psi)(x) = -(\det G) \sum_{j,k=1}^n g^{jk} \frac{\partial}{\partial x_j} \left(\frac{1}{\det G} \frac{\partial \psi}{\partial x_k} \right) \tag{3.6}$$

where $\psi = \psi(x)$ is again any $\mathbb{T}^n$-invariant smooth function. Given any Delzant polytope P and Kähler toric manifold (M_P, ω_P, J) defined by a "potential" g according to Theorem 2.8, the spectrum on $C^\infty(P)$ of the corresponding operator Δ, defined by (3.6), is the invariant spectrum of (M_P, ω_P, J) and will be denoted by

$$0 = \lambda_0 < \lambda_1(g) \leq \lambda_2(g) \leq \cdots \ .$$

The corresponding $\mathbb{T}^n$-invariant eigenfunctions are denoted by $\psi_{g,j} \in C^\infty(P)$.

One easily computes $\|\nabla\psi\|^2$ to be given pointwise by

$$\|\nabla\psi\|^2 = \sum_{j,k=1}^n \frac{\partial \psi}{\partial x_j} g^{jk} \frac{\partial \psi}{\partial x_k} = G^{-1}\left(\frac{\partial \psi}{\partial x}\right)$$

where $\frac{\partial\psi}{\partial x} = (\frac{\partial\psi}{\partial x_1}, \dots, \frac{\partial\psi}{\partial x_n})$ and $G^{-1}(\cdot)$ denotes the quadratic form on $\mathbb{R}^n$ defined by the symmetric matrix G^{-1}. The fact that in symplectic coordinates the volume form is always standard, makes the dependence of the Min-Max principle (3.5) on the metric much easier to understand. In particular, for the invariant eigenvalues we get

$$\lambda_j(g) = \inf_{\psi \in H_{g,j}} \frac{\displaystyle\int_P G^{-1}\left(\frac{\partial \psi}{\partial x}\right) dx}{\displaystyle\int_p \psi^2(x)\, dx}, \ j = 1, \dots , \tag{3.7}$$

where $H_{g,j} = \left\{\psi \in C^\infty(P) : \psi \neq 0 \text{ and } \int_P \psi \cdot \psi_{g,k} = 0,\ k = 0, \dots , j-1\right\}$. We see that the only major dependence on the metric is the G^{-1} that appears on the numerator (for $j = 1$ this is actually the only dependence on the metric because $H_{g,1} = \left\{\psi \in C^\infty(P) : \psi \neq 0 \text{ and } \int_P \psi = 0\right\}$ is independent of g). Hence, to increase the invariant eigenvalues $\lambda_j(g)$ one should try to increase the eigenvalues of G^{-1}, or equivalently decrease the eigenvalues of $G = \text{Hess}_x(g)$, always subject to the restrictions imposed by Theorem 2.8.

In [2] this idea is applied with some interesting results to the simplest possible example, the 2-sphere S^2. A theorem of J.Hersch [14] states that for any smooth metric on S^2, with total area equal to 4π, the first nonzero eigenvalue of the Laplace operator is less than or equal to 2 (this being the value for the standard round metric). It is natural to ask if, for S^1-invariant metrics, there is a similar upper bound for the first invariant eigenvalue? It turns out that the answer in general is no (see [10] and [2]). However, restricting to the much more rigid but still geometrically interesting class of surfaces of revolution in $\mathbb{R}^3$, one has the following

Theorem 3.3 ([2]) *Within the class of smooth S^1-invariant metrics g on S^2 with total area 4π and corresponding to a surface of revolution in $\mathbb{R}^3$, we have that*

$$\lambda_j(g) < \frac{\xi_j^2}{2}, \quad j = 1, \dots ,$$

where ξ_j is the $((j+1)/2)^{\text{th}}$ positive zero of the Bessel function J_0 if j is odd, and the $(j/2)^{\text{th}}$ positive zero of J_0' if j is even. These bounds are optimal.

In particular,

$$\lambda_1(g) < \frac{\xi_1^2}{2} \approx 2.89.$$

The proof of this theorem goes roughly as follows. Because in dimension 2 any metric is a Kähler metric, we have that any S^1-invariant metric on S^2 is a toric Kähler metric. Hence we can use (3.7) and try to maximize the single entry of the matrix G^{-1} ($n = 1$ in this S^2 case). It turns out that for surfaces of revolution in $\mathbb{R}^3$, which can be very explicitly characterized in this context, one accomplishes that in an optimal way by deforming the standard sphere, through a family of ellipsoids of revolution obtained by "pressing" the North and South poles against each other, towards the union of two flat discs of area 2π each (a singular surface). The estimates of Theorem 3.3 follow from the values of the invariant eigenvalues of the Euclidean Laplacian on a disc.

Hersch's theorem has been generalized by J.-P. Bourguignon, P. Li and S.T. Yau [3] to any Kähler metric on a projective complex manifold. Hence, it makes sense to generalize the question answered for S^1-invariant metrics on S^2 and ask:

under what conditions does there exist an upper bound for the first invariant eigenvalue of a Kähler toric manifold?

Borrowing from M. Kac, we can also ask the following attractive question on spectral geometry of toric manifolds:

can one hear the shape of a Delzant polytope?

As we have seen, associated to every Delzant polytope P we have a "canonical" Kähler toric manifold (M_P, ω_P, J_P) given by Theorems 2.3 and 2.5, and hence a "canonical" Laplace operator Δ_P, which by (3.6) can be explicitly written down for invariant functions as

$$\Delta_P(\psi) = -(\det G_P) \sum_{j,k=1}^{n} (g_P)^{jk} \frac{\partial}{\partial x_j} \left(\frac{1}{\det G_P} \frac{\partial \psi}{\partial x_k} \right), \ \psi \in C^\infty(P)\,.$$

Its spectrum, which coincides with the invariant spectrum of (M_P, ω_P, J_P) and will be denoted by

$$0 = \lambda_0 < \lambda_1(P) \leq \lambda_2(P) \leq \cdots \ ,$$

is then a sequence of numbers canonically associated to the Delzant polytope P, and the question is:

can we recover P from $\{\lambda_j(P)\}_{j=1}^\infty$?

Note that, besides being obviously invariant under translations of P in $\mathbb{R}^n$, both Δ_P and its spectrum are also invariant under $SL(n, \mathbb{Z})$ transformations of $\mathbb{R}^n$, i.e.

$$(\Delta_P(f)) \circ A^{-1} = \Delta_{A(P)}(f \circ A^{-1})\,, \ \forall A \in SL(n, \mathbb{Z})\,,$$

and so

$$\lambda_j(P) = \lambda_j(A(P))\,, \ j = 0, 1, \ldots\,, \ \forall A \in SL(n, \mathbb{Z})\,.$$

This follows easily from the fact that a $SL(n, \mathbb{Z})$ transformation of $\mathbb{R}^n$ simply corresponds to a $SL(n, \mathbb{Z})$ automorphism of the acting torus $\mathbb{T}^n$, i.e.

$$(M_{A(P)}, \omega_{A(P)}, J_{A(P)}) \cong (M_P, \omega_P, J_P)\,,$$

$$\tau_{A(P)} = \tau_P \circ A^T : \mathbb{T}^n \to \mathrm{Diff}(M_{A(P)}, \omega_{A(P)}, J_{A(P)})\,,$$

$$\phi_{A(P)} = A \circ \phi_P : M_{A(P)} \to A(P) \subset \mathbb{R}^n\,.$$

Hence, what we can hope to recover from the invariant spectrum $\{\lambda_j(P)\}_{j=1}^\infty$ is the equivalence class of P under translations and $SL(n, \mathbb{Z})$ transformations of $\mathbb{R}^n$.

3.3 Kähler toric geometry and combinatorics of Delzant polytopes. The application of geometry of toric varieties to combinatorics of convex polytopes has been very successful, with one of the most striking examples being the use of the Hard Lefshetz Theorem for simplicial toric varieties to prove McMullen's conjectures for the number of vertices, edges, faces, etc, of convex simplicial polytopes, in R. Stanley's paper [21]. As I. Dolgachev writes in his review of this paper [MR 81f:52014], "one must always look for toric varieties whenever one has a problem on convex polytopes".

Our goal in this subsection is to present, in the symplectic setting of §2, some tools from Kähler geometry that might be relevant to combinatorial problems on Delzant polytopes. A particular such problem, suggested by R. MacPherson during an informal conversation in 1997 and directly related to Stanley's result, is the following.

Let $P \subset \mathbb{R}^n$ be a Delzant polytope and denote by $f_j(P)$, $j = 0, \ldots, n$, the number of j-dimensional faces of P (with the convention that $f_n(P) = 1$). It is well-known (see [6] or [11]) that the Betti numbers of the toric manifold M_P associated to P are given by:

$$\begin{aligned} b^{2k+1}(M_P) &= 0,\ k = 0, \dots, n-1\,; \\ b^{2k}(M_P) &= \sum_{j=0}^{k} \binom{n-j}{n-k} (-1)^{k-j} f_{n-j}(P),\ k = 0, \dots, n\,. \end{aligned} \tag{3.8}$$

The $b^{2k}(M_P)$ are also denoted by $h^k(P),\ k = 0, \dots, n$, and called the h-numbers of P. Because we know that (M_P, ω_P) is a Kähler manifold, the Hard Lefshetz Theorem (HLT) applies and says that the map

$$\begin{aligned} L^{n-k}\ :\quad H^k(M_P) &\to H^{2n-k}(M_P),\ 0 \le k \le n\,, \\ \alpha &\mapsto \alpha \wedge (\omega_P)^{n-k} \end{aligned}$$

is an isomorphism. This immediately puts restrictions on the h-numbers of P, namely:

(i) $h^k(P) = h^{n-k}(P),\ 0 \le k \le n$;

(ii) $h^{k+1}(P) - h^k(P) \ge 0,\ k = 0, \dots, [n/2]$;

which are essentially McMullen's conjectures.

One would like to have a simple combinatorial proof of restrictions (i) and (ii) for the h-numbers of P. Of course, it would be even better if one could give a simple combinatorial proof of the full HLT. On a Kähler manifold the simplest proof of HLT is through Hodge theory, and the project suggested by MacPherson amounted to

develop on P a combinatorial version of Hodge theory on (M_P, ω_P, J_P).

The word "combinatorial" here means, at least to me, using data and analysis coming only from the polytope P. Differential forms, integration, differentiation, etc, are allowed provided they can be defined and performed directly on P.

Although MacPherson's project/problem is still very much open, the following might be a small initial step towards a solution. As we have seen in §2.3, associated to a Delzant polytope $P \subset \mathbb{R}^n$ we have the affine functions $\ell_r : \mathbb{R}^n \to \mathbb{R}$ defined by (2.7), the potential $g_P : P^\circ \to \mathbb{R}$ defined by (2.8), its Hessian $G_P = [(g_P)_{jk}] = \text{Hess}(g_P)$, and the "canonical" compatible toric complex structure J_P on (M_P, ω_P) given by Theorem 2.5. On any complex manifold, closed 2-forms of type $(1,1)$ can be written locally as $\partial\overline{\partial}$ of a smooth potential function. It is known that the cohomology of a Kähler toric manifold is generated by its $(1,1)$ part, and so one should understand how to represent the $\partial\overline{\partial}$ operator and the "generating" potentials on the polytope P. Note that, because harmonic forms are always invariant under isometries, we only need to understand $\mathbb{T}^n$-invariant forms and potentials for MacPherson's project.

In holomorphic (u, v) coordinates, and for a $\mathbb{T}^n$-invariant function $\nu = \nu(u)$, we have that

$$2i\partial\overline{\partial}\nu = \sum_{j,k=1}^{n} \frac{\partial^2 \nu}{\partial u_j \partial u_k}\, du_j \wedge dv_k\ .$$

Using (2.4) this means that in symplectic (x, y) coordinates, and for a $\mathbb{T}^n$-invariant function $\nu = \nu(x)$, we have

$$2i\partial\overline{\partial}\nu = \sum_{j,k,l=1}^{n} \frac{\partial}{\partial x_j}\left((g_P)^{kl}\frac{\partial \nu}{\partial x_l}\right) dx_j \wedge dy_k \ . \tag{3.9}$$

Remark 3.4 This formula becomes valid for any compatible toric complex structure J on (M_P, ω_P), determined by a "potential" $g \in C^\infty(P^\circ)$ of the form given in Theorem 2.8, by replacing g_P with g.

As to what the relevant "generating" potentials should be, Guillemin shows in [13] that there is one for each $(n-1)$-dimensional face of P and it is given by

$$\nu_r(x) = \log \ell_r(x),\ r = 1, \ldots, d \equiv f_{n-1}(P) \ . \tag{3.10}$$

He also shows that the differential forms

$$\alpha_r = \frac{1}{2\pi i}\left(\partial\overline{\partial}\nu_r\right) = -\frac{1}{4\pi}\sum_{j,k,l=1}^{n} \frac{\partial}{\partial x_j}\left((g_P)^{kl}\frac{\partial \log \ell_r}{\partial x_l}\right) dx_j \wedge dy_k \tag{3.11}$$

represent the Poincaré duals to the complex hypersurfaces $X_r \subset (M_P, J_P)$ defined by $\ell_r \circ \phi_P \equiv 0$ (i.e. the pre-images under the moment map ϕ_P of the $(n-1)$-dimensional faces of P), and are known to generate the full cohomology ring of M_P.

Exercise 3.5 (Symplectic potential)

(i) Using (3.9) and Remark 3.4 show that, for any toric complex structure J determined by a "potential" $g \in C^\infty(P^\circ)$, the function

$$f_g(x) = \sum_{m=1}^{n} x_m \frac{\partial g}{\partial x_m}(x) \ - \ g(x)$$

is the potential for the standard symplectic form

$$\omega_P = \sum_{j=1}^{n} dx_j \wedge dy_j \ .$$

Note: this also follows directly from (2.4) and (2.6).

(ii) Show that

$$f_{g_P}(x) = \frac{1}{2}\sum_{r=1}^{d} \lambda_r \log \ell_r(x) \ + \ \ell_\infty(x)\,,$$

where $\ell_\infty(x) = \sum_{r=1}^{d} \langle x, \mu_r\rangle$ is a smooth function on the whole P.

(iii) Conclude, using (3.10) and (3.11), that

$$\frac{1}{2\pi}\,[\omega_P] = -\sum_{r=1}^{d} \lambda_r \alpha_r \in H^2(M_P, \mathbb{R}) \ .$$

If one applies (3.8) to compute $b^2(M_P)$ one gets

$$b^2(M_P) \equiv h^1(P) = f_{n-1} - n \equiv d - n \ .$$

Although (3.10) and (3.11) seem to give more generators to $H^2(M_P)$ than are actually needed, the next proposition (stated on purpose in a self-contained and purely combinatorial manner) shows that is not the case.

Proposition 3.6 *Let $P \subset \mathbb{R}^n$ be a Delzant polytope defined by the set of inequalities $\ell_r(x) \geq 0,\ r = 1, \dots, d$, with*

$$\ell_r(x) = \langle x, \mu_r \rangle - \lambda_r \,,$$

each μ_r being a primitive element of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ and inward-pointing normal to the r-th $(n-1)$-dimensional face of P. Consider the function $g_P : P^\circ \to \mathbb{R}$ defined by

$$g_P(x) = \frac{1}{2} \sum_{r=1}^{d} \ell_r(x) \log \ell_r(x) \,,$$

its Hessian $G_P = \mathrm{Hess}(g_P) = [(g_P)_{jk}]$, the inverse $G_P^{-1} = \left[(g_P)^{jk}\right]$, and let $\Omega^2(P)$ be the real vector space of differential 2-forms generated over $C^\infty(P)$ by

$$dx_j \wedge dy_k,\ \ 1 \leq j, k \leq n \ .$$

Then, the subspace $H^2(P) \subset \Omega^2(P)$ generated over $\mathbb{R}$ by the differential forms

$$-4\pi\alpha_r = \sum_{j,k,l=1}^{n} \frac{\partial}{\partial x_j} \left((g_P)^{kl} \frac{\partial \log \ell_r}{\partial x_l} \right) dx_j \wedge dy_k \,,\ \ r = 1, \dots, d \,,$$

has real dimension $d - n$.

Proof Consider the linear map $R : \mathbb{R}^d \to H^2(P)$ defined by

$$R(a_1, \dots, a_d) = \sum_{r=1}^{d} a_r [-4\pi\alpha_r] \ .$$

R is clearly surjective and one easily checks that its kernel coincides with the image of the $(d \times n)$ matrix whose rows are the coordinates in $\mathbb{R}^n$ of the normals $\mu_r,\ r = 1, \dots, d$. Since, by condition (3) of Definition 2.2, the rank of such a matrix is n, we conclude that

$$\dim\, \ker(R) = n \,,$$

and so

$$\dim\, H^2(P) = d - n \,.$$

$\square$

Hence we do get a **unique** combinatorial differential form representing each cohomology class in $H^2(M_P, \mathbb{R})$. Although this has the flavour of Hodge theory, the forms α_r are in general not harmonic with respect to the "canonical" metric $\omega_P(\cdot, J_P\, \cdot)$. For example, the harmonic representative for $[\omega_P] \in H^2(M_P, \mathbb{R})$ is always the standard

$$\omega_P = \sum_{j=1}^{n} dx_j \wedge dy_j \,,$$

while we know from Exercise 3.5 that its representative in $H^2(P)$ is given by

$$-2\pi \sum_{r=1}^{d} \lambda_r \alpha_r \,.$$

It also follows easily from Exercise 3.5 that these two representatives are the same iff

$$\ell_\infty(x) = \sum_{r=1}^{d} \langle x, \mu_r \rangle \equiv \text{constant} \Leftrightarrow \sum_{r=1}^{d} \mu_r = \overline{0} \in \mathbb{R}^n \,,$$

and this is not always true for a Delzant polytope P (take for example the 4-gon or 5-gon in $\mathbb{R}^2$ corresponding to $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$). Even when this $\ell_\infty \equiv 0$ condition is satisfied, computations on simple examples (like $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$) quickly convince us that the individual forms α_r are almost never harmonic (it is likely that the only exceptions are projective spaces and their cartesian products).

In spite of this, Proposition 3.6 gives some hope that consideration of differential forms of the type given by (3.11) could be an initial step towards a combinatorial Hodge type theory on P. For a different approach, based on constant coefficient linear differential operators instead of differential forms, see the recent work of Timorin [22].

Appendix A Proof of Theorem 2.8

In this appendix we give a proof of Theorem 2.8. The idea is again to translate to symplectic Kähler geometry some well-known facts from complex Kähler geometry.

Let (M_P, ω_P, τ_P) be the toric symplectic manifold associated to a Delzant polytope $P \subset \mathbb{R}^n$, and J any compatible toric complex structure. As we know from section 2, Delzant's construction also provides (M_P, ω_P, τ_P) with a "canonical" compatible toric complex structure J_P. The next proposition explains why, in the statement of Theorem 2.8, we do not need to impose the condition of J being in the same diffeomorphism class as J_P.

Proposition A.1 *(M_P, J, τ_P) is equivariantly biholomorphic to (M_P, J_P, τ_P).*

Proof In the algebraic geometry context this is proved, for example, in [18], Chapter I, §2, Theorem 6. The key point of that proof is to show that every point of M_P admits an open invariant affine neighborhood. These neighborhoods are then patched together according to the combinatorics of a "fan" that uniquely determines the resulting normal variety. This same scheme can be applied to our symplectic context, with the role of the "fan" being played by the polytope P. We will now sketch how the argument goes.

Every point p of M_P admits an open invariant affine neighborhood U_p. The two extreme cases are when $p \in M_P^\circ$ and U_p is equivariantly biholomorphic to $\mathbb{C}^n/2\pi i\mathbb{Z}^n$, and when p is a fixed point of the action and (U_p, p) is equivariantly biholomorphic to $(\mathbb{C}^n, 0)$ with its standard $\mathbb{T}^n$-action. In all cases, the neighborhood itself only depends on the action and symplectic structure. The dependence on the complex structure appears only in the equivariant biholomorphism. The basic reason these neighborhoods exist is that we have $2n$ holomorphic vector fields in involution globally defined on M_P. In fact, let $\xi_1, \ldots, \xi_n$ be Hamiltonian vector fields on M_P induced by the action τ_P from a fixed standard basis of the Lie algebra of $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. Then $(\xi_1, \ldots, \xi_n, J\xi_1, \ldots, J\xi_n)$ (resp. $(\xi_1, \ldots, \xi_n, J_P\xi_1, \ldots, J_P\xi_n)$) are holomorphic vector fields on (M_P, J) (resp. (M_P, J_P)) in involution, since both J and J_P are integrable and $\mathbb{T}^n$-invariant, and the ξ_i's commute with each other. Moreover, on M_P° these vector fields are all non-zero and linearly independent (here the compatibility of J and J_P with ω_P is used) and so can be integrated to give the equivariant biholomorphism with $\mathbb{C}^n/2\pi i\mathbb{Z}^n$. On $M_P \setminus M_P^\circ = \phi_P^{-1}(\partial P)$, $\phi_P \equiv$ moment map, the vanishing of some of this vector fields is completely determined by the combinatorics of P, more precisely by the normals to its $(n-1)$-dimensional faces, and this information is sufficient to construct the required invariant affine neighborhoods of these points. For example, for a fixed point $p = \phi_P^{-1}(\text{vertex } v_p)$ we have that $U_p = \phi_P^{-1}(P^\circ \cup \{\text{faces of } P \text{ whose closure contains } v_p\})$ is (both for J and J_P) equivariantly biholomorphic to $(\mathbb{C}^n, 0)$.

Having established the existence of the required open invariant affine neighborhoods U_p for every point $p \in M_P$, one then checks that the patching of these neighborhoods is also completely determined by P and hence is the same for both (M_P, J, τ_P) and (M_P, J_P, τ_P). The simplest example of such patching is between $M_P^\circ \cong \mathbb{C}^n/2\pi i\mathbb{Z}^n$ and $U_p \cong (\mathbb{C}^n, 0)$, for a fixed point p corresponding to a vertex v_p of P for which the normals to the $(n-1)$-dimensional faces that meet at v_p are the standard basis vectors of $\mathbb{R}^n$. In this case the patching is simply given by

$$\begin{array}{rcl} \mathbb{C}^n/2\pi i\mathbb{Z}^n & \longrightarrow & \mathbb{C}^n \\ z = (z_1, \ldots, z_n) & \longmapsto & e^z = (e^{z_1}, \ldots, e^{z_n})\,. \end{array}$$

□

Let $\varphi_J : (M_P, J_P) \to (M_P, J)$ be such an equivariant biholomorphism. By the construction given in the above proof, we know that φ_J can be chosen so that it acts as the identity in cohomology. Then (M_P, ω_P, J) is equivariantly Kähler isomorphic to (M_P, ω_J, J_P), with $\omega_J = (\varphi_J)^*(\omega_P)$ and $[\omega_J] = [\omega_P] \in H^2(M_P)$. This means that there exists a $\mathbb{T}^n$-invariant smooth function $f_J \in C^\infty(M_P)$ such that

$$\omega_J = \omega_P + 2i\partial\overline{\partial} f_J\,, \tag{A.1}$$

where the ∂- and $\overline{\partial}$-operators are defined with respect to the complex structure J_P.

In the (x, y) symplectic coordinates of $M_P^\circ \cong P^\circ \times \mathbb{T}^n$, obtained via the "canonical" moment map $\phi_P : M_P \to \mathbb{R}^n$ with respect to ω_P, we then have the following. By Theorem 2.5, J_P is obtained from the "potential"

$$g_P(x) = \frac{1}{2}\sum_{r=1}^{d} \ell_r(x) \log \ell_r(x) \in C^\infty(P^\circ)\,,$$

where the $\ell_r : \mathbb{R}^n \to \mathbb{R},\ r = 1, \ldots, d$, are the affine functions defining P and given by (2.7). By (3.9), Exercise 3.5 and (A.1), we have that ω_J is given by

$$\omega_J = \omega_P + 2i\partial\overline{\partial} f_J = \sum_{j,k,l=1}^{n} \frac{\partial}{\partial x_j}\left((g_P)^{kl}\,\frac{\partial(f_P + f_J)}{\partial x_l}\right) dx_j \wedge dy_k \equiv 2i\partial\overline{\partial}\tilde{f}\,, \tag{A.2}$$

where $\tilde{f} \equiv f_P + f_J$, $f_J \in C^\infty(P)$ and

$$f_P(x) \equiv f_{g_P}(x) = \frac{1}{2}\sum_{r=1}^{d} \ell_r(x) \log \ell_r(x) + \ell_\infty(x)\ .$$

Note that, because $\omega_J(\cdot, J_P\,\cdot)$ is a Riemannian metric (in particular positive definite), there are restrictions on the function f_J and these will be specified and used below.

The proof of the first part of Theorem 2.8 consists now of the following three steps:

(i) write down on P a change of coordinates $\tilde{\varphi}_J : P \to P$, corresponding to the equivariant diffeomorphism $\varphi_J : M_P \to M_P$, that transforms the symplectic action/angle coordinates (x, y) for ω_P into symplectic action/angle coordinates $(\tilde{x} = \tilde{\varphi}_J(x), y)$ for ω_J;

(ii) find the "potential" $g = g(\tilde{x})$ for the transformed $J = (\tilde{\varphi}_J)_*(J_P)$ in this $(\tilde{x}, y)$ coordinates;

(iii) check that the function $h : P^\circ \to \mathbb{R}$, given by $h(\tilde{x}) = g(\tilde{x}) - g_P(\tilde{x})$ is actually smooth on the whole P.

The first step is easy. If one examines formula (A.2) for ω_J attentively, one is immediately led to the change of coordinates given in vector form by

$$\tilde{x} = \tilde{\varphi}_J(x) = x + G_P^{-1} \cdot \frac{\partial f_J}{\partial x} ,$$

where $\partial f_J / \partial x = (\partial f_J/\partial x_1, \ldots , \partial f_J / \partial x_n)^t \equiv$ column vector. Note that the behaviour of the matrix G_P^{-1} on the boundary of P, discussed in § 2.4, and the fact that ω_J given by (A.2) is nondegenerate, imply that $\tilde{\varphi}_J$ is a diffeomorphism of P that fixes its vertices and preserves each edge, 2-face, ..., n-face (this last one being the interior P° of P). The fact that $\omega_J(\cdot, J_P \cdot)$ is a Riemannian metric implies that

$$(d\tilde{\varphi}_J)\, G_P^{-1} \text{ is symmetric and positive definite on } P^\circ , \tag{A.3}$$

and so we must also have

$$\det (d\tilde{\varphi}_J) > 0 \text{ on the whole } P \tag{A.4}$$

($d\tilde{\varphi}_J$ denotes the Jacobian matrix of $\tilde{\varphi}_J$). Moreover, these conditions also imply that for a point x_0 belonging to the r-th $(n-1)$-dimensional face of P we have that

$$\langle (d\tilde{\varphi}_J)_{x_0} (\mu_r), \mu_r \rangle > 0 , \tag{A.5}$$

where μ_r is the inward pointing normal used in the definition of ℓ_r, and $\langle \cdot, \cdot \rangle$ denotes the standard euclidean inner product in $\mathbb{R}^n$.

For the second step, the simplest way is to use the Legendre duality between the potential for the symplectic form and the "potential" for the complex structure. In symplectic coordinates this duality is given by (2.6) (compare also with Exercise 3.5 (i)). In the $(\tilde{x}, y)$ symplectic coordinates for ω_J this says that

$$f(\tilde{x}) + g(\tilde{x}) = \sum_{m=1}^{n} \tilde{x}_m \frac{\partial g}{\partial \tilde{x}_m}(\tilde{x}) , \tag{A.6}$$

where f is the J-potential for the symplectic form $\omega_J = d\tilde{x} \wedge dy$, with the complex structure $J = (\tilde{\varphi}_J)_* (J_P)$ being given by

$$J = \begin{bmatrix} 0 & \vdots & -\left(\mathrm{Hess}_{\tilde{x}}(g)\right)^{-1} \\ \cdots & \cdots & \cdots \\ \mathrm{Hess}_{\tilde{x}}(g) & \vdots & 0 \end{bmatrix} .$$

The following two facts make (A.6) more explicit for our purposes:

(1) because being the Kähler potential of a $(1,1)$-form with respect to a complex structure is an invariant notion, we have that at $\tilde{x} = \tilde{\varphi}_J(x)$

$$f(\tilde{x}) = \tilde{f}(x) = f_P(x) + f_J(x) ;$$

(2) because $u = \partial g_P / \partial x$, $v = y$ and $\tilde{u} = \partial g / \partial \tilde{x}$, $\tilde{v} = y$ are holomorphic coordinates for J_P and $J = (\tilde{\varphi}_J)_* (J_P)$, the map $\tilde{u} = \tilde{u}(u)$, $\tilde{v} = v$ is a biholomorphism. Due to its very special form, it can only be the identity

plus a constant. We will assume, without any loss of generality for our purposes, that the constant is zero and so

$$\frac{\partial g}{\partial \tilde{x}_m}(\tilde{x}) = \left(\frac{\partial g_P}{\partial x_m} \circ \tilde{\varphi}_J^{-1}\right)(\tilde{x})\,.$$

Using these two facts, we can write (A.6) as

$$g(\tilde{x}) = \sum_{m=1}^{n} \tilde{x}_m \left(\frac{\partial g_P}{\partial x_m} \circ \tilde{\varphi}_J^{-1}\right)(\tilde{x}) - \left(f_P \circ \tilde{\varphi}_J^{-1}\right)(\tilde{x}) - \left(f_J \circ \tilde{\varphi}_J^{-1}\right)(\tilde{x})\,. \tag{A.7}$$

We can now address the third step. Since $\tilde{\varphi}_J$ is a smooth diffeomorphism of P, to prove that $h = g - g_P \in C^\infty(P)$ is equivalent to proving that $h \circ \tilde{\varphi}_J \in C^\infty(P)$. Using (A.7) and the fact that $f_J \in C^\infty(P)$, this means that what we need to show is that the function $\tilde{h} : P^\circ \to \mathbb{R}$ given by

$$\tilde{h}(x) \equiv h(\tilde{\varphi}_J(x)) + f_J(x) = \langle\, \tilde{\varphi}_J(x)\,, \frac{\partial g_P}{\partial x}(x)\,\rangle - f_P(x) - g_P(\tilde{\varphi}_J(x))$$

is smooth on the whole polytope P. A simple explicit computation gives

$$\tilde{h}(x) = \frac{1}{2}\left[\ell_\infty(\tilde{\varphi}_J(x)) - \ell_\infty(x) + \sum_{r=1}^{d} \ell_r(\tilde{\varphi}_J(x)) \log\left(\frac{\ell_r(x)}{\ell_r(\tilde{\varphi}_J(x))}\right)\right]\,,$$

and so we will be done if we can prove that the functions

$$\gamma_r(x) \equiv \frac{\ell_r(x)}{\ell_r(\tilde{\varphi}_J(x))}\,,\ r = 1, \ldots, d\,,$$

are smooth and strictly positive on P. The fact that they are smooth and strictly positive on P° is immediate. The question is, for each γ_r, what happens on the r-th $(n-1)$-dimensional face of P? At a point x_0 belonging to this face, and for its normal μ_r, we have that

$$d\left(\ell_r \circ \tilde{\varphi}_J\right)_{x_0}(\mu_r) = d\ell_r\left((d\tilde{\varphi}_J)_{x_0}(\mu_r)\right) = \langle\, \mu_r, (d\tilde{\varphi}_J)_{x_0}(\mu_r)\,\rangle > 0$$

by (A.5). Since $\tilde{\varphi}_J$ preserves the faces of P, we also have that $\ell_r(\tilde{\varphi}_J(x))$ can be written as $\ell_r(\tilde{\varphi}_J(x)) = \ell_r(x) \cdot \delta_r(x)$ for some smooth function δ_r (at least in a neighborhood of x_0). These two together mean that

$$0 < d\left(\ell_r \circ \tilde{\varphi}_J\right)_{x_0}(\mu_r) = \ell_r(x_0)(d\delta_r)_{x_0}(\mu_r) + \delta_r(x_0)(d\ell_r)_{x_0}(\mu_r) = \delta_r(x_0) \cdot \|\mu_r\|^2\,,$$

and so, in a neighborhood of x_0, the function

$$\gamma_r(x) = \frac{\ell_r(x)}{\ell_r(\tilde{\varphi}_J(x))} = \frac{1}{\delta_r(x)}$$

is smooth and strictly positive as desired.

The facts that $G = \mathrm{Hess}_{\tilde{x}}(g)$ is positive definite on P° and has determinant of the form

$$\det(G) = \left[\delta(\tilde{x}) \cdot \prod_{r=1}^{d} \ell_r(\tilde{x})\right]^{-1}\,,$$

with δ a smooth and strictly positive function on the whole P, follow from (A.3) and (A.4).

The second part of Theorem 2.8 can be proved either by reversing the reasoning we just did, or directly by showing that any complex structure J defined on $M_P^\circ \cong P^\circ \times \mathbb{T}^n$ by a "potential" g, satisfying the positivity and nondegeneracy conditions specified, compactifies to give a smooth compatible toric complex structure on (M_P, ω_P).

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