SYMMETRIC PERIODIC REEB ORBITS ON THE SPHERE

MIGUEL ABREU, HUI LIU, AND LEONARDO MACARINI

Abstract. A long standing conjecture in Hamiltonian Dynamics states that every contact form on the standard contact sphere S^{2n+1} has at least $n+1$ simple periodic Reeb orbits. In this work, we consider a refinement of this problem when the contact form has a suitable symmetry and we ask if there are at least $n+1$ simple symmetric periodic orbits. We show that there is at least one symmetric periodic orbit for any contact form and at least two symmetric closed orbits whenever the contact form is dynamically convex.

Contents

1. Introduction

Consider the unit sphere $S^{2n+1} = \{v \in \mathbb{R}^{2n+2};\ \|v\| = 1\}$ and the Liouville form $\lambda = \frac{1}{2}\sum_{i=1}^{n+1} q_i dp_i - p_i dq_i$. The standard contact structure on S^{2n+1} is given by $\xi = \ker \lambda|_{S^{2n+1}}$ and a contact form supporting this (cooriented) structure is a 1-form $\alpha = f\lambda|_{S^{2n+1}}$, where $f : S^{2n+1} \to \mathbb{R}$ is a positive function. Associated to α we have its Reeb vector field R_α uniquely characterized by the equations $\iota_{R_\alpha} d\alpha = 0$ and $\alpha(R_\alpha) = 1$. Reeb flows form a prominent class of Hamiltonian systems on energy levels and the study of Reeb flows on the standard contact sphere S^{2n+1} is equivalent to the study of Hamiltonian flows of proper homogeneous of degree two Hamiltonians $H : \mathbb{R}^{2n+2} \to \mathbb{R}$.

Let us denote by $\mathcal{P}$ the set of simple (i.e., non-iterated) closed Reeb orbits of α. A long standing conjecture in Hamiltonian Dynamics establishes that for every contact form α on S^{2n+1} we have that $\#\mathcal{P} \geqslant n+1$. It was proved for $n = 1$ by Cristofaro-Gardiner and

2010 *Mathematics Subject Classification.* 53D40, 37J10, 37J55.

Key words and phrases. Closed orbits, Reeb flows, dynamical convexity, equivariant symplectic cohomology.

MA and LM were partially funded by FCT/Portugal through UID/MAT/04459/2020 and PTDC/MAT-PUR/29447/2017. HL was partially supported by NSFC (No. 12022111).

Hutchings [17] (in the more general setting of Reeb flows in dimension three) and independently by Ginzburg, Hein, Hryniewicz and Macarini [21]; see also [28] where an alternate proof was given using a result from [21]. In higher dimensions, the conjecture is completely open without additional assumptions on α, such as convexity or certain index requirements or non-degeneracy of closed Reeb orbits. For instance, it is not know if every contact form on S^5 has at least two simple closed Reeb orbits (although the conjecture states that we should have at least three closed orbits).

In order to explain the convexity assumption, note that there is a natural bijection between contact forms α on (S^{2n+1}, ξ) and starshaped hypersurfaces Σ_α in $\mathbb{R}^{2n+2}$ given by

$$\alpha = f\lambda|_{S^{2n+1}} \longleftrightarrow \Sigma_\alpha = \{\sqrt{f(x)}x;\, x \in S^{2n+1}\}$$

and it satisfies the property that if Σ_α is the energy level of a homogeneous of degree two Hamiltonian $H_\alpha : \mathbb{R}^{2n+2} \to \mathbb{R}$ then the Hamiltonian flow of H_α on Σ_α is equivalent to the Reeb flow of α. We say that α is *convex* if Σ_α bounds a strictly convex domain.

When α is convex, Ekeland and Hofer proved in [19] that $\#\mathcal{P} \geqslant 2$. Later on, Long and Zhu in the remarkable work [31] showed that $\#\mathcal{P} \geqslant \lfloor (n+1)/2 \rfloor + 1$. This result was improved when n is even by Wang [37], furnishing the lower bound $\#\mathcal{P} \geqslant \lceil (n+1)/2 \rceil + 1$. In particular, $\#\mathcal{P} \geqslant n+1$ for every convex contact form on S^{2n+1} when $n = 2$ (it was proved before in [25]). In [38], Wang proved that it also holds when $n = 3$.

The notion of convexity is not natural from the point of view of Symplectic Topology since it is not a condition invariant under contactomorphisms. An alternative definition was introduced by Hofer, Wysocki and Zehnder [26], called *dynamical convexity*. A contact form on S^{2n+1} is dynamically convex if the Conley-Zehnder index of every closed orbit is bigger than or equal to $n+2$. It is not hard to see that every convex contact form is dynamically convex. Clearly, dynamical convexity is invariant under contactomorphisms. Moreover, dynamical convexity is more general than convexity: there are contact forms that are dynamically convex and are not contactomorphic to a convex one [11, 12]; see also [1, 22].

When α is dynamically convex, Ginzburg and Gürel [20] and, independently, Duan and Liu [18], proved that $\#\mathcal{P} \geqslant \lceil (n+1)/2 \rceil + 1$, generalizing the previous works of Long-Zhu and Wang.

In this work, we will consider a refinement of this conjecture when the contact form has a symmetry. More precisely, given an integer $p > 0$, consider the $\mathbb{Z}_p$-action on S^{2n+1}, regarded as a subset of $\mathbb{C}^{n+1}\backslash\{0\}$, generated by the map

$$\psi(z_0, \ldots, z_n) = \left(e^{\frac{2\pi\sqrt{-1}\ell_0}{p}} z_0, e^{\frac{2\pi\sqrt{-1}\ell_1}{p}} z_1, \ldots, e^{\frac{2\pi\sqrt{-1}\ell_n}{p}} z_n\right), \tag{1.1}$$

where $\ell_0, \ldots, \ell_n$ are integers called the *weights* of the action. Such an action is free when the weights are coprime with p and in that case we have a lens space obtained as the quotient of S^{2n+1} by the action of $\mathbb{Z}_p$. We denote such a lens space by $L_p^{2n+1}(\ell_0, \ell_1, \ldots, \ell_n)$. From now on, we will assume that the weights are coprime with p.

The question that we will address in this work is the following one. Given the action of a Lie group G on a manifold M and a vector field X on M invariant under this action, we say that a periodic orbit γ of X is *symmetric* if $g(\mathrm{Im}(\gamma)) = \mathrm{Im}(\gamma)$ for every $g \in G$.

Question: Let α be a contact form on S^{2n+1} invariant under the above $\mathbb{Z}_p$-action generated by ψ. Denote by $\mathcal{P}_s$ the set of simple *symmetric* closed orbits of the Reeb flow of α. Is it true that $\#\mathcal{P}_s \geqslant n+1$?

Note that when $p = 1$ the $\mathbb{Z}_p$-action is trivial and therefore every periodic orbit is symmetric. Thus, in this case the question is equivalent to the aforementioned problem if every contact form on S^{2n+1} carries at least $n+1$ closed orbits. Moreover, the irrational ellipsoids are invariant under this $\mathbb{Z}_p$-action for any p and weights $\ell_0, \ldots, \ell_n$ and carry precisely $n+1$ periodic orbits which are all symmetric.

Girardi proved in [23] that when $p = 2$, that is, when ψ is the antipodal map, we have that $\#\mathcal{P}_s \geqslant 1$ for any symmetric contact form. In the same work, it was proved that if Σ_α satisfies suitable pinching conditions then $\#\mathcal{P}_s \geqslant n+1$ when $p = 2$. This result was generalized by Abreu and Macarini in [1, Theorem 1.21] for any p assuming that α is convex and $\ell_0 = \cdots = \ell_n = 1$.

When α is convex and symmetric, for any p and weights $\ell_0, \ldots, \ell_n$, it was proved by Zhang [39] that $\#\mathcal{P}_s \geqslant 2$. This result was generalized to dynamically convex contact forms by Liu and Zhang [29] assuming that $p = 2$.

The main results in this work are the following generalizations of these results.

Theorem 1.1. *Let α be any contact form on S^{2n+1} invariant under the above $\mathbb{Z}_p$-action. Then α has at least one symmetric closed orbit.*

This result generalizes the results of Rabinowitz [34] (that corresponds to the case $p = 1$) and Girardi [23] (that corresponds to the case $p = 2$).

Theorem 1.2. *Let α be a dynamically convex contact form on S^{2n+1} invariant under the above $\mathbb{Z}_p$-action. Then α has at least two simple symmetric closed orbits.*

The last result generalizes the results of Zhang [39] (that corresponds to the case that α is convex) and Liu-Zhang [29] (that corresponds to the case $p = 2$).

Let β be the induced contact form on $L_p^{2n+1}(\ell_0, \ldots, \ell_n)$. A natural question is how the previous results can be seen in terms of periodic orbits of β. The simple symmetric closed orbits of α are in bijection with the simple closed orbits of β whose homotopy classes are generators of $\pi_1(L_p^{2n+1}(\ell_0, \ldots, \ell_n))$. Thus, Theorem 1.1 is equivalent to the statement that given any $a \in \pi_1(L_p^{2n+1}(\ell_0, \ldots, \ell_n))$ then every contact form β on $L_p^{2n+1}(\ell_0, \ldots, \ell_n)$ has at least one periodic orbit with homotopy class a. Theorem 1.2, in turn, is equivalent to the statement that given any $a \in \pi_1(L_p^{2n+1}(\ell_0, \ldots, \ell_n))$ and a dynamically convex contact form β on $L_p^{2n+1}(\ell_0, \ldots, \ell_n)$ (we say that β is dynamically convex if so is its lift to S^{2n+1}) we have at least two simple periodic orbits with homotopy class a, where a periodic orbit with homotopy class a is called simple if it is not an iterate of another periodic orbit with homotopy class a (although it can be the iterate of a closed orbit with another homotopy class).

Organization of the paper. The rest of the paper is organized as follows. The background on symplectic cohomology and Lusternik-Schnirelmann theory in Floer homology necessary for this work is presented in Section 2. Symplectic cohomology of orbifolds is discussed in Section 3, where there is also a brief explanation of Chen-Ruan cohomology in Section 3.1. The proofs of Theorems 1.1 and 1.2 are given in Sections 4 and 5 respectively.

Acknowledgements. The third author is grateful to Alexandru Oancea for useful discussions regarding this paper.

2. Symplectic cohomology and Lusternik-Schnirelmann theory

2.1. **Symplectic cohomology.** Let (M^{2n+1},ξ) be a contact manifold endowed with a strong symplectic filling given by a Liouville domain W such that $c_1(TW)|_{H_2(W,\mathbb{R})}=0$. The symplectic cohomology $\mathrm{SH}^*(W)$ of W is a symplectic invariant introduced by Cieliebak, Floer, Hofer and Viterbo [14, 36]. It is obtained as a direct limit of Floer cohomologies of a suitable class of Hamiltonians on W and a nice reference with full details can be found in [6].

Symplectic cohomology has a natural action filtration which allows us to define the *negative* and *positive* symplectic cohomologies $\mathrm{SH}^*_-(W)$ and $\mathrm{SH}^*_+(W)$. These cohomologies are related via the tautological exact triangle

$$\begin{array}{ccc} \mathrm{SH}^*_-(W) & \longrightarrow & \mathrm{SH}^*(W) \\ {\scriptstyle [+1]}\nwarrow & & \swarrow \\ & \mathrm{SH}^*_+(W) & \end{array} \tag{2.1}$$

Symplectic cohomology has another filtration given by the free homotopy classes a of loops in W and the previous exact triangle respects this filtration. Denote by $\mathrm{SH}^*_a(W)$ and $\mathrm{SH}^*_{a,\pm}(W)$ the corresponding cohomologies. The negative symplectic cohomology is generated by constant periodic orbits and therefore $\mathrm{SH}^*_-(W)=\mathrm{SH}^*_{0,-}(W)$. It turns out that $\mathrm{SH}^*_-(W)$ is isomorphic to $\mathrm{H}^*(W)$.

The action functional of an autonomous Hamiltonian is invariant under the natural S^1-action given by translations in the parametrization of the closed curves. An equivariant version of the symplectic cohomology, taking into account this symmetry, is the equivariant symplectic cohomology $\mathrm{SH}^*_{S^1}(W)$ introduced by Viterbo [36] and developed by Bourgeois and Oancea [7, 8, 9, 10]. As established in [9], the groups $\mathrm{SH}^*(W)$ and $\mathrm{SH}^*_{S^1}(W)$ fit into the Gysyn exact triangle

$$\begin{array}{ccc} \mathrm{SH}^{*+2}(W) & \xrightarrow{[-1]} & \mathrm{SH}^*_{S^1}(W) \\ \nwarrow & & \swarrow {\scriptstyle D} \\ & \mathrm{SH}^{*+2}_{S^1}(W) & \end{array} \tag{2.2}$$

where D is the so called *shift operator* [20].

Remark 2.1. To be more precise, the previous exact triangle is established by Bourgeois and Oancea for symplectic *homology* (with a different shift in the grading) but their arguments can be easily adapted to symplectic *cohomology*.

As in the non-equivariant case, the equivariant symplectic cohomology has as action filtration that allows us to define the positive/negative equivariant symplectic cohomology $\mathrm{SH}^*_{S^1,\pm}(W)$. It turns out that the positive equivariant symplectic cohomology with rational coefficients can be obtained as the cohomology of a cochain complex $\mathrm{CC}^*(\alpha)$ generated by the good closed Reeb orbits of a non-degenerate contact form α on M; c.f. [20, Proposition 3.3]. This complex is filtered by the action and graded by the Conley-Zehnder index. In this work, we will take the grading using sections of the determinant line bundle $\Lambda^{n+1}_{\mathbb{C}}TW$ following [32, 35]. If $c_1(TW)$ does not vanish, this grading is, in general, fractional; c.f. [1, Section 2.3]. The differential in the complex $\mathrm{CC}^*(\alpha)$, but not its cohomology, depends on several auxiliary choices, and the nature of the differential is not essential for our purposes. The complex $\mathrm{CC}^*(\alpha)$ is functorial in α in the sense that a symplectic cobordism equipped with a suitable extra structure gives rise to a map of complexes that induce an isomorphism in the cohomology. For the sake of brevity and to emphasize the obvious analogy with contact cohomology, we denote the cohomology of $\mathrm{CC}^*(\alpha)$ by $\mathrm{HC}^*(W)$ rather than $\mathrm{SH}^*_{S^1,+}(W)$. Furthermore, once we fix a free homotopy class of loops in W, the part of $\mathrm{CC}^*(\alpha)$ generated by good closed Reeb orbits in that class is a subcomplex. As a consequence, the entire complex $\mathrm{CC}^*(\alpha)$ breaks down into a direct sum of such subcomplexes indexed by free homotopy classes of loops in W.

A remarkable observation by Bourgeois and Oancea in [10, Section 4.1.2] is that under suitable additional assumptions on the indices of closed Reeb orbits the positive equivariant symplectic cohomology is defined even when M does not have a symplectic filling, using the symplectization of M, and therefore is a contact invariant. To be more precise, we assume that $c_1(\xi)|_{H_2(M,\mathbb{R})} = 0$ and that M admits a non-degenerate contact form α such that all of its closed *contractible* Reeb orbits have Conley–Zehnder index strictly greater than $3 - n$. Furthermore, under this assumption once again the equivariant symplectic cohomology of M with rational coefficients can be described as the cohomology of a complex $\mathrm{CC}^*(\alpha)$ generated by good closed Reeb orbits of α, graded by the Conley-Zehnder index and filtered by the action. The complex breaks down into the direct sum of subcomplexes indexed by free homotopy classes of loops *in* M. We will use the notation $\mathrm{HC}^*_a(M)$ to denote the cohomology of the complex generated by the orbits with free homotopy class a.

With the same assumption on α, we can also define a *non-equivariant* symplectic cohomology $\mathrm{SH}^*(M)$ in terms of the symplectization of M and that therefore does not require the existence of a filling W and is a contact invariant. The construction of this cohomology is analogous to the one in [10, Section 4.1.2]: the admissible Hamiltonians $H^c : (0,\infty) \times M \to \mathbb{R}$ are of the form $H^c(r,x) = h^c(r)$, with $h^c : (0,\infty) \to \mathbb{R}$ a convex increasing function such that $dh^c(r)/dr \to 0$, $r \to 0$, $dh^c(r)/dr = c$ for $r \gg 0$ and $d^2h^c(r)/dr^2 > 0$ whenever $dh^c(r)/dr < c$. (Here, of course, H^c has to be slightly perturbed to a time-dependent Hamiltonian in order to achieve nondegeneracy of its 1-periodic orbits.) The condition on the indices of the orbits prevents the bubbling off of planes in the concave end, so that the resulting Floer cohomology is defined. Then the argument developed in [9] to establish the Gysin exact triangle (2.2) works verbatim to prove the exact triangle

$$\begin{array}{ccc} \mathrm{SH}^{*+2}(M) & \xrightarrow{[-1]} & \mathrm{HC}^*(M)\,. \\ & \nwarrow \quad \swarrow_{D} & \\ & \mathrm{HC}^{*+2}(M) & \end{array} \tag{2.3}$$

These groups have a filtration by the free homotopy classes in M and this exact triangle respects this filtration.

Finally, let us briefly recall the definition of equivariant local symplectic cohomology. Given an isolated (possibly degenerate) closed orbit γ of α we have its equivariant local symplectic cohomology $\mathrm{HC}^*(\gamma)$ [20, 27]. It is supported in $[\mu(\gamma), \mu(\gamma) + \nu(\gamma)]$ (i.e. $\mathrm{HC}^k(\gamma) = 0$ for every $k \notin [\mu(\gamma), \mu(\gamma) + \nu(\gamma)]$), where $\mu(\gamma)$ is the Conley-Zehnder index of γ and $\nu(\gamma)$ is the nullity of γ, i.e., the geometric multiplicity of the eigenvalue 1 of the linearized Poincaré map. (The index of a degenerate periodic orbit here is defined as the lower semicontinuous extension of the Conley-Zehnder index for non-degenerate periodic orbits; c.f. [22].) If α carries finitely many simple closed orbits with free homotopy class a then if $\mathrm{HC}^k_a(M) \neq 0$ there exists a closed orbit γ with free homotopy class a such that $\mathrm{HC}^k(\gamma) \neq 0$; see [27]. In particular, this implies that $|\mu(\gamma) - k| \leqslant 2n$.

2.2. **Lusternik-Schnirelmann theory.** In what follows, we will explain briefly the results from Lusternik-Schnirelmann theory in Floer homology necessary for this work. We refer to [20] for details.

Let (M^{2n+1}, ξ) be a closed contact manifold and a a free homotopy class in M. Let α be a non-degenerate contact form on M such that every contractible closed Reeb orbit has index strictly bigger than $3-n$. Given numbers $0 < T_1 < T_2 \leqslant \infty$ we denote by $\mathrm{HC}^*_{a,(T_1,T_2)}(\alpha)$ the equivariant symplectic cohomology of α with free homotopy class a and action window (T_1, T_2). When α is degenerate and both T_1 and T_2 are not in the action spectrum of α we define $\mathrm{HC}^*_{a,(T_1,T_2)}(\alpha)$ as $\mathrm{HC}^*_{a,(T_1,T_2)}(\bar{\alpha})$ for some small non-degenerate perturbation $\bar{\alpha}$ of α. (Recall that the action spectrum of α is given by $\mathcal{A}(\alpha) = \{\int_\gamma \alpha;\, \gamma \text{ is a closed Reeb orbit of } \alpha\}$.) Given $T \in (0,\infty]$ we denote by $\mathrm{HC}^*_{a,T}(\alpha)$ the filtered equivariant symplectic cohomology $\mathrm{HC}^*_{a,(\epsilon,T)}(\alpha)$ for some $\epsilon > 0$ sufficiently small such that $\epsilon < \min\{T;\, T \in \mathcal{A}(\alpha)\}$.

Let α be a contact form on M such that every closed orbit with homotopy class a is isolated and $T \in (0,\infty]\backslash\mathcal{A}(\alpha)$. Given a non-trivial element $w \in \mathrm{HC}^k_{a,T}(\alpha)$ we have a spectral invariant given by

$$c_w(\alpha) = \inf\{T' \in (0,T)\backslash\mathcal{A}(\alpha);\, w \in \mathrm{Im}(i^{a,T'})\}$$

where $i^{a,T'} : \mathrm{HC}^*_{a,T'}(\alpha) \to \mathrm{HC}^*_{a,T}(\alpha)$ is the map induced in the cohomology by the inclusion of the complexes. It turns out that there exists a periodic orbit γ with action $c_w(\alpha)$ and free homotopy class a such that $\mathrm{HC}^k(\gamma) \neq 0$; c.f. [20, Corollary 3.9].

The shift operator $D : \mathrm{HC}^*_a(M) \to \mathrm{HC}^{*+2}_a(M)$ in the exact triangle (2.3) satisfies the property

$$c_w(\alpha) < c_{D(w)}(\alpha) \tag{2.4}$$

see [20, Theorem 1.1]. (Note here that the previous inequality is the reverse to the one in [20, Theorem 1.1]. This discrepancy comes from the fact that we deal with *co*homology instead of homology.) Suppose now that there exists $k_0 \in \mathbb{Q}$ such that $\mathrm{HC}^{k_0+2k}_a(M) \neq 0$ and $D : \mathrm{HC}^{k_0+2k}_a(M) \to \mathrm{HC}^{k_0+2k+2}_a(M)$ is an isomorphism for every $k \in \mathbb{N}_0$. Then it follows from (2.4) and the previous discussion that there exists an injective map $\psi : \mathbb{N}_0 \to \mathcal{P}_a$, where $\mathcal{P}_a$ is the set of closed orbits of α with homotopy class a, such that $\gamma_k := \psi(k)$ satisfies $\mathrm{HC}^{k_0+2k}(\gamma_k) \neq 0$. In particular, $|\mu(\gamma_k) - (k_0 + 2k)| \leqslant 2n$. The map ψ is called a *carrier map*; see [20].

3. Symplectic cohomology of orbifolds

Let (M, ξ) be a closed *smooth* contact manifold endowed with an exact *orbifold* symplectic filling W. Gironella and Zhou introduced in [24] a symplectic cohomology of W that will be fundamental in this work. Before we discuss this, we have to recall the definition of the Chen-Ruan cohomology.

3.1. **Chen-Ruan cohomology.** In this section, we will give a brief definition of Chen-Ruan cohomology enough for our purposes. In particular, we will not address its product structure since it will be not necessary here. We refer to [4] and [13] for details.

A *Lie groupoid* $\mathcal{G}$ consists of a manifold of objects G_0 and a manifold of arrows G_1 with the following structure maps: the source and target maps $s, t : G_1 \to G_0$, unit map $u : G_0 \to G_1$, inverse map $i : G_1 \to G_1$ and composition map $m : G_1\, {}_s\!\times_t G_1 \to G_1$. All these maps are required to be smooth and to obey the obvious properties: composition has to be associative and has to interact in the expected way with the unit and inverse maps. Moreover we require s, t to be submersions; this is needed so that $G_1 \,{}_s\!\times_t G_1$ is a manifold.

A Lie groupoid $\mathcal{G}$ is said to be proper if the map $(s,t) : G_1 \to G_0 \times G_0$ is proper. The Lie groupoid $\mathcal{G}$ is said to be étale if s and t are local diffeomorphisms; in this case we can define the dimension of $\mathcal{G}$ to be $\dim \mathcal{G} = \dim G_0 = \dim G_1$. An *orbifold groupoid* is an étale and proper Lie groupoid. Associated to it we have an underlying topological space: its orbit space $|\mathcal{G}| = G_0/\sim$ where $\sim$ is the equivalence relation defined by $x \sim y$ if and only if there is $g \in G_1$ such that $x = s(g)$ and $y = t(g)$.

A basic example is given by a *global quotient orbifold.* Let X be a manifold and G a finite group acting on it. The global quotient $[X/G]$ is given by the Lie groupoid $\mathcal{G} = G \ltimes X$ with $G_0 = X$, $G_1 = G \times X$, $s(g,x) = x$ and $t(g,x) = g(x)$.

Let $\mathcal{G}$ be an orbifold grupoid and consider the commutative diagram

$$\begin{array}{ccc} S_{\mathcal{G}} & \longrightarrow & G_1 \\ \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle (s,t)} \\ G_0 & \xrightarrow{\ \Delta\ } & G_0 \times G_0 \end{array}$$

where Δ is the diagonal map and $S_{\mathcal{G}} = \{g \in G_1;\ s(g) = t(g)\}$ is naturally equipped with the map $\beta : S_{\mathcal{G}} \to G_0,\ g \mapsto t(g) = s(g)$. Any $h \in G_1$ induces a diffeomorphism $\beta^{-1}(s(h)) \to \beta^{-1}(t(h))$ via the action by conjugation $h \cdot g = hgh^{-1}$ for $g \in \beta^{-1}(s(h))$. In particular, we can form the Lie groupoid $\mathcal{G} \ltimes S_{\mathcal{G}}$ with the manifold of objects given by $S_{\mathcal{G}}$ and manifold of arrows given by $G_1 \,{}_s\!\times_\beta S_{\mathcal{G}} = \{(h,g) \in G_1 \times S_{\mathcal{G}};\ s(h) = \beta(g)\}$, with source map $s(h,g) = g$ and target map $t(h,g) = hgh^{-1}$. This groupoid $\mathcal{G} \ltimes S_{\mathcal{G}}$ is called the *inertia grupoid* of $\mathcal{G}$ and denoted by $\Lambda\mathcal{G}$.

Example 3.1. Let $G \subset \mathrm{U}(n)$ be a finite subgroup and consider the global quotient orbifold $\mathcal{G} = G \ltimes \mathbb{C}^n$ with $G_0 = \mathbb{C}^n$, $G_1 = G \times \mathbb{C}^n$, $s(g,x) = x$ and $t(g,x) = g(x)$. Then

$$|\Lambda\mathcal{G}| = \{(x, (g)_{G_x});\ x \in \mathbb{C}^n \text{ and } g \in G_x\}$$

where G_x is the isotropy group of x and $(g)_{G_x}$ denotes the conjugacy class of g in G_x. If G is abelian and acts freely in $\mathbb{C}^n \backslash \{0\}$ then clearly

$$|\Lambda\mathcal{G}| \simeq \mathbb{C}^n \sqcup \{g \in G;\ g \neq 1\}.$$

The ungraded Chen-Ruan cohomology of an orbifold $\mathcal{G}$ is the just the (ungraded) singular cohomology of $|\Lambda\mathcal{G}|$. In order to describe its grading, we have to introduce a decomposition of $\Lambda\mathcal{G}$ into connected components.

For this, we define an equivalence relation $\simeq$ on $S_{\mathcal{G}}$ as follows. On each local uniformizer $G \ltimes U$, for $x, y \in U$, any two $g_1 \in G_x \subset G$ and $g_2 \in G_y \subset G$ are equivalent with respect to $\simeq$ if g_1, g_2 are conjugated in G. More generally, $g_1 \simeq g_2$ iff g_1, g_2 are connected by a sequence $\{h_1 = g_1, h_2, \ldots, h_{n-1}, h_n = g_2\} \subset S_{\mathcal{G}}$, such that h_i, h_{i+1} are equivalent in a local uniformizer. We denote by (g) the equivalence class of $g \in S_{\mathcal{G}}$ under the just defined relation $\simeq$, and by T the set $S_{\mathcal{G}}/\simeq$ of all such equivalence classes. Then $\Lambda\mathcal{G}$ can be decomposed as

$$\Lambda\mathcal{G} = \bigsqcup_{(g) \in T} \mathcal{G}_{(g)},$$

where $\mathcal{G}_{(g)}$ is the $\mathcal{G}$-inertia groupoid on the (g)-component of $S_{\mathcal{G}}$. Notice in particular that $\mathcal{G}_{(1)}$ is naturally isomorphic to $\mathcal{G}$. The orbifolds $\mathcal{G}_{(g)}$, for each $(g) \in T$, $g \neq 1$, are called the *twisted sectors* of $\mathcal{G}$ and $\mathcal{G}_{(1)}$ is called the *untwisted sector*.

Example 3.2. If $G \subset \mathrm{U}(n)$ is an abelian subgroup that acts freely in $\mathbb{C}^n \backslash \{0\}$ and $\mathcal{G}$ is the orbifold given in Example 3.1 then clearly $|\Lambda\mathcal{G}_1| = \mathbb{C}^n$ and $|\Lambda\mathcal{G}_g| = \{0\}$ for every $g \in G$ such that $g \neq 1$.

Now, assume that $\mathcal{G}$ admits an almost complex structure, i.e., an almost complex structure J on G_0 such that $s^*J = t^*J$. Let $g \in S_{\mathcal{G}}$ be an arrow with $s(g) = t(g) = x$. Choose an orbifold chart (U, G_x, ϕ) with U embedded in G_0. Then $g \in G_x$ is a map $g : U \to U$. Its differential at x is a map $(dg)_x : T_xG_0 \to T_xG_0$. The almost complex structure J_x endows T_xG_0 with the structure of a complex vector space, hence identifying T_xG_0 with $\mathbb{C}^n$ where $2n = \dim \mathcal{G}$; the condition that $s^*J = t^*J$ implies that $(dg)_x$ preserves the almost complex structure J_x on T_xG_0, so it can be regarded as a linear transformation of complex vector spaces or, equivalently, a matrix in $GL(n, \mathbb{C})$. Since g has finite order, say $m \in \mathbb{N}$, the eigenvalues of $(dg)_x$ (always regarded as a complex linear transformation) are of the form $e^{2\pi i\lambda_1}, \ldots, e^{2\pi i\lambda_n}$ where $\lambda_j \in \mathbb{Q}$ are such that $m\lambda_j \in \mathbb{Z}$. We then define a map age : $S_{\mathcal{G}} \to \mathbb{Q}$ by mapping $g \in S_{\mathcal{G}}$ to

$$\mathrm{age}(g) = \sum_{j=1}^{n} \{\lambda_j\}$$

where $\{\lambda_j\}$ denotes the fractional part of λ_j. Since eigenvalues do not change under conjugation, it is invariant with respect to the $\mathcal{G}$-action on $S_{\mathcal{G}}$ and thus induces a map (still called age) age : $|\Lambda\mathcal{G}| \to \mathbb{Q}$. By continuity of eigenvalues, the map is continuous and since it takes values in $\mathbb{Q}$ it must be locally constant. Hence, it is constant in each of the (un)twisted sectors, so we denote by age(g) the value that takes in the (un)twisted sector $|\mathcal{G}_{(g)}|$. In particular, it is clear that age$(1) = 0$.

Definition 3.3. *Let R be a field of characteristic zero. The* Chen-Ruan cohomology *of an orbifold $\mathcal{G}$ with coefficients in R is the $\mathbb{Q}$-graded vector space*

$$\mathrm{H}^*_{CR}(\mathcal{G}; R) = \bigoplus_{(g) \in T} \mathrm{H}^{*-2age(g)}(|\mathcal{G}_{(g)}|; R).$$

Example 3.4. If $G \subset \mathrm{U}(n)$ is a subgroup that acts freely in $\mathbb{C}^n \backslash \{0\}$ then clearly

$$\mathrm{H}^*_{CR}(\mathbb{C}^n/G; \mathbb{Q}) = \bigoplus_{(g) \in \mathrm{Conj}(G)} \mathbb{Q}[-2\mathrm{age}(g)], \tag{3.1}$$

where $-2\mathrm{age}(g)$ is the shift in the grading.

3.2. **Symplectic cohomology of orbifold fillings.** Let (M^{2n+1}, ξ) be a closed smooth contact manifold endowed with an exact orbifold symplectic filling W (see [24] for a definition of an exact orbifold filling). Suppose that the rational first Chern class $c_1^{\mathbb{Q}}(TW)$ vanishes. Gironella and Zhou introduced in [24] a symplectic cohomology of W that will play a fundamental role in this work.

As proved in [24, Theorem A] the tautological exact triangle (2.1) also holds for the symplectic cohomology of orbifolds, but now $\mathrm{SH}^*_-(W)$ is isomorphic to the Chen-Ruan cohomology $\mathrm{H}^*_{CR}(W)$, where from now on we are taking rational coefficients. Therefore, we have the exact triangle

$$\begin{array}{ccc} \mathrm{H}^*_{CR}(W) & \longrightarrow & \mathrm{SH}^*(W) \\ & {\scriptstyle [+1]}\nwarrow \quad \swarrow & \\ & \mathrm{SH}^*_+(W) & \end{array} \quad . \tag{3.2}$$

The symplectic cohomology of orbifolds is defined, as in the smooth case, as a limit of Floer cohomologies of a suitable class of Hamiltonians on W. The periodic orbits and Floer trajectories are orbifold maps $\gamma : S^1 \to \widehat{W}$ and $u : \mathbb{R} \times S^1 \to \widehat{W}$ respectively, where $\widehat{W}$ is the completion of W. Suppose, for instance, that W is a global quotient orbifold $[X/G]$. Then the periodic orbits and Floer trajectories correspond to smooth maps $\gamma : [0,1] \to X$ and $u : \mathbb{R} \times [0,1] \to X$ satisfying g-boundary conditions for $g \in G$; c.f. [33].

These symplectic homology groups have a filtration given by the free homotopy classes of W that correspond to the conjugacy classes of $\pi_1^{\mathrm{orb}}(W)$. Consider the case that $W = \mathbb{C}^{n+1}/G$, where $G \subset \mathrm{U}(n)$ is a finite subgroup. Then $\pi_1^{\mathrm{orb}}(W) \simeq G$ and, by definition, the Chen-Ruan cohomology $\mathrm{H}^*_{CR}(W)$ is filtered by the abelianization of G. It is easy to see from the proof of [24, Theorem A] that the exact triangle (3.2) respects this filtration.

4. Proof of Theorem 1.1

Let β be the induced contact form on $L_p^{2n+1}(\ell_0, \dots, \ell_n)$. As explained in the introduction, we have to show that given any $a \in \pi_1(L_p^{2n+1}(\ell_0, \dots, \ell_n))$ there exists a closed orbit of β with homotopy class a. Consider the discussion in Section 3.2. Suppose that $M = L_p^{2n+1}(\ell_0, \dots, \ell_n)$ and $W = \mathbb{C}^{n+1}/\mathbb{Z}_p$ where the $\mathbb{Z}_p$-action is the one generated by (1.1). Note that $c_1^{\mathbb{Q}}(TW) = 0$. Then, by the definition of the Chen-Ruan cohomology, $\mathrm{H}^*_{CR}(W)$ has a filtration by $(g) = g \in \mathbb{Z}_p$. Moreover, $\mathrm{SH}^*(W)$ and $\mathrm{SH}^*_+(W)$ also have filtrations by $\pi_1^{\mathrm{orb}}(W) \simeq \mathbb{Z}_p$. Denote these filtrations by

$$\mathrm{H}^*_{CR}(W) = \bigoplus_{k \in \mathbb{Z}_p} \mathrm{H}^*_{CR,k}(W),\ \mathrm{SH}^*(W) = \bigoplus_{k \in \mathbb{Z}_p} \mathrm{SH}^*_k(W) \text{ and } \mathrm{SH}^*_+(W) = \bigoplus_{k \in \mathbb{Z}_p} \mathrm{SH}^*_{k,+}(W).$$

As mentioned in Section 3.2, an inspection of the proof of [24, Theorem A] shows that the exact triangle (3.2) respects this filtration.

Recall that we are taking all these cohomology groups with rational coefficients. Thus, by [24, Theorem B], $\mathrm{SH}^*(W) = 0$ and therefore $\mathrm{H}^*_{CR}(W) \simeq \mathrm{SH}^*_+(W)$ respecting the aforementioned filtration. By (3.1), given $k \in \{0, \dots, p-1\} \simeq \mathbb{Z}_p$ we have that

$$\mathrm{H}^*_{CR,k}(W) = \begin{cases} \mathbb{Q} & \text{if } * = 2\sum_{i=0}^{n+1} \{k\ell_i/p\} \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Thus, $\mathrm{SH}^*_{k,+}(W) \neq 0$ for every k. By the definition of $\mathrm{SH}^*_{k,+}(W)$, it immediately implies that given any $a \in \pi_1(L_p^{2n+1}(\ell_0, \dots, \ell_n))$ we have that β has a closed orbit with homotopy class a.

5. Proof of Theorem 1.2

Let β be the induced contact form on $L_p^{2n+1}(\ell_0, \dots, \ell_n)$. We have to show that given any $a \in \pi_1(L_p^{2n+1}(\ell_0, \dots, \ell_n))$ there exist two simple closed orbits of β with homotopy class a. Assume, from now on, that β has finitely many simple periodic orbits with homotopy class a (otherwise, there is nothing to prove) and let $\mathcal{P}_a = \{\bar\gamma_1, \dots, \bar\gamma_m\}$ be the set of simple closed orbits with homotopy class a. By Theorem 1.1, $m \geqslant 1$. We have to show that $m \geqslant 2$.

As in the previous section, let $M = L_p^{2n+1}(\ell_0,\dots,\ell_n)$ and $W = \mathbb{C}^{n+1}/\mathbb{Z}_p$ be its orbifold filling, where the $\mathbb{Z}_p$-action is the one generated by (1.1). By [24, Theorem B], $\mathrm{SH}^*(W)=0$ (recall that we are taking rational coefficients) and therefore, by the exact triangle (3.2), $\mathrm{H}^*_{CR}(W) \simeq \mathrm{SH}^*_+(W)$. In particular, by (4.1), we have that $\mathrm{SH}^k_+(W)=0$ for every $k \geqslant 2n+2$.

The following result is well known to experts in the area. For the sake of completeness, we will provide a proof.

Proposition 5.1. *Let (M^{2n+1},ξ) be a smooth contact manifold endowed with an exact orbifold symplectic filling W such that $c_1^{\mathbb{Q}}(TW)=0$. Suppose that $c_1(\xi)|_{H_2(M,\mathbb{R})}=0$ and that M admits a non-degenerate contact form α supporting ξ such that every closed Reeb orbit γ contractible in W satisfies $\mu(\gamma) > 3-n$. Then we have an isomorphism between* $\mathrm{SH}^*(M)$ *and* $\mathrm{SH}^*_+(W)$ *respecting the grading.*

Proof. The proof is based on a stretch of the neck argument already used in [5, 15]. Consider an admissible Hamiltonian $H : S^1\times \widehat{W}\to\mathbb{R}$, where $\widehat{W}$ is the completion of W, used to define $\mathrm{SH}^*_+(W)$ (take, for instance, the type (II) of Hamiltonians considered in [24, Section 3.1], where H is C^2-small and autonomous on W and H depends only on the radius on $\widehat{W}\backslash W$). Consider the Floer trajectories joining periodic Reeb orbits (which are located in the symplectization part $\widehat{W}\backslash W$). Note that these are the Floer trajectories relevant in the definition of $\mathrm{SH}^*_+(W)$.

Now, degenerate the almost complex structure in stretch-of-the-neck manner near ∂W (see [5, 15]). We claim that, under the assumption on the indices, for a sufficiently stretched neck all Floer trajectories stay in the symplectization part and thus coincide with the Floer trajectories involved in the version of symplectic cohomology $\mathrm{SH}^*(M)$ defined using only the symplectization.

Arguing by contradiction, suppose that we would see in the limit a building whose top component, in the symplectization, is a curve which contains both punctures of the initial Floer trajectory and solves a Cauchy-Riemann equation with Hamiltonian perturbation. (That this top component curve contains both Floer punctures is a consequence of a maximum principle argument developed in [5, Pages 654-655] and [16, Lemma 2.3].) By our hypothesis of contradiction, it must have at least one negative puncture to which is attached a holomorphic building which end up eventually in the filling; this implies that the orbit in this negative puncture must be contractible in W.

This top component is regular because it solves a perturbed equation near the Floer punctures. But it has a positive Floer puncture with SFT degree k (the Symplectic Field Theory (SFT) degree is given by the Conley-Zehnder index plus $n-2$), a negative Floer puncture with SFT degree $k-1$ (these punctures are the punctures of the original Floer trajectory) and at least one other negative puncture with SFT degree strictly bigger than one by our index assumptions (note that the periodic orbit in this puncture must be contractible in W). Therefore, the Fredholm index of this top component curve, given by the SFT degree of the positive puncture minus the sum of the SFT degrees of the negative punctures, must be negative, contradicting the regularity. Such a curve therefore cannot exist.

Finally, note here that, since the the region where we stretch the neck is away from the singularities of W as well as the top component, the singular part of W does not play any role in this argument. □

Clearly, the lens space $M = L_p^{2n+1}(\ell_0,\dots,\ell_n)$ with its filling $W=\mathbb{C}^{n+1}/\mathbb{Z}_p$ satisfy the assumptions of the previous proposition. (Note here that $\pi_1^{\mathrm{orb}}(W)\simeq\pi_1(M)\simeq\mathbb{Z}_p$ and therefore a non-degenerate dynamically convex contact form on M has the property that every periodic orbit γ contractible in W satisfies $\mu(\gamma) > 3-n$.) Thus, by the previous proposition, we have that $\mathrm{SH}^*(M) \simeq \mathrm{SH}^*_+(W)$ and, therefore, $\mathrm{SH}^k(M) = 0$ for every $k \geqslant 2n+2$. Therefore, by the exact triangle (2.3), we have that $D : \mathrm{HC}^k(M) \to \mathrm{HC}^{k+2}(M)$ is an isomorphism for every $k \geqslant 2n+1$. As discussed before, D respects the filtration by the homotopy classes and consequently $D : \mathrm{HC}^k_a(M) \to \mathrm{HC}^{k+2}_a(M)$ is an isomorphism for every $k \geqslant 2n+1$. Now, we need the following result which has independent interest.

Proposition 5.2. *We have that*

$$\mathrm{HC}^*_a(M) = \begin{cases} \mathbb{Q} & \text{if } * = k_a + 2k \text{ for every } k \in \mathbb{N}_0 \\ 0 & \text{otherwise,} \end{cases} \tag{5.1}$$

where $k_a = \min\{k \in \mathbb{Q};\ \mathrm{HC}^k_a(M) \neq 0\}$.

Remark 5.3. The grade k_a is finite and can be easily computed from the weights of the action and p; see [1, Equation (1.2)].

Proof. As mentioned in Section 2.1, $\mathrm{HC}^*(M)$ can be described as the cohomology of a complex $\mathrm{CC}^*(\alpha)$ generated by good closed Reeb orbits, for some non-degenerate contact form α, graded by the Conley-Zehnder index. For lens spaces, which are very special toric contact manifolds, we can use toric contact geometry to explicitly write down non-degenerate toric Reeb vector fields with finitely many simple closed Reeb orbits and explicitly compute the Conley-Zehnder indices of all of them and of all of their iterates. For lens spaces with zero first Chern class, i.e. Gorenstein contact lens spaces, this has been done with all details in Section 5 of [2] with the following outcome:

- With an appropriate choice of toric contact form α and up to an arbitrarily large degree, $\mathrm{CC}^*(M)$ is generated by a single good simple closed Reeb orbit γ and its iterates.
- γ is a generator of the fundamental group $\pi_1(M)$ and

  $\mu_{CZ}\left(\gamma^{N+p}\right) = \mu_{CZ}\left(\gamma^{N}\right) + 2$ up to an arbitrarily large $N \in \mathbb{N}$ ($p = |\pi_1(M)|$).
- It immediately follows that the differential of the complex $\mathrm{CC}^*(\alpha)$ is zero and $\mathrm{HC}^*_a(M)$ satisfies the result stated in this proposition for any free homotopy class a.

Using the Conley-Zehnder index formula obtained in Section 4 of [3] and valid in the context of $\mathbb{Q}$-Gorenstein toric contact manifolds, i.e. toric contact manifolds with torsion first Chern class, the same approach can be adapted to general lens spaces as we will now explain.

Let $M = L^{2n+1}_p(\ell_0, \ldots, \ell_n)$ be an arbitrary lens space and assume, without any loss of generality, that $\ell_n = 1$. As a toric contact manifold, M is determined by a moment cone $C \subset \mathbb{R}^{n+1}$ whose facets have defining normals of the form

$$\nu_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k \\ m \end{bmatrix},\ \nu_j = \begin{bmatrix} \\ e_j \\ \\ 0 \\ m \end{bmatrix} \text{ for } j = 1, \ldots, n-1,\ \nu_n = \begin{bmatrix} -\ell_1 \\ \vdots \\ -\ell_{n-1} \\ q \\ m \end{bmatrix},$$

where

- $\{e_1, \ldots, e_{n-1}\}$ is the canonical basis of $\mathbb{Z}^{n-1}$;

- $m \in \{1, \dots, p-1\}$ is the minimal positive integer such that
$$m \cdot c_1(TM) = 0 \in H^2(M;\mathbb{Z}) \cong \mathbb{Z}_p \quad \text{(note that } m \text{ divides } p\text{);}$$
- $k \in \{0, \dots, p-1\}$ is the minimal non-negative integer such that
$$k \cdot (\ell_0 + \ell_1 + \cdots + \ell_n) \equiv \frac{p}{m} \mod p$$
(note that $c_1(M) \equiv \ell_0 + \ell_1 + \cdots + \ell_n \mod p$);
- $q = k \cdot (\ell_1 + \cdots + \ell_n) - p/m$.

In fact, the matrix $\beta = [\nu_0|\nu_1|\cdots|\nu_n]$ is such that
$$\begin{aligned}\det(\beta) &= \pm\left(km\left(\ell_1 + \cdots + \ell_{n-1} + 1\right) - mq\right)\\ &= \pm m\left(k\left(\ell_1 + \cdots + \ell_n\right) - k\left(\ell_1 + \cdots + \ell_n\right) + \frac{p}{m}\right)\\ &= \pm p\end{aligned}$$
and
$$\ker\left(\beta : \mathbb{T}^{n+1} \to \mathbb{T}^{n+1}\right)$$
is generated by
$$\left(-\frac{q}{p}, \frac{k\ell_1}{p}, \dots, \frac{k\ell_{n-1}}{p}, \frac{k}{p}\right).$$
Hence, the toric contact manifold determined by this moment cone $C \subset \mathbb{R}^{n+1}$ is
$$L_p^{2n+1}(-q, k\ell_1, \dots, k\ell_{n-1}, k).$$
Since
$$\begin{aligned}-q &= -k\left(\ell_1 + \cdots + \ell_n\right) + \frac{p}{m}\\ &\equiv -k\left(\ell_1 + \cdots + \ell_n\right) + k\left(\ell_0 + \ell_1 + \cdots + \ell_n\right) \mod p\\ &= k\ell_0\end{aligned}$$
we have that
$$\begin{aligned}L_p^{2n+1}(-q, k\ell_1, \dots, k\ell_{n-1}, k) &= L_p^{2n+1}(k\ell_0, k\ell_1, \dots, k\ell_{n-1}, k)\\ &= L_p^{2n+1}(\ell_0, \ell_1, \dots, \ell_{n-1}, 1)\\ &= M\,.\end{aligned}$$

Any positive linear combination of the normals $\nu_0, \nu_1, \dots, \nu_n$, gives rise to a toric Reeb vector field on M. In particular, we can consider toric Reeb vector fields of the form
$$R = \sum_{j=0}^{n-1} \varepsilon_j \nu_j + (1 - \varepsilon)\,\nu_n$$
for arbitrarily small $\varepsilon_0, \dots, \varepsilon_{n-1} > 0$ and $\varepsilon = \sum_{j=0}^{n-1} \varepsilon_j$. By choosing $\varepsilon_0, \dots, \varepsilon_{n-1}$ such that $\{\varepsilon_0, \dots, \varepsilon_{n-1}, 1\}$ are $\mathbb{Q}$-independent, we know that the Reeb flow of R has exactly $n+1$ simple closed orbits, one for each edge of the moment cone $C \subset \mathbb{R}^{n+1}$, and these are all non-degenerate and generators of the fundamental group $\pi_1(M)$ (cf. Section 2.2 of [2]). Moreover, since $R \approx \nu_n$, the n orbits corresponding to the n edges contained in the facet of C with normal ν_n, and all their iterates, have an arbitrarily large Conley-Zehnder index (for arbitrarily small $\varepsilon_0, \dots, \varepsilon_{n-1} > 0$). Hence, $\mathrm{HC}^*(M)$ can be completely determined by computing
$$\mu_{CZ}\left(\gamma^N\right) \text{ for all } N \in \mathbb{N} \text{ and arbitrarily small } \varepsilon_0, \dots, \varepsilon_{n-1} > 0,$$

where γ is the simple closed R-orbit corresponding to the edge of C determined by the n facets with normals $\nu_0, \ldots, \nu_{n-1}$.

This Conley-Zehnder index computation can be done in the following way:

(i) Complete $\{\nu_0, \ldots, \nu_{n-1}\}$ to a $\mathbb{Z}$-basis $\{\nu_0, \ldots, \nu_{n-1}, \eta\}$ of $\mathbb{Z}^{n+1}$ with the vector

$$\eta = (0, \ldots, 0, c, d)^t \,, \text{ where } c, d \in \mathbb{Z} \text{ are such that } kd - mc = 1.$$

A simple computation shows that

$$\nu_n = a_0\nu_0 + \sum_{j=1}^{n-1} (-\ell_j)\nu_j + p\eta \,, \text{ with } a_0 = \sum_{j=1}^{n} \ell_j - d\frac{p}{m} \,.$$

(ii) Write R in the basis $\{\nu_0, \ldots, \nu_{n-1}, \eta\}$:

$$\begin{aligned} R &= \sum_{j=0}^{n-1} \varepsilon_j \nu_j + (1-\varepsilon)\, \nu_n \\ &= \sum_{j=0}^{n-1} \varepsilon_j \nu_j + (1-\varepsilon) \left(a_0\nu_0 + \sum_{j=1}^{n-1} (-\ell_j)\nu_j + p\eta \right) \\ &= (\varepsilon_0 + (1-\varepsilon)a_0)\nu_0 + \sum_{j=1}^{n-1} (\varepsilon_j - (1-\varepsilon)\ell_j)\nu_j + (1-\varepsilon)p\eta \,. \end{aligned}$$

(iii) Use the Conley-Zehnder index formula obtained in Section 4 of [3] to find that

$$\mu_{CZ}\left(\gamma^N\right) = 2\left(\left\lfloor \frac{N(\varepsilon_0 + (1-\varepsilon)a_0)}{(1-\varepsilon)p} \right\rfloor + \sum_{j=1}^{n-1} \left\lfloor \frac{N(\varepsilon_j - (1-\varepsilon)\ell_j)}{(1-\varepsilon)p} \right\rfloor + N\frac{d}{m} \right) + n \,.$$

Since

$$\frac{\varepsilon_0 + (1-\varepsilon)a_0}{1-\varepsilon} = a_0 + \varepsilon' \quad \text{and} \quad \frac{\varepsilon_j - (1-\varepsilon)\ell_j}{1-\varepsilon} = -\ell_j + \varepsilon''$$

for arbitrarily small $\varepsilon', \varepsilon'' > 0$ and all $j = 1, \ldots, n-1$, we actually have that

$$\mu_{CZ}\left(\gamma^N\right) = 2\left(\left\lfloor \frac{Na_0}{p} \right\rfloor + \sum_{j=1}^{n-1} \left\lfloor \frac{-N\ell_j}{p} \right\rfloor + N\frac{d}{m} \right) + n \,,$$

up to an arbitrarily large $N \in \mathbb{N}$.

We can now finish the proof of this proposition by showing that

$$\mu_{CZ}\left(\gamma^{N+p}\right) = \mu_{CZ}\left(\gamma^N\right) + 2 \quad \text{up to an arbitrarily large } N \in \mathbb{N}.$$

In fact,

$$\begin{aligned}\mu_{CZ}\left(\gamma^{N+p}\right) &= 2\left(\left\lfloor \frac{Na_0}{p} + a_0 \right\rfloor + \sum_{j=1}^{n-1}\left\lfloor \frac{-N\ell_j}{p} - \ell_j \right\rfloor + N\frac{d}{m} + p\frac{d}{m}\right) + n \\ &= \mu_{CZ}\left(\gamma^{N}\right) + 2\left(a_0 + \sum_{j=1}^{n-1}(-\ell_j) + \frac{pd}{m}\right) \\ &= \mu_{CZ}\left(\gamma^{N}\right) + 2\left(\sum_{j=1}^{n}\ell_j - \frac{dp}{m} + \sum_{j=1}^{n-1}(-\ell_j) + \frac{pd}{m}\right) \\ &= \mu_{CZ}\left(\gamma^{N}\right) + 2(\ell_n) \\ &= \mu_{CZ}\left(\gamma^{N}\right) + 2 \quad \text{up to an arbitrarily large } N \in \mathbb{N}.\end{aligned}$$

□

Now, take $k_0 = \min\{k_a + 2k \geqslant 2n + 1; k \in \mathbb{N}_0\}$. From the discussion in Section 2.2, Proposition 5.2 and the fact that $D : \mathrm{HC}_a^{k_0+2k}(M) \to \mathrm{HC}_a^{k_0+2k+2}(M)$ is an isomorphism for every $k \in \mathbb{N}_0$, we have an injective map $\psi : \mathbb{N}_0 \to \mathcal{P}_a$ such that $\gamma_k := \psi(k)$ satisfies

$$|\mu(\gamma_k) - (k_0 + 2k)| \leqslant 2n. \tag{5.2}$$

(Note here that the periodic orbits with homotopy class a are isolated since we are assuming that there exist finitely many simple orbits with this homotopy class.) Consider the sequence of numbers

$$x(k) = \mu(\gamma_k).$$

By (5.2), its density $\delta(x) := \lim_{j\to\infty} \frac{1}{j}\#\{i; x(i) \leqslant j\}$ equals 1/2.

Consider the sequences of numbers

$$\bar{x}_j(i) = \mu(\bar{\gamma}_j^{ip+1})$$

for $j \in \{1, \ldots, m\}$ and $i \in \mathbb{N}_0$. Since $|\mu(\bar{\gamma}_j^{ip+1}) - (ip+1)\Delta(\bar{\gamma}_j)| \leqslant n$ for every $i \in \mathbb{N}_0$, we have that $\delta(\bar{x}_j) = 1/p\Delta(\bar{\gamma}_j)$, where $\Delta(\bar{\gamma}_j) = \lim_{k\to\infty} \frac{1}{k}\mu(\bar{\gamma}_j^k)$ is the mean index of $\bar{\gamma}_j$.

Now, note that each point in the sequence x belongs to one of the sequences $\bar{x}_j$ and no point in the sequences $\bar{x}_j$ can be used twice (because ψ is injective). So the density of x must be no more than the sum of the densities of $\bar{x}_j$:

$$\delta(x) \leqslant \sum_{j=1}^{m} \delta(\bar{x}_j) \quad \Longleftrightarrow \quad \frac{1}{2} \leqslant \sum_{j=1}^{m} \frac{1}{p\Delta(\bar{\gamma}_j)},$$

that is,

$$\frac{p}{2} \leqslant \sum_{j=1}^{m} \frac{1}{\Delta(\bar{\gamma}_j)}. \tag{5.3}$$

Now, suppose that $m = 1$, that is, there exists only one simple closed orbit γ with homotopy class a. Since α is dynamically convex, $\Delta(\gamma^p) > 2$ (see [30, Corollary 5.1]) which implies that $\Delta(\gamma) > 2/p$. Thus,

$$\frac{1}{\Delta(\gamma)} < \frac{p}{2}.$$

But, by (5.3),

$$\frac{1}{\Delta(\gamma)} \geqslant \frac{p}{2},$$

a contradiction.

Remark 5.4. The assumption of dynamical convexity of α is used to ensure that $\Delta(\gamma^p) > 2$ and that every contractible periodic orbit of β has index bigger than $3-n$ (so that we can define the equivariant symplectic cohomology of β in the symplectization of $L_p^{2n+1}(\ell_0, \ldots, \ell_n)$).

References

[1] M. Abreu, L. Macarini, *Dynamical implications of convexity beyond dynamical convexity.* Calc. Var. Partial Differential Equations **61** (2022), no. 3, Paper No. 116, 47 pp.

[2] M. Abreu, L. Macarini and M Moreira, *On contact invariants of non-simply connected Gorenstein toric contact manifolds.* Mathematical Research Letters **29** (2022), 1–42.

[3] M. Abreu, L. Macarini and M Moreira, *Contact invariants of $\mathbb{Q}$-Gorenstein toric contact manifolds, the Ehrhart polynomial and Chen-Ruan cohomology.* Preprint (2022), arXiv:2202.00442.

[4] A. Adem, J. Leida, Y. Ruan, Orbifolds and stringy topology, volume 171 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2007

[5] F. Bourgeois, A. Oancea, *An exact sequence for contact- and symplectic homology.* Invent. Math. **175** (2009), no. 3, 611–680.

[6] F. Bourgeois, A. Oancea, *Symplectic homology, autonomous Hamiltonians, and Morse-Bott moduli spaces.* Duke Math. J. **146** (2009), 71–174.

[7] F. Bourgeois, A. Oancea, *Fredholm theory and transversality for the parametrized and for the S^1-invariant symplectic action.* J. Eur. Math. Soc. (JEMS) **12** (2010), no. 5, 1181–1229.

[8] F. Bourgeois, A. Oancea, *The index of Floer moduli problems for parametrized action functionals.* Geom. Dedicata **165** (2013), 5–24.

[9] F. Bourgeois, A. Oancea, *The Gysin exact sequence for S^1-equivariant symplectic homology.* J. Topol. Anal. **5** (2013), no. 4, 361–407.

[10] F. Bourgeois, A. Oancea, *S^1-equivariant symplectic homology and linearized contact homology.* Int. Math. Res. Not. IMRN 2017, no. 13, 3849–3937.

[11] J. Chaidez, O. Edtmair, *3D convex contact forms and the Ruelle invariant.* Invent. Math. **229** (2022), no. 1, 243–301.

[12] J. Chaidez, O. Edtmair, *The Ruelle Invariant And Convexity In Higher Dimensions.* Preprint arXiv:2205.00935, 2022.

[13] W. Chen, Y. Ruan, *A new cohomology theory of orbifold.* Comm. Math. Phys. **248** (2004), 1–31.

[14] K. Cieliebak, A. Floer, and H. Hofer, *Symplectic homology. II. A general construction.* Math. Z. **218** (1995), 103–122.

[15] K. Cieliebak, U. Frauenfelder, and A. Oancea, *Rabinowitz Floer homology and symplectic homology.* Ann. Sci. Éc. Norm. Supér. (4) **43** (2010), no. 6, 957–1015.

[16] K. Cieliebak, A. Oancea, *Symplectic homology and the Eilenberg-Steenrod axioms.* Appendix written jointly with Peter Albers. Algebr. Geom. Topol. **18** (2018), no. 4, 1953–2130.

[17] D. Cristofaro-Gardiner, M. Hutchings, *From one Reeb orbit to two.* J. Differential Geom. **102** (2016), no. 1, 25–36.

[18] H. Duan, H. Liu, *Multiplicity and ellipticity of closed characteristics on compact star-shaped hypersurfaces in $\mathbb{R}^{2n}$.* Calc. Var. Partial Differential Equations **56** (2017), no. 3, Art. 65, 30 pp.

[19] I. Ekeland, H. Hofer, *Convex Hamiltonian energy surfaces and their periodic trajectories.* Comm. Math. Phys. **113** (1987), no. 3, 419–469.

[20] V. Ginzburg, B. Gürel, *Lusternik–Schnirelmann theory and closed Reeb orbits.* Math. Z. **295** (2020), 515–582.

[21] V. Ginzburg, D. Hein, U. Hryniewicz, L. Macarini, *Closed Reeb orbits on the sphere and symplectically degenerate maxima.* Acta Math. Vietnam. **38** (2013), no. 1, 55-78.

[22] V. Ginzburg, L. Macarini, *Dynamical convexity and closed orbits on symmetric spheres.* Duke Math. J. **170** (2021), 1201–1250.

[23] M. Girardi, *Multiple orbits for Hamiltonian systems on starshaped surfaces with symmetries.* Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 4, 285–294.

[24] F. Gironella, Z. Zhou, *Exact orbifold fillings of contact manifolds.* Preprint (2021), arXiv:2108.12247.

[25] X. Hu, Y. Long, W. Wang, *Resonance identity, stability, and multiplicity of closed characteristics on compact convex hypersurfaces.* Duke Math. J. **139** (2007), no. 3, 411–462.

[26] H. Hofer, K. Wysocki and E. Zehnder. *The dynamics of strictly convex energy surfaces in* $\mathbb{R}^4$. Ann. of Math. **148** (1998), 197–289.

[27] U. Hryniewicz, L. Macarini, *Local contact homology and applications.* J. Topol. Anal. **7** (2015), 167–238.

[28] H. Liu, Y. Long, *The existence of two closed characteristics on every compact star-shaped hypersurface in* $\mathbb{R}^4$. Acta Math. Sin. (Engl. Ser.) **32** (2016), no. 1, 40–53.

[29] H. Liu, L. Zhang, *Multiplicity of closed Reeb orbits on dynamically convex* $\mathbb{R}P^{2n-1}$ *for* $n \geqslant 2$. Discrete Contin. Dyn. Syst. **42** (2022), no. 4, 1801–1816.

[30] Y. Long, *Precise iteration formulae of the Maslov-type index theory and ellipticity of closed characteristics.* Adv. Math. **154** (2000), no. 1, 76–131.

[31] Y. Long, C. Zhu, *Closed characteristics on compact convex hypersurfaces in* $\mathbb{R}^{2n}$. Ann. of Math. **155** (2002), no. 2, 317–368.

[32] M. McLean, *Reeb orbits and the minimal discrepancy of an isolated singularity.* Invent. Math. **204** (2016), no. 2, 505–594.

[33] M. Moreira, *Floer homology for global quotient orbifolds.* Master's thesis. Instituto Superior Técnico, 2019. Available online at https://people.math.ethz.ch/~mimoreira/master_thesis.pdf

[34] P. Rabinowitz, *Periodic solutions of Hamiltonian systems.* Comm. Pure Appl. Math. **31** (1978), no. 2, 157–184.

[35] P. Seidel, *A biased view of symplectic cohomology.* In Current developments in mathematics, 2006, 211–253. Int. Press, Somerville, MA, 2008.

[36] C. Viterbo, *Functors and computations in Floer homology with applications. I.* Geom. Funct. Anal. **9** (1999), no. 5, 985–1033.

[37] W. Wang, *Existence of closed characteristics on compact convex hypersurfaces in* $\mathbb{R}^{2n}$. Calc. Var. Partial Differential Equations, **55** (2016), 1–25.

[38] W. Wang, *Closed characteristics on compact convex hypersurfaces in* $\mathbb{R}^8$. Adv. Math. **297** (2016), 93–148.

[39] D. Zhang, *P-cyclic symmetric closed characteristics on compact convex P-cyclic symmetric hypersurface in* $\mathbb{R}^{2n}$. Discrete Contin. Dyn. Syst. **33** (2013), no. 2, 947–964.

Center for Mathematical Analysis, Geometry and Dynamical Systems, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
E-mail address: mabreu@math.tecnico.ulisboa.pt

School of Mathematics and Statistics, Wuhan University, Wuhan 430072, Hubei, China
E-mail address: huiliu00031514@whu.edu.cn

Center for Mathematical Analysis, Geometry and Dynamical Systems, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
E-mail address: macarini@math.tecnico.ulisboa.pt