

REMARKS ON LAGRANGIAN INTERSECTIONS IN TORIC MANIFOLDS

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ABSTRACT. We consider two natural Lagrangian intersection problems in the context of symplectic toric manifolds: displaceability of torus orbits and of a torus orbit with the real part of the toric manifold. Our remarks address the fact that one can use simple cartesian product and symplectic reduction considerations to go from basic examples to much more sophisticated ones. We show in particular how rigidity results for the above Lagrangian intersection problems in weighted projective spaces can be combined with these considerations to prove analogous results for all monotone toric symplectic manifolds. We also discuss non-monotone and/or non-Fano examples, including some with a continuum of non-displaceable torus orbits.

1. INTRODUCTION

Let (M^{2n}, ω) be a toric symplectic manifold, i.e. a symplectic manifold equipped with an effective Hamiltonian \mathbb{T}^n -action generated by a moment map

$$\mu : M \rightarrow P := \mu(M) \subset (\mathbb{R}^n)^* ,$$

where the moment polytope P is defined by

$$(1) \quad \ell_i(x) := \langle x, \nu_i \rangle + a_i \geq 0, \quad i = 1, \dots, d.$$

Here the a_i 's are real numbers, each vector $\nu_i \in \mathbb{Z}^n$ is the primitive integral interior normal to the facet F_i of the polytope P and d is the number of facets of P .

Denote by $\tau : M \rightarrow M$ the canonical anti-symplectic involution, characterized by $\mu \circ \tau = \mu$, and let $R := M^\tau$ denote its fixed point set. R is a Lagrangian manifold, often called the real part of M .

Given $x \in \text{int}(P)$, let $T_x := \mu^{-1}(x)$ denote the corresponding \mathbb{T}^n -orbit, a Lagrangian torus in M . Since $\mu \circ \tau = \mu$, we also have that $\tau(T_x) = T_x$ and $(T_x)^\tau = T_x \cap R$. This last set, the real part of a regular \mathbb{T}^n -orbit, is discrete with 2^n points.

This can be seen quite explicitly in action-angle coordinates. Consider $\check{M} \subset M$ defined by $\check{M} = \mu^{-1}(\text{int}(P))$. One checks that \check{M} is an open dense subset of M , consisting of all the points where the \mathbb{T}^n -action is free. It can be described as

$$\check{M} \cong \text{int}(P) \times \mathbb{T}^n = \{(x_1, \dots, x_n, e^{i\theta_1}, \dots, e^{i\theta_n}) : x \in \text{int}(P) \subset \mathbb{R}^n, \theta \in \mathbb{R}^n/2\pi\mathbb{Z}^n\} ,$$

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where (x, θ) are symplectic action-angle coordinates for ω , i.e.

$$\omega|_{\check{M}} = dx \wedge d\theta = \sum_{j=1}^n dx_j \wedge d\theta_j .$$

In these action-angle coordinates the moment map and anti-symplectic involution are given by

$$\mu(x, e^{i\theta}) = x \quad \text{and} \quad \tau(x, e^{i\theta}) = (x, e^{-i\theta}) .$$

Hence, we have that

$$R \cap \check{M} = (x, \pm 1) \equiv (x_1, \dots, x_n, \pm 1, \dots, \pm 1), \quad x \in \text{int}(P),$$

and

$$\sharp(R \pitchfork T_x) = 2^n, \quad \forall x \in \text{int}(P).$$

Example 1.1. Consider $(\mathbb{C}P^n, \omega_{FS})$ with moment polytope $P \subset (\mathbb{R}^n)^*$ given by

$$P = \left\{ (x_1, \dots, x_n) : x_j + 1 \geq 0, \quad j = 1, \dots, n, \quad \text{and} \quad -\left(\sum_{j=1}^n x_j\right) + 1 \geq 0 \right\} .$$

Then

$$R = \mathbb{R}P^n \quad \text{and} \quad T_0 := \mu^{-1}(0) \equiv \text{Clifford } n\text{-torus}.$$

Example 1.2. Let (M^{2n}, ω) be a monotone toric symplectic manifold, i.e. such that $[\omega] = \lambda(2\pi c_1(\omega)) \in H^2(M)$ with $\lambda \in \mathbb{R}^+$. The corresponding moment polytope $P \subset (\mathbb{R}^n)^*$ is then a Fano Delzant polytope, meaning that it can be defined by (1) with

$$a_1 = \dots = a_d = \lambda .$$

In this case, the Lagrangian torus fiber over the origin $0 \in P$ will be called the *centered* or *special* torus fiber. It is the unique monotone torus fiber of the monotone toric symplectic manifold (M^{2n}, ω) .

Without any loss of generality, we will usually assume that $\lambda = 1$ (as we already did in Example 1.1).

In this context of toric symplectic manifolds, it is natural to consider the following Lagrangian intersection rigidity question: given $x \in \text{int}(P)$, does there exist $\psi \in \text{Ham}(M, \omega)$ such that

$$\psi(T_x) \cap T_x = \emptyset \quad \text{or} \quad \psi(T_x) \cap R = \emptyset \quad \text{or} \quad \psi(R) \cap R = \emptyset ?$$

Our remarks in this paper concern the first two of these Lagrangian intersection problems and will show how simple cartesian product (Section 2) and symplectic reduction (Section 3) considerations can be used to go from basic examples to more sophisticated ones.

To illustrate our point, consider the following theorem on the most basic example that can be considered in this context.

Theorem 1.3. *For $(\mathbb{C}P^n, \omega_{FS})$, as in Example 1.1, we have that:*

(i) [5, 8]

$$\psi(T_0) \cap T_0 \neq \emptyset \quad \text{and} \quad \sharp(\psi(T_0) \frown T_0) \geq 2^n, \quad \forall \psi \in \text{Ham}(\mathbb{C}P^n, \omega_{FS}).$$

(ii) [3, 4, 13, 20]

$$\psi(T_0^n) \cap \mathbb{R}P^n \neq \emptyset, \quad \forall \psi \in \text{Ham}(\mathbb{C}P^n, \omega_{FS}).$$

(iii) [1]

$$\sharp(\psi(T_0^{2n-1}) \frown \mathbb{R}P^{2n-1}) \geq 2^n, \quad \forall \psi \in \text{Ham}(\mathbb{C}P^{2n-1}, \omega_{FS}).$$

The following results follow by straightforward applications of our remarks.

Theorem 1.4. (Cf. Remark 2.10 and § 4.2.)

$$\sharp(\psi(T_0^{2n}) \frown \mathbb{R}P^{2n}) \geq 2^n, \quad \forall \psi \in \text{Ham}(\mathbb{C}P^{2n}, \omega_{FS}).$$

Theorem 1.5. Let (M^{2n}, ω) be a monotone toric symplectic manifold, as in Example 1.2, R its real part and T its special centered torus fiber. Let $\nu_1, \dots, \nu_d \in \mathbb{Z}^n$ denote the primitive integral interior normals to the facets of the corresponding Fano Delzant polytope $P \subset (\mathbb{R}^n)^*$.

(i) (Cf. Proposition 4.4 in § 4.3.) If $\sum_{i=1}^d \nu_i = 0$ then

$$\psi(T) \cap R \neq \emptyset, \quad \psi(T) \cap T \neq \emptyset \quad \text{and} \quad \sharp(\psi(T) \frown T) \geq 2^n, \quad \forall \psi \in \text{Ham}(M, \omega).$$

(ii) (Cf. Proposition 4.6 in § 4.5.) If $P \subset (\mathbb{R}^n)^*$ is symmetric, i.e. whenever $\nu \in \mathbb{Z}^n$ is the normal to a facet of P then $-\nu \in \mathbb{Z}^n$ is also the normal to a facet of P , then

$$\sharp(\psi(T) \frown R) \geq 2^n, \quad \forall \psi \in \text{Ham}(M, \omega).$$

Remark 1.6. A particular interesting example that fits both (i) and (ii) of this theorem is $M = \mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$, equipped with a monotone symplectic form (cf. § 4.5).

Consider now the following generalization of part (i) of Theorem 1.3, contained in the work of Woodward [22] and forthcoming work of Cho and Poddar [10].

Theorem 1.7. Let $\mathbb{C}P(1, \mathbf{m}) := \mathbb{C}P(1, m_1, \dots, m_n)$, $m_1, \dots, m_n \in \mathbb{N}$, denote the weighted projective space determined by the moment polytope $P_{\mathbf{m}} \subset (\mathbb{R}^n)^*$ given by

$$P = \left\{ (x_1, \dots, x_n) : x_j + 1 \geq 0, \quad j = 1, \dots, n, \quad \text{and} \quad - \left(\sum_{j=1}^n m_j x_j \right) + 1 \geq 0 \right\}$$

(i.e. the symplectic quotient of \mathbb{C}^{n+1} by the S^1 -action with weights $(1, m_1, \dots, m_n)$). Let $T_0 := \mu^{-1}(0)$ denote its special centered torus fiber, where $\mu : \mathbb{C}P(1, \mathbf{m}) \rightarrow P_{\mathbf{m}}$ is the moment map. Then

$$\psi(T_0) \cap T_0 \neq \emptyset \quad \text{and} \quad \sharp(\psi(T_0) \frown T_0) \geq 2^n, \quad \forall \psi \in \text{Ham}(\mathbb{C}P(1, \mathbf{m}), \omega_{\mathbf{m}}).$$

The following result, removing the zero-sum assumption in part (i) of Theorem 1.5 and originally due to Entov-Polterovich [13], Cho [9] and Fukaya-Oh-Ohta-Ono [15], follows from this Theorem 1.7 by a straightforward application of our symplectic reduction remark (cf. Proposition 4.9 in § 4.7).

Theorem 1.8. *Let (M, ω) be a compact monotone toric symplectic manifold, as in Example 1.2, and T its special centered torus fiber. Then*

$$\psi(T) \cap T \neq \emptyset \quad \text{and} \quad \sharp(\psi(T) \pitchfork T) \geq 2^n, \quad \forall \psi \in \text{Ham}(M, \omega).$$

Remark 1.9. *An appropriate generalization of part (ii) of Theorem 1.3 to weighted projective spaces would imply, by the same straightforward application of our symplectic reduction remark, that on any compact monotone toric symplectic manifold (M, ω) we also have that*

$$\psi(T) \cap R \neq \emptyset, \quad \forall \psi \in \text{Ham}(M, \omega).$$

This has been proved by Alston-Amorim [2] using the methods developed by Fukaya, Oh, Ohta and Ono in [14, 15, 16].

With the help of another basic example, i.e. the total space of the line bundle $\mathcal{O}(-1) \rightarrow \mathbb{C}\mathbb{P}^1$ (cf. Section 5), our remarks can also be used to prove interesting non-displaceability results on certain non-monotone and/or non-Fano examples, such as:

- a continuum of non-displaceable torus fibers on $M = \mathbb{C}\mathbb{P}^2 \sharp 2\overline{\mathbb{C}\mathbb{P}^2}$ with a certain non-monotone symplectic form (cf. Example 10.3 in [16] and § 5.4).
- a particular non-displaceable torus fiber on the family of non-Fano examples given by Hirzebruch surfaces $H_k := \mathbb{P}(\mathcal{O}(-k) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$, with $2 \leq k \in \mathbb{N}$ (cf. Example 10.1 in [16] and § 6.1).
- a continuum of non-displaceable torus fibers on a certain non-Fano toric symplectic 4-manifold considered by McDuff (cf. § 2.1 in [18] and § 6.2).

For all non-displaceable torus fibers in these examples, we can also use our remarks to obtain an appropriate optimal lower bound for the number of transversal intersection points, which for these Lagrangian 2-tori is 4.

Remark 1.10. *Regarding this last example, McDuff shows in [18] that it gives rise, under a repeated wedge construction, to a monotone symplectic toric 12-manifold with a continuous interval of Lagrangian torus fibers that cannot be displaced by her method of probes [17]. Hence, as she points out, these fibers “may perhaps be non-displaceable by Hamiltonian isotopies, even though, according to [14], their Floer homology vanishes”. Although we can use the remarks in this paper to prove non-displaceability of the relevant torus fibers in the non-Fano toric symplectic 4-manifold (cf. § 6.2), so far we have not been able to do the same for the corresponding monotone toric symplectic 12-manifold.*

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2. FIRST REMARK: CARTESIAN PRODUCT

This first remark, on cartesian products, is motivated by Alston's result in part (iii) of Theorem 1.3 and can be used to remove its dimension restriction, i.e. prove Theorem 1.4. We will also use it in combination with our second remark, on symplectic reduction.

2.1. Combinatorial Floer Invariant. Let $P \subset (\mathbb{R}^n)^*$ be a moment polytope defined by

$$\ell_i(x) := \langle x, \nu_i \rangle + a_i \geq 0, \quad i = 1, \dots, d.$$

Each vector $\nu_i = ((\nu_i)_1, \dots, (\nu_i)_n) \in \mathbb{Z}^n$ is the primitive integral normal to the facet F_i of the polytope P , and d is the number of facets of P . We will say that P is **even** if d is even.

Let CF^n be the vector space of dimension 2^n generated over \mathbb{Z}_2 by the following 2^n symbols:

$$(\varepsilon_1, \dots, \varepsilon_n) \text{ with } \varepsilon_k = \pm 1, \quad k = 1, \dots, n.$$

Consider the linear map

$$\partial_P : CF^n \rightarrow CF^n$$

defined on basis elements by

$$\partial_P(\varepsilon) = \sum_{i=1}^d (-1)^{\nu_i} \varepsilon$$

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ and

$$(-1)^{\nu_i} \varepsilon = ((-1)^{(\nu_i)_1} \varepsilon_1, \dots, (-1)^{(\nu_i)_n} \varepsilon_n).$$

Proposition 2.2. $\partial_P^2 = 0$ if d is even and $\partial_P^2 = \text{id}$ if d is odd.

Proof.

$$\begin{aligned} \partial_P(\partial_P \varepsilon) &= \partial_P \left(\sum_{i=1}^d (-1)^{\nu_i} \varepsilon \right) = \sum_{i=1}^d \partial_P ((-1)^{\nu_i} \varepsilon) \\ &= \sum_{i,j=1}^d (-1)^{\nu_j} ((-1)^{\nu_i} \varepsilon) = \sum_{i,j=1}^d (-1)^{\nu_i + \nu_j} \varepsilon \\ &= 2 \left(\sum_{i < j} (-1)^{\nu_i + \nu_j} \varepsilon \right) + \sum_{i=1}^d (-1)^{2\nu_i} \varepsilon \\ &= 0 + \sum_{i=1}^d \varepsilon = d\varepsilon \end{aligned}$$

□

Definition 2.3. The Floer invariant $HF(P)$ of an even integral polytope P is defined by

$$HF(P) := \dim \ker(\partial_P) - \dim \text{im}(\partial_P).$$

Proposition 2.4. *If $P = P' \times P''$, with P' and P'' even integral polytopes, then*

$$HF(P) = HF(P') \cdot HF(P'').$$

Proof. Suppose $P' \subset \mathbb{R}^{n'}$ and $P'' \subset \mathbb{R}^{n''}$ have normal vectors

$$\nu'_1, \dots, \nu'_{d'} \in \mathbb{Z}^{n'} \quad \text{and} \quad \nu''_1, \dots, \nu''_{d''} \in \mathbb{Z}^{n''}.$$

Then $P = P' \times P'' \subset \mathbb{R}^{n'} \times \mathbb{R}^{n''} = \mathbb{R}^n$, $n = n' + n''$, has normal vectors

$$(\nu'_1, 0), \dots, (\nu'_{d'}, 0), (0, \nu''_1), \dots, (0, \nu''_{d''}) \in \mathbb{Z}^{n'} \times \mathbb{Z}^{n''} = \mathbb{Z}^n.$$

Hence, the linear map ∂_P on $CF^n = CF^{n'} \otimes CF^{n''}$ is given by

$$\partial_P(\varepsilon' \otimes \varepsilon'') = (\partial_{P'}\varepsilon') \otimes \varepsilon'' + \varepsilon' \otimes (\partial_{P''}\varepsilon'').$$

The result of the proposition follows by a standard application of the Künneth formula in this context. \square

Using this proposition and the fact that $P \times P$ always has an even number of facets, we can define the Floer invariant of any integral polytope.

Definition 2.5. *The Floer invariant $HF(P)$ of an integral polytope P is defined by*

$$HF(P) := \sqrt{\dim \ker(\partial_{P \times P}) - \dim \text{im}(\partial_{P \times P})}.$$

2.6. Relation with Lagrangian Floer Homology. Suppose now that P is a Fano Delzant polytope and let M_P denote its associated smooth Fano toric variety. This means that when defining $P \subset (\mathbb{R}^n)^*$ by

$$\ell_i(x) := \langle x, \nu_i \rangle + a_i \geq 0, \quad i = 1, \dots, d.$$

we can assume that $a_1 = \dots = a_d = 1$ and (M_P, ω_P) is monotone: $[\omega_P] = 2\pi c_1(M_P)$.

Denote by R_P the Lagrangian real part of (M_P, ω_P) and by T_P the Lagrangian torus over $0 \in \text{int}(P)$. Assuming that the Lagrangian Floer homology $HF(R_P, T_P; \mathbb{Z}_2)$ is well defined, we have the following theorem.

Theorem 2.7.

$$\dim HF(R_P, T_P; \mathbb{Z}_2) = HF(P).$$

Corollary 2.8.

$$\sharp(\psi(R_P) \cap T_P) \geq HF(P), \quad \forall \psi \in \text{Ham}(M_P, \omega_P).$$

Proof of the Corollary. If P is a Fano Delzant polytope, then $P \times P$ is an even Fano Delzant polytope to which Theorem 2.7 applies. Suppose that

$$\sharp(\psi(R_P) \cap T_P) < HF(P).$$

Then $(\psi \times \psi) \in \text{Ham}(M_P \times M_P, \omega_P \times \omega_P)$ would be such that

$$\sharp[(\psi \times \psi)(R_P \times R_P) \cap (T_P \times T_P)] = (\sharp(\psi(R_P) \cap T_P))^2 < (HF(P))^2 = HF(P \times P),$$

which would imply that

$$\dim HF(R_{P \times P}, T_{P \times P}; \mathbb{Z}_2) < HF(P \times P).$$

This contradicts Theorem 2.7. As before, we are assuming that $HF(R_{P \times P}, T_{P \times P}; \mathbb{Z}_2)$ is well defined. \square

Remark 2.9. *The meaning of $HF(R_P, T_P; \mathbb{Z}_2)$ being well defined depends on the technical tools one is willing to use. If one uses only a basic version of Lagrangian Floer homology, then the condition that the minimal Maslov number of R_P is bigger than two has to be enforced. Under this assumption, it can be proved as in [1, 8, 11, 19] that, for the standard complex structure J on M_P , the boundary of the moduli space of holomorphic strips with dimension one is given by broken strips and holomorphic disks with Maslov index two and boundary in T_P . Moreover, the linearized Cauchy-Riemann operator $D_u \bar{\partial}_J$ is surjective for all J -holomorphic strips u of Maslov index one or two. It follows from this that $HF(R_P, T_P; \mathbb{Z}_2)$ is well defined whenever d is even. As a matter of fact, if d is even then the holomorphic disks with index two and boundary in T_P counted by ∂_P^2 cancel out when we work with \mathbb{Z}_2 coefficients; this is the key point in the proof of Proposition 2.2. In particular, $HF(R_{P \times P}, T_{P \times P}; \mathbb{Z}_2)$ is well defined whatever is the parity of d . Indeed, the minimal Maslov number of $R_{P \times P}$ equals the minimal Maslov number of R_P .*

With this hypothesis on R_P the set of examples that are covered by Corollary 2.8 is a bit restrictive. If one uses a sophisticated version of Lagrangian Floer homology, such as the one developed by Fukaya, Oh, Ohta and Ono in [14, 15, 16], then Corollary 2.8 covers a lot more ground (see [2]).

Remark 2.10. *Even with only basic Lagrangian Floer homology, $HF(R_{P \times P}, T_{P \times P}; \mathbb{Z}_2)$ is indeed well defined when P is a simplex, i.e. $M_P = \mathbb{C}P^n$, and Corollary 2.8 can be combined with a combinatorial computation of $HF(P \times P)$ to prove Theorem 1.4, hence removing the dimension restriction of Alston's result. We will not present that combinatorial computation here since (i) it is presented in [2] and (ii), as we will see, our remark on symplectic reduction can also be used to easily remove this restriction.*

The proof of Theorem 2.7 is a simple combination of the following two ingredients:

- (i) the characterization by Cho [8] and Cho-Oh [11] of the holomorphic discs on Fano toric manifolds, that are relevant for the differential on the Lagrangian Floer complex, as Blaschke products;
- (ii) the existence of *global homogeneous coordinates* on any smooth toric variety M_P , not just on $\mathbb{C}P^n$ (see §4.4 of Cox [12]).

We will now discuss some details of (ii), and how they contribute to the proof of Theorem 2.7.

From the well known quotient representation

$$M_P = (\mathbb{C}^d \setminus Z_P) / G_P$$

we get a map

$$\begin{aligned} \mathbb{C}^d \setminus Z_P &\longrightarrow M_P \\ (z_1, \dots, z_d) &\longmapsto [z_1, \dots, z_d] \end{aligned}$$

defining homogeneous coordinates on M_P , where homogeneous here is with respect to the action of the torus G_P . Note that the canonical anti-holomorphic involution $\tau_P : M_P \rightarrow M_P$ is given in these homogeneous coordinates as

$$(2) \quad \tau_P([z_1, \dots, z_d]) = [\bar{z}_1, \dots, \bar{z}_d].$$

Let m_1, \dots, m_ℓ be the integral points in P , i.e.

$$\{m_1, \dots, m_\ell\} = P \cap \mathbb{Z}^n,$$

and consider the map

$$\phi : \mathbb{C}^d \setminus Z_P \rightarrow \mathbb{C}P^{\ell-1}$$

given by

$$(3) \quad \phi(z_1, \dots, z_d) = [z^{[m_1]}, \dots, z^{[m_\ell]}],$$

where

$$z^{[m]} = \prod_{i=1}^d z_i^{\ell_i(m)}.$$

It turns out that ϕ induces a well defined embedding

$$\begin{aligned} M_P &\longrightarrow \mathbb{C}P^{\ell-1} \\ [z_1, \dots, z_d] &\longmapsto [z^{[m_1]}, \dots, z^{[m_\ell]}]. \end{aligned}$$

If one restricts this map to $(\mathbb{C}^*)^n \subset M_P$, one gets an embedding

$$\varphi : (\mathbb{C}^*)^n \rightarrow \mathbb{C}P^{\ell-1}$$

given in the standard coordinates of $(\mathbb{C}^*)^n$ as

$$(4) \quad \varphi(t_1, \dots, t_n) = [t^{m_1}, \dots, t^{m_\ell}]$$

where

$$t^{m_k} = \prod_{i=1}^n t_i^{(m_k)_i}.$$

Since

$$R_P \cap (\mathbb{C}^*)^n = (\mathbb{R}^*)^n \quad \text{and} \quad T_P = (S^1)^n \subset (\mathbb{C}^*)^n$$

we get that

$$R_P \cap T_P = \{(\varepsilon_1, \dots, \varepsilon_n) : \varepsilon_i = \pm 1\}.$$

Any such intersection point can be written in homogeneous coordinates as

$$[\varepsilon_1, \dots, \varepsilon_d], \quad \varepsilon_i = \pm 1.$$

As Alston [1] does in the case of $\mathbb{C}P^n$, using the work of Cho [8] and Cho-Oh [11], the characterization of relevant holomorphic discs as Blaschke products shows that the Lagrangian Floer differential is given in homogeneous coordinates by

$$\partial_P([\varepsilon_1, \dots, \varepsilon_d]) = \sum_{i=1}^d [\varepsilon_1, \dots, -\varepsilon_i, \dots, \varepsilon_d].$$

We can now use the maps ϕ and φ , defined by (3) and (4), to compute $\partial_P(\varepsilon_1, \dots, \varepsilon_n)$ directly. Note that if

$$\phi([\varepsilon_1, \dots, \varepsilon_i, \dots, \varepsilon_d]) = [\varepsilon^{[m_1]}, \dots, \varepsilon^{[m_\ell]}]$$

then

$$\begin{aligned} \phi([\varepsilon_1, \dots, -\varepsilon_i, \dots, \varepsilon_d]) &= [(-1)^{\ell_i(m_1)} \varepsilon^{[m_1]}, \dots, (-1)^{\ell_i(m_\ell)} \varepsilon^{[m_\ell]}] \\ &= [(-1)^{\langle m_1, \nu_i \rangle + a_i} \varepsilon^{[m_1]}, \dots, (-1)^{\langle m_\ell, \nu_i \rangle + a_i} \varepsilon^{[m_\ell]}] \\ &= [(-1)^{\langle m_1, \nu_i \rangle} \varepsilon^{[m_1]}, \dots, (-1)^{\langle m_\ell, \nu_i \rangle} \varepsilon^{[m_\ell]}]. \end{aligned}$$

On the other hand, if

$$\varphi(t_1, \dots, t_n) = [t^{m_1}, \dots, t^{m_\ell}] = [\varepsilon^{[m_1]}, \dots, \varepsilon^{[m_\ell]}]$$

then

$$\begin{aligned} \varphi((-1)^{(\nu_i)_1} t_1, \dots, (-1)^{(\nu_i)_n} t_n) &= [(-1)^{\langle m_1, \nu_i \rangle} t^{m_1}, \dots, (-1)^{\langle m_\ell, \nu_i \rangle} t^{m_\ell}] \\ &= [(-1)^{\langle m_1, \nu_i \rangle} \varepsilon^{[m_1]}, \dots, (-1)^{\langle m_\ell, \nu_i \rangle} \varepsilon^{[m_\ell]}] \\ &= \phi([\varepsilon_1, \dots, -\varepsilon_i, \dots, \varepsilon_d]). \end{aligned}$$

Hence,

$$\begin{aligned} \partial_P([\varepsilon_1, \dots, \varepsilon_d]) &= \sum_{i=1}^d [\varepsilon_1, \dots, -\varepsilon_i, \dots, \varepsilon_d] \\ \Leftrightarrow \partial_P(\varepsilon_1, \dots, \varepsilon_n) &= \sum_{i=1}^d ((-1)^{(\nu_i)_1} \varepsilon_1, \dots, (-1)^{(\nu_i)_n} \varepsilon_n). \end{aligned}$$

2.11. Examples.

Example 2.12. Let $P = [-1, 1] \subset \mathbb{R}$ with $M_P = \mathbb{C}P^1 = S^2$. In this case R_P is a meridian circle and T_P is the equator. P has normals $\nu_1 = 1$ and $\nu_2 = -1$. CF is a 2-dimensional vector space with basis $e_1 = (1)$ and $e_2 = (-1)$. The differential ∂_P is given by

$$\partial_P(\varepsilon) = ((-1)^1 \varepsilon) + ((-1)^{-1} \varepsilon) = 2(-\varepsilon) = 0.$$

Hence,

$$HF(P) = 2.$$

Proposition 2.13. *If P is an even polytope such that whenever ν is a normal $-\nu$ is also a normal, i.e. such that P is symmetric, then $HF(P) = 2^n$.*

Proof. As in the example above, we always have $\partial_P = 0$ under these conditions. \square

Remark 2.14. *Under the assumption that $HF(R_P, T_P; \mathbb{Z}_2)$ is well defined, it follows from this proposition that if P is symmetric then*

$$\sharp(\psi(T_P) \cap R_P) \geq 2^n \quad \text{for any } \psi \in \text{Ham}(M_P, \omega_P).$$

In fact, as we will see in Proposition 4.6, one can easily use our remark on symplectic reduction to remove the assumption and give another proof of this result.

Example 2.15. When $n = 2$ there are exactly two even Fano polytopes with the property of Proposition 2.13: the Fano square, corresponding to $\mathbb{C}P^1 \times \mathbb{C}P^1$, and the Fano hexagon, corresponding to $\mathbb{C}P^2$ blown up at 3 points.

Example 2.16. Let $P \subset \mathbb{R}^2$ be the Fano simplex corresponding to $M_P = \mathbb{C}P^2$. Since P has 3 facets, an odd number, we will consider the even Fano Delzant polytope $P \times P \subset \mathbb{R}^4$, whose 6 facets have normals

$$\begin{aligned} \nu_1 = (1, 0, 0, 0), \nu_2 = (0, 1, 0, 0), \nu_3 = (-1, -1, 0, 0), \nu_4 = (0, 0, 1, 0), \nu_5 = (0, 0, 0, 1) \\ \text{and } \nu_6 = (0, 0, -1, -1). \end{aligned}$$

This means that $\partial_{P \times P} : CF^4 \rightarrow CF^4$ is given by

$$\begin{aligned} \partial_{P \times P}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = & (-\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) + (\varepsilon_1, -\varepsilon_2, \varepsilon_3, \varepsilon_4) + (-\varepsilon_1, -\varepsilon_2, \varepsilon_3, \varepsilon_4) \\ & + (\varepsilon_1, \varepsilon_2, -\varepsilon_3, \varepsilon_4) + (\varepsilon_1, \varepsilon_2, \varepsilon_3, -\varepsilon_4) + (\varepsilon_1, \varepsilon_2, -\varepsilon_3, -\varepsilon_4). \end{aligned}$$

With this explicit formula it is not hard to check that

$$HF(P \times P) = \dim \ker(\partial_{P \times P}) - \dim \text{im}(\partial_{P \times P}) = 10 - 6 = 4.$$

Hence,

$$HF(P) = \sqrt{4} = 2$$

and applying Corollary 2.8 we conclude that

$$\sharp(\psi(\mathbb{R}P^2) \cap \mathbb{T}^2) \geq 2,$$

for any $\psi \in \text{Ham}(\mathbb{C}P^2)$. This estimate is known to be optimal (see for example the end of section 5 in [2]).

3. SECOND REMARK: SYMPLECTIC REDUCTION

Here we will state some elementary general facts in the particular context of symplectic toric manifolds.

Let $(\tilde{M}, \tilde{\omega})$ be a symplectic toric manifold of dimension $2N$ with $\tilde{\mathbb{T}}$ -action generated by a moment map

$$\tilde{\mu} : \tilde{M} \rightarrow \tilde{P} \subset (\mathbb{R}^N)^*.$$

As before, given $x \in \text{int}(\tilde{P})$, let $\tilde{T}_x := \tilde{\mu}^{-1}(x)$ denote the corresponding $\tilde{\mathbb{T}}$ -orbit, a Lagrangian torus in \tilde{M} , and let \tilde{R} denote the real part of \tilde{M} , i.e. the Lagrangian submanifold given by the fixed point set of the canonical anti-symplectic involution $\tilde{\tau} : \tilde{M} \rightarrow \tilde{M}$, characterized by $\tilde{\mu} \circ \tilde{\tau} = \tilde{\mu}$. Recall that

$$\tilde{\tau}(\tilde{\theta} \cdot \tilde{p}) = -\tilde{\theta} \cdot (\tilde{\tau}(\tilde{p})), \quad \forall \tilde{\theta} \in \tilde{\mathbb{T}}, \tilde{p} \in \tilde{M}, \quad \text{and} \quad \tilde{\tau}(\tilde{T}_x) = \tilde{T}_x.$$

Moreover, $(\tilde{T}_x)^{\tilde{\tau}} = \tilde{T}_x \cap \tilde{R}$ and this set, the real part of a regular $\tilde{\mathbb{T}}$ -orbit, is discrete with 2^N points.

Let $K \subset \tilde{\mathbb{T}}$ be a subtorus of dimension $N - n$ determined by an inclusion of Lie algebras

$$\iota : \mathbb{R}^{N-n} \rightarrow \mathbb{R}^N.$$

The moment map for the induced action of K on \tilde{M} is

$$\tilde{\mu}_K = \iota^* \circ \tilde{\mu} : \tilde{M} \rightarrow (\mathbb{R}^{N-n})^*.$$

Let $c \in \tilde{\mu}_K(\tilde{M}) \subset (\mathbb{R}^{N-n})^*$ be a regular value and assume that K acts freely on the level set

$$Z := \tilde{\mu}_K^{-1}(c) \subset \tilde{M}.$$

Then, the reduced space $(M := Z/K, \omega)$ is a symplectic toric manifold of dimension $2n$ with $\mathbb{T} := \tilde{\mathbb{T}}/K$ -action generated by a moment map

$$\mu : M \rightarrow P \subset (\mathbb{R}^n)^* \cong \ker(\iota^*)$$

that fits in the commutative diagram

$$\begin{array}{ccc} \tilde{M} \supset Z & \xrightarrow{\tilde{\mu}} & \tilde{P} \subset (\mathbb{R}^N)^* \\ \pi \downarrow & & \uparrow \\ M & \xrightarrow{\mu} & P \subset (\mathbb{R}^n)^* \end{array}$$

where π is the quotient projection and the vertical arrow on the right is the inclusion $(\mathbb{R}^n)^* \cong \ker(\iota^*) \subset (\mathbb{R}^{N-n})^*$.

Recall that the reduced symplectic form ω is characterized by $\pi^*\omega = \tilde{\omega}|_Z$. Note that given $T_x := \mu^{-1}(x)$, with $x \in \text{int}(P) \subset \text{int}(\tilde{P})$, we have that

$$\pi^{-1}(T_x) = \tilde{T}_x.$$

Moreover, $\tilde{\tau}(Z) = Z$, $Z^{\tilde{\tau}} = Z \cap \tilde{R}$ and $\tilde{\tau}$ induces the canonical anti-symplectic involution $\tau : M \rightarrow M$ via

$$\pi \circ \tilde{\tau} = \tau \circ \pi.$$

Let $p \in R := M^\tau$. Then a simple counting argument shows that

$$\sharp(\pi^{-1}(p) \cap \tilde{R}) = 2^{N-n}.$$

Lemma 3.1. *Let $\psi \in \text{Ham}(M, \omega)$. Then there is $\tilde{\psi} \in \text{Ham}(\tilde{M}, \tilde{\omega})$ such that $\tilde{\psi}(Z) = Z$ and*

$$\pi(\tilde{\psi}(\tilde{p})) = \psi(\pi(\tilde{p})), \quad \forall \tilde{p} \in Z.$$

Proof. Given a time dependent hamiltonian $h_t : M \rightarrow \mathbb{R}$ let $\tilde{h}_t : \tilde{M} \rightarrow \mathbb{R}$ be any smooth extension to \tilde{M} of $h_t \circ \pi : Z \rightarrow \mathbb{R}$. The Hamiltonian flow generated by \tilde{h}_t has the desired properties. \square

Proposition 3.2. *Let $\psi \in \text{Ham}(M, \omega)$, $\tilde{\psi} \in \text{Ham}(\tilde{M}, \tilde{\omega})$ a lift given by the previous lemma and $x \in \text{int}(P) \subset \text{int}(\tilde{P})$. Then*

- (i) $\sharp(\psi(T_x) \cap R) = r \Rightarrow \sharp(\tilde{\psi}(\tilde{T}_x) \cap \tilde{R}) = r2^{N-n}$.
- (ii) $\psi(T_x) \cap T_x = \emptyset \Rightarrow \tilde{\psi}(\tilde{T}_x) \cap \tilde{T}_x = \emptyset$.

Moreover

- (iii) $\sharp(\psi(T_x) \pitchfork T_x) = r \Rightarrow \exists \tilde{\varphi} \in \text{Ham}(\tilde{M}, \tilde{\omega})$ such that $\sharp(\tilde{\varphi}(\tilde{T}_x) \pitchfork \tilde{T}_x) = r2^{N-n}$.

Remark 3.3. *McDuff's method of probes [17] can be seen as a particular case of (ii).*

Proof. To prove (i), suppose that $\tilde{q} \in \tilde{\psi}(\tilde{T}_x) \cap \tilde{R}$ and let $\tilde{p} = \tilde{\psi}^{-1}(\tilde{q}) \in \tilde{T}_x$. Then $p = \pi(\tilde{p}) \in T_x$ and

$$\psi(p) = \psi(\pi(\tilde{p})) = \pi(\tilde{\psi}(\tilde{p})) = \pi(\tilde{q}) = \pi(\tilde{\tau}(\tilde{q})) = \tau(\pi(\tilde{q})) = \tau(q) = \tau(\psi(p)).$$

Hence, $q = \psi(p) \in R$ and so $q \in \psi(T_x) \cap R$.

Moreover, given $q \in \psi(T_x) \cap R$ let $\tilde{q} \in \pi^{-1}(q) \cap \tilde{R}$. Let $p = \psi^{-1}(q) \in T_x$ and $\tilde{p} = \tilde{\psi}^{-1}(\tilde{q})$. Then

$$\begin{aligned} \pi(\tilde{p}) &= \pi(\tilde{\psi}^{-1}(\tilde{q})) = \psi^{-1}(\pi(\tilde{q})) = \psi^{-1}(q) = p \\ \Rightarrow \tilde{p} &\in \pi^{-1}(p) \subset \pi^{-1}(T_x) = \tilde{T}_x \\ \Rightarrow \tilde{q} &= \tilde{\psi}(\tilde{p}) \in \tilde{\psi}(\tilde{T}_x). \end{aligned}$$

Hence, $\tilde{q} \in \tilde{\psi}(\tilde{T}_x) \cap \tilde{R}$ and there are exactly 2^{N-n} such \tilde{q} 's.

To prove (ii), suppose that $\tilde{q} \in \tilde{\psi}(\tilde{T}_x) \cap \tilde{T}_x$ and let $\tilde{p} = \tilde{\psi}^{-1}(\tilde{q}) \in \tilde{T}_x$. Then $p = \pi(\tilde{p}), q = \pi(\tilde{q}) \in T_x$ and

$$\psi(p) = \psi(\pi(\tilde{p})) = \pi(\tilde{\psi}(\tilde{p})) = \pi(\tilde{q}) = q$$

which implies that $q \in \psi(T_x) \cap T_x$.

To prove (iii), note that since T_x and $\psi(T_x)$ intersect transversely at r points in M , say $p_1, \dots, p_r \in M$, we have that \tilde{T}_x and $\tilde{\psi}(\tilde{T}_x)$ intersect in Z in a Morse-Bott way along the r orbits of the subtorus $K \subset \tilde{\mathbb{T}}$ given by $\pi^{-1}(p_1), \dots, \pi^{-1}(p_r) \subset \tilde{M}$. Standard equivariant neighborhood theorems in symplectic geometry imply that a sufficiently small neighborhood $\tilde{\mathcal{U}} \subset \tilde{M}$ of each of these isotropic tori is K -equivariantly symplectomorphic to $\mathcal{V}_1 \times \mathcal{V}_2 \subset (\mathbb{R}^{2n}, \omega_{\text{st}}) \times (T^*K, \omega_{\text{can}})$, where $\mathcal{V}_1 \subset \mathbb{R}^{2n}$ is a neighborhood of the origin, $\mathcal{V}_2 \subset T^*K$ is a neighborhood of the 0-section, and

$$\omega_{\text{st}} = du \wedge dv = \sum_{j=1}^n du_j \wedge dv_j \quad \text{and} \quad \omega_{\text{can}} = -d\lambda_{\text{can}}$$

are the usual symplectic forms on \mathbb{R}^{2n} and T^*K respectively. Moreover, we can identify \mathcal{V}_1 with a neighborhood $\mathcal{U} \subset M$ of the point in $\psi(T_x) \pitchfork T_x$ under consideration and require that

$$T_x \cap \mathcal{U} \cong \{(u, v) \in \mathcal{V}_1 : v = 0\} \quad \text{and} \quad \psi(T_x) \cap \mathcal{U} \cong \{(u, v) \in \mathcal{V}_1 : u = 0\}.$$

Let $\varphi \in \text{Ham}^c(\mathcal{V}_2, \omega_{\text{can}})$ be such that $\sharp(\varphi(0\text{-section}) \pitchfork (0\text{-section})) = 2^{N-n}$ (one can clearly construct such optimal displacing Hamiltonians supported in arbitrarily small neighborhoods of the 0-section in T^*K). We can then consider $\text{id} \times \varphi : \mathcal{V}_1 \times \mathcal{V}_2 \rightarrow \mathcal{V}_1 \times \mathcal{V}_2$, extend as the identity to \tilde{M} and compose with $\tilde{\psi}$ to obtain a Hamiltonian that perturbs the relevant intersection K -orbit into 2^{N-n} transversal intersection points. By doing that at each of the r points in $\psi(T_x) \pitchfork T_x$ we obtain the desired $\tilde{\varphi} \in \text{Ham}(\tilde{M}, \tilde{\omega})$. \square

Corollary 3.4.

(i) If $\sharp(\tilde{\psi}(\tilde{T}_x) \cap \tilde{R}) \geq m$ for any $\tilde{\psi} \in \text{Ham}(\tilde{M}, \tilde{\omega})$ then

$$\sharp(\psi(T_x) \cap R) \geq \frac{m}{2^{N-n}}, \quad \forall \psi \in \text{Ham}(M, \omega).$$

(ii) If $\tilde{T}_x \subset (\tilde{M}, \tilde{\omega})$ is non-displaceable, then $T_x \subset (M, \omega)$ is also non-displaceable.

(iii) If $\sharp(\tilde{\psi}(\tilde{T}_x) \pitchfork \tilde{T}_x) \geq m$ for any $\tilde{\psi} \in \text{Ham}(\tilde{M}, \tilde{\omega})$ then

$$\sharp(\psi(T_x) \pitchfork T_x) \geq \frac{m}{2^{N-n}}, \quad \forall \psi \in \text{Ham}(M, \omega).$$

Remark 3.5. *This idea of using symplectic reduction to prove intersection properties of Lagrangian submanifolds was used by Tamarkin in [20]. It is also present in the work of Borman [6] on reduction properties of quasi-morphisms and quasi-states (see also [7]).*

3.6. Symplectic Reduction Construction of Toric Manifolds. Recall that any symplectic toric manifold (M^{2n}, ω) can be constructed as a symplectic reduction of

$$(\tilde{M} = \mathbb{R}^{2d} \cong \mathbb{C}^{2d}, \tilde{\omega} = \sum_{j=1}^d dx_j \wedge dy_j),$$

where d is the number of facets of the corresponding polytope $P \subset (\mathbb{R}^n)^*$. This reduction is with respect to the natural action of a subtorus $K \subset \mathbb{T} = \mathbb{T}^d$ of dimension $d - n$, whose Lie algebra $\text{Lie}(K) \subset \mathbb{R}^d = \text{Lie}(\mathbb{T}^d)$ is determined as the kernel of the linear map

$$(5) \quad \beta : \mathbb{R}^d \rightarrow \mathbb{R}^n, \quad \beta(e_j) = \nu_j, \quad j = 1, \dots, d,$$

where $\{e_1, \dots, e_d\}$ is the canonical basis of \mathbb{R}^d and $\nu_1, \dots, \nu_d \in \mathbb{Z}^n \subset \mathbb{R}^n$ are the primitive integral interior normals to the facets of the moment polytope P .

When $K = K_1 \times K_2 \subset \mathbb{T}^d$, corresponding to a splitting $\text{Lie}(K) = \text{Lie}(K_1) \times \text{Lie}(K_2) \subset \mathbb{R}^d$, recall that the principle of reduction in stages tells us that, at appropriate level sets, reduction with respect to the action of $K \subset \mathbb{T}^d$ is equivalent to

- first reducing with respect to $K_1 \subset \mathbb{T}^d$, obtaining a symplectic manifold (M_1, ω_1) with Hamiltonian action of \mathbb{T}^d/K_1 ,
- then reducing (M_1, ω_1) with respect to $K_2 \subset \mathbb{T}^d/K_1$.

This principle will be used repeatedly in the applications considered in the next sections. It is also the main ingredient in the proof of the following proposition, which in turn will be used in the proof of Theorem 1.8 (cf. Proposition 4.9 in § 4.7).

Proposition 3.7. *Let (M^{2n}, ω) be a symplectic toric manifold and $\nu_1, \dots, \nu_d \in \mathbb{Z}^n$ the primitive integral interior normals to the facets of its moment polytope $P \subset (\mathbb{R}^n)^*$. Let $m_1, \dots, m_d \in \mathbb{N}$ be such that*

$$(6) \quad \sum_{j=1}^d m_j \nu_j = 0.$$

Then (M^{2n}, ω) can be obtained as a symplectic reduction of the weighted projective space $\mathbb{C}P(m_1, \dots, m_d)$.

Proof. (M^{2n}, ω) can be obtained as the symplectic reduction of $(\mathbb{R}^{2d}, dx \wedge dy)$ by the action of a subtorus $K \subset \mathbb{T}^d$ with $\text{Lie}(K) = \ker \beta$, where the linear map β is given by (5). This means in particular that

$$(k_1, \dots, k_d) \in \text{Lie}(K) \subset \mathbb{R}^d \Leftrightarrow \sum_{j=1}^d k_j \nu_j = 0,$$

which together with (6) implies that K can be written as $K = K_1 \times K_2$ with $\text{Lie}(K_1) = \text{span}\{(m_1, \dots, m_d)\}$. Since the weighted projective space $\mathbb{C}P(m_1, \dots, m_d)$ is obtained as the symplectic reduction of $(\mathbb{R}^{2d}, dx \wedge dy)$ by the action of K_1 , one can use the principle of reduction in stages to conclude the proof. \square

4. FIRST APPLICATION: MONOTONE CASES

We will use the following results stated in Theorem 1.3:

- (i) The Clifford torus $T^n \subset \mathbb{C}P^n$ is non-displaceable and $\sharp(\psi(T_0) \frown T_0) \geq 2^n$, $\forall \psi \in \text{Ham}(\mathbb{C}P^n, \omega_{FS})$ [5, 8].
- (ii) The pair $(\mathbb{R}P^{2n-1}, T^{2n-1})$ is non-displaceable in $\mathbb{C}P^{2n-1}$ and $\sharp(\psi(T^{2n-1}) \frown \mathbb{R}P^{2n-1}) \geq 2^n$ for any $\psi \in \text{Ham}(\mathbb{C}P^{2n-1})$ [1].

Definition 4.1. *Recall from Example 1.2 that any monotone toric symplectic manifold has a unique monotone torus fiber, called the centered or special torus fiber, which is the Lagrangian torus orbit over the “center” of its moment polytope. A symplectic reduction of a monotone toric symplectic manifold at a level containing its centered torus fiber, i.e. through the special “center” of its moment polytope, will be called a centered symplectic reduction.*

4.2. Application 1. One can easily extend Alston’s result to $\mathbb{C}P^{2n}$. In fact, $\mathbb{C}P^{2n}$ can be obtained as a centered symplectic reduction of $\mathbb{C}P^{2n+1}$ (cf. Figure 1 for the $n = 1$ case) and we get that

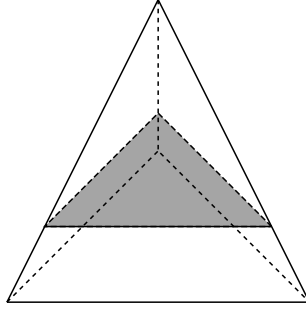
$$\sharp(\psi(T^{2n}) \frown \mathbb{R}P^{2n}) \geq \frac{2^{n+1}}{2} = 2^n \quad \text{for any } \psi \in \text{Ham}(\mathbb{C}P^{2n}).$$

Note that the fact that this estimate is known to be optimal for $\mathbb{C}P^2$ implies that Alston’s bound for $\mathbb{C}P^3$ is also optimal.

4.3. Application 2. Let (M^{2n}, ω) be a monotone toric manifold, R its real part and T its special centered torus fiber. Let $\nu_1, \dots, \nu_d \in \mathbb{Z}^n$ denote the primitive integral interior normals to the facets of the moment polytope of M .

Proposition 4.4. *If $\sum_{i=1}^d \nu_i = 0$ then T and the pair (R, T) are non-displaceable. Moreover,*

$$\sharp(\psi(T) \frown T) \geq 2^n, \quad \forall \psi \in \text{Ham}(M, \omega).$$

FIGURE 1. $\mathbb{C}P^2$ as reduction of $\mathbb{C}P^3$.

Proof. Proposition 3.7 and the zero-sum condition on the normals imply that the standard symplectic reduction construction of M from \mathbb{C}^d factors through $\mathbb{C}P^{d-1}$. The monotone condition implies that this is a centered symplectic reduction and we can apply Corollary 3.4. \square

4.5. **Application 3.** The monotone $M = \mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$, i.e. the monotone blow-up of $\mathbb{C}P^2$ at three points, fits the context of Application 2 and can be obtained as a centered symplectic reduction of $\mathbb{C}P^5$. This means that

$$\sharp(\psi(T) \frown R) \geq \frac{2^3}{2^3} = 1 \quad \text{for any } \psi \in \text{Ham}(M).$$

It can also be obtained as a centered symplectic reduction of $\mathbb{C}P^2 \times \mathbb{C}P^2$. Since our Floer combinatorial invariant of $\mathbb{C}P^2 \times \mathbb{C}P^2$ is 4, this gives the same bound:

$$\sharp(\psi(T) \frown R) \geq \frac{4}{2^2} = 1 \quad \text{for any } \psi \in \text{Ham}(M).$$

However, if one sees M as a centered symplectic reduction of $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, cf. Figure 2, one improves the bound to

$$\sharp(\psi(T) \frown R) \geq \frac{2^3}{2} = 4 \quad \text{for any } \psi \in \text{Ham}(M),$$

which is optimal and coincides with the value of the Floer combinatorial invariant of the hexagon.

Let us describe the details of this reduction construction. The cube at the left of Figure 2, corresponding to the moment polytope of a monotone $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, can be described by the following inequalities:

$$\begin{array}{lll} x_1 + 1 \geq 0 & x_2 + 1 \geq 0 & x_3 + 1 \geq 0 \\ -x_1 + 1 \geq 0 & -x_2 + 1 \geq 0 & -x_3 + 1 \geq 0 \end{array}$$

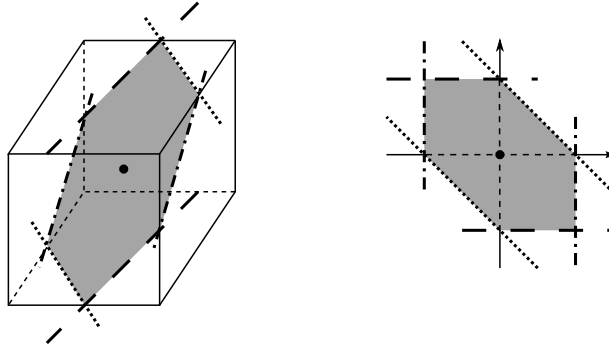


FIGURE 2. $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ as reduction of $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$.

The monotone $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ can be obtained from this toric manifold by centered symplectic reduction with respect to the circle $S^1 \subset \mathbb{T}^3$ determined by the Lie algebra vector $(-1, -1, 1) \in \mathbb{R}^3 = \text{Lie}(\mathbb{T}^3)$, at level $-x_1 - x_2 + x_3 = 0$. The quotient 2-torus \mathbb{T}^3/S^1 acts on the reduced manifold and, with respect to its Lie algebra basis given by $(1, 0, 0), (0, 1, 0) \in \text{Lie}(\mathbb{T}^3/S^1) \cong \mathbb{R}^3/\{(-1, -1, 1)\}$, the resulting moment polytope is described by the above inequalities with $x_3 = x_1 + x_2$, i.e. the hexagon at the right of Figure 2.

In fact, the monotone $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ is just a particular case of the following more general proposition.

Proposition 4.6. *Let (M^{2n}, ω) be a monotone toric manifold, R its real part and T its special centered torus fiber. Suppose that the corresponding moment polytope $P \subset \mathbb{R}^n$ is symmetric, i.e. if $\nu \in \mathbb{Z}^n$ is the interior normal to a facet of P then $-\nu$ is also the interior normal to a facet of P . Then*

$$\#(\psi(T) \frown R) \geq 2^n \quad \text{for any } \psi \in \text{Ham}(M) \text{ and this bound is optimal.}$$

Proof. The fact that the polytope P is symmetric implies that M can be obtained as a symplectic reduction of the product of d copies of $\mathbb{C}P^1$, where $2d$ is the number of facets of P . The fact that M is monotone implies that all the $\mathbb{C}P^1$'s have the same area and that this is a centered symplectic reduction. Hence, we get from Corolary 3.4 that

$$\#(\psi(T) \frown R) \geq \frac{2^d}{2^{d-n}} = 2^n, \quad \forall \psi \in \text{Ham}(M, \omega).$$

Since $\#(T \frown R) = 2^n$ the bound is indeed optimal. \square

4.7. Application 4. Let (M^{2n}, ω) be a compact monotone symplectic toric manifold and T its special centered torus fiber. Denote by $\nu_1, \dots, \nu_d \in \mathbb{Z}^n$ the primitive integral interior normals to the facets of its Delzant polytope $P \subset (\mathbb{R}^n)^*$.

Lemma 4.8. *There exists $k \in \{1, \dots, d\}$ such that*

$$\nu_k + \sum_{j=1, j \neq k}^d m_j \nu_j = 0$$

with all $m_j \in \mathbb{N}$.

Proof. Since P is the moment polytope of a compact toric manifold, the support of its associated fan is the whole \mathbb{R}^n . In particular, every lattice vector $\nu \in \mathbb{Z}^n$ belongs to a cone of the fan determined by some vertex of P , which means that it can be written as a non-negative integral linear combination of the primitive integral interior normals to the n facets that meet at that vertex. The Lemma follows by taking

$$\nu = - \sum_{j=1}^d \nu_j.$$

□

Proposition 4.9. *On any compact monotone toric symplectic manifold (M, ω) the special centered Lagrangian torus fiber T is non-displaceable. Moreover,*

$$\sharp(\psi(T) \pitchfork T) \geq 2^n, \quad \forall \psi \in \text{Ham}(M, \omega).$$

Proof. Using the previous Lemma, and after a possible re-ordering of the normals, we can assume that

$$\sum_{j=1}^{d-1} m_j \nu_j + \nu_d = 0 \quad \text{with } m_1, \dots, m_{d-1} \in \mathbb{N}.$$

This condition and Proposition 3.7 imply that the standard symplectic reduction construction of M from \mathbb{C}^d factors through the weighted projective space $\mathbb{C}\mathbb{P}(1, m_1, \dots, m_{d-1})$. The monotone condition implies that this factorization goes through the special centered non-displaceable torus fiber of this weighted projective space (cf. Theorem 1.7) and we can apply Corollary 3.4. □

As a particular example, consider the monotone $M = \mathbb{C}\mathbb{P}^2 \sharp \overline{\mathbb{C}\mathbb{P}^2}$, i.e. the monotone blow-up of $\mathbb{C}\mathbb{P}^2$ at one point, with polytope $P \subset (\mathbb{R}^2)^*$ given by

$$\begin{array}{ll} x_1 + 1 \geq 0 & (\nu_1 = (1, 0)) & -x_1 - x_2 + 1 \geq 0 & (\nu_3 = (-1, -1)) \\ x_2 + 1 \geq 0 & (\nu_2 = (0, 1)) & x_1 + x_2 + 1 \geq 0 & (\nu_4 = (1, 1)). \end{array}$$

We have that

$$(\nu_1 + \nu_2 + 2\nu_3) + \nu_4 = 0$$

and M can be obtained as a “centered” symplectic reduction of $\mathbb{C}\mathbb{P}(1, 1, 1, 2)$. In fact, the polytope of $\mathbb{C}\mathbb{P}(1, 1, 1, 2)$ is given by

$$\begin{array}{ll} x_1 + 1 \geq 0 & x_3 + 1 \geq 0 \\ x_2 + 1 \geq 0 & -x_1 - x_2 - 2x_3 + 1 \geq 0 \end{array}$$

and its reduction with respect to the circle $S^1 \subset \mathbb{T}^3$ determined by $(1, 1, 1) \in \mathbb{R}^3 = \text{Lie}(\mathbb{T}^3)$ at level

$$x_1 + x_2 + x_3 = 0, \text{ i.e. } x_3 = -x_1 - x_2,$$

gives rise to the above polytope for $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

5. SECOND APPLICATION: NON-MONOTONE FANO CASES

For the applications in this section we will assume that some form of the following general result is true:

- If T_i or the pair (T_i, R_i) have $HF \neq 0$ or are non-displaceable in (M_i, ω_i) , $i = 1, 2$, then the same is true for the corresponding $T_1 \times T_2$ and $(T_1 \times T_2, R_1 \times R_2)$ in $(M_1 \times M_2, \omega_1 \times \omega_2)$.

Remark 5.1. *For Lagrangian torus orbits in toric symplectic manifolds, the set-up of Woodward [22] applies and proves a result of this form (cf. [21]).*

Moreover, in some of the applications below we will also use the following result:

- In the total space of the line bundle $\mathcal{O}(-1) \rightarrow \mathbb{C}P^1$, the special torus T sitting over the origin in the polygon on the right side of Figure 3 is non-displaceable. This has been proved by Woodward (cf. Example 1.3 in [22]) and can also be seen as a consequence of a result of Cho in [9] (cf. polygon on the left side of Figure 3).

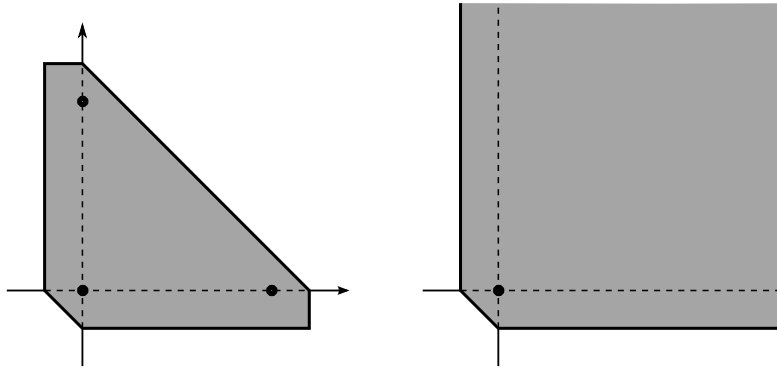


FIGURE 3. Cho's result on the left and its limit on the right, giving a non-displaceable torus fiber in the total space of the line bundle $\mathcal{O}(-1) \rightarrow \mathbb{C}P^1$.

5.2. Application 5. Consider $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, i.e. the blow-up of $\mathbb{C}P^2$ at one point.

Figure 4 illustrates how one can obtain two non-displaceable torus fibers when the exceptional divisor is small, i.e. smaller than monotone. On the left, one thinks of M as a symplectic reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1$, with $\mathbb{C}P^2$ “smaller” than $\mathbb{C}P^1$, to show that the torus fiber over the origin is non-displaceable. On the right, one thinks of M as a symplectic

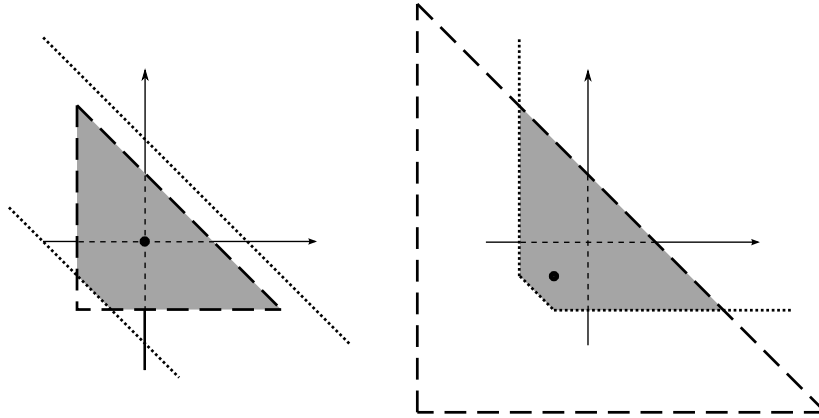


FIGURE 4. $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ as reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1$ (on the left, $\mathbb{C}P^2$ “smaller” than $\mathbb{C}P^1$) and of $\mathcal{O}(-1) \times \mathbb{C}P^2$ (on the right).

reduction of $\mathcal{O}(-1) \times \mathbb{C}P^2$ to show that a torus fiber “close” to the exceptional divisor is non-displaceable.

Figure 5 illustrates how one can obtain one non-displaceable torus fiber when the exceptional divisor is big, i.e. bigger than monotone. One thinks again of M as a symplectic reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1$, but now with $\mathbb{C}P^2$ “bigger” than $\mathbb{C}P^1$.

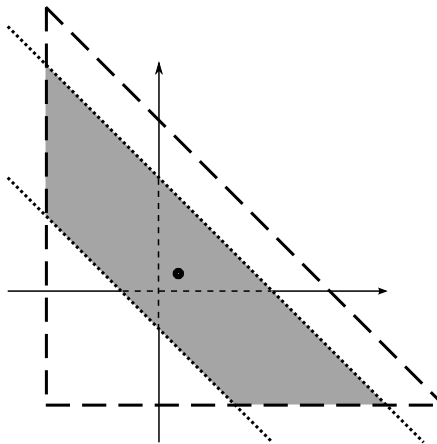


FIGURE 5. $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ as reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1$, now with $\mathbb{C}P^2$ “bigger” than $\mathbb{C}P^1$.

Note that the monotone case with just one non-displaceable torus fiber over the special central point can be obtained as a limit of any of these.

5.3. Application 6. A very similar idea applies to $M = \mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2} \cong (\mathbb{C}P^1 \times \mathbb{C}P^1) \# \overline{\mathbb{C}P^2}$, i.e. the equal blow-up of $\mathbb{C}P^2$ at two points which can also be thought of as the blow-up of $\mathbb{C}P^1 \times \mathbb{C}P^1$ at one point. One recovers the results of Fukaya-Oh-Ohta-Ono [14, 15] and Woodward [22] illustrated in Figures 6 and 7.

Figure 6 illustrates how one can obtain two non-displaceable torus fibers in the big two-point blow-up of $\mathbb{C}P^2$, which is equivalent to a small one-point blow-up of $\mathbb{C}P^1 \times \mathbb{C}P^1$. On the left, one thinks of M as a symplectic reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, with “big” $\mathbb{C}P^2$, to show that the torus fiber over the origin is non-displaceable. On the right, one thinks of M as a symplectic reduction of $\mathcal{O}(-1) \times \mathbb{C}P^2$ to show that a torus fiber “close” to the blown-up point on $\mathbb{C}P^1 \times \mathbb{C}P^1$ is non-displaceable.

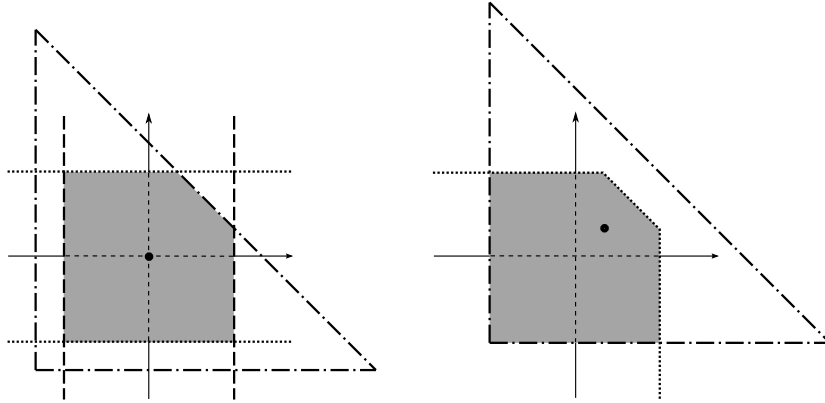


FIGURE 6. $(\mathbb{C}P^1 \times \mathbb{C}P^1) \# \overline{\mathbb{C}P^2}$ (small blow-up), equivalently $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$ (big blow-ups), as reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ (on the left) and of $\mathcal{O}(-1) \times \mathbb{C}P^2$ (on the right).

Figure 7 illustrates how one can obtain three non-displaceable torus fibers in the small two-point blow-up of $\mathbb{C}P^2$, which is equivalent to a big one-point blow-up of $\mathbb{C}P^1 \times \mathbb{C}P^1$. On the left, one thinks of M as a symplectic reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, with “small” $\mathbb{C}P^2$ and large $\mathbb{C}P^1$ ’s, to show that a torus fiber “close” to the origin is non-displaceable. On the right, one thinks of M as a symplectic reduction of $\mathcal{O}(-1) \times \mathbb{C}P^1 \times \mathbb{C}P^1$ to show that there is a non-displaceable torus fiber “close” to each blown-up point on $\mathbb{C}P^2$.

Again, note that the monotone case with just one non-displaceable torus fiber over the special central point can be obtained as a limit of any of these. It could also be obtained using Proposition 4.9, as was illustrated in § 4.7 for the monotone one-point blow-up $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

5.4. Application 7. Here we will use the same idea to understand an example Fukaya, Oh, Ohta and Ono [14, 15], presented in Figure 8 (see Example 10.3 in [16]). The symplectic manifold is $M = \mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$ with blow-ups of different sizes, one smaller than monotone and the other bigger than monotone, and they obtain a closed interval of non-displaceable

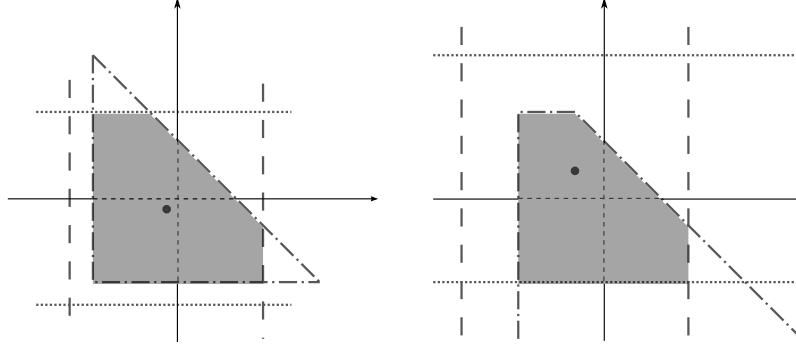


FIGURE 7. $(\mathbb{CP}^1 \times \mathbb{CP}^1) \# \overline{\mathbb{CP}^2}$ (big blow-up), equivalently $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ (small blow-ups), as reduction of $\mathbb{CP}^2 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ (on the left) and of $\mathcal{O}(-1) \times \mathbb{CP}^1 \times \mathbb{CP}^1$ (on the right).

torus fibers. This can also be obtained by considering M as the symplectic reduction of $\mathcal{O}(-1) \times \mathbb{CP}^1 \times \mathbb{CP}^1$ (or the compact $(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}) \times \mathbb{CP}^1 \times \mathbb{CP}^1$) as shown in Figure 8.

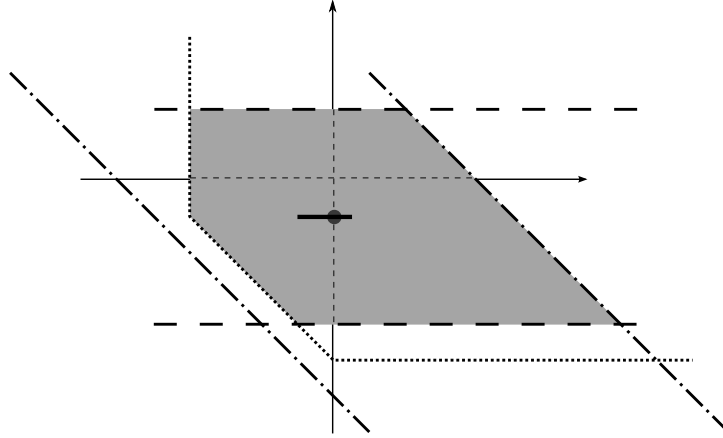


FIGURE 8. $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ as reduction of $\mathcal{O}(-1) \times \mathbb{CP}^1 \times \mathbb{CP}^1$.

The details are as follows. Consider $(\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}, \omega_\alpha)$ given by the Delzant polytope $P_\alpha \subset (\mathbb{R}^2)^*$ determined by the following inequalities:

$$\begin{aligned} x_1 + 1 &\geq 0 & x_1 + x_2 + 1 + \alpha &\geq 0 \\ x_2 + 1 &\geq 0 & -x_2 + (1 - 2\alpha) &\geq 0 \\ -(x_1 + x_2) + 1 &\geq 0 & & \text{with } 0 < \alpha < 1. \end{aligned}$$

The non-displaceable torus fibers are over the points with coordinates $(-\alpha + \lambda, -\alpha)$ for $0 < \lambda < 3\alpha/2$ (Figure 8 corresponds to $\alpha = \lambda = 1/4$). To prove that for each such pair of real numbers α and λ , we consider $\mathcal{O}(-1) \times \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ with moment polytope given by the cartesian product of the following polytopes:

- the one for the $\mathcal{O}(-1)$ factor is given by the inequalities

$$x_1 + 1 \geq 0, \quad x_1 + x_2 + 1 + \alpha \geq 0 \quad \text{and} \quad x_2 + 1 + \lambda \geq 0,$$

having a non-displaceable torus fiber over the point with coordinates $x_1 = -\alpha + \lambda$ and $x_2 = -\alpha$.

- the one for the first $\mathbb{C}\mathbb{P}^1$ factor is given by the inequalities

$$x_3 + 1 \geq 0 \quad \text{and} \quad -x_3 + (1 - 2\alpha) \geq 0,$$

having a non-displaceable torus fiber over the point with coordinate $x_3 = -\alpha$.

- the one for the second $\mathbb{C}\mathbb{P}^1$ factor is given by the inequalities

$$x_4 + 1 + 4\alpha - 2\lambda \geq 0 \quad \text{and} \quad -x_4 + 1 \geq 0,$$

having a non-displaceable torus fiber over the point with coordinate $x_4 = -2\alpha + \lambda$.

We can now do symplectic reduction with respect to the 2-torus $\mathbb{T}^2 \subset \mathbb{T}^4$ determined by the Lie algebra vectors $(0, -1, 1, 0), (-1, -1, 0, 1) \in \mathbb{R}^4 = \text{Lie}(\mathbb{T}^4)$ at the level given by

$$x_3 = x_2 \quad \text{and} \quad x_4 = x_1 + x_2,$$

to obtain the polytope P_α with non-displaceable torus fiber over the point with coordinates $x_1 = -\alpha + \lambda$ and $x_2 = -\alpha$.

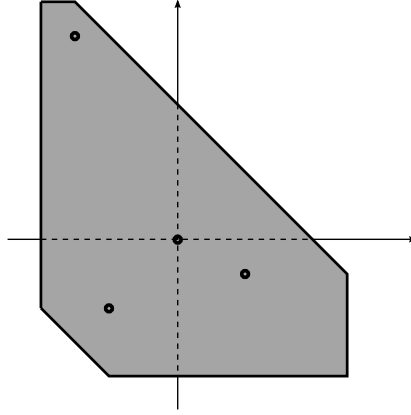
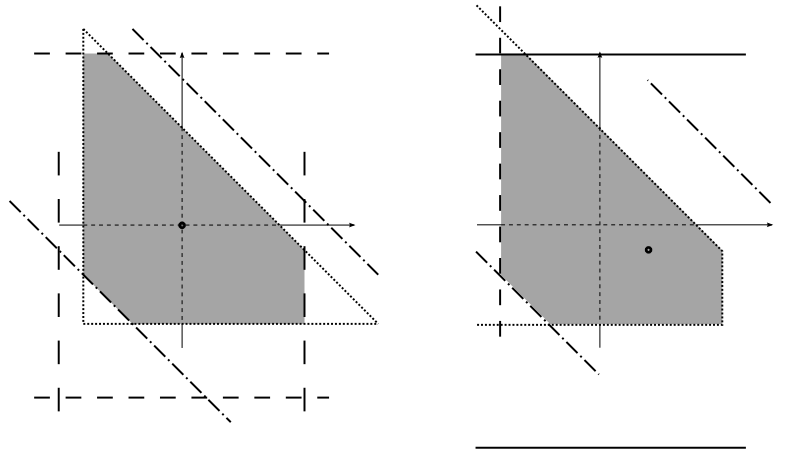
5.5. Application 8. One can use the same idea to understand non-displaceable torus fibers on $M = \mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$ for all possible sizes of blown-up points. Figure 9 illustrates the case of three small size blow-ups, where one gets four non-displaceable torus fibers.

The center fiber is obtained by seeing M as the symplectic reduction of $\mathbb{C}\mathbb{P}^2 \times \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$, as in the left side of Figure 10, while each of the off-center fibers is obtained by seeing M as the symplectic reduction of $\mathcal{O}(-1) \times \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$, as in the right side of Figure 10.

Note that we can also get a closed interval of non-displaceable torus fibers for $M = \mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$, e.g. by blowing up the example in Application 6 at the lower right corner of Figure 8.

6. THIRD APPLICATION: NON-FANO CASES

Here we will use the basic non-displaceability result for the central torus fiber of a weighted projective space stated in Theorem 1.7 and show how it implies non-displaceability of at least one torus fiber on any Hirzebruch surface, a result of Fukaya-Oh-Ohta-Ono [14] (see Example 10.1 in [16]), and a continuous interval of non-displaceable torus fibers on a particular non-Fano toric surface considered by McDuff [18].

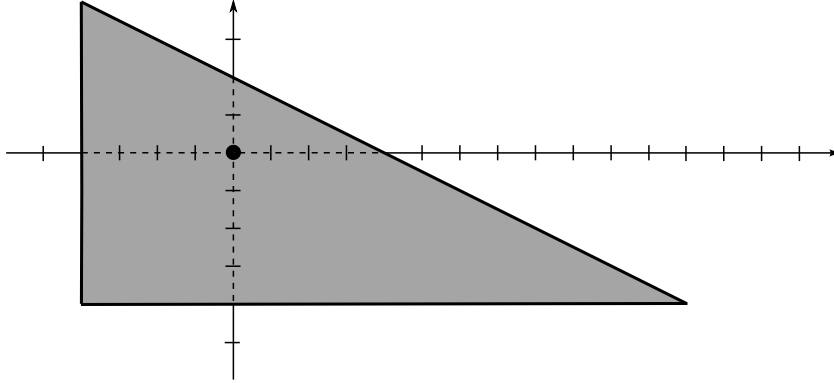
FIGURE 9. Four non-displaceable torus fibers on $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$.FIGURE 10. $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ as reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ (on the left) and of $\mathcal{O}(-1) \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ (on the right).

6.1. **Application 9.** Consider the weighted projective space $\mathbb{C}P(1, 1, k)$, the symplectic quotient of \mathbb{C}^3 by the S^1 -action with weights $(1, 1, k)$, with moment polytope $P_k \subset \mathbb{R}^2$ given by

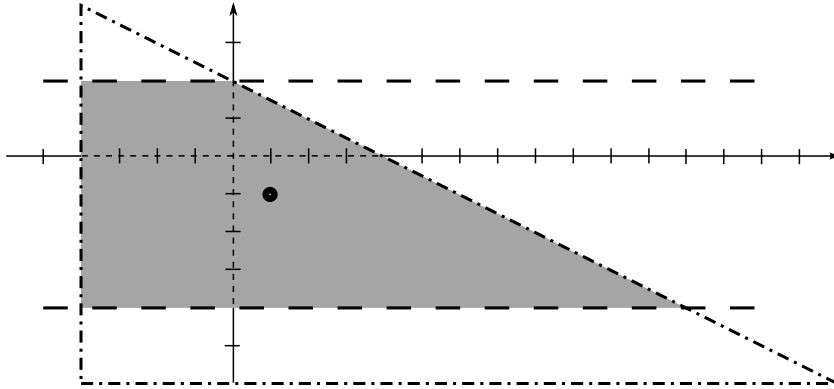
$$x_1 + 1 \geq 0, \quad x_2 + 1 \geq 0 \quad \text{and} \quad -x_1 - kx_2 + 1 \geq 0.$$

The special torus sitting over the origin is non-displaceable. Figure 11 illustrates the $k = 2$ case.

Now let $H_k := \mathbb{P}(\mathcal{O}(-k) \oplus \mathbb{C}) \rightarrow \mathbb{C}P^1$ be a Hirzebruch surface, with $2 \leq k \in \mathbb{N}$. Each of these Hirzebruch surfaces can be seen as a centered symplectic reduction of $\mathbb{C}P(1, 1, k) \times$

FIGURE 11. $\mathbb{CP}(1, 1, 2)$.

\mathbb{CP}^1 and that implies at least one non-displaceable torus fiber on any H_k . Figure 12 illustrates the $k = 2$ case.

FIGURE 12. The Hirzebruch surface H_2 as reduction of $\mathbb{CP}(1, 1, 2) \times \mathbb{CP}^1$.

6.2. Application 10. Here we will consider the non-Fano toric symplectic 4-manifold described by McDuff in § 2.1 of [18]. We already pointed out its potential significance in Remark 1.10.

Consider the Delzant polytope $P \subset (\mathbb{R}^2)^*$ determined by the following inequalities:

$$\begin{aligned} x_1 + 1 &\geq 0 & -x_1 - 3x_2 + 3 &\geq 0 \\ x_2 + 1 &\geq 0 & -x_1 - 2x_2 + 3 &\geq 0 \\ -x_2 + 1 &\geq 0 & & \end{aligned}$$

The torus fibers over the points with coordinates $x_1 = \lambda$ and $x_2 = 0$, with $1 < \lambda < 2$, are non-displaceable by probes. We can show that all these fibers are in fact non-displaceable by considering this toric 4-manifold as a symplectic reduction of $\mathbb{C}P(1, 1, 2) \times \mathbb{C}P^1 \times \mathcal{O}(-1)$, as shown in Figure 13 (where $\lambda = 3/2$).

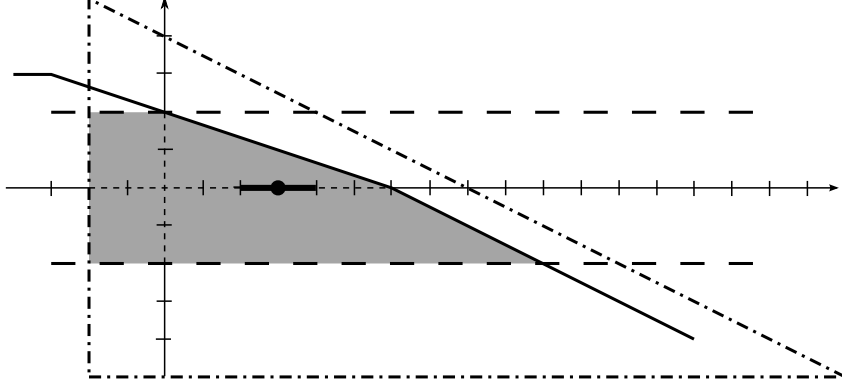


FIGURE 13. Non-Fano toric surface with interval on non-displaceable torus fibers.

More precisely, for each $1 < \lambda < 2$, consider $\mathbb{C}P(1, 1, 2) \times \mathbb{C}P^1 \times \mathcal{O}(-1)$ with moment polytope given by the cartesian product of the following polytopes:

- the one for the $\mathbb{C}P(1, 1, 2)$ factor is given by the inequalities

$$x_1 + 1 \geq 0, \quad x_2 + 1 + \lambda \geq 0 \quad \text{and} \quad -x_1 - 2x_2 + 1 + 2\lambda \geq 0,$$

having a non-displaceable torus fiber over the point with coordinates $x_1 = \lambda$ and $x_2 = 0$.

- the one for the $\mathbb{C}P^1$ factor is given by the inequalities

$$x_3 + 1 \geq 0 \quad \text{and} \quad -x_3 + 1 \geq 0,$$

having a non-displaceable torus fiber over the point with coordinate $x_3 = 0$.

- the one for the $\mathcal{O}(-1)$ factor is given by the inequalities

$$x_4 + 3 \geq 0, \quad x_5 + 3 - \lambda \geq 0 \quad \text{and} \quad x_4 + x_5 + 3 \geq 0,$$

having a non-displaceable torus fiber over the point with coordinate $x_4 = -\lambda$ and $x_5 = 0$.

We can now do symplectic reduction with respect to the 3-torus $\mathbb{T}^3 \subset \mathbb{T}^5$ determined by the Lie algebra vectors $(0, -1, 1, 0, 0), (-1, -2, 0, 1, 0), (0, -1, 0, 0, 1) \in \mathbb{R}^5 = \text{Lie}(\mathbb{T}^5)$ at the level given by

$$x_3 = x_2, \quad x_4 = -x_1 - 2x_2 \quad \text{and} \quad x_5 = -x_2,$$

to obtain the polytope P with non-displaceable torus fiber over the point with coordinates $x_1 = \lambda$ and $x_2 = 0$.

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