22 Centuries Measuring Area

Miguel Abreu

IST - Technical University of Lisbon and Portuguese Mathematical Society

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Introduction and Goals

Based on a Klein Vignette written in collaboration with Ana Cannas da Silva for the Mathematics Klein Project in Portuguese, a brazilian initiative with portuguese cooperation, within the Klein Project of the International Mathematical Union (IMU) and International Commission on Mathematical Instruction (ICMI).

Give a symplectic explanation for the formula

$$\text{Volume } B^{2n}(R) = \frac{\pi^n R^{2n}}{n!} = (2\pi)^n \cdot \frac{(R^2/2)^n}{n!} .$$

Note: Volume $B^N(R) = \pi^{N/2} R^N / \Gamma(N/2 + 1)$.

Publicize the Klein Project, in particular its vignettes.

Archimedes favorite Theorem (287–212 AC)

"The surface area of a sphere between two parallel planes depends only on the distance between those planes and not on the height where they intersect the sphere"

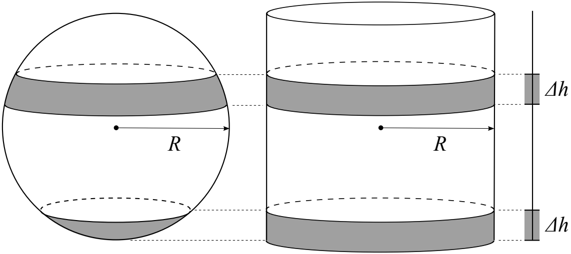


Figure: Spherical and cylindrical strips with the same area.

Archimedes favorite Theorem (287–212 AC)

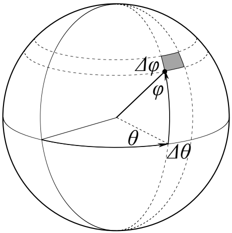


Figure: Longitude: $0 \leq \theta < 2\pi$; latitude: $-\pi/2 \leq \varphi \leq \pi/2$.

Approximate area of shadowed rectangle:

$$\text{height} \times \text{width} \simeq (R \cdot \Delta\varphi)(R \cdot \cos\varphi \cdot \Delta\theta) = \Delta\theta \cdot R^2 \cos\varphi \cdot \Delta\varphi\,.$$

Approx. area of spherical strip between parallels φ and $\varphi + \Delta\varphi$:

$$\text{area}_{\text{sph}} \simeq 2\pi R^2 \cos\varphi \cdot \Delta\varphi\,.$$

Archimedes favorite Theorem (287–212 AC)

Since

$$\begin{aligned}\Delta h &= R \cdot \sin(\varphi + \Delta\varphi) - R \cdot \sin\varphi \\ &= R \cdot (\sin\varphi \cdot \cos\Delta\varphi + \cos\varphi \cdot \sin\Delta\varphi) - R \cdot \sin\varphi \\ &= R \cdot \sin\varphi \cdot (\cos\Delta\varphi - 1) + R \cdot \cos\varphi \cdot \sin\Delta\varphi\end{aligned}$$

and

$$\cos\Delta\varphi - 1 \simeq 0\,,\ \sin\Delta\varphi \simeq \Delta\varphi \quad \text{when } \Delta\varphi \simeq 0,$$

we have that

$$\Delta h \simeq R \cdot \cos\varphi \cdot \Delta\varphi\,.$$

This implies that

$$\text{area}_{\text{sph}} \simeq 2\pi R^2 \cdot \cos\varphi \cdot \Delta\varphi = 2\pi R \cdot \Delta h = \text{area}_{\text{cyl}}\,.$$

Archimedes favorite Theorem (287–212 AC)

Hence, the use of these (h, θ) coordinates significantly simplifies the computation of the area of certain spherical regions.

These coordinates are a first example of action-angle coordinates. These are used very often in symplectic geometry.

Symplectic Geometry in dimension 2 is nothing but the study of area preserving transformations. Archimedes favorite Theorem can be considered its first result.

Action-angle coordinates for the disk

$$h = \frac{1}{2} r^2 \quad \text{and} \quad \theta$$

with

$$\theta = \text{angle measured from a coordinate ray}$$
$$r = \text{distance to the origin.}$$

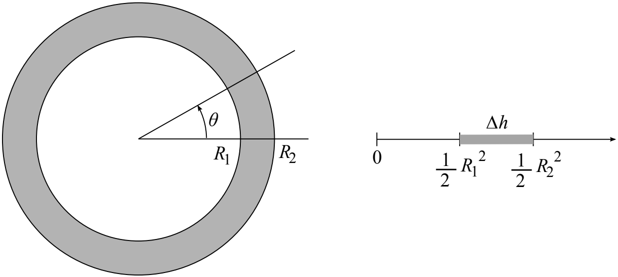


Figure: In action-angle coordinates, area of the annulus $= 2\pi \cdot \Delta h$.

Action-angle coordinates for $B^4 \subset \mathbb{R}^4$

$$(h_1 = r_1{}^2/2, \theta_1, h_2 = r_2{}^2/2, \theta_2) \in \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$$

$$B^4(R) = \left\{ (h_1, \theta_1, h_2, \theta_2) \ : \ h_1 + h_2 \leq \frac{1}{2} R^2 \right\}$$

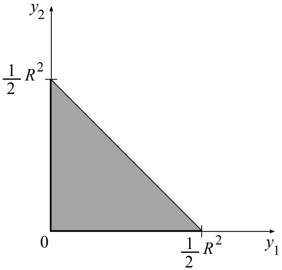
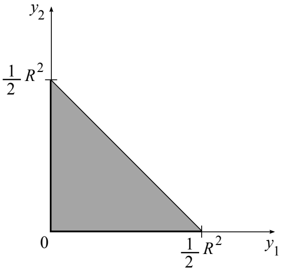


Figure: Triangle that encodes the 4-dimensional ball of radius R.

Volume for $B^4 \subset \mathbb{R}^4$



$$\text{Volume } B^4(R) = (2\pi)^2 \times \text{area of shadowed triangle}$$
$$= (2\pi)^2 \times \frac{(R^2/2)^2}{2}$$
$$= \frac{\pi^2 \cdot R^4}{2}$$

Action-angle coordinates and volume for $B^{2n} \subset \mathbb{R}^{2n}$

$$(h_1 = {r_1}^2/2, \theta_1, \ldots, h_n = {r_n}^2/2, \theta_n) \in \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 = \mathbb{R}^{2n}$$

$$B^{2n}(R) = \left\{ (h_1, \theta_1, \ldots, h_n, \theta_n) \ : \ h_1 + \cdots + h_n \leq \frac{1}{2} R^2 \right\}$$

$$\begin{aligned} \text{Volume } B^{2n}(R) &= (2\pi)^n \times \text{volume of the corresponding } n\text{-simplex} \\ &= (2\pi)^n \times \frac{(R^2/2)^n}{n!} \\ &= \frac{\pi^n \cdot R^{2n}}{n!} \end{aligned}$$

Klein Project

- Felix Klein's original book on Elementary Mathematics from an Advanced Standpoint (1908): "a book for upper secondary teachers that communicates the breadth and vitality of the research discipline of mathematics and connects it to the senior secondary school curriculum".

- Goal of ICMI-IMU Klein Project: revisit and update, 100 years later, Klein's original intent.

- Klein Vignettes: should start with an inspiring example or problem and say something about mathematics and its development in the 20th century ($4-5$ pages). There are already 13 currently available in english at http://wikis.zum.de/dmuw/Klein_Vignettes