Instituto Superior Técnico
Departamento de Matemática

CÁLCULO DIFERENCIAL E INTEGRAL I
LMAC, MEBIOM, MEFT – 1º SEM. 2014/15

1ª FICHA DE EXERCÍCIOS

0. Desigualdades e Módulos

1. Mostre que:

1.1. $\left\{x \in \mathbb{R} : \frac{x-1}{x+3} < 2\right\} =]-\infty, -7[\cup]-3, +\infty[$

1.2. $\left\{x \in \mathbb{R} : \frac{x-1}{x+3} \geq 2\right\} = [-7, -3[$

1.3. $\left\{x \in \mathbb{R} : \frac{x+1}{x-3} \leq 2\right\} =]-\infty, 3[\cup [7, +\infty[$

1.4. $\left\{x \in \mathbb{R} : \frac{x+1}{x-3} > 2\right\} =]3, 7[$

1.5. $\left\{x \in \mathbb{R} : \frac{x-2}{x+1} < 3\right\} = \left]-\infty, -\frac{5}{2}\right[\cup]-1, +\infty[$

1.6. $\left\{x \in \mathbb{R} : \frac{x-2}{x+1} \geq 3\right\} = \left[-\frac{5}{2}, -1\right[$

1.7. $\left\{x \in \mathbb{R} : \frac{x+2}{x-1} \leq 3\right\} =]-\infty, 1[\cup \left[\frac{5}{2}, +\infty\right[$

1.8. $\left\{x \in \mathbb{R} : \frac{x+2}{x-1} > 3\right\} = \left]1, \frac{5}{2}\right[$

1.9. $\left\{x \in \mathbb{R} : \frac{x-3}{x+2} < 1\right\} =]-2, +\infty[$

1.10. $\left\{x \in \mathbb{R} : \frac{x-3}{x+2} \geq 1\right\} =]-\infty, -2[$

1.11. $\left\{x \in \mathbb{R} : \frac{x+3}{x-2} \leq 1\right\} =]-\infty, 2[$

1.12. $\left\{x \in \mathbb{R} : \frac{x+3}{x-2} > 1\right\} =]2, +\infty[$

1.13. $\left\{x \in \mathbb{R} : \frac{x-2}{x+3} < 2x\right\} =]-3, -2[\cup \left]-\frac{1}{2}, +\infty\right[$

1.14. $\left\{x \in \mathbb{R} : \frac{x-2}{x+3} \geq 2x\right\} =]-\infty, -3[\cup \left[-2, -\frac{1}{2}\right]$

1.15. $\left\{x \in \mathbb{R} : \frac{x+2}{3-x} > 2x\right\} = \left]-\infty, \frac{1}{2}\right[\cup]2, 3[$

1.16. $\left\{x \in \mathbb{R} : \frac{x+2}{3-x} \leq 2x\right\} = \left[\frac{1}{2}, 2\right] \cup]3, +\infty[$

1.17. $\left\{x \in \mathbb{R} : \frac{x+4}{x-1} < 3x\right\} = \left]-\frac{2}{3}, 1\right[\cup]2, +\infty[$

1.18. $\left\{x \in \mathbb{R} : \frac{x+4}{x-1} \geq 3x\right\} = \left]-\infty, -\frac{2}{3}\right] \cup]1, 2]$

1.19. $\left\{x \in \mathbb{R} : \frac{4-x}{x+1} > 3x\right\} =]-\infty, -2[\cup \left]-1, \frac{2}{3}\right[$

1.20. $\left\{x \in \mathbb{R} : \frac{4-x}{x+1} \leq 3x\right\} = [-2, -1[\cup \left[\frac{2}{3}, +\infty\right[$

1.21. $\left\{x \in \mathbb{R} : \frac{x-7}{x-3} < 3x\right\} = \left]1, \frac{7}{3}\right[\cup]3, +\infty[$

1.22. $\left\{x \in \mathbb{R} : \frac{x-7}{x-3} \geq 3x\right\} =]-\infty, 1] \cup \left[\frac{7}{3}, 3\right[$

1.23. $\left\{x \in \mathbb{R} : \frac{x+7}{x+3} \leq -3x\right\} =]-\infty, -3[\cup \left[-\frac{7}{3}, -1\right]$

1.24. $\left\{x \in \mathbb{R} : \frac{x+7}{x+3} > -3x\right\} = \left]-3, -\frac{7}{3}\right[\cup]-1, +\infty[$

2. Mostre que:

2.1. $\{x \in \mathbb{R} : |x+2| = 3\} = \{-5, 1\}$
2.2. $\{x \in \mathbb{R} : |x+2| \leq 1\} = [-3, -1]$
2.3. $\{x \in \mathbb{R} : |3-x| > 2\} =]-\infty, 1[\cup]5, +\infty[$
2.4. $\{x \in \mathbb{R} : 2 < |x| < 3\} =]-3, -2[\cup]2, 3[$
2.5. $\{x \in \mathbb{R} : 3 < 2|x-1| \leq 5\} = \left[-\frac{3}{2}, -\frac{1}{2}\right[\cup \left]\frac{5}{2}, \frac{7}{2}\right]$
2.6. $\{x \in \mathbb{R} : |x-3| > 2 \ \wedge \ x \geq 0\} = [0, 1[\cup]5, +\infty[$
2.7. $\{x \in \mathbb{R} : |x+2| \leq 3 \ \wedge \ x+1 > 0\} =]-1, 1]$
2.8. $\{x \in \mathbb{R} : |3-4x| < 1\} = \left]\frac{1}{2}, 1\right[$
2.9. $\{x \in \mathbb{R} : |4x+3| > 1\} =]-\infty, -1[\cup \left]-\frac{1}{2}, +\infty\right[$
2.10. $\{x \in \mathbb{R} : |5-3x| \leq 2\} = \left[1, \frac{7}{3}\right]$
2.11. $\{x \in \mathbb{R} : |5+3x| \geq 2\} = \left]-\infty, -\frac{7}{3}\right] \cup [-1, +\infty[$
2.12. $\{x \in \mathbb{R} : |4x+1| > 5\} = \left]-\infty, -\frac{3}{2}\right[\cup]1, +\infty[$
2.13. $\{x \in \mathbb{R} : |1-4x| < 5\} = \left]-1, \frac{3}{2}\right[$
2.14. $\{x \in \mathbb{R} : |5x+2| \geq 3\} =]-\infty, -1] \cup \left[\frac{1}{5}, +\infty\right[$
2.15. $\{x \in \mathbb{R} : |2-5x| \leq 3\} = \left[-\frac{1}{5}, 1\right]$
2.16. $\{x \in \mathbb{R} : |3x-4| \leq 1\} = \left[1, \frac{5}{3}\right]$
2.17. $\{x \in \mathbb{R} : |3x+4| \geq 1\} = \left]-\infty, -\frac{5}{3}\right] \cup [-1, +\infty[$
2.18. $\{x \in \mathbb{R} : |2x+3| > 5\} =]-\infty, -4[\cup]1, +\infty[$
2.19. $\{x \in \mathbb{R} : |3-2x| < 5\} =]-1, 4[$
2.20. $\{x \in \mathbb{R} : |2-3x| < 1\} = \left]\frac{1}{3}, 1\right[$
2.21. $\{x \in \mathbb{R} : |2+3x| > 1\} =]-\infty, -1[\cup \left]-\frac{1}{3}, +\infty\right[$
2.22. $\{x \in \mathbb{R} : |5x-4| \leq 1\} = \left[\frac{3}{5}, 1\right]$
2.23. $\{x \in \mathbb{R} : |5x+4| \geq 1\} =]-\infty, -1] \cup \left[-\frac{3}{5}, +\infty\right[$
2.24. $\{x \in \mathbb{R} : |5-2x| < 1\} =]2, 3[$
2.25. $\{x \in \mathbb{R} : |2x+5| > 1\} =]-\infty, -3[\cup]-2, +\infty[$
2.26. $\{x \in \mathbb{R} : |5-6x| \leq 1\} = \left[\frac{2}{3}, 1\right]$
2.27. $\{x \in \mathbb{R} : |6x-5| > 1\} = \left]-\infty, \frac{2}{3}\right[\cup]1, +\infty[$
2.28. $\{x \in \mathbb{R} : |9-2x| < 1\} =]4, 5[$
2.29. $\{x \in \mathbb{R} : |2x-9| \geq 1\} =]-\infty, 4] \cup [5, +\infty[$
2.30. $\{x \in \mathbb{R} : |4-3x| < 8\} = \left]-\frac{4}{3}, 4\right[$
2.31. $\{x \in \mathbb{R} : |3x-4| \geq 8\} = \left]-\infty, -\frac{4}{3}\right] \cup [4, +\infty[$
2.32. $\{x \in \mathbb{R} : |3-4x| \leq 7\} = \left[-1, \frac{5}{2}\right]$
2.33. $\{x \in \mathbb{R} : |4x-3| > 7\} =]-\infty, -1[\cup \left]\frac{5}{2}, +\infty\right[$
2.34. $\{x \in \mathbb{R} : |7-2x| \leq 1\} = [3, 4]$
2.35. $\{x \in \mathbb{R} : |2x-7| > 1\} =]-\infty, 3[\cup]4, +\infty[$
2.36. $\{x \in \mathbb{R} : |5-2x| < 9\} =]-2, 7[$
2.37. $\{x \in \mathbb{R} : |2x-5| \geq 9\} =]-\infty, -2] \cup [7, +\infty[$
2.38. $\{x \in \mathbb{R} : |5-3x| < 1\} = \left]\frac{4}{3}, 2\right[$
2.39. $\{x \in \mathbb{R} : |3x-5| \geq 1\} = \left]-\infty, \frac{4}{3}\right] \cup [2, +\infty[$
2.40. $\{x \in \mathbb{R} : 2 < 3|x+1| \leq 5\} = \left[-\frac{8}{3}, -\frac{5}{3}\right[\cup \left]-\frac{1}{3}, \frac{2}{3}\right]$

3. Mostre que:

3.1. $\{x \in \mathbb{R} : \; |3-2x| \geq |x+2|\} = \left]-\infty, \frac{1}{3}\right] \cup [5, +\infty[$
3.2. $\{x \in \mathbb{R} : \; |x| = |x-2|\} = \{1\}$
3.3. $\{x \in \mathbb{R} : \; |x| \leq |x-2|\} =]-\infty, 1]$
3.4. $\{x \in \mathbb{R} : \; |2x-5| \geq |1-x|\} =]-\infty, 2] \cup [4, +\infty[$
3.5. $\{x \in \mathbb{R} : \; |6x-5| < |1-8x|\} =]-\infty, -2[\cup \left]\frac{3}{7}, +\infty\right[$
3.6. $\{x \in \mathbb{R} : \; |5-6x| \geq |8x-1|\} = \left[-2, \frac{3}{7}\right]$
3.7. $\{x \in \mathbb{R} : \; |2x-9| < |1-8x|\} = \left]-\infty, -\frac{4}{3}\right[\cup]1, +\infty[$
3.8. $\{x \in \mathbb{R} : \; |9-2x| \geq |8x-1|\} = \left[-\frac{4}{3}, 1\right]$
3.9. $\{x \in \mathbb{R} : \; |3x-4| \leq |8-9x|\} = \left]-\infty, \frac{2}{3}\right] \cup [1, +\infty[$
3.10. $\{x \in \mathbb{R} : \; |4-3x| > |9x-8|\} = \left]\frac{2}{3}, 1\right[$
3.11. $\{x \in \mathbb{R} : \; |4x-3| < |7-6x|\} =]-\infty, 1[\cup]2, +\infty[$
3.12. $\{x \in \mathbb{R} : \; |3-4x| \geq |6x-7|\} = [1, 2]$
3.13. $\{x \in \mathbb{R} : \; |2x-7| < |1-6x|\} = \left]-\infty, -\frac{3}{2}\right[\cup]1, +\infty[$
3.14. $\{x \in \mathbb{R} : \; |7-2x| \geq |6x-1|\} = \left[-\frac{3}{2}, 1\right]$
3.15. $\{x \in \mathbb{R} : \; |2x-5| \leq |9-4x|\} =]-\infty, 2] \cup \left[\frac{7}{3}, +\infty\right[$
3.16. $\{x \in \mathbb{R} : \; |5-2x| > |4x-9|\} = \left]2, \frac{7}{3}\right[$
3.17. $\{x \in \mathbb{R} : \; |3x-5| \leq |1-4x|\} =]-\infty, -4] \cup \left[\frac{6}{7}, +\infty\right[$
3.18. $\{x \in \mathbb{R} : \; |5-3x| > |4x-1|\} = \left]-4, \frac{6}{7}\right[$
3.19. $\{x \in \mathbb{R} : \; 3|2-x| \leq |x|\} = \left[\frac{3}{2}, 3\right]$
3.20. $\{x \in \mathbb{R} : \; 3|x-2| > |x|\} = \left]-\infty, \frac{3}{2}\right[\cup]3, +\infty[$
3.21. $\{x \in \mathbb{R} : \; |4x-9| \geq |6-x|\} =]-\infty, 1] \cup [3, +\infty[$
3.22. $\{x \in \mathbb{R} : \; |9-4x| < |6-x|\} =]1, 3[$
3.23. $\{x \in \mathbb{R} : \; |3x+4| \leq |x+8|\} = [-3, 2]$
3.24. $\{x \in \mathbb{R} : \; |3x+4| > |x+8|\} =]-\infty, -3[\cup]2, +\infty[$
3.25. $\{x \in \mathbb{R} : \; |5x-2| \geq |x+2|\} =]-\infty, 0] \cup [1, +\infty[$
3.26. $\{x \in \mathbb{R} : \; |2-5x| < |x+2|\} =]0, 1[$
3.27. $\{x \in \mathbb{R} : \; |7-4x| \leq |2x+1|\} = [1, 4]$
3.28. $\{x \in \mathbb{R} : \; |4x-7| > |2x+1|\} =]-\infty, 1[\cup]4, +\infty[$
3.29. $\{x \in \mathbb{R} : \; |5x-4| \geq |x+4|\} =]-\infty, 0] \cup [2, +\infty[$
3.30. $\{x \in \mathbb{R} : \; |4-5x| < |x+4|\} =]0, 2[$
3.31. $\{x \in \mathbb{R} : \; |7-2x| \leq |x+1|\} = [2, 8]$
3.32. $\{x \in \mathbb{R} : \; |2x-7| > |x+1|\} =]-\infty, 2[\cup]8, +\infty[$
3.33. $\{x \in \mathbb{R} : \; |5-2x| < |x-1|\} =]2, 4[$
3.34. $\{x \in \mathbb{R} : \; |2-x| \geq |3+2x|\} = \left[-5, -\frac{1}{3}\right]$
3.35. $\{x \in \mathbb{R} : \; |3-5x| < |7x-6|\} = \left]-\infty, \frac{3}{4}\right[\cup \left]\frac{3}{2}, +\infty\right[$
3.36. $\{x \in \mathbb{R} : \; |5x-3| \geq |6-7x|\} = \left[\frac{3}{4}, \frac{3}{2}\right]$
3.37. $\{x \in \mathbb{R} : \; |3x-2| > |4-9x|\} = \left]\frac{1}{3}, \frac{1}{2}\right[$
3.38. $\{x \in \mathbb{R} : \; |2-3x| \leq |9x-4|\} = \left]-\infty, \frac{1}{3}\right] \cup \left[\frac{1}{2}, +\infty\right[$
3.39. $\{x \in \mathbb{R} : \; |2x-5| > |4-x|\} =]-\infty, 1[\cup]3, +\infty[$
3.40. $\{x \in \mathbb{R} : \; |5-2x| \leq |x-4|\} = [1, 3]$

4. Mostre que:

4.1. $\{x \in \mathbb{R} : \ 4 < x^2 < 9\} = \,]-3,-2[\, \cup \,]2,3[$

4.2. $\{x \in \mathbb{R} : \ 9 \leq (x-1)^2 < 25\} = \,]-4,-2] \cup [4,6[$

4.3. $\{x \in \mathbb{R} : \ x^2 - 1 > 0 \ \wedge \ x - 3 \leq 0\} = \,]-\infty,-1[\, \cup \,]1,3]$

4.4. $\{x \in \mathbb{R} : \ x^2 - 4 \leq 0 \ \wedge \ x + 1 > 0\} = \,]-1,2]$

4.5. $\{x \in \mathbb{R} : \ x^2 - 2x - 3 \geq 0\} = \,]-\infty,-1] \cup [3,+\infty[$

4.6. $\{x \in \mathbb{R} : \ 2 - x - x^2 > 0\} = \,]-2,1[$

4.7. $\{x \in \mathbb{R} : \ |x^2 - 2| \leq 1\} = \left[-\sqrt{3},-1\right] \cup \left[1,\sqrt{3}\right]$

4.8. $\{x \in \mathbb{R} : \ |3 - 2x + x^2| = 5\} = \left\{1-\sqrt{3}, 1+\sqrt{3}\right\}$

4.9. $\{x \in \mathbb{R} : \ |3 - 2x + x^2| < 5\} = \left]1-\sqrt{3}, 1+\sqrt{3}\right[$

4.10. $\{x \in \mathbb{R} : \ |15 + 2x - x^2| \geq 9\} = \,]-\infty,-4] \cup \left[1-\sqrt{7}, 1+\sqrt{7}\right] \cup [6,+\infty[$

4.11. $\{x \in \mathbb{R} : \ |x^2 + 2x - 15| < 9\} = \left]-6,-1-\sqrt{7}\right[\cup \left]-1+\sqrt{7},4\right[$

4.12. $\{x \in \mathbb{R} : \ |4x - 3x^2| > 1\} = \left]-\infty, \frac{2-\sqrt{7}}{3}\right[\cup \left]\frac{1}{3},1\right[\cup \left]\frac{2+\sqrt{7}}{3},+\infty\right[$

4.13. $\{x \in \mathbb{R} : \ |3x^2 + 4x| \leq 1\} = \left[\frac{-2-\sqrt{7}}{3},-1\right] \cup \left[-\frac{1}{3}, \frac{-2+\sqrt{7}}{3}\right]$

4.14. $\{x \in \mathbb{R} : \ |3x^2 - 5x + 1| \geq 1\} = \,]-\infty,0] \cup \left[\frac{2}{3},1\right] \cup \left[\frac{5}{3},+\infty\right[$

4.15. $\{x \in \mathbb{R} : \ |3x^2 + 5x + 1| < 1\} = \left]-\frac{5}{3},-1\right[\cup \left]-\frac{2}{3},0\right[$

4.16. $\{x \in \mathbb{R} : \ |x^2 + 4x - 3| > 2\} = \,]-\infty,-5[\, \cup \left]-2-\sqrt{5},-2+\sqrt{5}\right[\cup \,]1,+\infty[$

4.17. $\{x \in \mathbb{R} : \ |3 + 4x - x^2| \leq 2\} = \left[-1, 2-\sqrt{5}\right] \cup \left[2+\sqrt{5},5\right]$

4.18. $\{x \in \mathbb{R} : \ |2x^2 - 5x| \geq 3\} = \left]-\infty,-\frac{1}{2}\right] \cup \left[1,\frac{3}{2}\right] \cup [3,+\infty[$

4.19. $\{x \in \mathbb{R} : \ |2x^2 + 5x| < 3\} = \left]-3,-\frac{3}{2}\right[\cup \left]-1,\frac{1}{2}\right[$

4.20. $\{x \in \mathbb{R} : \ |1 + 4x - 3x^2| > 1\} = \left]-\infty, \frac{2-\sqrt{10}}{3}\right[\cup \left]0,\frac{4}{3}\right[\cup \left]\frac{2+\sqrt{10}}{3},+\infty\right[$

4.21. $\{x \in \mathbb{R} : \ |3x^2 + 4x - 1| \leq 1\} = \left[\frac{-2-\sqrt{10}}{3},-\frac{4}{3}\right] \cup \left[0,\frac{-2+\sqrt{10}}{3}\right]$

4.22. $\{x \in \mathbb{R} : \ |x^2 + 3x - 2| \geq 2\} = \,]-\infty,-4] \cup [-3,0] \cup [1,+\infty[$

4.23. $\{x \in \mathbb{R} : \ |2 + 3x - x^2| < 2\} = \,]-1,0[\, \cup \,]3,4[$

4.24. $\{x \in \mathbb{R} : \ |x^2 - 5x + 2| \geq 2\} = \,]-\infty,0] \cup [1,4] \cup [5,+\infty[$

4.25. $\{x \in \mathbb{R} : \ |x^2 + 5x + 2| < 2\} = \,]-5,-4[\, \cup \,]-1,0[$

4.26. $\{x \in \mathbb{R} : \ |2x^2 - 3x - 1| > 1\} = \left]-\infty,-\frac{1}{2}\right[\cup \left]0,\frac{3}{2}\right[\cup \,]2,+\infty[$

4.27. $\{x \in \mathbb{R} : \ |2x^2 + 3x - 1| \leq 1\} = \left[-2,-\frac{3}{2}\right] \cup \left[0,\frac{1}{2}\right]$

4.28. $\{x \in \mathbb{R} : \ |2x^2 + 4x - 3| > 3\} = \,]-\infty,-3[\, \cup \,]-2,0[\, \cup \,]1,+\infty[$

4.29. $\{x \in \mathbb{R} : \ |3 + 4x - 2x^2| \leq 3\} = [-1,0] \cup [2,3]$

4.30. $\{x \in \mathbb{R} : \ |x^2 + 3x - 7| \geq 3\} = \,]-\infty,-5] \cup [-4,1] \cup [2,+\infty[$

4.31. $\{x \in \mathbb{R} : \ |x^2 - 3x - 7| < 3\} = \,]-2,-1[\, \cup \,]4,5[$

4.32. $\{x \in \mathbb{R} : \ |4 - x - x^2| \geq 2\} = \,]-\infty,-3] \cup [-2,1] \cup [2,+\infty[$

4.33. $\{x \in \mathbb{R} : \ |x^2 - x - 4| < 2\} = \,]-2,-1[\, \cup \,]2,3[$

4.34. $\{x \in \mathbb{R} : \ |3x^2 + 2x - 3| > 2\} = \left]-\infty,-\frac{5}{3}\right[\cup \left]-1,\frac{1}{3}\right[\cup \,]1,+\infty[$

4.35. $\{x \in \mathbb{R} : \ |3 + 2x - 3x^2| \leq 2\} = \left[-1,-\frac{1}{3}\right] \cup \left[1,\frac{5}{3}\right]$

4.36. $\left\{x \in \mathbb{R} : \ |5x^2 + 4x - \frac{1}{2}| > \frac{1}{2}\right\} = \,]-\infty,-1[\, \cup \left]-\frac{4}{5},0\right[\cup \left]\frac{1}{5},+\infty\right[$

4.37. $\left\{x \in \mathbb{R} : \ |5x^2 - 4x - \frac{1}{2}| \leq \frac{1}{2}\right\} = \left[-\frac{1}{5},0\right] \cup \left[\frac{4}{5},1\right]$

4.38. $\{x \in \mathbb{R} : \ |5x^2 + 4x - 5| \geq 4\} = \left]-\infty,-\frac{9}{5}\right] \cup \left[-1,\frac{1}{5}\right] \cup [1,+\infty[$

4.39. $\{x \in \mathbb{R} : \ |5 + 4x - 5x^2| < 4\} = \left]-1,-\frac{1}{5}\right[\cup \left]1,\frac{9}{5}\right[$

5. Mostre que:

5.1. $\{x \in \mathbb{R} : |x(x-3)| = |1-3x|\} = \left\{-1, 3-2\sqrt{2}, 1, 3+2\sqrt{2}\right\}$

5.2. $\{x \in \mathbb{R} : |x(x-3)| > |1-3x|\} =]-\infty, -1[\cup]3-2\sqrt{2}, 1[\cup]3+2\sqrt{2}, +\infty[$

5.3. $\left\{x \in \mathbb{R} : |x^2+x| \leq |x+\frac{3}{4}|\right\} = \left[\frac{-3}{2}, \frac{-\sqrt{3}}{2}\right] \cup \left[\frac{-1}{2}, \frac{\sqrt{3}}{2}\right]$

5.4. $\left\{x \in \mathbb{R} : |x-x^2| \leq |x-\frac{3}{4}|\right\} = \left[\frac{-\sqrt{3}}{2}, \frac{1}{2}\right] \cup \left[\frac{\sqrt{3}}{2}, \frac{3}{2}\right]$

5.5. $\{x \in \mathbb{R} : |3x+4| > |x^2+3x|\} =]-3-\sqrt{5}, -2[\cup]-3+\sqrt{5}, 2[$

5.6. $\{x \in \mathbb{R} : |4-3x| > |3x-x^2|\} =]-2, 3-\sqrt{5}[\cup]2, 3+\sqrt{5}[$

5.7. $\{x \in \mathbb{R} : |2x^2-5x| \leq |5x-8|\} = [-2, 1] \cup [2, 4]$

5.8. $\{x \in \mathbb{R} : |2x^2+5x| \leq |5x+8|\} = [-4, -2] \cup [-1, 2]$

5.9. $\{x \in \mathbb{R} : |2x-x^2| < |1-2x|\} =]-1, 2-\sqrt{3}[\cup]1, 2+\sqrt{3}[$

5.10. $\{x \in \mathbb{R} : |x^2+2x| < |2x+1|\} =]-2-\sqrt{3}, -1[\cup]-2+\sqrt{3}, 1[$

5.11. $\{x \in \mathbb{R} : |5x+4| > |4x^2+5x|\} =]-2, -1[\cup]-\frac{1}{2}, 1[$

5.12. $\{x \in \mathbb{R} : |5x-4| > |4x^2-5x|\} =]-1, \frac{1}{2}[\cup]1, 2[$

5.13. $\{x \in \mathbb{R} : |3-2x| \geq |2x-x^2|\} = [-\sqrt{3}, 1] \cup [\sqrt{3}, 3]$

5.14. $\{x \in \mathbb{R} : |2x+3| \geq |x^2+2x|\} = [-3, -\sqrt{3}] \cup [-1, \sqrt{3}]$

5.15. $\{x \in \mathbb{R} : |x^2+3x| \leq |3x+5|\} = [-5, -\sqrt{5}] \cup [-1, \sqrt{5}]$

5.16. $\{x \in \mathbb{R} : |x^2-3x| \leq |3x-5|\} = [-\sqrt{5}, 1] \cup [\sqrt{5}, 5]$

5.17. $\{x \in \mathbb{R} : |2x^2+3x| < |3x+4|\} =]-2, -\sqrt{2}[\cup]-1, \sqrt{2}[$

5.18. $\{x \in \mathbb{R} : |2x^2-3x| < |3x-4|\} =]-\sqrt{2}, 1[\cup]\sqrt{2}, 2[$

5.19. $\{x \in \mathbb{R} : |2x^2+x| > |2x+1|\} =]-\infty, -1[\cup]1, +\infty[$

5.20. $\{x \in \mathbb{R} : |x-2x^2| \geq |1-2x|\} =]-\infty, -1] \cup \left\{\frac{1}{2}\right\} \cup [1, +\infty[$

5.21. $\{x \in \mathbb{R} : |3x^2+x| \leq |3x+1|\} = [-1, 1]$

5.22. $\{x \in \mathbb{R} : |x-3x^2| < |1-3x|\} =]-1, \frac{1}{3}[\cup]\frac{1}{3}, 1[$

5.23. $\{x \in \mathbb{R} : |3x^2+4x| \geq |3x+2|\} =]-\infty, -2] \cup [-1, -\frac{1}{3}] \cup [\frac{2}{3}, +\infty[$

5.24. $\{x \in \mathbb{R} : |4x-3x^2| > |2-3x|\} =]-\infty, -\frac{2}{3}[\cup]\frac{1}{3}, 1[\cup]2, +\infty[$

5.25. $\{x \in \mathbb{R} : 3|x+1| \leq 2|x^2+2x|\} =]-\infty, -3] \cup [-\frac{3}{2}, -\frac{1}{2}] \cup [1, +\infty[$

5.26. $\{x \in \mathbb{R} : 3|1-x| < 2|2x-x^2|\} =]-\infty, -1[\cup]-\frac{1}{2}, \frac{3}{2}[\cup]3, +\infty[$

5.27. $\{x \in \mathbb{R} : 8|x^2+x| \geq 3|2x+1|\} =]-\infty, -\frac{3}{2}] \cup [-\frac{3}{4}, -\frac{1}{4}] \cup [\frac{1}{2}, +\infty[$

5.28. $\{x \in \mathbb{R} : 8|x^2-x| > 3|1-2x|\} =]-\infty, -\frac{1}{2}[\cup]\frac{1}{4}, \frac{3}{4}[\cup]\frac{3}{2}, +\infty[$

5.29. $\{x \in \mathbb{R} : 3|x+6| \leq |x^2+4x|\} = \left]-\infty, \frac{-1-\sqrt{73}}{2}\right] \cup \left[\frac{-1+\sqrt{73}}{2}, +\infty\right[$

5.30. $\{x \in \mathbb{R} : 3|6-x| < |4x-x^2|\} = \left]-\infty, \frac{1-\sqrt{73}}{2}\right[\cup \left]\frac{1+\sqrt{73}}{2}, +\infty\right[$

I. Indução Matemática

1. Demonstre por indução as relações seguintes (entre parentesis, cada relação é escrita usando o símbolo de somatório, cf. exercícios do grupo II).

(a) $1 + 2 + 3 + \cdots + n = n(n+1)/2$ para qualquer $n \in \mathbb{N}$.
$(\sum_{k=1}^{n} k = n(n+1)/2)$

(b) $1 + 3 + 5 + \cdots + (2n-1) = n^2$ para qualquer $n \in \mathbb{N}$.
$(\sum_{k=1}^{n} (2k-1) = n^2)$

(c) $1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ para qualquer $n \in \mathbb{N}$.
$(\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6)$

(d) $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$ para qualquer $n \in \mathbb{N}$.
$(\sum_{k=1}^{n} k^3 = (\sum_{k=1}^{n} k)^2)$

(e) $0^3 + 1^3 + \cdots + (n-1)^3 < n^4/4 < 1^3 + 2^3 + \cdots + n^3$ para qualquer $n \in \mathbb{N}$.
$(\sum_{k=1}^{n} (k-1)^3 < n^4/4 < \sum_{k=1}^{n} k^3)$

(f) $1/\sqrt{1} + 1/\sqrt{2} + \cdots + 1/\sqrt{n} > \sqrt{n}$ para qualquer $n \in \mathbb{N}$ tal que $n \geq 2$.
$\left(\sum_{k=1}^{n} 1/\sqrt{k} > \sqrt{n} \right)$

2. Seja $P(n)$ a proposição: $n^2 + 3n + 1$ é par para todo o $n \in \mathbb{N}$.

(a) Mostre que se $P(k)$ é verdadeira para um dado $k \in \mathbb{N}$, então $P(k+1)$ também é verdadeira.

(b) Critique a afirmação: "Por indução fica provado que $P(n)$ é verdadeira para todo o $n \in \mathbb{N}$".

(c) Prove que $n^2 + 3n + 1$ é ímpar para todo o $n \in \mathbb{N}$.

3. Seja $P(n)$ a proposição: $1 + 2 + 3 + \cdots + n = (2n+1)^2/8$ para todo o $n \in \mathbb{N}$.

(a) Mostre que se $P(k)$ é verdadeira para um dado $k \in \mathbb{N}$, então $P(k+1)$ também é verdadeira.

(b) Critique a afirmação: "Por indução fica provado que $P(n)$ é verdadeira para todo o $n \in \mathbb{N}$".

(c) Modifique $P(n)$, mudando a igualdade para uma desigualdade que seja verdadeira para todo o $n \in \mathbb{N}$.

4. Mostre a **desigualdade de Bernoulli**, i.e. $(1 + x)^n \geq 1 + nx$ para qualquer $n \in \mathbb{N}$ e qualquer $x \in \mathbb{R}$ tal que $x \geq -1$.

II. Símbolo de Somatório

Dado $n \in \mathbb{N}$ e uma sequência de números reais $a_1, a_2, \ldots, a_n \in \mathbb{R}$, o símbolo de somatório $\sum_{k=1}^{n} a_k$ define-se por recorrência da seguinte forma:

$$\sum_{k=1}^{n} a_k = a_1 \quad \text{se } n = 1\,, \quad \sum_{k=1}^{n} a_k = \left(\sum_{k=1}^{n-1} a_k\right) + a_n \quad \text{se } n > 1\,.$$

Resolva os exercícios seguintes com base nesta definição.

1. Determine os valores numéricos das seguintes somas:

(a) $\sum_{i=1}^{8} (2i - 3)$; (b) $\sum_{k=1}^{7} (k-4)^2$; (c) $\sum_{j=1}^{4} j(j+1)(j+2)$; (d) $\sum_{i=1}^{4} 6$;

(e) $\sum_{j=1}^{3} j^{2j}$; (f) $\sum_{k=1}^{7} (-1)^k (2k-3)$; (g) $\sum_{n=1}^{5} \frac{1}{n(n+1)}$.

2. Demonstre as seguintes propriedades do somatório:
(a) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$ (propriedade aditiva);
(b) $\sum_{k=1}^{n} (c\, a_k) = c \sum_{k=1}^{n} a_k$ para qualquer constante $c \in \mathbb{R}$ (homogeneidade);
(c) $\sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0$ (propriedade telescópica).

3. Utilizando os resultados do Exercício I.1 e as propriedades anteriores do somatório, calcule:

(a) $\sum_{k=1}^{18} (k+1)$; (b) $\sum_{k=1}^{20} (2k-1)^2$; (c) $\sum_{k=1}^{15} (k-3)^3$;

(d) $\sum_{k=1}^{20} \left(\frac{1}{k+1} - \frac{1}{k}\right)$; (e) $\sum_{k=1}^{20} \left(3^k - 3^{k+2}\right)$.

4. Mostre que para qualquer $n \in \mathbb{N}$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

pelos seguintes dois métodos distintos:
(a) usando indução.
(b) observando que $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ e usando as propriedades do Exercício 2.

5. Mostre que para qualquer $n \in \mathbb{N}$ e quaisquer números reais $a, b \in \mathbb{R}$ é válida a igualdade

$$a^n - b^n = (a-b) \sum_{k=1}^{n} a^{n-k} b^{k-1}\,.$$

6. Mostre que para quaisquer $n \in \mathbb{N}$ e $r \in \mathbb{R}$ com $r \neq 1$

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

pelos seguintes dois métodos distintos:

(a) usando indução.

(b) aplicando as propriedades do Exercício 2 a $(1 - r) \sum_{k=0}^{n} r^k$.

A que é igual a soma quando $r = 1$?

Nota: por definição, $r^0 = 1$.

7. O símbolo $n!$, designado por n-**factorial**, define-se por recorrência da seguinte forma:

$$0! = 1 \qquad \text{e} \qquad n! = n \cdot (n-1)!\,, \text{ para qualquer } n \in \mathbb{N}\,.$$

Observe que $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$. Dados inteiros $0 \leq k \leq n$, o **coeficiente binomial** $\binom{n}{k}$ (às vezes também representado por C_k^n) é definido por

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}\,.$$

(a) Mostre que

$$\binom{n}{k} = \binom{n}{n-k} \qquad \text{e} \qquad \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}\,.$$

Esta última fórmula é a chamada **lei do triângulo de Pascal**, permitindo o cálculo rápido dos sucessivos coeficientes binomiais.

(b) Prove por indução a **fórmula do desenvolvimento do binómio de Newton**:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\,, \text{ para quaisquer } a, b \in \mathbb{R} \text{ e } n \in \mathbb{N}_0\,.$$

(c) Use a fórmula anterior para estabelecer as igualdades

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \quad \text{e} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0\,, \text{ para qualquer } n \in \mathbb{N}_0\,.$$

8. Usando a desigualdade triangular ($|x + y| \leq |x| + |y|$) e o método de indução, mostre que para todo o $n \in \mathbb{N}$ e quaisquer números reais $x_1, \ldots, x_n \in \mathbb{R}$ é válida a desigualdade

$$\left| \sum_{k=1}^{n} x_k \right| \leq \sum_{k=1}^{n} |x_k|\,.$$

III. Indução e Somatórios

Use indução para mostrar que, para qualquer $n \in \mathbb{N}$:

1.
$$\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!} \ .$$

2.
$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1} \ .$$

3.
$$\sum_{k=1}^{n} k(3k-1) = n^2(n+1) \ .$$

4.
$$\sum_{k=1}^{n} k(3k+1) = n(n+1)^2 \ .$$

5.
$$\sum_{k=1}^{n} (k-1)(k+2) = \frac{(n-1)n(n+4)}{3} \ .$$

6.
$$\sum_{k=1}^{n} (k-1)(3k+2) = (n-1)n(n+2) \ .$$

7.
$$\sum_{k=1}^{n} (k+1)2^k = n2^{n+1} \ .$$

8.
$$\sum_{k=1}^{n} (k+1)2^{k-1} = n2^n \ .$$

9.
$$\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n} \ .$$

10.
$$\sum_{k=1}^{n} \frac{k}{2^{k+1}} = 1 - \frac{n+2}{2^{n+1}} \ .$$

11.
$$\sum_{k=1}^{n} k(k+3) = \frac{n(n+1)(n+5)}{3} \ .$$

12.
$$\sum_{k=1}^{n} k(3k+5) = n(n+1)(n+3) \ .$$
13.
$$\sum_{k=1}^{n} (2k+1)3^k = n3^{n+1} \ .$$
14.
$$\sum_{k=1}^{n} (2k+1)3^{k-1} = n3^{n} \ .$$
15.
$$\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = 1 - \frac{1}{(n+1)^2} \ .$$
16.
$$\sum_{k=1}^{n} \frac{5-2k}{3^k} = 1 + \frac{n-1}{3^n} \ .$$
17.
$$\sum_{k=1}^{n} \frac{2k-1}{3^k} = 1 - \frac{n+1}{3^n} \ .$$
18.
$$\sum_{k=1}^{n} k(k+2)2^k = (n^2+1)2^{n+1} - 2 \ .$$
19.
$$\sum_{k=1}^{n} k(k+2)2^{k-1} = (n^2+1)2^{n} - 1 \ .$$
20.
$$\sum_{k=1}^{n} \frac{(k-2)^2}{2^k} = 2 - \frac{n^2+2}{2^n} \ .$$
21.
$$\sum_{k=1}^{n} \frac{(k-3)^2}{2^k} = 3 - \frac{(n-1)^2+2}{2^n} \ .$$
22.
$$\sum_{k=1}^{n} \frac{(k-2)3^{k-1}}{(k+1)!} = 1 - \frac{3^n}{(n+1)!} \ .$$
23.
$$\sum_{k=1}^{n} \frac{(k-3)3^{k-1}}{k!} = 1 - \frac{3^n}{n!} \ .$$

IV. Funções Elementares

1) Esboce os gráficos dos polinómios $f(x) = x$ e $g(x) = x^3$, assinalando de forma conveniente os seus três pontos de intersecção.

2) Esboce os gráficos dos polinómios $f(x) = x^2 - 2$ e $g(x) = 2x^2 + 4x + 1$, assinalando de forma conveniente os seus dois pontos de intersecção.

3) Seja $f(x) = \sum_{k=0}^{n} c_k x^k$ um polinómio de grau $n \in \mathbb{N}$. Prove cada uma das seguintes proposições.
(a) Se $n \geq 1$ e $f(0) = 0$, então $f(x) = xg(x)$ com g um polinómio de grau $n-1$.
(b) Para cada $a \in \mathbb{R}$, a função p dada por $p(x) = f(x+a)$ é também um polinómio de grau n.
(c) Se $n \geq 1$ e $f(a) = 0$ para um dado $a \in \mathbb{R}$, então $f(x) = (x-a)h(x)$ com h um polinómio de grau $n-1$. [Sugestão: considere $p(x) = f(x+a)$.]
(d) Se $f(x) = 0$ para $(n+1)$ valores distintos de $x \in \mathbb{R}$, então $c_k = 0\,,\ k = 0, \ldots, n$, e portanto $f(x) = 0\,,\ \forall x \in \mathbb{R}$.
(e) Seja $g(x) = \sum_{k=0}^{m} b_k x^k$ um polinómio de grau $m \in \mathbb{N}$, com $m \geq n$. Se $g(x) = f(x)$ para $(m+1)$ valores distintos de $x \in \mathbb{R}$, então $m = n$, $b_k = c_k\,,\ k = 0, \ldots, n$, e portanto $g(x) = f(x)\,,\ \forall x \in \mathbb{R}$.

4) Em cada caso, determine todos os polinómios p de grau ≤ 2 satisfazendo as condições dadas.

(a) $p(0) = p(1) = p(2) = 1$
(b) $p(0) = p(1) = 1\,,\ p(2) = 2$
(c) $p(0) = p(1) = 1$
(d) $p(0) = p(1)$

5) Em cada caso, determine todos os polinómios p de grau ≤ 2 satisfazendo as condições dadas para qualquer $x \in \mathbb{R}$.

(a) $p(x) = p(1-x)$ (b) $p(x) = p(1+x)$ (c) $p(2x) = 2p(x)$ (d) $p(3x) = p(x+3)$

6) Considere as seguintes propriedades fundamentais das funções **seno**, $\sin : \mathbb{R} \to \mathbb{R}$, e **coseno**, $\cos : \mathbb{R} \to \mathbb{R}$:
1. $\cos(0) = \sin(\pi/2) = 1$ e $\cos(\pi) = -1$.
2. Para quaisquer $x, y \in \mathbb{R}$ tem-se que
$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)\ .$$
3. Para $0 < x < \pi/2$ tem-se que
$$0 < \cos(x) < \frac{\sin(x)}{x} < \frac{1}{\cos(x)}\ .$$

Prove a partir delas as seguintes propriedades importantes das funções seno e coseno. [Sugestão: Apostol, Vol. I, §2.5.]
(a) $\sin^2(x) + \cos^2(x) = 1\,,\ \forall x \in \mathbb{R}$.
(b) $\sin(0) = \cos(\pi/2) = \sin(\pi) = 0$.
(c) $\sin(-x) = -\sin(x)$ e $\cos(-x) = \cos(x)\,,\ \forall x \in \mathbb{R}$ (i.e. o seno é uma função ímpar e o coseno uma função par).
(d) $\sin(x + \pi/2) = \cos(x)$ e $\cos(x + \pi/2) = -\sin(x)\,,\ \forall x \in \mathbb{R}$.

(e) $\sin(x+2\pi) = \sin(x)$ e $\cos(x+2\pi) = \cos(x)$, $\forall x \in \mathbb{R}$ (i.e. o seno e o coseno são funções periódicas).
(f) Para quaisquer $x, y \in \mathbb{R}$ tem-se que

$$\begin{aligned} \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \ , \\ \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \ . \end{aligned}$$

(g) Para quaisquer $a, b \in \mathbb{R}$ tem-se que

$$\begin{aligned} \sin(a) - \sin(b) &= 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right) \ , \\ \cos(a) - \cos(b) &= -2\sin\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right) \ . \end{aligned}$$

(h) No intervalo $[0, \pi/2]$, o seno é estritamente crescente e o coseno é estritamente decrescente.

7) Com base nas propriedades das funções seno e coseno listadas no exercício anterior, mostre que:
(a) $\sin(x) = 0 \Leftrightarrow x = k\pi$ com $k \in \mathbb{Z}$.
(b) $\cos(x) = 0 \Leftrightarrow x = k\pi + \pi/2$ com $k \in \mathbb{Z}$.
(c) $\sin(x+\pi) = -\sin(x)$ e $\cos(x+\pi) = -\cos(x)$, $\forall x \in \mathbb{R}$.
(d) $\cos(2x) = \cos^2(x) - \sin^2(x)$ e $\sin(2x) = 2\sin(x)\cos(x)$, $\forall x \in \mathbb{R}$.
(e) $2\cos(x)\cos(y) = \cos(x-y) + \cos(x+y)$, $\forall x, y \in \mathbb{R}$.
(f) $2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$, $\forall x, y \in \mathbb{R}$.
(g) $2\sin(x)\cos(y) = \sin(x-y) + \sin(x+y)$, $\forall x, y \in \mathbb{R}$.
(h) Para quaisquer $x, y \in \mathbb{R}$ e $h \neq 0$ tem-se que

$$\begin{aligned} \frac{\sin(x+h) - \sin(x)}{h} &= \frac{\sin(h/2)}{h/2}\cos(x+h/2) \ , \\ \frac{\cos(x+h) - \cos(x)}{h} &= -\frac{\sin(h/2)}{h/2}\sin(x+h/2) \ . \end{aligned}$$

8) Considere as funções **seno hiperbólico**, $\sinh : \mathbb{R} \to \mathbb{R}$, e **coseno hiperbólico**, $\cosh : \mathbb{R} \to \mathbb{R}$, definidas por

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \text{e} \quad \cosh(x) = \frac{e^x + e^{-x}}{2} \ .$$

Mostre que:
(a) $\cosh^2(x) - \sinh^2(x) = 1$, $\forall x \in \mathbb{R}$.
(b) $\sinh(0) = 0$ e $\cosh(0) = 1$.
(c) $\sinh(-x) = -\sinh(x)$ e $\cosh(-x) = \cosh(x)$, $\forall x \in \mathbb{R}$.
(d) para quaisquer $x, y \in \mathbb{R}$ tem-se que

$$\begin{aligned} \cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \ , \\ \sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y) \ . \end{aligned}$$

(e) $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$ e $\sinh(2x) = 2\sinh(x)\cosh(x)$, $\forall x \in \mathbb{R}$.
(f) $\cosh(x) + \sinh(x) = e^x$ e $\cosh(x) - \sinh(x) = e^{-x}$, $\forall x \in \mathbb{R}$.

9) Determine o domínio das funções definidas pelas seguintes expressões.

(a) $f(x) = \tan\frac{x}{2} - \cot\frac{x}{2}$ (b) $f(x) = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}$ (c) $f(x) = \frac{x}{\sqrt{4-x^2}}$

(d) $f(x) = \log(\log x)$ (e) $f(x) = \log\left(1 + x^{3/2}\right)$ (f) $f(x) = \log\left(1 - x^{2/3}\right)$

(g) $f(x) = \log\left(\frac{x^2-1}{x^2+1}\right)$ (h) $f(x) = \log\left(1 + \sqrt{x+1}\right)$

V. Limites Elementares

1) Calcule os seguintes limites.

(a) $\lim_{x\to 2} \frac{x^2-4}{x-2}$ (b) $\lim_{x\to 1} \frac{2x^2-3x+1}{x-1}$ (c) $\lim_{x\to 0^+} \frac{\sqrt{x^2}}{x}$

(d) $\lim_{x\to 0^-} \frac{\sqrt{x^2}}{x}$ (e) $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2}$ (f) $\lim_{x\to -2} \frac{x^3+8}{x^2-4}$

(g) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

2) Usando o caso notável

$$\lim_{x\to 0} \frac{\sin x}{x} = 1\ ,$$

mostre que:

(a) $\lim_{x\to 0} \frac{\sin(2x)}{x} = 2$ (b) $\lim_{x\to 0} \frac{\sin(5x)}{\sin x} = 5$ (c) $\lim_{x\to 0} \frac{\sin(5x)-\sin(3x)}{x} = 2$

(d) $\lim_{x\to a} \frac{\sin x - \sin a}{x-a} = \cos a$ (e) $\lim_{x\to 0} \frac{\tan(2x)}{\sin x} = 2$ (f) $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

3) Calcule os seguintes limites.

(a) $\lim_{t\to 0} \frac{\sin(\tan t)}{\sin(t)}$ (b) $\lim_{x\to \frac{\pi}{2}} \frac{\sin(\cos x)}{\cos x}$ (c) $\lim_{t\to \pi} \frac{\sin(t-\pi)}{t-\pi}$

(d) $\lim_{x\to 1} \frac{\sin(x^2-1)}{x-1}$ (e) $\lim_{x\to +\infty} x \sin\frac{1}{x}$ (f) $\lim_{x\to 0} \frac{1-\cos(2x)}{x^2}$

4) Seja $D = [0, +\infty[\setminus \{1\}$ e considere a função $f : D \to \mathbb{R}$ definida por

$$f(x) = \frac{\sqrt{x}}{x-1} \qquad \text{para} \qquad x \in D\ .$$

Calcule

$$\lim_{x\to +\infty} f(x)\ , \qquad \lim_{x\to 1^-} f(x) \qquad \text{e} \qquad \lim_{x\to 1^+} f(x)\ .$$

5) Calcule os limites quando $x \to 0^+$, $x \to 0^-$, $x \to +\infty$ e $x \to -\infty$ das seguintes funções definidas em $\mathbb{R} \setminus \{0\}$.

(a) $e^{1/x}$ (b) $\sinh(1/x)$ (c) $\cosh(1/x)$

(d) e^{1/x^2} (e) $\sinh(1/x^2)$ (f) $\cosh(1/x^2)$

6) Calcule os limites quando $x \to 0$, $x \to +\infty$ e $x \to -\infty$ das funções definidas pelas seguintes expressões.

(a) $\dfrac{\sin(3x)}{x}$ (b) $x\cos\left(\dfrac{1}{x}\right)$ (c) $\cos\left(\dfrac{2x+\pi}{x^2+1}\right)$ (d) $\cos\left(\dfrac{2x-\pi}{x^2+1}\right)$

(e) $\sin\left(\dfrac{x-\pi}{x^2+2}\right)$ (f) $\cos\left(\dfrac{x+\pi}{x^2+2}\right)$ (g) $\cos\left(\dfrac{x+\pi}{x^2+2}\right)$ (h) $\cos\left(\dfrac{x-\pi}{x^2+2}\right)$

(i) $\sin\left(\dfrac{x+\pi}{x^2+4}\right)$ (j) $\sin\left(\dfrac{x-\pi}{x^2+4}\right)$ (k) $\sin\left(\dfrac{\pi x}{2x-1}\right)$ (l) $\cos\left(\dfrac{\pi x}{x+1}\right)$

(m) $\sin\left(\dfrac{\pi x}{\sqrt{4x^2+1}}\right)$ (n) $\cos\left(\dfrac{\pi x}{\sqrt{4x^2+1}}\right)$

7) Calcule os limites quando $x \to 0^+$ e $x \to +\infty$ das funções definidas pelas seguintes expressões.

(a) $\log\left(\dfrac{1}{1+\sqrt{x}}\right)$ (b) $\log\left(\dfrac{\sqrt{x}}{1+x}\right)$ (c) $\log\left(\dfrac{\sqrt{x}}{1+\sqrt{x}}\right)$ (d) $\log\left(\dfrac{x}{1+\sqrt{x}}\right)$

(e) $\log\left(\dfrac{1+x}{1+\sqrt{x}}\right)$ (f) $\log\left(\dfrac{1+\sqrt{x}}{1+x}\right)$ (g) $\log\left(\dfrac{1}{1+x^2}\right)$ (h) $\log\left(\dfrac{x}{1+x^2}\right)$

(i) $\log\left(\dfrac{x^2}{1+x^2}\right)$ (j) $\log\left(\dfrac{x}{\sqrt{1+x^2}}\right)$ (k) $\log\left(\dfrac{x^2}{\sqrt{1+x^2}}\right)$ (l) $\log\left(\dfrac{1+\sqrt{x}}{\sqrt{1+x}}\right)$

8) Calcule os limites quando $x \to 0^+$ e $x \to +\infty$ das funções definidas pelas seguintes expressões.

(a) $e^{\frac{1}{\sqrt{x}}}$ (b) $e^{-\frac{1}{\sqrt{x}}}$ (c) $e^{\frac{\sqrt{x}}{1+\sqrt{x}}}$ (d) $e^{\frac{1-x}{\sqrt{x}}}$ (e) $e^{\frac{x}{1+\sqrt{x}}}$ (f) $e^{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$

(g) $e^{\frac{1-x^2}{x}}$ (h) $e^{\frac{x^2}{1+x}}$ (i) $e^{\frac{x^2}{1+x^2}}$ (j) $e^{\frac{x}{\sqrt{1+x^2}}}$ (k) $e^{\frac{1-\sqrt{x}}{\sqrt{1+x}}}$ (l) $e^{\frac{x^2}{\sqrt{1+x^2}}}$