The Essential Boundary in Hilbert Spaces of Polyanalytic Functions.

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Abstract

A Fredholm symbolic calculus is constructed for poly-Toeplitz operators with continuous symbol and I will explain how such symbol requires the notion of j-essential boundary. The symbol calculus is well known for Bergman-Toeplitz operators, in which setting the removal boundary is a subset of the boundary having zero transfinite diameter. Some surprising differences between the analytical and the poly-analytical case will be presented.

Poly-Bergman spaces

$U \subset \mathbb{C}$ open connected set ; $dA(z) = dxdy$ Lebesgue area measure

$$\partial_{\bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \partial_z := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

Definition (Poly-Bergman spaces)

$f \in \mathcal{A}^2_j(U)$ if $f \in L^2(U, dA)$, f is smooth and

$$\partial^j_{\bar{z}} f = 0 \text{ and } \partial^{-j}_z f = 0, \text{ respectively if } j \in \mathbb{Z}_+ \text{ and } j \in \mathbb{Z}_- \qquad (1.1)$$

f is j-analytic function if is smooth and satisfies (1.1)

Poly-Bergman spaces

Poly-Bergman spaces are **reproducing kernel Hilbert spaces**.

$$\left|f(z)\right| \leq \frac{|j|}{\sqrt{\pi}\, d_z}\left\|f\right\|_{L^2(U)} \quad ; \quad f \in \mathcal{A}_j^2(U),\ j \in \mathbb{Z}_\pm,\ d_z := \operatorname{dist}(z;\partial U)$$

Definition (Poly-Bergman kernel and projection)

$K_{U,j}(z,w)$, $j \in \mathbb{Z}_\pm$ is the j-Poly-Bergman reproducing kernel for U, i.e. the unique function such that $K_{U,j}(z,w) := \overline{k}_{U,j,z}(w)$ and

$$f(z) = \langle f, k_{U,z}\rangle \ ;\ f \in \mathcal{A}_j^2(U),\ z \in U\,.$$

$B_{U,j}$ is the **orthogonal projections** from $L^2\,(U, d\,A)$ onto $\mathcal{A}_j^2\,(U)$.

$B_{U,j}$ is an integral operator with kernel given by $K_{U,j}$, i.e.

$$B_{U,j}f(z) = \int_U K_{U,j}(z,w)f(w)dA(w)\ ;\ f \in L^2(U, d\,A)$$

Density of Polyanalytic functions on $\overline{U}$

- Next, the results will focus on bounded domains without constrains on the boundary
- The bounded hypothesis is relevant is the majority of the proofs and is relevant in some results
- Some results in smooth bounded finitely connected domains U are important, e.g. to prove the local type property of poly-Bergman projection. This is the aim of the following slides.

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Dzhuraev's Formulas

- Beurling transform (unitary on $L^2(\mathbb{C})$) and its compression to $L^2(U)$

$$Sf(z) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(w)}{(w-z)^2}\,dA(w) \quad \text{and} \quad S_U := \chi_U S \chi_U$$

- Dzhuraev's Operators (for $j \in \mathbb{Z}_+$)

$$D_{U,j} = I - (S_U)^j (S_U^*)^j \quad \text{and} \quad D_{U,-j} = I - (S_U^*)^j (S_U)^j$$

Lemma (Vékua)

$U \subset \mathbb{C}$ a bounded finitely connected domain; ∂U smooth; $f \in L^2(U)$

- *If f is a smooth function on U then $S_U f$ and $S_U^* f$ are smooth and*

$$\partial_{\bar{z}} S_U f = \partial_z f \quad , \quad \partial_z S_U^* f = \partial_{\bar{z}} f. \tag{2.1}$$

- *The space of smooth functions on $\overline{U}$ is invariant under S_U and S_U^*.*

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- *The space of smooth functions on $\overline{U}$ is invariant under S_U and S_U^*.*

Some Remarks on Dzhuraev's Operators

- If U is bounded finitely connected, ∂U is smooth then

$$B_{U,j} - D_{U,j} \in \mathcal{K} \quad (j \in \mathbb{Z}_{\pm}).$$

- The exact Dzhuraev's formulas are valid for domains Möbius equivalente to the a disk ($\mathbb{D}, \Pi$ and Ω) ([P-13])
- The existence of Dzhuraev's formulas are strongly dependent on the regularity of the boundary ([KP-08, P-Sub.])

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Density of Polyanalytic functions on $\overline{U}$

In this slide U is a smooth bounded finitely connected domain

$$\mathcal{A}_j^2(\overline{U}) := \mathcal{A}_j^2(U) \cap C^\infty(\overline{U}) \ , \ j \in \mathbb{Z}_\pm .$$

- From Vekua derivation formulas $\operatorname{Im} D_{U,j} \subset \mathcal{A}_j^2(U)$
- from previous Lemma we can prove $\mathcal{A}_j^2(\overline{U})$ is dense in $\operatorname{Im} D_{U,j}$
- we can also prove that $\ker D_{U,j} \cap \mathcal{A}_j^2(U) \subset \mathcal{A}_j^2(\overline{U})$.

Theorem ([P2-Sub.])

Let $U \subset \mathbb{C}$ be a bounded finitely connected domain with smooth boundary. For every $j \in \mathbb{Z}_\pm$, one has that $\mathcal{A}_j^2(\overline{U})$ is dense in $\mathcal{A}_j^2(U)$.

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Density of Polyanalytic functions on $\overline{U}$

$\mathrm{Rat}(X)$ the set of rational functions with poles out of $X \subset \mathbb{C}$ compact.

Proposition ([P2-Sub.])

$U \subset \mathbb{C}$ a bounded finitely connected; ∂U smooth; $j \in \mathbb{Z}_+$. Then

$$\Big\{ \sum_{k=0}^{j-1} \overline{z}^k r_k(z) : r_k \in \mathrm{Rat}(\overline{U}) \Big\} \quad \text{and} \quad \Big\{ \sum_{k=0}^{j-1} z^k \overline{r}_k(z) : r_k \in \mathrm{Rat}(\overline{U}) \Big\}$$

is dense in the poly-Bergman space $\mathcal{A}^2_j(U)$ and $\mathcal{A}^2_{-j}(U)$, respectively.

- Bergman case: classical results of Farrell, Markusevic, Mergeljan
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Berger-Shaw Theorem

$$H_{\phi,j} : \mathcal{A}_j^2(U) \to \left[\mathcal{A}_j^2(U)\right]^{\perp} \quad , \quad H_{\phi,j}(g) = (I - B_{U,j})(\phi g)$$

$$[B_{U,j}, \phi I] = H^*_{\overline{\phi},j}(I - B_{U,j}) - H_{\phi,j} B_{U,j} \quad \text{and} \quad H^*_{\overline{z},j} H_{\overline{z},j} = \left[T^*_{z,j}, T_{z,j}\right]$$

$$T_{\phi,j} : \mathcal{A}_j^2(U) \mapsto \mathcal{A}_j^2(U) \quad , \quad T_{\phi,j}(g) := B_{U,j}(\phi g).$$

Proposition

$U \subset \mathbb{C}$ a bounded domain; $j \in \mathbb{Z}_{\pm}$. Then, $B_{U,j}$ is an operator of local type if and only if the self-commutator of $T_{z,j}$ is compact.

$$T \in \mathcal{B}(\mathcal{H}) \text{ is } j\text{-multicyclic if } \mathcal{H} = \text{cl span}\,\{r(T)v_k : r \in \text{Rat}(\sigma(T));\ k = 1, \cdots, j\}$$

$$T \in \mathcal{B}(\mathcal{H}) \text{ is hyponormal if } [T^*, T] \geq 0$$

Theorem (Berger-Shaw)

If $T \in \mathcal{B}(\mathcal{H})$ is hyponormal and j-multicyclic, then $\text{Tr}\,[T^*, T] \leq \frac{j}{\pi}|\sigma(T)|$.

If U is a smooth bounded finitely connected domain, then it follows that the self-commutator of $T_{z,j}$ is in the trace class.

Variation of the domain

For an arbitrary bounded domain we consider the variation of the domain technique.

Definition (Inner exhaustive sequence [P-Sub.])

Let $U \subset \mathbb{C}$ be a domain. $\{U_n\}_{n\in\mathbb{N}}$ is a *Inner exhaustive sequence* for U if

$$U_n \subset U_{n+1} \subset U \quad ; \quad \cup_{n\in\mathbb{N}} U_n = U.$$

Theorem (Inner variation of the domain [P-Sub.])

If $\{U_n\}_{n\in\mathbb{N}}$ is a Inner exhaustive sequence for U then

$$B_{U,j} = \operatorname{s-lim}_n \chi_U B_{U_n,j} \chi_U.$$

Proposition ([P2-Sub.])

Let $U \subset \mathbb{C}$ be a bounded domain and let j be a non-zero integer. The self-commutator $[T^*_{z,j}, T_{z,j}]$ is a trace class operator and

$$\mathrm{Tr}\,[T^*_{z,j}, T_{z,j}] \leq |j||U|/\pi.$$

The Allan-Douglas local principle

Corollary ([P2-Sub.])

Let $U \subset \mathbb{C}$ be a bounded domain and let j be a non-zero integer. The poly-Bergman projection $B_{U,j}$ is an operator of local type.

- $\mathfrak{U}_j := \operatorname{alg}\left\{B_{U,j}, aI : a \in C(\overline{U})\right\} \subset \mathcal{B}(L^2(U))$

Proposition ([P2-Sub.])

Let $U \subset \mathbb{C}$ be a bounded domain and let j be a non-zero integer. The C^*-algebra $\mathfrak{U}_j$ is irreducible. Furthermore, $\mathfrak{U}_j$ contains $\mathcal{K}(L^2(U))$.

- Apply local principles to the commutative C^*-algebra $\mathfrak{U}_j^\pi := \mathfrak{U}_j/\mathcal{K}$ over some of its $*$-subalgebra
- $\mathfrak{U}_j^\pi$ is a commutative C^*-algebra

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The Allan-Douglas local principle

$\mathcal{A}$ a C^*-algebra with identity e; $\mathcal{Z} \subset \mathcal{A}$ a central $*$-subalgebra; $e \in \mathcal{Z}$; $\mathcal{M}(\mathcal{Z})$ the maximal ideal space; I_x the closed two-sided ideal of $\mathcal{A}$ generated by $x \in \mathcal{M}(\mathcal{Z})$; $\mathcal{A}_x := \mathcal{A}/I_x$; $\pi_x : \mathcal{A} \to \mathcal{A}_x$.

Theorem (Allan-Douglas)

(i) *a is invertible in $\mathcal{A}$ iff $a_x := \pi_x(a)$ is invertible in $\mathcal{A}_x$, for $x \in \mathcal{M}(\mathcal{Z})$.*

(ii) *$\mathcal{M}(\mathcal{Z}) \ni x \mapsto \|a_x\| \in \mathbb{R}_0^+$ is USC and $\|a\| = \max_{x \in \mathcal{M}(\mathcal{Z})} \|a_x\|$.*

$A^\pi := A + \mathcal{K}$; the local algebra $\mathfrak{U}^\pi_{j,z} := \mathfrak{U}^\pi_j / I^\pi_{U,z}$, for $z \in \overline{U}$; $\pi_z : \mathfrak{U}^\pi_j \to \mathfrak{U}^\pi_{j,z}$; $A^\pi_z := \pi_z(A^\pi)$, for $A \in \mathfrak{U}_j$

Proposition

If $U \subset \mathbb{C}$ is a bounded domain then $(B_{U,j})^\pi_z = 0$, for $z \in U$ and $j \in \mathbb{Z}_\pm$.

Since $K_{U,j}(z, w) \in C^\infty(U \times U)$ then the previous Proposition is evident.

The Bergman removal boundary

Definition (S. Axler, J. B. Conway, G. MacDonald)

- $w \in \partial_{2-r} U$ if $w \in \partial U$ and every function in $\mathcal{A}^2(U)$, for some $\delta > 0$, can be extended to an analytic function on $U \cup D(w, \delta)$
- The essential boundary
$$\partial_{2-e} U := \partial U \ominus \partial_{2-r} U$$

Theorem (S. Axler, J. B. Conway, G. MacDonald)

Let $U \subset \mathbb{C}$ be a bounded domain and let $w \in \partial U$. Then $w \in \partial_{2-r} U$ iff there exists $\delta > 0$ such that $\partial U \cap \overline{D}(w, \delta)$ as zero transfinite diameter.

Definition of j-removal boundary

- K compact set has zero logarithmic capacity iff $\mathcal{A}^2(\mathbb{C}\backslash K) = \{0\}$ (L. Carleson, Selected Problems on Exceptional Sets, 67)
- Different proof in David R. Adams, Lars Inge Hedberg 96 (Potencial Theory); see also Conway, Functions of one Complex Variabel II; Kouchekian 03
- $K \subset \mathbb{C}$ compact set as zero logarithmic capacity iff as zero transfinite diameter

$$\lim_{n} \max_{z_1,\ldots,z_n \in K} \left(\prod_{z_j \neq z_k} |z_j - z_k| \right)^{\frac{n(n-1)}{2}}$$

- Definition also possible by means of Chebichev polynomials (K infinite) $\lim \max_K |T_{K,n}(z)|^{1/n}$

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The points at which B_U is locally equivalent to zero

Theorem (S. Axler, J. B. Conway, G. MacDonald)

Let $U \subset \mathbb{C}$ be a bounded domain. If $f \in C(\overline{U})$ then T_f is compact if and only if $f(\partial_{2-e} U) = \{0\}$.

By localization is follows straightforwardly a criterion for the Bergman projection to be locally equivalent to zero at some point $w \in \partial U$.

Theorem ([P2-Sub.])

Let $U \subset \mathbb{C}$ be a bounded domain and let $w \in \partial U$. Then $w \in \partial_{2-r} U$ if and only if $(B_U)_w^\pi = 0$, i.e. $(B_U)_w^\pi = 0$, $w \in \partial U$ iff there exists $\delta > 0$ such that $\partial U \cap D(w, \delta)$ as zero transfinite diameter.

Definition of j-removal boundary

Definition (j-Removal boundary [P2-Sub.])

- $w \in \partial_r^j U$ if $w \in \partial U$ and $(B_{U,j})_w^\pi = 0$;
- the *j-essential boundary* is defined by

$$\partial_e^j U := \partial U \ominus \partial_r^j U.$$

Proposition ([P2-Sub.])

$$\partial_r^j U = \partial_r^{|j|} U \quad \text{and} \quad \partial_r^j U \subset \partial_r U = \partial_{2-r} U.$$

Proposition ([P2-Sub.])

The set $\partial_e^j U$ is closed, $U \cup \partial_r^j U$ is open and connected and $\partial \overline{U} \subset \partial_e^j U$.

$$U_r^j := U \cup \partial_r^j U.$$

Local algebras

Proposition

Let $U \subset \mathbb{C}$ be a bounded domain and let $j \in \mathbb{Z}_{\pm}$. If $z \in U_r^j$, then $\mathfrak{U}^{\pi}_{j,z} \cong \mathbb{C}$. For every $a \in C(\overline{U})$, the $*$-isomorphism $\Phi_{U,z}$ is given by

$$(B_{U,j})^{\pi}_z \mapsto 0 \quad \textit{and} \quad (aI)^{\pi}_z \mapsto a(z).$$

Proposition

Let $U \subset \mathbb{C}$ be a bounded domain and let $j \in \mathbb{Z}_{\pm}$. If $z \in \partial U_e^j$, then $\mathfrak{U}^{\pi}_{j,z} \cong \mathbb{C}^2$. For every $a \in C(\overline{U})$, the $*$-isomorphism $\Phi_{U,z}$ is given by

$$(B_{U,j})^{\pi}_z \mapsto (1,0) \quad \textit{and} \quad (aI)^{\pi}_z \mapsto (a(z), a(z)).$$

The C^*-algebra $\mathfrak{U}_j^\pi$

Theorem ([P2-Sub.])

Let $U \subset \mathbb{C}$ be a bounded domain and let $j \in \mathbb{Z}_\pm$. Then

$$\mathfrak{U}_j^\pi \cong C(\overline{U}) \oplus C(\partial_e^j U) \quad \text{by} \quad (aI + bB_{U,j})^\pi \overset{\Phi_U}{\longmapsto} a \oplus (a+b)_{|\partial_e^j U}.$$

Let $j \in \mathbb{Z}_\pm$. The poly-Toeplitz C*-algebra $\mathfrak{T}_j(U)$ is defined as follows

$$\mathfrak{T}_j(U) := \mathrm{alg}\left\{T_{f,j} : f \in C(\overline{U})\right\}.$$

Theorem ([P2-Sub.])

Let $U \subset \mathbb{C}$ be a bounded domain and let $j \in \mathbb{Z}_\pm$. Then

$$\mathfrak{T}_j^\pi(U) \cong C(\partial_e^j U) \quad \text{by} \quad (T_{f,j})^\pi \longmapsto f_{|\partial_e^j U}.$$

Structure of the j-removal boundary

U bounded domain; $w \in U$; $U_w := U \backslash \{w\}$

Proposition ([P-13])

Let $U \subset \mathbb{C}$ be a bounded domain, let $w \in U$ and let $j = 2, \ldots$. Then

$$\mathcal{A}_j^2(U_w) = \mathrm{span}\left\{\psi, \frac{(\overline{z}-\overline{w})^k}{(z-w)^l} : \psi \in \mathcal{A}_j^2(U);\ k = 1, \cdots, j-1;\ l = 1, \cdots k\right\}$$

The Hilbert space $\mathcal{A}_j^2(U_\xi) \ominus \mathcal{A}_j^2(U)$ has finite dimension $j(j-1)/2$.

Corollary ([P2-Sub.])

Let $j \in \mathbb{Z}_\pm$. If w is an isolated point of ∂U then $w \in \partial_r^j U$.

Structure of the j-removal boundary

Theorem ([P2-Sub.])

Let U be a bounded domain and let $j \in \mathbb{Z}_{\pm}$. Then $\partial_e^j U = \sigma_e(T_{z,j})$ and $w \in \partial_e^j U$ if and only if $\operatorname{Im} T_{\phi_w,j}$ is not closed. Moreover,

$$\operatorname{Ind} T_{\phi_w,j} = -\mathrm{codim}\, T_{\phi_w,j} = -j \ , \ w \in U_r^j.$$

Theorem ([P2-Sub.])

Let $U \subset \mathbb{C}$ be a bounded domain, let $j \in \mathbb{Z}_{\pm}$ and let $w \in \partial U$. Then, $w \in \partial_r^j U$ iff there exists $\delta > 0$ such that every function $f \in \mathcal{A}_j^2(U)$ can be extended to a function in the poly-Bergman space over $U \cup D_w(w, \delta)$.

Structure of the j-removal boundary

$w_n \in \partial U$, $n \in \mathbb{N}$ such that $w_n \neq w_m$, $n \neq m$ and $\lim w_n = w$

$$f(z) = \sum_n 2^{-n} \frac{\overline{z} - \overline{w}_n}{z - w_n}$$

Theorem ([P2-Sub.])

Let $U \subset \mathbb{C}$ be a bounded domain. If $j \neq \pm 1$ then the removal boundary $\partial_r^j U$ coincides with the set of all isolated points of ∂U.

Corollary ([P2-Sub.])

Let $j, k \in \mathbb{Z}_{\pm}$. If $j, k = \pm 1$ or $j, k \neq \pm 1$, then $\partial_r^j U = \partial_r^k U$.

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Structure of the j-removal boundary

- if $j = \pm 1$ then $w \in \partial_r^j U$ iff $w \in \partial U$ and there exists $\delta > 0$ such that $c(\partial U \cap \overline{D}(w, \delta) = 0$;
- if $j \neq \pm 1$ then $w \in \partial_r^j U$ iff w is isolated point of ∂U.
- if $j = \pm 1$ then $w \in \partial_r^j U$ can be uncountable;
- if $j \neq \pm 1$ then $w \in \partial_r^j U$ is countable;
- It is easily seen that $(U_r)_r = U_r$;
- The equality $(U_r^j)_r^j = U_r^j$ does not necessarily hold.

Structure of the j-removal boundary

Proposition ([P2-Sub.])

Let U be a bounded domain and let $j \in \mathbb{Z}_\pm$. Thus,

$$\mathcal{A}_j^2(U) = \mathcal{A}_j^2(U_r^j) \oplus E_j^2(U).$$

The space $E_j^2(U)$ is a separable Hilbert space, which is finite-dimensional space if and only $j = \pm 1$ or if the set $\partial_r^j U$ is finite, in which case

$$\dim E_j^2(U) = \#\left(\partial_r^j U\right) |j|(|j| - 1)/2.$$

Corollary ([P2-Sub.])

Let U be a bounded domain and let $j = \pm 2, \pm 3, \dots$. Then $B_{U,j}^\pi = B_{U_r^j,j}^\pi$ and $B_{U,j} = B_{U_r^j,j}$ if and only if $\partial_r^j U$ is finite and $\partial_r^j U = \emptyset$, respectively.

Classical Cantor-Bendixson rank and ∂U_r^j

Consider the transfinite sequence of domains

$$\mathcal{U}_0 := U \quad ; \quad \mathcal{U}_{\alpha+1} := (\mathcal{U}_\alpha)_r^j \quad ; \quad \mathcal{U}_\lambda := \bigcup_{\alpha<\lambda} \mathcal{U}_\alpha, \quad \lambda \text{ is limit ordinal.}$$

$X := \partial U$ and let X' denote the set of cluster points of X.
The Cantor-Bendixson derivatives X^α are defined as follows

$$X^0 := X \quad ; \quad X^{\alpha+1} := (X^\alpha)' \quad ; \quad X^\lambda := \bigcap_{\alpha<\lambda} X^\alpha, \qquad \text{if } \lambda \text{ is limit ordinal.}$$

$X^\alpha = \partial\mathcal{U}_\alpha$. there exists a countable ordinal α_0 such that $X^\alpha = X^{\alpha_0}$, for $\alpha \geq \alpha_0$ The least such ordinal α_0 is denoted by $\rho(X)$ and is said to be the Cantor-Bendixson rank of X. Now we define the domain $U_\infty^j := \mathcal{U}_{\rho(X)}$.

Classical Cantor-Bendixson rank and ∂U_r^j

Theorem

Let U be a bounded domain and let $j \in \mathbb{Z}_{\pm}$. Thus,

$$\mathcal{A}_j^2(U) = \mathcal{A}_j^2(U_\infty^j) \oplus \mathcal{E}_j^2(U).$$

If $j = \pm 1$, then $\mathcal{E}_j^2(U) = \{0\}$. If $j \neq \pm 1$, then $\mathcal{E}_j^2(U)$ is a finite dimensional space if and only if $\partial_r^j U$ is finite, in which case $\dim \mathcal{E}_j^2(U) = \dim E_j^2(U)$. Furthermore, the j-removal boundary of the domain U_∞^j is the empty set.

What can one say about the structure of the spaces $E_j^2(U)$ and $\mathcal{E}_j^2(U)$?

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