Algebras Generated by a Finite Number of Poly and Anti-Poly-Bergman Projections and by the Multiplication Operators by Piecewise Continuous Functions

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1 COEFFICIENTS and PROJECTIONS

* U is an arbitrary domain in $\mathbb{C}$ and Π the upper half plane

* $L^2(U, dA)$ - the Lebesgue measure space of square integrable functions with respect to the area measure in U

* POLY and ANTI-POLY-BERGMAN SPACES

  $\mathcal{A}_n^2(U)$, $\overset{\sim}{\mathcal{A}}{}_n^2(U)$ - the subspaces of $L^2(U, dA)$ respectively defined by n-differentiable functions verifying
$$\frac{\partial^n f}{\partial \overline{z}^n} = 0, \frac{\partial^n f}{\partial z^n} = 0, \quad n \in \mathbb{N},$$
  where the operators $\partial/\partial\overline{z}$ and $\partial/\partial z$ are defined in the usuall way by
$$\partial/\partial\overline{z} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}), \quad \partial/\partial z = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}).$$

* Using *Dzhuraev* results on multiconected bounded domains with smooth boundary we have
$$\sup_{z\in K} |f(z)| \leq C_K \|f\|, f \in \mathcal{A}_n^2(U) \cup \overset{\sim}{\mathcal{A}}{}_n^2(U), K \text{ a compact subset.}$$

* $\mathcal{A}_n^2(U)$ and $\overset{\sim}{\mathcal{A}}{}_n^2(U)$ are functional Hilbert spaces .

* Using *Riez Representation Theorem* we define **Poly-Bergman kernel** and **anti-poly-Bergman kernel** of order n or reproducing kernels for $\mathcal{A}_n^2(U)$ and $\overset{\sim}{\mathcal{A}}{}_n^2(U)$
$$K_{U,n}(z, w) = \overline{k}_{U,n,z}(w)$$
$$\widetilde{K}_{U,n}(z, w) = \overline{\widetilde{k}}_{U,n,z}(w)$$

* POLY and ANTI-POLY-BERGMAN PROJECTIONS $B_{U,n}$ and $\widetilde{B}_{U,n}, n \in \mathbb{N}$ are respectively defined to be the orthogonal projections of $L^2(U, dA)$ onto its closed subspaces $\mathcal{A}_n^2(U)$ and $\widetilde{\mathcal{A}}_n^2(U)$. In the case $n = 1$, i.e. the case of Bergman and anti-Bergman projections, notation will be abbreviated for B_U and $\widetilde{B}_U$ respectively.

* COEFFICIENTS - $PC(\mathcal{L}) \subset L^\infty$ is the class of piecewise continuous functions in $\overline{\dot{\Pi}}$ with discontinuities in the system of curves $\mathcal{L}$. We impose the following conditions on $\mathcal{L}$

  ($\mathcal{L}$1) for each $z \in \Pi \cap \mathcal{L}$ there exist numbers $r_z > 0$ and $n_z \in \mathbb{N}$ such that every disk $D(z, r)$ of radius $r \in (0, r_z)$ centered at z is divided by $\mathcal{L}$ into n_z domains with z as a common limit point;

  ($\mathcal{L}$2) for each $z \in \mathcal{L} \cap \mathbb{R}$ there exists a neighborhood V_z of z such that $V_z \cap \mathcal{L}$ consists of a finite number of Lyapunov arcs having only the point z in common and forming at this point pairwise distinct non-zero angles with $\mathbb{R}_+$ less than π;

  ($\mathcal{L}$3) if $\infty \in \mathcal{L}$, then there exists a neighborhood V_∞ of the point $z = \infty$ such that the set $\{-1/\zeta : \zeta \in V_\infty \cap \mathcal{L}\}$ consists of a finite number of Lyapunov arcs having only the origin in common and forming at this point pairwise distinct non-zero angles with $\mathbb{R}_+$ less that π.

* If $B_1, ..., B_n$ are elements in a given C*-algebra we will write alg $\{B_1, ..., B_n\}$ to denote the C*-algebra generated by the elements $B_1, ..., B_n$, and, if $B_1, ..., B_n \in$

$\mathcal{B}(L^2(U))$ and $\mathcal{A}$ is a closed C*-subalgebra of $L^\infty(U)$, we will write alg $\{B_1, ..., B_n; \mathcal{A}\}$ to denote the C*-algebra generated by the operators $B_1, ..., B_n$, and by multiplication operators by elements in $\mathcal{A}$. Abbreviate alg $\{B_1, ..., B_n; PC(\mathcal{L})\}$ for alg $\{B_1, ..., B_n; \mathcal{L}\}$

We will considerer the C*-algebras of operators

$$\mathfrak{A}_{n,m} := \text{alg}\left\{B_{\Pi,1}, ..., B_{\Pi,n}, \widetilde{B}_{\Pi,1}, ..., \widetilde{B}_{\Pi,m}; \mathcal{L}\right\}$$

2 SOME NECESSARY PROPERTIES OF POLY AND ANTI-POLY BERGMAN PROJECTIONS

Considerer the bi-dimensional Singular Integral Operators on $L^2(U)$, given by

$$\begin{aligned}(S_{U,2n}f)(z) &= \frac{(-1)^n|n|}{\pi}\int_U \frac{e^{i2n\theta}}{|w-z|^2}f(w)dA(w),\\ \theta &= arg(w-z), n \in \mathbb{Z}.\end{aligned}$$

Denote the operator $S_{U,-2}$ by S_U and $S_{\mathbb{C},2n}$ by $S_{2n}, n \in \mathbb{Z}$. One has

$$\begin{aligned}S_{U,2n} &= \chi_U S_{2n}\chi_U,\\ S_{2n} &= F^{-1}\left(\frac{\xi}{\overline{\xi}}\right)^n F, \quad n \in \mathbb{Z}.\end{aligned}$$

$*$ If $U \subset \mathbb{C}$ is a multiconnected domain with smooth boundary then Dzhuraev obtained

$$\begin{aligned}B_{U,n} &= I - S_{U,-2n}S_{U,2n} + T_{U,n}\\ \widetilde{B}_{U,n} &= I - S_{U,2n}S_{U,-2n} + \overline{T}_{U,n},\end{aligned}$$

where $T_{U,n}$ is a compact integral operator, and $\overline{T}_{U,n}f = \overline{T_{U,n}\overline{f}}$.

* In the case of unit disk one has $T_{\mathbb{D},n} = 0$.

* J. Ramirez and I. Spitkovsky obtained the following equalities

  $$B_{\Pi,n} = I - S_{\Pi,-2n}S_{\Pi,2n} \quad \text{and} \quad \widetilde{B}_{\Pi,n} = I - S_{\Pi,2n}S_{\Pi,-2n}.$$

* ANALYTICAL SHIFT OPERATOR - Given two conformally equivalent domains U and V and the analytic bijection

  $$\varphi : U \to V,$$

  then the analytical shift

  $$W_\varphi : L^2(V) \to L^2(U), (W_\varphi f)(z) = f(\varphi(z))\varphi'(z)$$

  is an unitary operator.

Lemma 1 *Suppose that* φ *is a Möbius transform, i.e* $\varphi(z) = (az+b)/(cz+d)$, *with* $\Delta := ad - bc \neq 0$, *and that* U *and* V *are domains of* $\mathbb{C}$ *such* $\varphi(U) = V$. *Thus, with*

$$c_n(z) = (\frac{\overline{cz} + \overline{d}}{cz + d})^n (\frac{\Delta}{\overline{\Delta}})^{n/2}, \quad n \in \mathbb{Z},$$

one has

$$W_\varphi S_{V,2n} W_\varphi^* = c_{n+1} S_{U,2n} c_{n-1}.$$

$\square$

* LIMIT OPERATORS - Consider the dilations $d_y(w) = yw$ $(y > 0,\ w \in \Pi)$. Given $A \in \mathcal{B}(L^2(\Pi))$ such that the strong limits

  $$A_0 := \operatorname*{s-lim}_{y \to +0} (W_{d_y} A W_{d_y}^*), \quad A_\infty := \operatorname*{s-lim}_{y \to +\infty} (W_{d_y} A W_{d_y}^*)$$

exist, we say that A_0 and A_∞ are the *limit operators* of A (with respect to the family $\{W_{d_y}\}_{y>0}$ as $y \to 0$ and $y \to \infty$, respectively).

$$A_0, A_\infty \in \mathcal{B}(L^2(\Pi))$$

$$\begin{aligned}
\|A_0\| &\leq \liminf_{y\to+0} \|W_{d_y} A W_{d_y}^*\| = \|A\|, \\
\|A_\infty\| &\leq \liminf_{y\to+\infty} \|W_{d_y} A W_{d_y}^*\| = \|A\|.
\end{aligned}$$

Proposition 2.1 (Properties) *Suppose* $A, B, A_n \in \mathcal{B}(L^2(\Pi)), n \in \mathbb{N}$, *and* $\tau \in \{0, \infty\}$.

(i) *If* A_τ, B_τ *exist, then for all* $\lambda_1, \lambda_2 \in \mathbb{C}$, *the limit operators* $(\lambda_1 A + \lambda_2 B)_\tau$, $(AB)_\tau$ *also exist and* $(\lambda_1 A + \lambda_2 B)_\tau = \lambda_1 A_\tau + \lambda_2 B_\tau$, $(AB)_\tau = A_\tau B_\tau$;

(ii) *If* A_n *converges uniformly to* A *and* $(A_n)_\tau$ *exist for all* $n \in \mathbb{N}$, *then* A_τ *exists and* $A_\tau = \lim_{n\to\infty} (A_n)_\tau$;

(iii) *if* K *is a compact operator in* $L^2(\Pi)$, *then* $K_\tau = 0$;

(iv) *if* $a \in C(\dot{\overline{\Pi}})$ *and* $a(\tau) = 0$, *then* $(aI)_\tau = 0$.

□

In particular, considering the application

$$\varphi : \Pi \to \mathbb{D}, \varphi(z) = \frac{z-i}{z+i},$$

that has jacobian given by $J_\varphi(z) = |\varphi'(z)|^2$ one has

$$\begin{aligned}
W_\varphi S_{\mathbb{D},2n} W_\varphi^* &= (-1)^n c_{n+1} S_{\Pi,2n} c_{n-1}, \\
W_\varphi S_{\mathbb{D},2n}^* W_\varphi^* &= (-1)^n c_{-n+1} S_{\Pi,2n}^* c_{-n-1},
\end{aligned}$$

Dzhuraev obtained

$$S_{\mathbb{D},-2n} = S_{\mathbb{D}}^n, S_{\mathbb{D},2n} = S_{\mathbb{D}}^{*n}, n \in \mathbb{N}.$$

If one multiplies the above equations, on the right by W_φ and on the left by W_φ^*, one gets

$$c_{-n+1} S_{\Pi,-2n} c_{-n-1} = (S_\Pi c_{-2})^n, \quad c_{n+1} S_{\Pi,2n} c_{n-1} = (c_2 S_\Pi^*)^n.$$

Calculate the limite operators of both sides of the last equalities to obtain

$$\begin{aligned} S_{\Pi,-2n} &= (u_{-n+1} S_{\Pi,-2n} u_{-n-1})_0 = (S_\Pi u_{-2})_0^n = S_\Pi^n, \\ S_{\Pi,2n} &= (u_{n+1} S_{\Pi,2n} u_{n-1})_0 = (u_2 S_\Pi^*)_0^n = S_\Pi^{*n}, \end{aligned}$$

and thus

$$\begin{aligned} B_{\Pi,n} &= I - S_\Pi^n S_\Pi^{*n}, \\ \widetilde{B}_{\Pi,n} &= I - S_\Pi^{*n} S_\Pi^n. \end{aligned}$$

$*$ From *Vasilevski* work we have the decomposition

$$L^2(\Pi) = \left(\bigoplus_{k=0}^{\infty} \mathcal{A}_{(k)}^2(\Pi) \right) \bigoplus \left(\bigoplus_{k=0}^{\infty} \widetilde{\mathcal{A}}_{(k)}^2(\Pi) \right).$$

$*$ In particular

$$B_{\Pi,n} \widetilde{B}_{\Pi,n} = 0.$$

By the above we easily get

Proposition 2

$*$ *The operators*

$$\begin{aligned} S_\Pi^k &: \mathcal{A}_{(j)}^2(\Pi) \to \mathcal{A}_{(j+k)}^2(\Pi) \\ S_\Pi^k &: \widetilde{\mathcal{A}}_{(j+k)}^2(\Pi) \to \widetilde{\mathcal{A}}_{(j)}^2(\Pi) \end{aligned}$$

are unitary.

$$* \; L^2(\Pi) = \left(\bigoplus_{k=0}^{\infty} S_\Pi^k\big(\mathcal{A}^2(\Pi)\big) \right) \oplus \left(\bigoplus_{k=0}^{\infty} \big(S_\Pi^*\big)^k\big(\widetilde{\mathcal{A}}^2(\Pi)\big) \right).$$

□

3 LOCALIZATION

From relations with Singular Integral Operators one has:

Proposition 3 *For any function $a \in C(\dot{\overline{\Pi}})$, the commutators $aB_{\Pi,n} - B_{\Pi,n}aI$ and $a\widetilde{B}_{\Pi,n} - \widetilde{B}_{\Pi,n}aI$ are compact on the space $L^2(\Pi)$.*

□

- $*$ $\mathcal{K}$ the ideal of compact operators on $L^2(\Pi)$
- $*$ $\mathfrak{A}_{n,m}$ contains all compact operators
- $*$ $\mathfrak{A}^\pi_{n,m} := \mathfrak{A}_{n,m}/\mathcal{K}$

Theorem 3.1 [Allan/Douglas] *Let $\mathcal{A}$ be a unital C^*-algebra satisfying the conditions mentioned above.*

(i) *If $a \in \mathcal{A}$, then a is invertible in $\mathcal{A}$ if and only if for every $x \in \mathcal{M}(\mathcal{Z})$ the coset $a_x := \pi_x(a)$ is invertible in $\mathcal{A}_x$.*

(ii) *For every $a \in \mathcal{A}$, the function*

$$\mathcal{M}(\mathcal{Z}) \to \mathbb{R}_+, \quad x \mapsto \|a_x\|$$

is upper semi-continuous. If $a \in \mathcal{A}$ and the coset $a_{x_0} \in \mathcal{A}_{x_0}$ is invertible in $\mathcal{A}_{x_0}$ for some $x_0 \in \mathcal{M}(\mathcal{Z})$, then the cosets a_x are invertible in $\mathcal{A}_x$ for all x in a neighborhood of x_0.

(iii) *For every* $a \in \mathcal{A}$,

$$\|a\| = \max_{x \in \mathcal{M}(\mathcal{Z})} \|a_x\|.$$

$\square$

* $\mathcal{Z}^\pi := \left\{ cI + \mathcal{K} : \ c \in C(\dot{\overline{\Pi}}) \right\} \subset \mathfrak{A}^\pi_{n,m}$
* The ideal generated by the cosets of functions in $C(\dot{\overline{\Pi}})$ and vanishing at z is given by

$$J^\pi_z = \left\{ (cA)^\pi : \ c \in C(\dot{\overline{\Pi}}), \ c(z) = 0, \ A \in \mathfrak{A}_{n,m} \right\}.$$

* $\mathfrak{A}^\pi_{n,m,z} := \mathfrak{A}^\pi_{n,m} / J^\pi_z$
* $A^\pi = A + \mathcal{K}, A \in \mathfrak{A}^\pi_{n,m}, \ A^\pi \stackrel{z}{\sim} B^\pi$ if $A^\pi - B^\pi \in J^\pi_z$, $A, B \in \mathfrak{A}_{n,m}, z \in \mathcal{M}(\mathcal{Z}^\pi) = \dot{\overline{\Pi}}$.

Corollary 3.2 *An operator* $A \in \mathfrak{A}_{n,m}$ *is Fredholm on the space* $L^2(\Pi)$ *if and only if for every* $z \in \dot{\overline{\Pi}}$ *the coset* $A^\pi_z := A^\pi + J^\pi_z$ *is invertible in the quotient algebra* $\mathfrak{A}^\pi_{n,m,z}$.

$\square$

* The projections $B_{U,n}, \widetilde{B}_{U,n}$ are integral operators with kernels respectively given by $K_{U,n}(z,w)$ and $\widetilde{K}_{U,n}(z,w)$. So, we have the formulas

$$\begin{aligned}(B_{U,n}f)(z) &= \int_U K_{U,n}(z,w) f(w) dA(w), \\ (\widetilde{B}_{U,n}f)(z) &= \int_{\mathbb{D}} \widetilde{K}_{U,n}(z,w) f(w) dA(w), f \in L^2.\end{aligned}$$

$*$ If $\{e_{n,k}\}_{k\in\mathbb{N}}$ and $\{\widetilde{e}_{n,k}\}_{k\in\mathbb{N}}$ are respectively Hilbert bases of $\mathcal{A}_n^2(U)$ and $\underset{\sim}{\mathcal{A}}_n^2(U)$ then

$$\begin{aligned} K_{U,n}(z,w) &= \sum_{k=0}^{\infty} e_{n,k}(z)\overline{e}_{n,k}(w), \\ \widetilde{K}_{U,n}(z,w) &= \sum_{k=0}^{\infty} \widetilde{e}_{n,k}(z)\overline{\widetilde{e}}_{n,k}(w), \end{aligned}$$

For fix z, both previous series are uniformly convergent in compact subsets and the same is valid if we change z by w.

Lemma 3.3 $B^{\pi}_{\Pi,n} \overset{z}{\sim} 0$ *and* $\widetilde{B}^{\pi}_{\Pi,n} \overset{z}{\sim} 0, z \in \Pi.$

□

From the last lemma we easily achieve:

Proposition 4 *We have the following assertions:*
i) if $z \in \Pi\backslash\mathfrak{L}$ *then* $\mathfrak{A}^{\pi}_{n,m,z} \simeq \mathbb{C}$;
ii) if $z \in \Pi \cap \mathfrak{L}$ *then* $\mathfrak{A}^{\pi}_{n,m,z} \simeq \mathbb{C}^{n_z}$*, where* $n_z \in \mathbb{N}$ *is given by condition (*$\mathfrak{L}$*1) on* $\mathfrak{L}$*;*
iii) if $z \in \partial\Pi$ *and there is no curve of* $\mathfrak{L}$ *converging to* z *then* $\mathfrak{A}^{\pi}_{n,m,z} \simeq \mathbb{C}^{n+m+1}$.

□

In the last proposition it's useful to considerer true poly and anti-poly Bergman projections. The true poly Bergman projections is the orthogonal projection on the closed subspace of L^2 given by

$$\begin{aligned} \mathcal{A}^2_{(n)}(U) &= \mathcal{A}^2_n(U) \cap [\mathcal{A}^2_{n-1}(U)]^{\perp}, n > 1 \\ \mathcal{A}^2_{(1)}(U) &= \mathcal{A}^2_1(U) := \mathcal{A}^2(U). \end{aligned}$$

For the case of anti-poly-Bergman we have analogous definitions. By induction we easily see that

$$\mathcal{A}^2_n(U) = \oplus_{k=1}^n \mathcal{A}^2_{(k)}(U), \qquad \widetilde{\mathcal{A}}^2_n(U) = \oplus_{k=1}^n \widetilde{\mathcal{A}}^2_{(k)}(U),$$

and by definition we get

$$B_{U,n} = \sum_{k=1}^n B_{U,(k)}, \quad \widetilde{B}_{U,n} = \sum_{k=1}^n \widetilde{B}_{U,(k)}.$$

If ones calculate $B_{U,n} - B_{U,n-1}$ by application of previous formula we get

$$B_{U,(n)} = B_{U,n} - B_{U,n-1}, \quad \widetilde{B}_{U,(n)} = \widetilde{B}_{U,n} - \widetilde{B}_{U,n-1}.$$

4 LOCAL ALGEBRAS AT POINTS OF $\dot{\mathbb{R}} \cap \mathcal{L}$

* If φ is an analytic bijection of Π onto itself, consider the C^*-algebra isomorphism

$$\mu_\varphi : \mathcal{B}(L^2(\Pi)) \to \mathcal{B}(L^2(\Pi)), \quad A \mapsto W_\varphi A W_\varphi^*,$$

* $\mathcal{B}^\pi(L^2(\Pi))$ is the Calkin algebra $\mathcal{B}(L^2(\Pi)) + \mathcal{K}$, and given $A \in \mathcal{B}(L^2(\Pi))$ define $A^\pi \in \mathcal{B}^\pi := A + \mathcal{K}$. The application μ_φ is a quotient C^*-algebra isomorphism

$$\mu_\varphi^\pi : \mathcal{B}^\pi(L^2(\Pi)) \to \mathcal{B}^\pi(L^2(\Pi)), \quad A^\pi \mapsto \big(W_\varphi A W_\varphi^*\big)^\pi.$$

* $\mathfrak{A}_{n,m,\varphi} := \mu_\varphi(\mathfrak{A}_{n,m})$ is a C^*-algebra

* Considerer the bigger C^*-algebra Λ of all operators of local type in $\mathcal{B} = \mathcal{B}(L^2(\Pi))$ and the quotient

$$\Lambda^\pi := \Lambda/\mathcal{K}.$$

* To every point $z \in \dot{\overline{\Pi}}$ assign the closed two-sided ideal
$$\widehat{J}^\pi_z := \Big\{(cA)^\pi : \ c \in C(\dot{\overline{\Pi}}), \ c(z) = 0, \ A \in \Lambda\Big\}.$$
* Introduce the quotient C^*-algebras $\Lambda^\pi_z := \Lambda^\pi / \widehat{J}^\pi_z$.
* Suppose $\varphi(0) = z, z \in \dot{\overline{\Pi}}$ and define the C^*-algebra
$$\widehat{\mathfrak{A}}^\pi_{n,m,\varphi,0} := \Big\{A^\pi + \widehat{J}^\pi_0 : \ A \in \mathfrak{A}_{n,m,\varphi}\Big\}$$

Lemma 4.1 $\mathfrak{A}^\pi_{n,m,z} \cong \widehat{\mathfrak{A}}^\pi_{n,m,\varphi,0}$

□

* If one applies above result in case $\varphi(w) = w + z$, if $z \in \mathbb{R}$ and $\varphi(w) = -w^{-1}$, if $z = \infty$, we always get local algebras at the origin.
* The behavior of singular integral operator $S_{\mathbb{D}}$ after some class of change of variable, is characterized in the following result.
* A homeomorphism between two domains $\alpha : U \to V$ is called *quasiconformal* if α has locally integrable generalized first derivatives satisfying the inequality
$$\Big|\frac{\partial \alpha}{\partial \overline{z}}\Big| \leq k \Big|\frac{\partial \alpha}{\partial z}\Big| \quad \text{where} \quad k = \text{const} < 1.$$
We will use the notation
$$\beta_w := \frac{\partial \alpha}{\partial z}(w), \quad \gamma_w := \frac{\partial \alpha}{\partial \overline{z}}(w).$$
* If α is a diffeomorphism of U onto itself we define the unitary shift operator $\widehat{W}_\alpha \in \mathcal{B}(L^2(U))$ given by
$$\widehat{W}_\alpha f = \big|J_\alpha\big|^{1/2}(f \circ \alpha).$$

Analogously one can define the unitary shift operator $\widehat{W}_{\widehat{\alpha}}$ associated to $\widehat{\alpha}$ a piecewise diffeomorphism bijection of $\dot{\overline{\Pi}}$ onto itself.

Lemma 4.2 *If $\widetilde{\alpha}$ is a quasiconformal diffeomorphism of the closed unit disk $\overline{\mathbb{D}}$ onto itself such that the partial derivatives satisfy the Hölder condition in $\overline{\mathbb{D}}$, then the operators*

$$\widehat{W}_{\widetilde{\alpha}} S_{\mathbb{D}} \widehat{W}_{\widetilde{\alpha}}^{-1} - \frac{J_w}{\beta_w^2} \sum_{n=1}^{\infty} \left(\frac{\gamma_w}{\beta_w}\right)^{n-1} \left(S_{\mathbb{D}}\right)^n + \frac{\overline{\gamma}_w}{\beta_w} I,$$

$$\widehat{W}_{\widetilde{\alpha}} S_{\mathbb{D}}^* \widehat{W}_{\widetilde{\alpha}}^{-1} - \frac{J_w}{\overline{\beta_w^2}} \sum_{n=1}^{\infty} \left(\frac{\overline{\gamma_w}}{\overline{\beta_w}}\right)^{n-1} \left(S_{\mathbb{D}}^*\right)^n + \frac{\gamma_w}{\overline{\beta}_w} I$$

are compact on the space $L^2(\mathbb{D})$.

$\square$

Note that the previous results was obtained jointly with V.A. Mozel.

For $z \in \dot{\mathbb{R}} \cap \mathcal{L}$ choose a neighborhood $V_z \subset \Pi$ satisfying $(\mathfrak{L}2)$, if $z \in \mathbb{R} \cap \mathcal{L}$ or $(\mathfrak{L}3)$ if $z = \infty$. Then $V_z \setminus \mathfrak{L} = \bigcup_{k=1}^{n_z} \Omega_k$ where $n_z - 1$ is the number of curves in $\mathfrak{L}$ with the endpoint z and Ω_k are the connected components of $V_z \setminus \mathfrak{L}$.

* Let $\mathfrak{L}_z, z \in \dot{\overline{\Pi}}$ denote the set of rays in $\dot{\overline{\Pi}}$ outgoing from the origin and being parallel to tangents to the curves of $\mathfrak{L}$ at the point z and $\Pi \setminus \mathfrak{L}_z = \bigcup_{k=1}^{n_z} R_k^z$ where R_k^z are sectors of Π with vertex at the origin

which correspond to the domains Ω_k (the sets Ω_k and R_k^z are numerated counterclockwise).

Lemma 4.3 *There exits $r_0 > 0$ and $\widehat{\alpha}$, a piecewise diffeomorphism bijection of $\dot{\overline{\Pi}}$ onto itself such that $\widehat{\alpha}(z,\overline{z}) = z$ for $z \in \dot{\overline{\Pi}} \setminus \overline{D^+_{r_0}}$, $\widehat{\alpha}|_{\overline{D^+_{r_0}}}$ is a quasiconformal diffeomorphism of $\overline{D^+_{r_0}}$ onto itself whose partial derivatives $\dfrac{\partial\widehat{\alpha}}{\partial z}$, $\dfrac{\partial\widehat{\alpha}}{\partial \overline{z}}$ satisfy the Hölder condition in $\overline{D^+_{r_0}}$, the Jacobian $J_{\widehat{\alpha}}$ is positive and separated from zero, and*

$$\widehat{\alpha}(z,\overline{z}) = z, \quad \frac{\partial\widehat{\alpha}}{\partial z}(z,\overline{z}) = 1, \quad \frac{\partial\widehat{\alpha}}{\partial \overline{z}}(z,\overline{z}) = 0 \quad \textit{for} \quad z \in \mathbb{R}.$$

For such $\widehat{\alpha}$ one has

$$\left(\widehat{W}_{\widehat{\alpha}} S_\Pi \widehat{W}_{\widehat{\alpha}}^{-1} - S_\Pi\right)^\pi \in \widehat{J}_0^\pi, \quad \left(\widehat{W}_{\widehat{\alpha}} S_\Pi^* \widehat{W}_{\widehat{\alpha}}^{-1} - S_\Pi^*\right)^\pi \in \widehat{J}_0^\pi.$$

If condition $(\mathfrak{L}2)$ is fulfilled, then the map $\widehat{\alpha}$ can be chosen such that

$$R_k^z \cap \overline{D^+_{r_0}} = \widehat{\alpha}(t_{-z}(\Omega_k) \cap \overline{D^+_{r_0}}).$$

$\square$

$*$ Consider the "linearization" of $\mathfrak{A}_{n,m}$ at $z \in \dot{\mathbb{R}} \cap \mathcal{L}$

$\mathfrak{A}_{n,m}^z := \text{alg}\left\{B_{\Pi,1}, ..., B_{\Pi,n}, \widetilde{B}_{\Pi,1}, ..., \widetilde{B}_{\Pi,m}; \mathfrak{L}_z\right\} \subset \mathcal{B}(L^2(\Pi))$,

Since $\mathcal{Z}^\pi$ is a central subalgebra of the C^*-algebra

$$\mathfrak{A}_{n,m}^{z,\pi} := \mathfrak{A}_{n,m}^z / \mathcal{K},$$

we can define the C^*-subalgebras of Λ_0^π

$$\begin{aligned} \widehat{\mathfrak{A}}_{n,m,0}^{z,\pi} &:= \left\{A^\pi + \widehat{J}_0^\pi : \ A \in \mathfrak{A}_{n,m}^z\right\}, z \in \mathbb{R} \cap \mathcal{L}, \\ \widehat{\mathfrak{A}}_{n,m,\infty}^{\infty,\pi} &:= \left\{A^\pi + \widehat{J}_\infty^\pi : \ A \in \mathfrak{A}_{n,m}^\infty\right\}. \end{aligned}$$

Proposition 4.4
I) $\mathfrak{A}^{\pi}_{n,m,z} \cong \widehat{\mathfrak{A}}^{z,\pi}_{n,m,0}$ *by an *-isomorphism given by*

$$\begin{array}{l}\nu^{\pi}_{z}(B^{\pi}_{\Pi,j} + J^{\pi}_{z}) = B^{\pi}_{\Pi,j} + \widehat{J}^{\pi}_{0}, \quad j = 1, ..., n \\ \nu^{\pi}_{z}(\widetilde{B}^{\pi}_{\Pi,j} + J^{\pi}_{z}) = \widetilde{B}^{\pi}_{\Pi,j} + \widehat{J}^{\pi}_{0}, \quad j = 1, ..., m \\ \nu^{\pi}_{z}((aI)^{\pi} + J^{\pi}_{z}) = (a_z I)^{\pi} + \widehat{J}^{\pi}_{0}, a \in PC(\mathcal{L})\end{array}$$

where

$$a_z = \sum_{k=1}^{n_z} c_k \chi_{R^z_k}, \quad c_k = \lim_{\zeta \to z,\, \zeta \in D_k} a(\zeta) \ \ (k = 1, 2, \ldots, n_z).$$

II) $\mathfrak{A}^{\pi}_{n,m,\infty} \cong \widehat{\mathfrak{A}}^{\infty,\pi}_{n,m,\infty}$ *by an *-isomorphism given by*

$$\begin{array}{rcl}\nu^{\pi}_{n,m,\infty}(B^{\pi}_{\Pi,j} + J^{\pi}_{\infty}) & = & B^{\pi}_{\Pi,j} + \widehat{J}^{\pi}_{\infty}, j = 1, ..., n, \\ \nu^{\pi}_{n,m,\infty}(\widetilde{B}^{\pi}_{\Pi,j} + J^{\pi}_{\infty}) & = & \widetilde{B}^{\pi}_{\Pi,j} + \widehat{J}^{\pi}_{\infty}, j = 1, ..., m, \\ \nu^{\pi}_{n,m,\infty}((aI)^{\pi} + J^{\pi}_{\infty}) & = & (a_{\infty} I)^{\pi} + \widehat{J}^{\pi}_{\infty},\end{array}$$

where

$$a_{\infty} = \sum_{k=1}^{n_{\infty}} c_k \chi_{R^{\infty}_k}, \quad c_k = \lim_{\zeta \to \infty,\, \zeta \in \Omega_k} a(\zeta) \ \ (k = 1, 2, \ldots, n_{\infty}).$$

$\square$

$*$ Introduce the operator C^*-algebras

$$\mathcal{O}_{n,m,\omega} = \operatorname{alg}\left\{B_{\Pi,1}, ..., B_{\Pi,n}, \widetilde{B}_{\Pi,1}, ..., \widetilde{B}_{\Pi,m}, \chi_{R_1} I \ldots \chi_{R_n} I\right\}$$

where

$$\omega = (\omega_1, \ldots, \omega_{n-1}), \omega_k = e^{i\theta_k}, 0 < \theta_1 < \ldots < \theta_{n-1}$$

and the sectors R_k are defined by

$$\{re^{i\theta} : \theta_{k-1} < \theta < \theta_k\}, \theta_0 = 0, \theta_n = \pi.$$

* Because operators in $\mathcal{O}_{n,m,\omega}$ are of homogeneous type and by means of limit operators we prove that the applications, in case $z \in \mathbb{R} \cap \mathfrak{L}$, given by

$$\Psi_0 : \mathcal{O}_{n,m,\omega} \to \widehat{\mathfrak{A}}^{z,\pi}_{n,m,0}, \quad A \mapsto A^\pi + \widehat{J}^\pi_0,$$

  and if $\infty \in \mathfrak{L}$,

$$\Psi_\infty : \mathcal{O}_{n,m,\omega} \to \widehat{\mathfrak{A}}^{\infty,\pi}_{n,m,\infty}, \quad A \mapsto A^\pi + \widehat{J}^\pi_\infty.$$

  are *-isomorphisms. So, we have:

Theorem 4.5 *If $z \in \dot{\mathbb{R}} \cap \mathfrak{L}$, then the C^*-algebra $\mathfrak{A}^\pi_{n,m,z}$ is isomorphic to the C^*-algebra $\mathcal{O}_{n,m,\omega}$, with $n = n_z$ and $R_k = R^z_k$ $(k = 1, \ldots, n)$.*

□

Observe that the previous result could be obtained by already old arguments of Simonenko and Chin Min, on local type homogeneous operators and applied to ours results of local replacement of curves by tangent lines.

5 SYMBOL CALCULUS FOR $\mathfrak{A}_{n,m}$

* For $z \in \mathbb{R} \cup \{\infty\}$ we have

  $\mathfrak{A}^\pi_{n,m,z} \cong \text{alg}\left\{B_{\Pi,1}, ..., B_{\Pi,n}, \widetilde{B}_{\Pi,1}, ..., \widetilde{B}_{\Pi,m}, \chi_{R_1} I \ldots \chi_{R_k} I\right\}$

  where R_l $(l = 1, \ldots, k)$ are sectors between $k-1$ rays in Π outgoing from the origin.

* $\mathcal{H}_\infty$ denote the C^*-algebra of essentially bounded homogeneous functions on $\mathbb{R}^2$.

* By Dzhuraev formulas $\mathcal{O}_{n,m,\omega}$ is the C^*-subalgebra of $\mathcal{B}(L^2(\mathbb{R}^2))$ generated by the two-dimensional singular integral operator S and by the operators of multiplication by functions in $\mathcal{H}_\infty$. Since $\overline{\xi}/\xi \in \mathcal{H}_\infty$ thus $\mathcal{O}_{n,m,\omega}$ is a closed *-subalgebra of

$$\mathcal{R} := \left\{a(x)I \quad , F^{-1}b(\xi)F \quad : a, b \in \mathcal{H}_\infty\right\}$$

* Let $\mathbb{T}$ be the unit circle and Ω be the C^*-algebra of bounded norm-continuous operator-valued functions

$$U : \mathbb{R} \to \mathcal{B}(L^2(\mathbb{T})), \quad \lambda \mapsto U(\lambda) \in \Omega_\lambda,$$

with the norm $\|U\| = \sup_{\lambda\in\mathbb{R}} \|U(\lambda)\|$.

* For $u \in C^\infty(\mathbb{T})$ and $\lambda \in \mathbb{C}$ such that $\operatorname{Im}\lambda > 0$ and $\lambda \neq ik,\ k = 1, 2, \ldots$, define

$$(E(\lambda)u)(\tau) = \gamma(\lambda) \int_{\mathbb{T}} (-\tau\cdot\omega + i0)^{-i\lambda-1} u(\omega) d\omega, \quad \tau \in \mathbb{T},$$

where $d\omega$ is the length measure on $\mathbb{T}$,

$$\gamma(\lambda) = \frac{1}{2\pi}\, \Gamma(1 + i\lambda)\, e^{\pi(i-\lambda)/2}$$

and the expression $(t \pm i0)^\mu$ for $t \in \mathbb{R}$ and $\mu \in \mathbb{C}$ is defined by:

$$(t \pm i0)^\mu = \lim_{y\to 0^+} (t \pm iy)^\mu.$$

For $\operatorname{Im}\lambda \leq 0$ the integral in $E(\lambda)$ is understood in the sense of analytic continuation, since for every $u \in C^\infty(\mathbb{T})$ the function $\lambda \mapsto E(\lambda)u(t)$ admits analytic continuation in the complex plane minus the poles $\lambda = ik$ $(k = 1, 2, \ldots)$ of the Γ-function. Using Plamenevski decomposition

$$F = (M^{-1} \otimes I)(V \otimes I)(I \otimes_\lambda E(\lambda))(M \otimes I),$$

and the relations

$$\begin{aligned}(M \otimes I)(a(x)I)(M^{-1} \otimes I) &= I \otimes a(t)I, \\ (M \otimes I)(F^{-1}b(\xi)F)(M^{-1} \otimes I) &= I \otimes_\lambda (E(\lambda)^{-1}b(w)E(\lambda)),\end{aligned}$$

its possible to construct an *-isomorphism between the C^*-algebra $\mathcal{R}$ and a C^*-subalgebra of Ω. The isomorphism is given on the generators of $\mathcal{R}$ by

$$\begin{aligned}a(x)I &\mapsto (a(t)I)_{\lambda\in\mathbb{R}}, \\ F^{-1}b(\xi)F &\mapsto (E(\lambda)^{-1}b(w)E(\lambda))_{\lambda\in\mathbb{R}}.\end{aligned}$$

So we have the relations

$$\begin{aligned}(M \otimes I)B_{\Pi,j}(M^{-1} \otimes I) = I \otimes_\lambda B_j(\lambda), \\ (M \otimes I)\widetilde{B}_{\Pi,j}(M^{-1} \otimes I) = I \otimes_\lambda \widetilde{B}_j(\lambda)\end{aligned}$$

where the operators $B_j(\lambda), \widetilde{B}_j(\lambda) \in \mathcal{B}(L^2(\mathbb{T})), j \in \mathbb{N}^+$ are given by

$$\begin{aligned}B_j(\lambda) &= \chi_+\Big(I - E(\lambda)^{-1}\big(\frac{\overline{w}}{w}\big)^j E(\lambda)\chi_+ E(\lambda)^{-1}\big(\frac{w}{\overline{w}}\big)^j E(\lambda)\Big)\chi_+ I, \\ \widetilde{B}_j(\lambda) &= \chi_+\Big(I - E(\lambda)^{-1}\big(\frac{w}{\overline{w}}\big)^j E(\lambda)\chi_+ E(\lambda)^{-1}\big(\frac{\overline{w}}{w}\big)^j E(\lambda)\Big)\chi_+ I,\end{aligned}$$

and χ_+ is the characteristic function of the upper half-circle $\mathbb{T}_+$.

Further, the sectors R_l of the partition of Π by the rays $\gamma_1, \ldots, \gamma_{k-1}$ outgoing from the origin have the form

$$R_l = \big\{z \in \Pi : \ \theta_{l-1} < \arg z < \theta_l\big\}, \quad l = 1, \ldots, k,$$

where $\theta_0 = 0 < \theta_1 < \ldots < \theta_{k-1} < \theta_k = \pi$.

Theorem 5.1 *The C^*-algebra $\mathcal{O}_{n,m,\omega}$ is isomorphic to the C^*-subalgebra of $\mathcal{A}$ generated by the norm-continuous operator functions*

$$\lambda \mapsto B_i(\lambda), \quad \lambda \mapsto \widetilde{B}_j(\lambda), \quad \lambda \mapsto \chi_l I$$

where

$$i = 1, \ldots, n; j = 1, \ldots, m; l = 1, \ldots, k.$$

$\square$

Next aim is to study the C$^*-$algebra $\mathcal{A}(\lambda)$ defined by

$$\text{alg}\left\{B_i(\lambda), \widetilde{B}_j(\lambda), \chi_{R_l} I : i = 1, ..., n; j = 1, ..., m; l = 1, \ldots, k\right\}.$$

The following result on algebras generated by orthogonal projections with relations will do to our purposes. Note that the same result is valid if one substitute the one-dimensional projection P_k by arbitrary finite dimensional ones.

Theorem 5 *Let H be a Hilbert space and $Q_i, P_l (i = 1, ..., \eta_1\ , l = 1, ..., \eta_2)$ be projections in $\mathcal{B}(H)$ verifying the conditions:*

(i) $Q_i Q_j = \delta_{ij} Q_i \ (i, j = 1, ...\eta_1)$;

(ii) $\sum_{i=1}^{\eta_1} Q_i = I$;

(iii) $P_l \ (l = 1, ..., \eta_2)$ *are one-dimensional projections ;*

(iv) $P_k P_l = \delta_{kl} P_k$.

(v) $\cap_{l=1}^{\eta_2} (ImP_l)^\perp \cap ImQ_i \neq \{0\}, i = 1, ..., \eta_1$;

(vi) if $v_1, ..., v_{\eta_2}$ are norm one generators of $ImP_1, ..., ImP_{\eta_2}$ respectively, then the vectors $Q_i v_1, ..., Q_i v_{\eta_2}$ are linearly independent for every $i = 1, ..., \eta_1$.

Let $\mathcal{A}$ be the C^*-subalgebra of $\mathcal{B}(H)$ generated by the projections Q_i $(i = 1, ..., \eta_1)$ and P_l $(l = 1, ..., \eta_2)$, let $S = diag\,\{S_i\}_{i=1}^{\eta_1}$, where S_i are invertible matrices in $\mathbb{C}^{\eta_2 \times \eta_2}$ that transform the system

$$\nu_i = \{Q_i v_1, ..., Q_i v_{\eta_2}\}$$

of linearly independent vectors in H into orthonormal systems, and let $\mathfrak{S}$ be the C^*-subalgebra of $\mathbb{C}^{\eta_1\eta_2 \times \eta_1\eta_2}$ generated by the matrices

$$\widetilde{Q}_i = diag\,\{\delta_{i,j} I_{\eta_2}\}_{j=1}^{\eta_1} \text{ and } S\widetilde{P}_l S^{-1} \ (i = 1, ..., \eta_1;\ l = 1, ..., \eta_2),$$

where

$$\widetilde{P}_l = \left(diag\,\{\delta_{l,j}\}_{j=1}^{\eta_2} E_{s,i}\right)_{s,i=1}^{\eta_1},\ l = 1, ..., \eta_2,$$

and

$$E_{s,i} = \begin{bmatrix} \langle Q_i v_1, Q_i v_1\rangle & \langle Q_i v_2, Q_i v_1\rangle & ... & \langle Q_i v_{\eta_2}, Q_i v_1\rangle \\ \langle Q_i v_1, Q_i v_2\rangle & \langle Q_i v_2, Q_i v_2\rangle & ... & \langle Q_i v_{\eta_2}, Q_i v_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle Q_i v_1, Q_i v_{\eta_2}\rangle & \langle Q_i v_2, Q_i v_{\eta_2}\rangle & ... & \langle Q_i v_{\eta_2}, Q_i v_{\eta_2}\rangle \end{bmatrix} \in \mathbb{C}^{\eta_2 \times \eta_2}.$$

Then the map σ, defined on generators of $\mathcal{A}$ by

$$\begin{aligned} Q_i &\mapsto (\delta_{i,1} \oplus \delta_{i,2} \oplus ... \oplus \delta_{i,\eta_1}) \oplus \widetilde{Q}_i \ (i = 1, ..., \eta_1), \\ P_l &\mapsto (0 \oplus 0 \oplus ... \oplus 0) \oplus S\widetilde{P}_l S^{-1} \ (l = 1, ..., \eta_2), \end{aligned}$$

extends to a C^*-algebra isomorphism of the C^*-algebra $\mathcal{A}$ onto the C^*-algebra $\mathbb{C}^{\eta_1} \oplus \mathfrak{S}$

$\square$

Lemma 5.2 *Let $z \in \Pi$. For the functions*

$$f_m(z) = \frac{(z-i)^m}{(z+i)^{m+2}} \quad (m = 0, 1, 2, \ldots)$$

and every $k = 1, 2, \ldots$, we have the formulas

$$(S_\Pi^k f_m)(z) = \frac{(-1)^k}{(k-1)!} \frac{\partial^{k-1}}{\partial w^{k-1}} \left[\frac{(w-\overline{z})^{k-1}(w-i)^m}{(w+i)^{m+2}} \right]\Bigg|_{w=z} +$$

$$\frac{(-1)^k}{(m+1)!} \frac{(z-i)^m}{(z+i)^{m+2}} \frac{\partial^{m+1}}{\partial w^{m+1}} \left[w^{k-1}(1-w)^2 \Big(w - \frac{\overline{z}-i}{z-i} \Big)^m \right]\Bigg|_{w=\frac{\overline{z}+i}{z+i}}.$$

where $z \in \Pi$. In particular, for $m = 0$,

$$(S_\Pi^k f_0)(z) = \frac{(-1)^k}{(z+i)^2} \left[(k+1) \Big(\frac{\overline{z}+i}{z+i} \Big)^k - k \Big(\frac{\overline{z}+i}{z+i} \Big)^{k-1} \right].$$

$\square$

Proposition 5.3 $\operatorname{Im} B_{(k)}(\lambda)$ *and* $\operatorname{Im} \widetilde{B}_{(k)}(\lambda)$ *are one-dimensional with norm one generators respectively given by*

$$g_{\lambda,k} = S_\Pi^{k-1}(\lambda) g_{\lambda,1} \quad \textit{and} \quad \widetilde{g}_{\lambda,k} = (S_\Pi^*)^{k-1}(\lambda) \widetilde{g}_{\lambda,1}.$$

Also

$$\begin{aligned} g_{\lambda,k}(t) &= (-1)^{k-1} G(\lambda)\, t^{i\lambda-1} F(1-k, 1-i\lambda; 1; 1-t^{-2}), \\ \widetilde{g}_{\lambda,k}(t) &= (-1)^{k-1} G(-\lambda)\, t^{-i\lambda+1} F(1-k, 1+i\lambda; 1; 1-t^2), \end{aligned}$$

where $t \in \mathbb{T}_+$,

$$G(\lambda) = \begin{cases} \Big(\dfrac{2\lambda}{1-e^{-2\pi\lambda}} \Big)^{1/2} & \textit{if} \quad \lambda \in \mathbb{R} \setminus \{0\}, \\ \pi^{-1/2} & \textit{if} \quad \lambda = 0, \end{cases}$$

and $F(-m, b; c; z)$ is the $(2,1)$-hypergeometric function given for $m = 0, 1, 2, \ldots$ and $b, c, z \in \mathbb{C}$ by

$$F(-m, b; c; z) = \sum_{n=0}^{m} \frac{(-m)_n (b)_n}{(c)_n n!} z^n.$$

with

$$(a)_n = a(a+1)\dots(a+n+1)$$

$\square$

Theorem 5.4 *If $z \in \dot{\mathbb{R}} \cap \mathfrak{L}$ is a common endpoint of $n_z - 1$ arcs of $\mathfrak{L}$, then the local C^*-algebra $\mathfrak{A}^\pi_{n,m,z}$ is isomorphic to a C^*-subalgebra $\mathbb{C}^k \oplus \mathfrak{C}_z$ of $\mathbb{C}^k \oplus C_b(\mathbb{R}, \mathbb{C}^{k(n+m)\times k(n+m)})$, where $k = n_z$. The isomorphism is given by*

$$\begin{aligned}
B_{\Pi,(j)} &\mapsto \big(0,\dots,0\big) \oplus \big(\lambda \mapsto M_{n,m}(\lambda, j)\big), \quad j = 1,2,\dots,n,\\
\widetilde{B}_{\Pi,(l)} &\mapsto \big(0,\dots,0\big) \oplus \big(\lambda \mapsto \widetilde{M}_{n,m}(\lambda, l)\big), \quad l = 1,2,\dots,m,\\
aI &\mapsto \big(a_1(z),\dots,a_k(z)\big) \oplus \Big(\lambda \mapsto \mathrm{diag}\big\{a_j(z) I_{(n+m)}\big\}_{j=1}^{k}\Big),
\end{aligned}$$

where $a_l(z)$ ($l = 1,2,\dots,k$) is the limit of the function $a \in PC(\mathfrak{L})$ at the point z in the sector R_l, and the matrix functions $M_{n,m}(\cdot, j)$, $\widetilde{M}_{n,m}(\cdot, l) \in C_b\big(\mathbb{R}, \mathbb{C}^{[k(n+m)]_z,[k(n+m)]_z}\big)$ are defined by the mentioned above theorem on algebras generated by orthogonal projections.

$\square$

Theorem 5.5 *The C^*-algebra*

$$\mathfrak{A}^\pi_{n,m} = \mathrm{alg}\left\{B_{\Pi,(1)},\dots,B_{\Pi,(n)}, \widetilde{B}_{\Pi,(1)},\dots,\widetilde{B}_{\Pi,(m)}; \mathfrak{L}\right\}$$

is isomorphic to the C^*-subalgebra $\Phi(\mathfrak{A}_{n,m})$ of the C^*-algebra

$$\Phi_{\mathfrak{A}_{n,m}} := \left(\bigoplus_{z\in\dot{\overline{\Pi}}} \mathbb{C}^{n_z}\right) \oplus \left(\bigoplus_{z\in\dot{\mathbb{R}}\setminus\mathfrak{L}} \mathbb{C}^{n+m}\right) \oplus \left(\bigoplus_{z\in\dot{\mathbb{R}}\cap\mathfrak{L}} \mathfrak{C}_z\right),$$

and the isomorphism $\Phi : \mathfrak{A}^{\pi}_{n,m} \to \Phi(\mathfrak{A}^{\pi}_{n,m})$ *is given by*

$$\begin{aligned}
\Phi(B^{\pi}_{\Pi,(j)}) &:= \left(\bigoplus_{z\in\dot{\overline{\Pi}}} (0,\dots,0)\right) \oplus \left(\bigoplus_{z\in\dot{\mathbb{R}}\setminus\mathfrak{L}} \widetilde{e}_j\right) \\
&\oplus \left(\bigoplus\nolimits_{z\in\dot{\mathbb{R}}\cap\mathfrak{L}} \Big(\lambda\mapsto M_{n,m}(\lambda,j)\Big)\right), \\
\Phi(\widetilde{B}^{\pi}_{\Pi,(l)}) &:= \left(\bigoplus_{z\in\dot{\overline{\Pi}}} (0,\dots,0)\right) \oplus \left(\bigoplus_{z\in\dot{\mathbb{R}}\setminus\mathfrak{L}} \widetilde{e}_{n+l}\right) \\
&\oplus \left(\bigoplus\nolimits_{z\in\dot{\mathbb{R}}\cap\mathfrak{L}} \Big(\lambda\mapsto \widetilde{M}_{n,m}(\lambda,l)\Big)\right),
\end{aligned}$$

$$\begin{aligned}
\Phi((aI)^{\pi}) &:= \left(\bigoplus_{z\in\dot{\overline{\Pi}}} \big(a_1(z),\dots,a_{n_z}(z)\big)\right) \oplus \left(\bigoplus_{z\in\dot{\mathbb{R}}\setminus\mathfrak{L}} \big(a(z),a(z)\big)\right) \\
&\oplus \left(\bigoplus_{z\in\dot{\mathbb{R}}\cap\mathfrak{L}} \Big(\lambda\mapsto \mathrm{diag}\big\{a_j(z)I_{n+m}\big\}_{j=1}^{n_z}\Big)\right),
\end{aligned}$$

where $\widetilde{e}_k = (0,\dots,0,1,0,\dots,0) \in \mathbb{C}^{n+m}$ *with the unit at the* k*-entry,* $k = 1,2,\dots,n+m$*,* $\mathfrak{C}_z$ *is the* C^**-subalgebra of* $C_b\big(\mathbb{R},\mathbb{C}^{n_z(n+m)\times n_z(n+m)}\big)$ *determined in theorem* C^**-algebras generated by orthogonal projections with relations,* n_z *is the number of connected components* D_k *of the set* $(V_z \cap \Pi)\setminus\mathfrak{L}$ *for a sufficiently small neighborhood* V_z *of a point* $z \in \dot{\overline{\Pi}}$*, and*

$$a_j(z) := \lim_{\zeta\to z,\,\zeta\in D_j} a(\zeta) \ \ (j = 1,2,\dots,n_z).$$

An operator $A \in \mathfrak{A}_{n,m}$ *is Fredholm on the space* $L^2(\Pi)$ *if and only if its symbol* $\Phi(A)$ *is invertible in the* C^**-algebra*

$\Phi_{\mathfrak{A}_{n,m}}$, *that is, if for the corresponding block (j,j)-entries we have*

$$\begin{aligned}
\big([\Phi(A)](z)\big)_j \neq 0 & \quad \textit{for all } z \in \dot{\overline{\Pi}} \ \textit{ and all } j = 1, 2, \ldots, n_z; \\
\big([\Phi(A)](z)\big)_j \neq 0 & \quad \textit{for all } z \in \dot{\mathbb{R}} \setminus \mathfrak{L} \textit{ and all } j = 1, 2; \\
\det\big([\Phi(A)](z)\big)(\lambda) \neq 0 & \quad \textit{for all } z \in \dot{\mathbb{R}} \cap \mathfrak{L} \textit{ and all } \lambda \in \overline{\mathbb{R}}.
\end{aligned}$$

$\square$

If one consider only Bergman and anti-Bergman projections we obtain simpler matrices. Define

$$\begin{aligned}
\alpha_{11}^l(\lambda) &= \langle \chi_l g_\lambda, \chi_l g_\lambda \rangle, \alpha_{12}^l(\lambda) = \langle \chi_l g_\lambda, \chi_l \widetilde{g}_\lambda \rangle, \\
\alpha_{21}^l(\lambda) &= \overline{\alpha_{12}^l(\lambda)}, \quad \alpha_{22}^l(\lambda) = \langle \chi_l \widetilde{g}_\lambda, \chi_l \widetilde{g}_\lambda \rangle.
\end{aligned}$$

Theorem 5.6 *$M(\cdot)$, $\widetilde{M}(\cdot) \in C(\overline{\mathbb{R}}, \mathbb{C}^{2n\times 2n})$ and*

$$\begin{aligned}
\alpha_{11}^l(\lambda) &= \frac{e^{-2\lambda\theta_l} - e^{-2\lambda\theta_{l-1}}}{e^{-2\lambda\pi} - 1}, \alpha_{12}^l(\lambda) = \frac{\lambda}{\sinh(\pi\lambda)} \frac{e^{-2i\theta_l} - e^{-2i\theta_{l-1}}}{-2i}, \\
\alpha_{21}^l(\lambda) &= \frac{\lambda}{\sinh(\pi\lambda)} \frac{e^{2i\theta_l} - e^{2i\theta_{l-1}}}{2i}, \alpha_{22}^l(\lambda) = \frac{e^{2\lambda\theta_l} - e^{2\lambda\theta_{l-1}}}{e^{2\lambda\pi} - 1},
\end{aligned}$$

with $\lambda \in \mathbb{R} \setminus 0$. For $\lambda = 0$,

$$\begin{aligned}
\alpha_{11}^l(0) &= \frac{\theta_l - \theta_{l-1}}{\pi}, \alpha_{12}^l(0) = \frac{e^{-2i\theta_l} - e^{-2i\theta_{l-1}}}{-2\pi i}, \\
\alpha_{21}^l(0) &= \frac{e^{2i\theta_l} - e^{2i\theta_{l-1}}}{2\pi i}, \alpha_{22}^l(0) = \frac{\theta_l - \theta_{l-1}}{\pi}.
\end{aligned}$$

$$\big[M(\lambda)\big]_{s,j} = \begin{cases} \sqrt{\alpha_{11}^{l}(\lambda)\alpha_{11}^{r}(\lambda)} & ,j = 2l-1,\ s = 2r-1, \\ 0 & ,\text{otherwise} \end{cases}$$

$$\big[\widetilde{M}(\lambda)\big]_{s,j} = \begin{cases} \alpha_{12}^{l}(\lambda)\overline{\alpha_{12}^{r}(\lambda)}/\sqrt{\alpha_{11}^{l}(\lambda)\alpha_{11}^{r}(\lambda)} & ,\ \ j = 2l-1,\ s = 2r-1, \\ \big(\alpha_{12}^{l}(\lambda)/\sqrt{\alpha_{11}^{l}(\lambda)}\big)\|f_{2r}(\lambda)\| & ,\ \ j = 2l-1,\ s = 2r, \\ \big(\overline{\alpha_{12}^{r}(\lambda)}/\sqrt{\alpha_{11}^{r}(\lambda)}\big)\|f_{2l}(\lambda)\| & ,\ \ j = 2l,\ s = 2r-1, \\ \|f_{2l}(\lambda)\|\,\|f_{2r}(\lambda)\| & ,\ \ j = 2l,\ s = 2r, \end{cases}$$

□

6 Isomorphism with analog C*-algebra of unit disk case

Considerer the special case of Bergman and anti-Bergman projection.

Theorem 6.1 *Consider the application*

$$\varphi : \Pi \to \mathbb{D}, \varphi(z) = \frac{z-i}{z+i}.$$

Thus

$$\operatorname{alg}\big\{B_{\mathbb{D}}, \widetilde{B}_{\mathbb{D}}; \mathfrak{L}\big\}/\mathcal{K} \cong \operatorname{alg}\big\{B_{\Pi}, \widetilde{B}_{\Pi}; \varphi^{-1}(\mathfrak{L})\big\}/\mathcal{K},$$

by a C^*-algebra isomorphism acting on generators by the rule

$$B_{\mathbb{D}}^{\pi} \mapsto B_{\Pi}^{\pi}, \quad \widetilde{B}_{\mathbb{D}}^{\pi} \mapsto \widetilde{B}_{\Pi}^{\pi}, \quad \textit{and} \quad (aI)^{\pi} \mapsto \big((a\circ\varphi)I\big)^{\pi}$$

□

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