FERMAT AND THE NUMBER OF FIXED POINTS OF PERIODIC FLOWS

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ABSTRACT. We obtain lower bounds for the number of fixed points of a circle action on a compact almost complex manifold M^{2n} with nonempty fixed point set, provided the Chern number $c_1c_{n-1}[M]$ vanishes. The proofs combine techniques originating in equivariant K-theory with celebrated number theory results on polygonal numbers, stated by Fermat. These lower bounds depend only on n and, in some cases, are better than existing bounds. If the fixed point set is discrete, we also prove divisibility properties for the number of fixed points, improving similar statements obtained by Hirzebruch in 1999. Our results apply, for example, to a class of manifolds which do not support any Hamiltonian circle action, namely those for which the first Chern class is torsion. This includes, for instance, all symplectic Calabi Yau manifolds.

1. INTRODUCTION

Finding the minimal positive number of fixed points of a circle action on a compact almost complex manifold is, in general, an unsolved problem in equivariant geometry[1]. It is also connected with the question of whether there exists a symplectic non-Hamiltonian S^1-action on a compact symplectic manifold with nonempty and discrete fixed point set. Much of the activity concerning this problem originated in a result by T. Frankel [Fr59] for Kähler manifolds, in which he showed that a Kähler S^1-action on a compact Kähler manifold M is Hamiltonian if and only if it has fixed points. In this case, this implies that the action has at least $\frac{1}{2}\dim M+1$ fixed points, since they coincide with the critical points of the corresponding Hamiltonian function (a perfect Morse-Bott function). For the larger class of unitary[2] manifolds (see Remark 2.4), a conjecture in this direction was made by Kosniowski [K79] in 1979 and is still open in general.

Conjecture 1 (Kosniowski '79)**.** *There exists a linear function $f(\cdot)$ such that, for every $2n$-dimensional compact unitary S^1-manifold M with isolated fixed points which is not equivariantly unitary cobordant with the empty set, the number of fixed points is greater than $f(n)$. In particular, $f(x) = x/2$ should satisfy this condition, implying that number of fixed points is expected to be at least $\lfloor n/2\rfloor + 1$.*

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[1] In the terminology of dynamical systems, circle actions are regarded as *periodic flows* and the fixed points of the action correspond to the equilibrium points of the flow.

[2] A unitary (or weakly almost complex) manifold is a smooth manifold endowed with a fixed complex structure on the stable tangent bundle of M. If S^1 acts on a unitary manifold M preserving the given complex structure on the stable tangent bundle, then M is called a unitary S^1-manifold.

Several other lower bounds were obtained in the literature, by retrieving information from a nonvanishing Chern number of the manifold. For example, Hattori [Ha85] showed that a unitary S^1-manifold for which $c_1^n[M]$ does not vanish (implying that c_1 is not torsion), must have at least $n+1$ fixed points (see Theorem 2.3). Since then many other results followed [PT11, LL10, CKP12, J14]; we review these in Section 2.

It is therefore natural to study the situation in which the first Chern class is torsion. In the symplectic case this condition automatically implies that the manifold cannot support any Hamiltonian circle action (see Proposition 2.15), and is, for instance, satisfied by the important family of symplectic Calabi-Yau manifolds, for which we have $c_1 = 0$. Since the existence of a symplectic manifold admitting a non-Hamiltonian circle action with discrete fixed point set is still unknown, and there is very little information on the required topological properties of the possible candidates, our results shed some light on this problem.

In this note we make the weaker assumption that the Chern number $c_1c_{n-1}[M]$ of an almost complex S^1-manifold M is zero (cf. Section 2.3). The choice of this Chern number is motivated by its expression in terms of numbers of fixed points obtained in [GS12, Theorem 1.2] (see Theorem 4.1). Interestingly, if M is a 6-dimensional compact symplectic manifold satisfying $c_1c_2[M] = 0$, then M does not admit any Hamiltonian S^1-action and, if $c_1c_2[M] \neq 0$, then all symplectic circle actions are Hamiltonian (cf. Proposition 2.14).

Using the expression for $c_1c_{n-1}[M]$ given in Theorem 4.1, we obtain *divisibility results* for the number of fixed points $|M^{S^1}|$ of a circle action on an almost complex manifold with $c_1c_{n-1}[M] = 0$ when the fixed points set is nonempty and discrete. Our methods do not generalize to unitary S^1-manifolds (cf. Remark 4.3).

Theorem A. *Let (M, J) be a $2n$-dimensional compact connected almost complex manifold equipped with a J-preserving S^1-action with nonempty, discrete fixed point set M^{S^1} and such that $c_1c_{n-1}[M] = 0$. Let m be such that $n = 2m$ ($m \geqslant 1$) when n is even, and $n = 2m+3$ ($m \geqslant 0$) when n is odd. If $r = \gcd(m, 12)$, then*

$$|M^{S^1}| \equiv 0 \pmod{\frac{12}{r}} \quad \text{if } n \text{ is even} \tag{1.1}$$

and

$$|M^{S^1}| \equiv 0 \pmod{\frac{24}{r}} \quad \text{if } n \text{ is odd.} \tag{1.2}$$

Remark 1.1 As usual, we assume $\gcd(0, a) = a$ for every positive integer a. ⊘

Note that the divisors of $|M^{S^1}|$ that are given by Theorem A are always factors of 24. The proof of this theorem can be found in Section 7.

When the fixed point set is discrete, the number of fixed points coincides with the Euler characteristic of the a.c. manifold. Coincidently, in a letter to V. Gritsenko, Hirzebruch also obtains divisibility results for the Euler characteristic of an almost complex manifold M satisfying $c_1c_{n-1}[M] = 0$ [Hi99]. Theorem A gives exactly the same divisibility factors when $\dim M \equiv 0 \pmod 6$ but, when $\dim M \not\equiv 0 \pmod 6$, it adds the additional information that the Euler characteristic (or equivalently $|M^{S^1}|$) must be a multiple of 3, leading to greater divisors (cf. Theorem G in Section 9). Under the stronger condition that $c_1 = 0$ in integer cohomology, we can combine Hirzebruch's results with ours obtaining, in some cases, greater divisors for the number of fixed points (see Theorem H in Section 9). For example, when $\dim M = 4$ and $c_1 = 0$, we prove that the number of fixed points is always a multiple of 24. This will be true, in general, whenever $\dim M \equiv 4 \pmod{16}$ and $\dim M \not\equiv 0 \pmod 6$.

The factors obtained in Theorem A already give us lower bounds $d(n)$ for the number of fixed points that depend on the dimension of the manifold. We will see that they can sometimes be improved to *lower bounds* $\mathcal{B}(n) = \ell(n)d(n)$, where $\ell(n)$ is an integer between 1 and 7. These are obtained from the minimum values of certain integer-valued functions restricted to a set of integer points in a specific hyperplane. The corresponding minimization problems are solved in Theorems E and F in Sections 5 and 6, using celebrated number theory results on the possible representations of a positive integer number as a sum of polygonal numbers. We recall that polygonal numbers are those of the form

$$\frac{(s-2)k^2+(4-s)k}{2}, \quad \text{with } s \geqslant 3 \text{ and } k \geqslant 1$$

(represented by regular polygons as in Figure 1.1). In this paper we will only use results about squares and triangular numbers (i.e. the numbers obtained with $s = 4$ and $s = 3$).[3] These were originally stated by Fermat in 1640 and proved by Legendre, Lagrange, Euler, Gauss and Ewell (see Section 3).

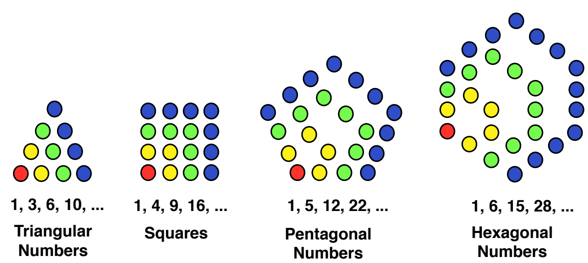

FIGURE 1.1. Some Polygonal Numbers.

The lower bounds obtained are summarized in the following theorem which combines the solutions of the minimization problems listed in Theorems E and F (in sections 5 and 6) with the fact that the number of fixed points is at least 4 when $\dim M \geqslant 8$ (see Theorem 2.8). Its proof can be found in Section 7. Some examples of the lower bounds obtained are listed in Table 1.1 and Figure 1.2 shows the lower bounds for $\dim M \leqslant 300$.

Theorem B. *Let (M, J) be a $2n$-dimensional compact connected almost complex manifold equipped with a J-preserving S^1-action with nonempty fixed point set and such that $c_1c_{n-1}[M] = 0$. Then the number of fixed points of the S^1-action is at least $\mathcal{B}(n)$, where $\mathcal{B}(n)$ is given as follows.*

For $n = 2m$ $(m \geqslant 1)$ and $r := \gcd(m, 12)$,

(i) *if $r = 1$ then $\mathcal{B}(n) = 12$;*

[3]In many references, k is allowed to be zero so that 0 is a polygonal number for every s (see sequences A000290 and A000217 in OEIS).

(ii) *if* $r = 2$ *then*
- $\mathcal{B}(n) = 6$ *if* $m \not\equiv 14 \pmod{16}$,
- $\mathcal{B}(n) = 12$ *otherwise;*

(iii) *if* $r = 3$ *then*
- $\mathcal{B}(n) = 4$ *if all prime factors of* $\frac{m}{3}$ *congruent to* $3 \pmod 4$ *occur with even exponent,*
- $\mathcal{B}(n) = 8$ *otherwise;*

(iv) *if* $r = 4$ *then*
- $\mathcal{B}(n) = 6$ $m \neq 4^k(16t + 14)\, \forall k, t \in \mathbb{Z}_{\geqslant 0}$,
- $\mathcal{B}(n) = 9$ *otherwise;*

(v) *if* $r = 6$ *then*
- $\mathcal{B}(n) = 4$ *if all prime factors of* $\frac{m}{3}$ *congruent to* $3 \pmod 4$ *occur with even exponent,*
- $\mathcal{B}(n) = 6$ *if at least one prime factor of* $\frac{m}{3}$ *congruent to* $3 \pmod 4$ *occurs with an odd exponent and* $m \not\equiv 14 \pmod{16}$,
- $\mathcal{B}(n) = 8$ *otherwise;*

(vi) *if* $r = 12$ *then*
- $\mathcal{B}(n) = 4$ *if* m *is a square or all prime factors of* $\frac{m}{3}$ *congruent to* $3 \pmod 4$ *occur with even exponent,*
- $\mathcal{B}(n) = 6$ *if none of the above holds and* $m \neq 4^k(16t + 14)\, \forall k, t \in \mathbb{Z}_{\geqslant 0}$,
- $\mathcal{B}(n) = 7$ *otherwise.*

For $n = 2m + 3$ *(*$m \geqslant 0$*) and* $r := \gcd(m, 12)$,

(i) *if* $r \leqslant 4$ *then* $\mathcal{B}(n) = \frac{24}{r}$;

(ii) *if* $r = 6$ *then*
- $\mathcal{B}(n) = 4$ *if every prime factor of* $\frac{2}{3}m + 1$ *congruent to* $3 \pmod 4$ *occurs with even exponent,*
- $\mathcal{B}(n) = 8$ *otherwise;*

(iii) *if* $r = 12$ *then*
- $\mathcal{B}(n) = 2$ *if* $m = 0$,
- $\mathcal{B}(n) = 4$ *if* $m \neq 0$ *and every prime factor of* $\frac{2}{3}m + 1$ *congruent to* $3 \pmod 4$ *occurs with even exponent,*
- $\mathcal{B}(n) = 6$ *otherwise.*

Remark 1.2 It is easy to see that, in many cases, we have some "periodicity" of $\mathcal{B}(n)$. For example, one can easily show that

$$\mathcal{B}(n) = 24 \quad \text{if and only if} \quad n \equiv 1 \pmod{12} \quad \text{or} \quad n \equiv 5 \pmod{12}$$

(note that here $\mathcal{B}(n) = d(n)$). Moreover,

$$\begin{aligned} \mathcal{B}(n) = 12 \quad & \text{if and only if} \\ & n \equiv 2 \pmod{12}, \quad \text{or} \quad n \equiv 10 \pmod{12}, \text{ or} \\ & n \equiv 7 \pmod{24}, \quad \text{or} \quad n \equiv 23 \pmod{24}, \text{ or} \\ & n \equiv 28 \pmod{96} \quad \text{or} \quad n \equiv 92 \pmod{96}. \end{aligned}$$

Here we also have $\mathcal{B}(n) = d(n)$ except when $n \equiv 28$ or $92 \pmod{96}$, where $\mathcal{B}(n) = 2\, d(n)$. All these cases can be easily observed in Figure 1.2. ⊘

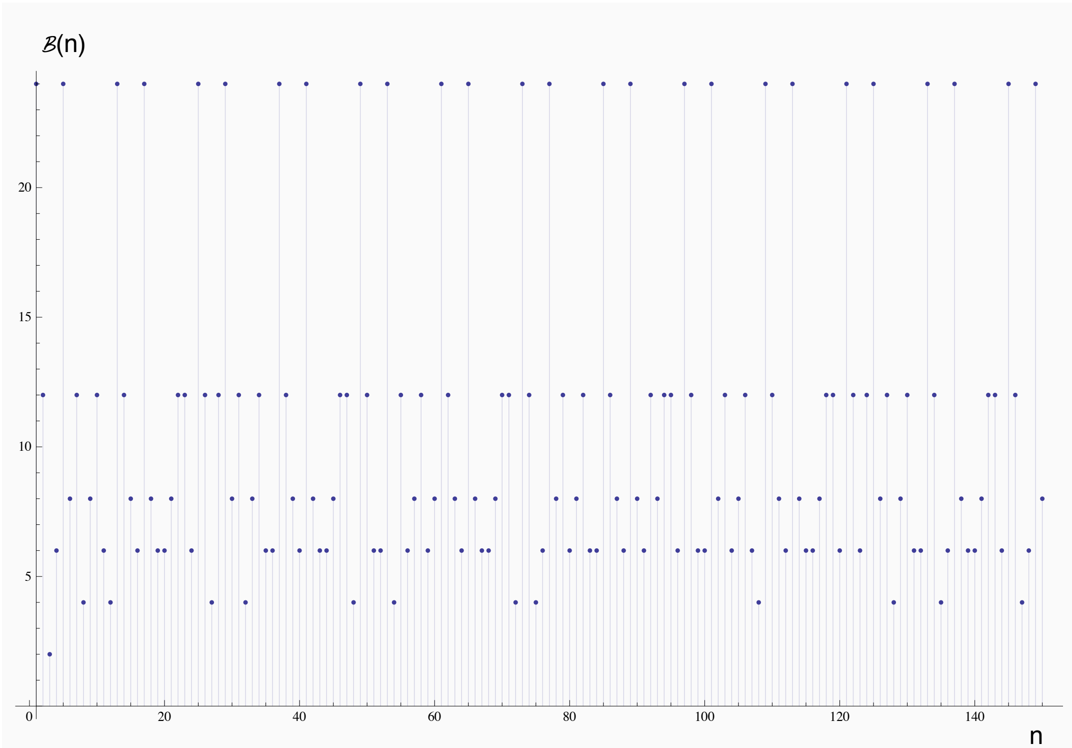

FIGURE 1.2. Values of $\mathcal{B}(n)$ obtained from Theorem B for $n \leqslant 150$.

In some dimensions, the lower bounds obtained are greater than $\lfloor n/2 \rfloor + 1$ (the lower bound proposed by Kosniowski) and, in some cases, they are even better than $n+1$, the existing lower bound for Hamiltonian actions and some almost complex S^1-manifolds. We give a complete list of these dimensions in Propositions 8.1 and 8.2. However, since the lower bounds obtained are at most equal to 24, our results do not support Kosniowski's hypothesis that there should exist a lower bound that depends linearly on the dimension of the manifold.

If $c_1 = 0$ in integer cohomology, we can again combine Hirzebruch's results with ours obtaining, in some cases, a better lower bound for the number of fixed points (see Theorem I in Section 9). For example, when $\dim M = 4$ and $c_1 = 0$ we prove that the number of fixed points is at least 24. This will be true, in general, whenever $\dim M \equiv 4 \pmod{16}$ and $\dim M \not\equiv 0 \pmod{6}$.

If we restrict to Hamiltonian actions on symplectic manifolds with $c_1 c_{n-1}[M] = 0$, then we can also use our methods to obtain lower bounds for the corresponding number of fixed points, which improve the existing lower bound of $\frac{1}{2}\dim M + 1$. These are summarized in the following theorem, whose proof can be found in Section 7.

Theorem C. *Let M be a $2n$-dimensional compact connected symplectic manifold with $c_1 c_{n-1}[M] = 0$. Then the number of fixed points of a Hamiltonian S^1-action* *on M is at least*

- $(n+1)(n+2)$, *if n is even;*
- $n^2+6n+17+\dfrac{24}{\gcd\left(\frac{n-3}{2},12\right)}$, *if $n>3$ is odd.*

Remark 1.3 Note that 6-dimensional symplectic manifolds with $c_1c_2[M]=0$ do not admit any Hamiltonian S^1-action with a discrete fixed point set (cf. Proposition 2.14). A 4-dimensional example of a Hamiltonian S^1-action on a symplectic manifold M satisfying $c_1^2[M]=0$ is given in Example 10.1. Its number of fixed points is 12, the lower bound given by Theorems B and C. $\oslash$

On the contrary, if we restrict to symplectic actions that are not Hamiltonian, then the lower bounds that we obtain by our method remain the same as those that are listed in Theorem B (cf. Remarks 5.4 and 6.5).

In Section 10, we provide several examples that show how some of the lower bounds obtained are sharp and illustrate our divisibility results for the number of fixed points. In particular, we give examples where the number of fixed points is actually equal to our lower bound $\mathcal{B}(n)$ in dimensions $4, 6, 10, 12$ and 18. It would be interesting to know the answer to the following question.

Question 1.4 *Does there exist a compact almost complex S^1-manifold M of dimension 8 with $c_1c_3[M]=0$ and exactly 6 fixed points?* $\oslash$

The existence of such a manifold would also guarantee the existence of a 14 dimensional example with exactly 12 fixed points, the lower bound given by Theorem B (see Remark 10.4).

In Table 1.1 we illustrate some of the results obtained in this work. In the first part of the table, almost all the lower bounds $\mathcal{B}(n)$ given by Theorem B and listed in the second column, coincide with the ones obtained from Theorem A (except for $n=8$ and $n=12$). This is no longer the case in the second part of the table.

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$\frac{1}{2}\dim M$	A priori possible values of $\lvert M^{S^1}\rvert$ if $c_1c_{n-1}[M]=0$		Kosniowski's conjectural lower bound	Lower bound Ham. actions
n	general	Ham. actions	$\lfloor n/2\rfloor+1$	$n+1$
2	$12^*, 24, 36, \ldots$	$12, 24, 36, \ldots$	2	3
3	$2, 4, 6, \ldots$	—	2	4
4	$6, 12, 18, \ldots$	$30, 36, 42, \ldots$	3	5
5	$24, 48, 72, \ldots$	$96, 120, 144, \ldots$	3	6
6	$4, 8, 12, \ldots$	$56, 60, 64, \ldots$	4	7
7	$12, 24, 36, \ldots$	$120, 132, 144, \ldots$	4	8
8	$6, 9, 12, \ldots$	$90, 93, 96, \ldots$	5	9
9	$8, 16, 24, \ldots$	$160, 168, 176, \ldots$	5	10
10	$12^*, 24, 36, \ldots$	$132, 144, 156, \ldots$	6	11
11	$6, 12, 18, \ldots$	$210, 216, 222, \ldots$	6	12
12	$4, 6, 8, \ldots$	$182, 184, 186, \ldots$	7	13
13	$24, 48, 72, \ldots$	$288, 312, 336, \ldots$	7	14
14	$12, 24, 36, \ldots$	$240, 252, 264, \ldots$	8	15
15	$4, 8, 12, \ldots$	$336, 340, 344, \ldots$	8	16
...				
18	$8,\ 12,\ 16,\ \ldots$	$380, 384, 388, \ldots$	10	19
28	$12,\ 18,\ 24,\ \ldots$	$870, 876, 882, \ldots$	15	29
99	$6,\ 8,\ 10,\ \ldots$	$10414, 10416, \ldots$	50	100
112	$9, 12, 15, \ldots$	$12882, 12885, \ldots$	57	113
144	$6,\ 7,\ 8,\ \ldots$	$21170, 21171, \ldots$	73	145
252	$8,\ 10,\ 12,\ \ldots$	$64262, 64264, \ldots$	127	253
1008	$7,\ 8,\ 9,\ \ldots$	$1019090, 1019091, \ldots$	505	1009

* if $c_1=0$ then, a priori, the possible values of $\lvert M^{S^1}\rvert$ are $24, 48, 72, \ldots$

TABLE 1.1. Some of the results obtained in Theorems A, B and C.

2. Preliminaries

We review some results which are relevant for this article, including some which we will need in the proofs.

2.1. **Origins.** It has been a long standing problem to estimate the minimal number of fixed points of a circle action on a compact almost complex manifold with nonempty fixed point set. If the manifold is symplectic, i.e. if it admits a closed, non-degenerate two-form $\omega \in \Omega^2(M)$ (*symplectic form*), we say that an S^1-action on (M,ω) is *symplectic* if it preserves ω. If $\mathcal{X}_M$ is the vector field induced by the S^1 action then we say that the action is *Hamiltonian* if the 1-form $\iota_{\mathcal{X}_M}\omega := \omega(\mathcal{X}_M,\cdot)$

is exact, that is, if there exists a smooth map $\mu\colon M \to \mathbb{R}$ such that $-\,\mathrm{d}\mu = \iota_{\mathcal{X}_M}\omega$. The map μ is called a *momentum map*. If a symplectic manifold is equipped with a Hamiltonian S^1-action then the following fact is well-known.

Proposition 2.1. *A Hamiltonian S^1-action on a $2n$-dimensional compact symplectic manifold has at least $n+1$ fixed points.*

This follows from the fact that, when the fixed point set is discrete, the momentum map is a perfect Morse function whose critical set is equal to the fixed point set. The Morse inequalities then become equalities and the number of fixed points is equal to the sum of the betti numbers. Since the classes $[\omega^k] \in H^{2k}(M,\mathbb{R})$ are non trivial for $k = 0,\dots,n$, the number of fixed points is at least $n+1$.

This lower bound holds on all Kähler S^1-manifolds with a nonempty fixed point set [Fr59].

Theorem 2.2 (Frankel '59)**.** *A Kähler S^1-action on a $2n$-dimensional compact connected Kähler manifold is Hamiltonian if and only if it has fixed points, in which case it has at least $n+1$ fixed points.*

The same lower bound was obtained by Hattori [Ha85, Corollary 3.8] on a particular class of unitary manifolds.

Theorem 2.3 (Hattori)**.** *If M is a $2n$-dimensional unitary S^1-manifold such that $c_1^n[M]$ does not vanish, then the number of fixed points is at least $n+1$.*

Remark 2.4 A *unitary* (or weakly almost complex) manifold M is a smooth manifold endowed with a fixed complex structure on the stable tangent bundle of M [M99]. If the complex structure is given on the tangent bundle, M is called an almost complex manifold. If S^1 acts on a unitary (resp. an almost complex) manifold preserving the given complex structure on the stable tangent bundle (resp. tangent bundle), then M is called a unitary (resp. almost complex) S^1-manifold. Hence every S^1-symplectic manifold is an S^1-almost complex manifold, and a unitary S^1-manifold. Moreover, each component of the fixed point set of a unitary S^1-manifold is again a unitary S^1-manifold of even codimension, and its normal bundle in M is a complex S^1-vector bundle with the complex structure induced from the one on the stable tangent bundle. In particular, the tangent space T_pM at an isolated fixed point is a complex S^1-module. It has two possible orientations: the one induced from the orientation of M and the other induced from the complex structure on T_pM. They coincide whenever M is almost complex, but may be different otherwise (see for example [M99, Section 4]).

Let M be a $2n$ dimensional unitary manifold with stable tangent bundle E. Since E is a complex vector bundle, one can consider the Chern classes $c_j \in H^{2j}(M,\mathbb{Z})$ of E as well as any Chern number $(c_1^{i_1}c_2^{i_2}\cdots c_n^{i_n})[M]$.

Moreover, one says that M *bounds* if it is unitary cobordant with the empty set, meaning that it can be realized as the oriented boundary of a unitary oriented $2n+1$-manifold with boundary W such that the induced unitary structure of ∂W is isomorphic to the unitary structure of M. In particular, this is the case if and only if all Chern numbers of M are equal to zero. $\oslash$

Still working with unitary manifolds, Kosniowski [K79, Theorem 5] obtains the following results.

Theorem 2.5 ([K79]). *Let M be a unitary S^1-manifold with two fixed points. Then M is either a boundary or $\dim M$ is equal to 2 or 6. Moreover, if M is an almost complex S^1-manifold with two fixed points, then $\dim M$ is either 2 or 6.*

Corollary 2.6 ([K79]). *If M is an almost complex S^1-manifold with $\dim M \geqslant 8$ and a nonempty fixed point set, then the number of fixed points is at least 3.*

Kosniowski further proposes the existence of a linear function $f : \mathbb{R} \to \mathbb{R}_+$ such that, for every $2n$-dimensional compact unitary S^1-manifold M with isolated fixed points which does not bound equivariantly, the number of fixed points is greater than $f(n)$; in particular, he expects that f can be taken to be $f(x) = \frac{x}{2}$, leading to the conjecture that the number of fixed points is at least $\left\lfloor \frac{n}{2} \right\rfloor + 1$ (Conjecture 1).

Remark 2.7 Since, for an almost complex S^1-manifolds with non-empty discrete fixed point set, the Euler characteristic $c_n[M]$ is equal to the number of fixed points, these manifolds cannot bound, satisfying the conditions in Kosniowski's conjecture.

Note that the condition that M bounds cannot be removed as one can construct examples in any dimension of unitary S^1-manifolds that are boundaries and have exactly two fixed points (see [K79, Theorem 3]). For example, one can take S^{2k} with the circle action induced from the inclusion in $\mathbb{C}^k \times \mathbb{R}$ and the unitary structure induced from the further inclusion in $\mathbb{C}^k \times \mathbb{C}$. This action has exactly two fixed points but the two possible orientations on the corresponding tangent spaces (as described in Remark 2.4) agree for one of the fixed points and disagree for the other. ⊘

The lower bound of Corollary 2.6 can be further improved.

Theorem 2.8. *If M is an almost complex S^1-manifold with $\dim M \geqslant 8$ and a nonempty fixed point set, then the number of fixed points is at least 4.*

Proof. When the manifold is symplectic, this result is an immediate consequence of Theorem 1.1 of [J14]. To prove this particular part of his theorem, Jang uses an analog of Theorem 2.5 for symplectic manifolds, the fact that the total sum of the isotropy weights at all fixed points is equal to zero [Ha85, Proposition 2.11] and the Atiyah-Bott and Berline-Vergne localization formula in equivariant cohomology. Since all these results still hold for almost complex manifolds, the claim follows. □

2.2. **Other recent contributions.** Following Kosniowski's conjecture and the theorems of Frankel, Hattori and Kosniowski, many results have appeared in recent works for symplectic S^1-manifolds and for almost complex S^1-manifolds, which can be easily extended to unitary S^1-manifolds.

Using the Atiyah-Bott and Berline-Vergne localization formula in equivariant cohomology, Pelayo and Tolman [PT11] proved the following result

Theorem 2.9 (Pelayo-Tolman). *Let M^{2n} be a compact symplectic S^1-manifold and let $c_1^{S^1}(M)\colon M^{S^1} \to \mathbb{Z}$ be the map given by the sum of the weights of the S^1-isotropy representation T_pM. If $c_1^{S^1}(M)$ is somewhere injective[4], then the S^1-action has at least $n+1$ fixed points.*

[4]Let $f\colon X \to Y$ be a map between sets then f is *somewhere injective* if there exists $y \in Y$ such that $f^{-1}(\{y\})$ is a singleton.

Remark 2.10 The map $c_1^{S^1}(M)\colon M^{S^1} \to \mathbb{Z}$ is usually called the *Chern class map* and can be naturally identified with the restriction of the first S^1-equivariant Chern class of TM to each fixed point $p \in M^{S^1}$. Note that it can also be defined when M is unitary, if one takes the first S^1-equivariant Chern class of the stable tangent bundle of M. Similarly, one can define other maps $c_\ell^{S^1}(M)\colon M^{S^1} \to \mathbb{Z}$ for $\ell = 1, \dots, n$, by considering the restrictions of the S^1-equivariant Chern classes $c_\ell^{S^1}$ of TM at each fixed point $p \in M^{S^1}$. ⊘

Following this result, Ping Li and Kefeng Liu generalized Theorem 2.3 [LL10].

Theorem 2.11 (Li-Liu)**.** *Let M^{2mn} be an almost-complex manifold. If there exist positive integers $\lambda_1, \dots, \lambda_u$ with $\sum_{i=1}^u \lambda_i = m$ such that the corresponding Chern number $(c_{\lambda_1} \cdots c_{\lambda_u})^n[M]$ is nonzero, then any S^1-action on M must have at least $n+1$ fixed points.*

This was further generalized by Cho, Kim and Park [CKP12] to include other non vanishing Chern numbers.

Theorem 2.12 (Cho-Kim-Park)**.** *Let M be a $2n$-dimensional unitary S^1-manifold and let $i_1, i_2, \dots, i_n$ be non-negative integers satisfying $i_1 + 2i_2 + \cdots + ni_n = n$. If M does not bound equivariantly and $c_1^{i_1} c_2^{i_2} \cdots c_n^{i_n}[M] \neq 0$, then M must have at least max$\{i_1, \cdots, i_n\} + 1$ fixed points.*

All these results use the Atiyah-Bott and Berline-Vergne localization formula. The crucial hypothesis for the establishment of the lower bound is the existence of non-negative integers $i_1, i_2, \dots, i_n$ and a value k of one of the maps $c_\ell^{S^1}(M)$, for which

$$\sum_{\substack{p \,\in\, M^{S^1} \\ c_\ell^{S^1}(M)(p) = k}} \frac{\prod_{j \neq \ell} c_j^{i_j}(M)(p)}{\Lambda_p} \neq 0,$$

where Λ_p is the product of the weights in the isotropy representation T_pM. This is trivially achieved with $\ell = 1$ and $i_2 = \cdots = i_n = 0$, whenever the Chern class map is somewhere injective [PT11] or when $c_1^n[M] \neq 0$; when $c_1^{i_1} c_2^{i_2} \cdots c_n^{i_n}[M] \neq 0$ with $i_1 + 2i_2 + \cdots + ni_n = n$, then it is true, for example for ℓ such that $i_\ell = \max\{i_1, \cdots, i_n\}$ (giving Theorems 2.11 and 2.12). These techniques can be generalized to unitary S^1-manifolds as in [CKP12].

2.3. **The hypothesis** $c_1 c_{n-1}[M] = 0$**.** Contrary to the results in Section 2.2, in this work we will focus on the situation in which a particular Chern number vanishes. Moreover, we do not use the Atiyah-Bott and Berline-Vergne localization formula and most of the techniques used will only hold for almost complex S^1-manifolds and cannot be generalized to unitary manifolds (see Remark 4.3). Therefore, from now on, we will always assume to be working with almost complex S^1-manifolds.

The only known expressions of Chern numbers in terms of number of fixed points concern $c_n[M]$ (which equals this number and the Euler characteristic, when the fixed point set is discrete) and $c_1 c_{n-1}[M]$ [GS12, Theorem 1.2] (see Theorem 4.1). Thus the natural candidate is $c_1 c_{n-1}[M] = 0$.

Note that $c_1 c_{n-1}[M] = 0$ is satisfied under the stronger condition that c_1 or c_{n-1} are torsion in integer cohomology. In particular, in the case in which c_1 is torsion, Theorem 2.3 [Ha85] cannot be applied; moreover, the same holds for Theorem 2.9 [PT11] as we can see from the following lemma.

Lemma 2.13. *Let (M,J) be a compact almost complex manifold such that c_1 is a torsion element in $H^2(M,\mathbb{Z})$. If M admits a J-preserving circle action with a discrete fixed point set, then the Chern class map $c_1^{S^1}(M) : M^{S^1} \to \mathbb{Z}$ is identically zero.*

Proof. Since c_1 is a torsion element in $H^2(M,\mathbb{Z})$, there exists $k \in \mathbb{Z}$ such that $kc_1 = 0$. Then the restriction of the equivariant extension $k\,c_1^{S^1} \in H^2_{S^1}(M,\mathbb{Z})$ to the fixed point set is constant, implying that the Chern class map is constant. Since $c_1^{S^1}(M)(p)$ coincides with the sum of the isotropy weights at $p \in M^{S^1}$, and the total sum of all the isotropy weights at all fixed points is equal to zero [Ha85, Proposition 2.11], we have

$$\sum_{p\in M^{S^1}} c_1^{S^1}(M)(p) = 0,$$

and so this constant must be zero. □

Note that, if M is a 6-dimensional compact connected symplectic manifold, the action is Hamiltonian if and only if $c_1c_2[M] \neq 0$. Indeed, we have the following proposition.

Proposition 2.14. *Suppose that S^1 acts symplectically on a compact connected 6-dimensional symplectic manifold M with nonempty discrete fixed point set. Then the S^1-action is Hamiltonian if and only if $c_1c_2[M] \neq 0$.*

Proof. This follows from a result of Feldman [Fe01] which states that the Todd genus associated to M is either 1 or 0, according to whether the action is Hamiltonian or not, and the fact that, when $\dim(M) = 6$, one has

$$\mathrm{Todd}(M) = \int_M \frac{c_1c_2}{24}.$$

□

In general, if the manifold is symplectic and c_1 is torsion in integer cohomology, then, necessarily, the action is non-Hamiltonian. We thank one of the anonymous referees for suggesting the proof of this result in the case of non isolated fixed points.

Proposition 2.15. *Let (M,ω) be a compact symplectic manifold such that c_1 is torsion in integer cohomology. Then M does not admit any Hamiltonian circle action.*

Proof. The case of isolated fixed points follows immediately from Feldman's result [Fe01] and the fact that, for unitary S^1-manifolds with isolated fixed points, if c_1 is torsion, then the Todd genus is zero [Ha85, Proposition 3.21]. Alternatively, this is also an easy consequence of Lemma 2.13 since, if the action is Hamiltonian, then $c_1^{S^1}(M)(p) \neq 0$ at both the minimum and the maximum points of the momentum map.

If the fixed point set is not discrete and the action is Hamiltonian, consider the S^1-equivariant map $i\colon S^2 \to M$ whose image is a gradient sphere from the minimum to the maximum of the moment map. Then the integral of $i^*(c_1)$ on S^2 would be non-zero by the Atiyah-Bott and Berline-Vergne localization formula in equivariant cohomology. However, if c_1 is torsion, this integral should vanish. □

Therefore, the lower bounds we obtain, naturally apply to a class of compact symplectic manifolds that do not support any Hamiltonian circle action with isolated fixed points, namely symplectic manifolds whose first Chern class is torsion. For example, these results apply to symplectic Calabi Yau manifolds, i.e. symplectic manifolds with $c_1 = 0$ [FP09].

We finish this section with a property which gives a way of producing infinitely many manifolds with $c_1 c_{n-1}[M] = 0$ (see Section 10).

Lemma 2.16. *Let M^{2m} and N^{2n} be compact almost complex manifolds satisfying $c_1 c_{m-1}[M] = c_1 c_{n-1}[N] = 0$. Then $c_1 c_{m+n-1}[M \times N] = 0$.*

Proof. This follows from the fact that if, for any almost complex manifold M^{2m} with $c_m[M] \neq 0$, we set

$$\gamma(M) := \frac{c_1 c_{m-1}[M]}{c_m[M]},$$

we have $\gamma(M \times N) = \gamma(M) + \gamma(N)$ (see [S96, Section 3]). □

3. Fermat's statements

In 1640 Fermat stated (without proof) that every positive integer can be represented as a sum of 4 squares and as a sum of 3 triangular numbers, where square and triangular numbers are those respectively described by k^2 and $\frac{k(k+1)}{2}$, with $k = 0, 1, 2, 3, \ldots$ (here we consider 0 to be a square, as well as a triangular number).

Lagrange, in 1770, proved the part of Fermat's theorem regarding squares, obtaining his celebrated Four Squares Theorem [D52, p. 279].

Theorem 3.1 (Lagrange's Four Squares Theorem)**.** *Every nonnegative integer can be represented as the sum of 4 (or fewer) squares.*

In 1798 Legendre proved a much deeper statement which described exactly which numbers needed all four squares [D52, p. 261].

Theorem 3.2 (Legendre's Three Squares Theorem)**.** *The set of positive integers that cannot be represented as sums of three (or fewer) squares is the set*

$$\left\{ m \in \mathbb{Z}_{>0} : \ m = 4^k(8t+7), \quad \text{for some} \quad k, t \in \mathbb{Z}_{\geqslant 0} \right\}.$$

After this, it was natural to think which numbers could be written as a sum of two squares. A complete answer to this question was given by Euler [D52, p. 230].

Theorem 3.3 (Euler)**.** *A positive integer $m > 1$ can be written as a sum of two squares if and only if every prime factor of m which is congruent to 3 (mod 4) occurs with even exponent.*

Example 3.4 The integer $m = \mathbf{245} = 5 \cdot 7^2$ can be written as a sum of two squares. In particular, $245 = 4 \cdot 7^2 + 7^2 = 14^2 + 7^2$. As the number $m = \mathbf{105}$ is not divisible by 4 and is congruent to $1 \pmod 8$, one concludes that it can be written as the sum of 3 or fewer squares. However, since $105 = 3 \cdot 5 \cdot 7$ has a prime factor congruent to $3 \pmod 4$ occurring with odd exponent, it cannot be written as a sum of 2 squares.

For instance, we have $105 = 10^2 + 2^2 + 1^2$. Since $m = \mathbf{60} = 4 \cdot 15 = 4 \cdot (8 + 7)$, we know from Theorem 3.2 that it cannot be represented as a sum of 3 or fewer squares so we really need 4 squares. For example, $60 = 6^2 + 4^2 + 2^2 + 2^2$. ⊘

Let us now see what happens with triangular numbers. The part of Fermat's statement regarding these numbers was first proved by Gauss [D52, p. 17].

Theorem 3.5 (Gauss)**.** *Every nonnegative number can be written as the sum of three (or fewer) triangular numbers.*

After this result, Ewell [E92] gave a simple description of the numbers that are sums of two triangular numbers.

Theorem 3.6 (Ewell)**.** *A positive integer m can be represented as a sum of two triangular numbers if and only if every prime factor of $4m + 1$ which is congruent to $3 \pmod 4$ occurs with even exponent.*

Example 3.7 Taking $m = \mathbf{106}$ one obtains $4m + 1 = 425 = 5^2 \cdot 17$ and so m can be written as a sum of two triangular numbers. For instance, $106 = 105 + 1 = \frac{14 \cdot 15}{2} + \frac{1 \cdot 2}{2}$. On the other hand, if one takes $m = \mathbf{59}$, then $4m + 1 = 237 = 3 \cdot 79$ and so, by Theorem 3.6, m cannot be written as a sum of 2 triangular numbers. For instance we have $59 = 28 + 21 + 10 = \frac{7 \cdot 8}{2} + \frac{6 \cdot 7}{2} + \frac{4 \cdot 5}{2}$. ⊘

4. A minimization problem

4.1. **Tools.** Let us then see how to obtain a lower bound for the number of fixed points of a J-preserving circle action on an almost complex manifold (M, J) satisfying $c_1 c_{n-1}[M] := \int_M c_1 c_{n-1} = 0$, where c_1 and c_{n-1} are respectively the first and the $(n-1)$ Chern classes of M.

The first result that we need is the expression of $c_1 c_{n-1}[M]$ in terms of numbers of fixed points.

Theorem 4.1 ([GS12])**.** *Let (M, J) be a $2n$-dimensional compact connected almost complex manifold equipped with an S^1-action which preserves the almost complex structure J and has a nonempty discrete fixed point set. For every $i = 0, \ldots, n$, let N_i be the number of fixed points with exactly i negative weights in the isotropy representation T_pM. Then*

$$c_1 c_{n-1}[M] = \sum_{i=0}^{n} N_i \Big(6i(i-1) + \frac{5n - 3n^2}{2} \Big). \tag{4.1}$$

Remark 4.2 If M is a $2n$-dimensional symplectic S^1-manifold and the S^1-action is Hamiltonian, then the number N_i of fixed points with exactly i negative weights in the corresponding isotropy representations coincides with the $2i$-th Betti number $b_{2i}(M)$ of M. Consequently, the expression for $c_1 c_{n-1}[M]$ given in (4.1) becomes

$$c_1 c_{n-1}[M] = \sum_{i=0}^{n} b_{2i}(M) \Big(6i(i-1) + \frac{5n - 3n^2}{2} \Big). \tag{4.2}$$

For example, if $\dim M = 4$, equation (4.2) gives

$$c_1^2[M] = 10 b_0(M) - b_2(M), \tag{4.3}$$

where we used the fact that $b_0(M) = b_4(M)$. ⊘

Remark 4.3 The equality in (4.1) is obtained by considering the expression (in Theorem 2 of [S96]) of $c_1c_{n-1}[M]$ in terms of derivatives of the Hirzebruch genus, noting that the coefficients of this genus are equal to the numbers N_k of fixed points with exactly k negative isotropy weights. Although the S^1-equivariant Hirzebruch genus can be generalized to unitary S^1-manifolds to a rigid equivariant elliptic genus, the coefficients of the corresponding (non-equivariant) genus will no longer be the numbers N_k (as in the case of almost complex S^1-manifolds). Instead, they will be the numbers h_k defined in [HM05, Section 3] which depend on the choice of orientations of the tangent spaces T_pM of the fixed points with k negative weights [HM05, Proposition 3.8] (note that $h_k = N_k$ when the manifold is almost complex). Even if the expressions of the Chern numbers $c_n[M]$ and $c_1c_{n-1}[M]$ in terms of derivatives of the Hirzebruch genus hold for unitary manifolds, they will depend on the numbers h_k and cannot be used when counting the total number of fixed points. Indeed, even the absolute value of h_k can be different from N_k as the contribution of one fixed point with k negative weights might be canceled with one of opposite orientation. ⊘

4.2. **The minimization problem.** For each $m \in \mathbb{Z}_{\geqslant 0}$ let us consider the functions $F_1, F_2, G_1, G_2 : \mathbb{Z}^{m+1} \to \mathbb{Z}$ defined by

$$F_1(N_0, \dots, N_m) \quad := \quad N_m + 2\sum_{k=1}^{m} N_{m-k}; \tag{4.4}$$

$$F_2(N_0, \dots, N_m) \quad := \quad 2\sum_{k=0}^{m} N_{m-k}; \tag{4.5}$$

$$G_1(N_0, \dots, N_m) \quad := \quad -mN_m + 2\sum_{k=1}^{m} (6k^2 - m)\, N_{m-k}; \tag{4.6}$$

$$G_2(N_0, \dots, N_m) \quad := \quad 2\sum_{k=0}^{m} \Big(6k(k+1) - (m-1)\Big) N_{m-k}. \tag{4.7}$$

Moreover, for $i \in \{1, 2\}$, let

$$\mathcal{Z}_i := \big\{(N_0, \dots, N_m) \in (\mathbb{Z}_{\geqslant 0})^{m+1} \setminus \{0\} \mid G_i(N_0, \dots, N_m) = 0\big\}. \tag{4.8}$$

Then Theorem 4.1 can be restated as follows.

Theorem 4.4. *Let (M, J) be a $2n$-dimensional compact connected almost complex manifold equipped with an S^1-action which preserves the almost complex structure J and has a nonempty discrete fixed point set. For every $i = 0, \dots, n$, let N_i be the number of fixed points p with exactly i negative weights in the isotropy representation T_pM. Moreover, let $G_1, G_2 : \mathbb{Z}^{m+1} \to \mathbb{Z}$ be the functions defined in (4.6) and (4.7). Then*

$$c_1c_{n-1}[M] = \begin{cases} G_1(N_0, \dots, N_m) & \text{if } n \text{ is even;} \\ G_2(N_0, \dots, N_m) & \text{if } n \text{ is odd,} \end{cases} \tag{4.9}$$

where c_1 and c_{n-1} are respectively the first and the $(n-1)$ Chern classes of M.

Proof. Consider the map $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by

$$g(i,n) = 6i(i-1) + \frac{5n - 3n^2}{2}.$$

In [Ha85, Proposition 2.11] Hattori shows that $N_i = N_{n-i}$ for every $i \in \mathbb{Z}$. Hence, by (4.1), if $n = 2m$, we have

$$\begin{aligned} c_1 c_{n-1}[M] &= \sum_{i=0}^{n} N_i \, g(i,n) = -mN_m + \sum_{k=1}^{m} \Big(g(m-k, 2m) + g(m+k, 2m) \Big) N_{m-k} \\ &= -mN_m + 2\sum_{k=1}^{m} (6k^2 - m) \, N_{m-k} = G_1(N_0, \dots, N_m). \end{aligned}$$

Analogously, if $n = 2m+1$, we have

$$\begin{aligned} c_1 c_{n-1}[M] &= \sum_{i=0}^{n} N_i \, g(i,n) = \sum_{k=0}^{m} N_{m-k} \Big(g(m-k, 2m+1) + g(m+k+1, 2m+1) \Big) \\ &= 2\sum_{k=0}^{m} \Big(6k(k+1) - m + 1 \Big) \, N_{m-k} = G_2(N_0, \dots, N_m). \end{aligned}$$

□

Using this, one obtains the following minimization problem.

Theorem D. *Let (M, J) be a $2n$-dimensional compact connected almost complex manifold such that $c_1 c_{n-1}[M] = 0$, equipped with a J-preserving S^1-action with nonempty, discrete fixed point set.*

For $m := \left\lfloor \frac{n}{2} \right\rfloor$, let F_1, $F_2 : \mathbb{Z}^{m+1} \to \mathbb{Z}$ be the functions defined respectively in (4.4) and (4.5), and let $\mathcal{Z}_1, \mathcal{Z}_2$ be the sets given in (4.8). Then the S^1-action has at least $\mathcal{B}(n)$ fixed points, where

$$\mathcal{B}(n) := \begin{cases} \min_{\mathcal{Z}_1} F_1 & \text{if } n \text{ is even;} \\ \min_{\mathcal{Z}_2} F_2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let N_i be the number of fixed points with exactly i negative weights in the corresponding isotropy representations. Since the total number of fixed points is

$$\sum_{k=0}^{n} N_k$$

and $N_i = N_{n-i}$ for every $i \in \mathbb{Z}$ [Ha85, Proposition 2.11], it follows that $F_1(N_0, \dots, N_m)$ and $F_2(N_0, \dots, N_m)$ count the total number of fixed points when $n = 2m$ and $n = 2m+1$ respectively. Moreover, since the fixed point set is nonempty, we must have $(N_0, \dots, N_m) \neq 0$.

Since we are assuming that $c_1 c_{n-1}[M] = 0$, the constraints $G_1 = 0$ and $G_2 = 0$ are obtained from Theorem 4.4, according to whether n is odd or even. □

5. A lower bound when n is even

Here we compute the minimal value $\mathcal{B}(n)$ of the function F_1 restricted to $\mathcal{Z}_1$, obtaining a lower bound for the number of fixed points of the S^1-action when n is *even*.

Theorem E. *Let $n = 2m$ $(m \geqslant 1)$ be an even positive integer and let $\mathcal{B}(n)$ be the minimum of the function F_1 restricted to $\mathcal{Z}_1$, where F_1 and $\mathcal{Z}_1$ are respectively defined by (4.4) and (4.8). Then $\mathcal{B}(n)$ can take all values in the set $\{2,3,4,6,7,8,9,12\}$. In particular, if $r := \gcd\left(\frac{n}{2}, 12\right)$ $(= \gcd(m,12))$, we have that:*

(i) *if $r=1$ then $\mathcal{B}(n)=12$;*

(ii) *if $r=2$ then*
- $\mathcal{B}(n)=6$ *if $m \not\equiv 14 \pmod{16}$,*
- $\mathcal{B}(n)=12$ *otherwise;*

(iii) *if $r=3$ then*
- $\mathcal{B}(n)=4$ *if all prime factors of $\frac{m}{3}$ congruent to $3 \pmod 4$ occur with even exponent,*
- $\mathcal{B}(n)=8$ *otherwise;*

(iv) *if $r=4$ then*
- $\mathcal{B}(n)=3$ *if m is a square,*
- $\mathcal{B}(n)=6$ *if m is not a square and $m \neq 4^k(16t+14)\,\forall k,t \in \mathbb{Z}_{\geqslant 0}$,*
- $\mathcal{B}(n)=9$ *otherwise;*

(v) *if $r=6$ then*
- $\mathcal{B}(n)=2$ *if $\frac{m}{6}$ is a square,*
- $\mathcal{B}(n)=4$ *if $\frac{m}{6}$ is not a square and all prime factors of $\frac{m}{3}$ congruent to $3 \pmod 4$ occur with even exponent,*
- $\mathcal{B}(n)=6$ *if $\frac{m}{6}$ is not a square, at least one prime factor of $\frac{m}{3}$ congruent to $3 \pmod 4$ occurs with an odd exponent and $m \not\equiv 14 \pmod{16}$,*
- $\mathcal{B}(n)=8$ *otherwise;*

(vi) *if $r=12$ then*
- $\mathcal{B}(n)=2$ *if $\frac{m}{6}$ is a square,*
- $\mathcal{B}(n)=3$ *if m is a square,*
- $\mathcal{B}(n)=4$ *if none of the above holds and all prime factors of $\frac{m}{3}$ congruent to $3 \pmod 4$ occur with even exponent,*
- $\mathcal{B}(n)=6$ *if none of the above holds and $m \neq 4^k(16t+14)\,\forall k,t\in\mathbb{Z}_{\geqslant 0}$,*
- $\mathcal{B}(n)=7$ *otherwise.*

Proof. A point $(N_0,\dots,N_m) \in (\mathbb{Z}_{\geqslant 0})^{m+1} \setminus \{0\}$ is in $\mathcal{Z}_1$ if and only if

$$G_1 := -mN_m + 2\sum_{k=1}^{m}(6k^2-m)N_{m-k} = 0,$$

which is equivalent to

$$N_m = 2\sum_{k=1}^{m}\left(\frac{6k^2}{m}-1\right)N_{m-k}. \tag{5.1}$$

Hence, to find $\min_{\mathcal{Z}_1} F_1$, we start by substituting (5.1) in (4.4), obtaining

$$F_1 = \frac{12}{m}\sum_{k=1}^{m} k^2 N_{m-k}. \tag{5.2}$$

Since F_1 is integer valued on $\mathbb{Z}^{m+1}$, we have

$$\frac{12}{m}\sum_{k=1}^{m} k^2 N_{m-k} \in \mathbb{Z}.$$

As $N_0, \dots, N_{m-1} \in \mathbb{Z}$, this is equivalent to having

$$\sum_{k=1}^{m} k^2 N_{m-k} \equiv 0 \pmod{\frac{m}{r}},$$

with $r := \gcd(m, 12) = \gcd(\frac{n}{2}, 12) \in \{1, 2, 3, 4, 6, 12\}$. This implies that

$$F_1 \equiv 0 \pmod{\frac{12}{r}}. \tag{5.3}$$

Remark 5.1 Condition (5.3) proves Theorem A when n is even. $\oslash$

We then want to find the smallest positive value of

$$\sum_{k=1}^{m} k^2 N_{m-k}$$

which is a multiple of $\frac{m}{r}$ and such that

$$\sum_{k=1}^{m} \left(\frac{6k^2}{m} - 1\right) N_{m-k} \geqslant 0, \tag{5.4}$$

so that the expression on the right-hand-side of (5.1) is a non-negative integer. Then, by (5.2), the minimum $\mathcal{B}(n)$ of F_1 on $\mathcal{Z}_1$ is obtained by multiplying this value by $\frac{12}{m}$.

Remark 5.2 Note that, when $m \leqslant 6$, condition (5.4) is always satisfied. Hence, the smallest multiple of $\frac{m}{r}$ that satisfies all the required conditions is $\frac{m}{r}$ itself (taking for instance $N_{m-1} = \frac{m}{r}$, $N_m = \frac{2(6-m)}{r}$ and all other N_i's equal to 0), leading to

$$\mathcal{B}(n) = \frac{m}{r} \cdot \frac{12}{m} = \frac{12}{r}, \quad \text{whenever} \quad n = 2m \quad \text{with} \quad m \leqslant 6.$$

$\oslash$

In general, we see that (5.4) is equivalent to

$$\sum_{k=1}^{m} k^2 N_{m-k} \geqslant \frac{m}{6}\sum_{k=1}^{m} N_{m-k},$$

so our goal is to find the smallest positive multiple of $\frac{m}{r}$ which can be written as

$$\sum_{k=1}^{m} k^2 N_{m-k}$$

and is greater or equal to

$$\frac{m}{6}\sum_{k=1}^{m} N_{m-k}.$$

In other words, for each m, we want to find the smallest value of $\ell \in \mathbb{Z}_{>0}$ such that

$$\ell \cdot \frac{m}{r} = \sum_{k=1}^{m} k^2 N_{m-k} \geqslant \frac{m}{6}\sum_{k=1}^{m} N_{m-k}. \tag{5.5}$$

Note that the first sum in (5.5) is a sum of squares, possibly with repetitions (whenever one of the N_{m-k}s is greater than 1), and that the sum on the right hand side of (5.5) is precisely the number of squares used in this representation of $\ell \cdot \frac{m}{r}$ as a sum of squares. We then want to find the smallest value of $\ell \in \mathbb{Z}_{>0}$ such that

$$\sum_{k=1}^{m} N_{m-k} \leqslant \frac{6\ell}{r}, \tag{5.6}$$

where $\sum_{k=1}^{m} N_{m-k}$ is the smallest number of squares that is needed to represent the positive integer $\ell \cdot \frac{m}{r}$ as a sum of squares of numbers smaller or equal than m. We can then use the results in Section 3.

When $r = 1$, condition (5.6) becomes

$$\sum_{k=1}^{m} N_{m-k} \leqslant 6\ell. \tag{5.7}$$

This can be achieved with $\ell = 1$ since, by Theorem 3.1, the positive integer $\frac{m}{r} = m$ can be written as a sum of 4 or fewer squares (necessarily of numbers $\leqslant m$), and then

$$\sum_{k=1}^{m} N_{m-k} \leqslant 4 \leqslant 6 = 6\ell.$$

We conclude that, when $r = 1$, we always have $\mathcal{B}(n) = \frac{12}{m} \cdot \frac{m}{r} = 12$.

When $r = 2$, condition (5.6) becomes

$$\sum_{k=1}^{m} N_{m-k} \leqslant 3\ell. \tag{5.8}$$

Hence, if $\frac{m}{r} = \frac{m}{2}$ can be written as a sum of 3 or fewer squares (necessarily of numbers $\leqslant m$), (5.8) can be achieved with $\ell = 1$. Otherwise $\ell = 2$ suffices, since then, by Theorem 3.1, the number $\frac{2m}{r} = m$ can be written as a sum of 4 or fewer squares (necessarily of numbers $\leqslant m$) and then

$$\sum_{k=1}^{m} N_{m-k} \leqslant 4 \leqslant 6 = 3\ell.$$

Note that, since $r = 2$, the number $\frac{m}{2}$ cannot be a multiple of 4 and so the condition

$$\frac{m}{2} \neq 4^k(8t+7) \quad \text{for all} \quad k, t \in \mathbb{Z}_{\geqslant 0}$$

in Theorem 3.2 is, in this situation, equivalent to

$$\frac{m}{2} \neq 8t+7 \quad \text{for all} \quad t \in \mathbb{Z}_{\geqslant 0}$$

which, in turn, is equivalent to $m \not\equiv 14 \pmod{16}$. Hence, by Theorem 3.2, we conclude that, when $r = 2$, we have $\mathcal{B}(n) = \frac{12}{m} \cdot \frac{m}{2} = 6$ if $m \not\equiv 14 \pmod{16}$ and $\mathcal{B}(n) = \frac{12}{m} \cdot \frac{2m}{2} = 12$ otherwise.

When $r = 3$, condition (5.6) becomes

$$\sum_{k=1}^{m} N_{m-k} \leqslant 2\ell. \tag{5.9}$$

Hence, if $\frac{m}{r}$ is a square or a sum of 2 squares (necessarily of numbers $\leqslant m$), (5.9) can be achieved with $\ell = 1$. Otherwise $\ell = 2$ suffices, since then, by Theorem 3.1, the number $\frac{2m}{r} = \frac{2m}{3}$ can be written as a sum of 4 or fewer squares (necessarily of numbers $\leqslant m$). Hence, by Theorem 3.3, we conclude that, when $r = 3$, $\mathcal{B}(n) = \frac{12}{m} \cdot \frac{m}{3} = 4$ if all prime factors of $\frac{m}{3}$ congruent to $3 \pmod 4$ occur with even exponent and $\mathcal{B}(n) = \frac{12}{m} \cdot \frac{2m}{3} = 8$ otherwise.

When $r = 4$, condition (5.6) becomes

$$\sum_{k=1}^{m} N_{m-k} \leqslant \frac{3\ell}{2}. \tag{5.10}$$

Hence, if $\frac{m}{r} = \frac{m}{4}$ is a square (or, equivalently, if m is a square), (5.10) can be achieved with $\ell = 1$. Otherwise, if $\frac{2m}{r} = \frac{m}{2}$ can be written as a sum of 3 or fewer squares (necessarily of numbers $\leqslant m$), (5.10) can be achieved with $\ell = 2$. Otherwise, $\ell = 3$ suffices since then, by Theorem 3.2, the number $\frac{3m}{3} = \frac{3m}{4}$ can be written as a sum of 4 or fewer squares (necessarily of numbers $\leqslant m$).

Hence, by Theorem 3.2, we conclude that, when $r = 4$, we have $\mathcal{B}(n) = \frac{12}{m} \cdot \frac{m}{4} = 3$ if m is a square, $\mathcal{B}(n) = \frac{12}{m} \cdot \frac{2m}{4} = 6$ if m is not a square and $\frac{m}{2} \neq 4^k(8t+7)$ for all $k, t \in \mathbb{Z}_{\geqslant 0}$, and $\mathcal{B}(n) = \frac{12}{m} \cdot \frac{3m}{4} = 9$ in all other cases.

When $r = 6$, condition (5.6) becomes

$$\sum_{k=1}^{m} N_{m-k} \leqslant \ell. \tag{5.11}$$

Hence, if $\frac{m}{r} = \frac{m}{6}$ is a square, then (5.11) can be achieved with $\ell = 1$. Otherwise, if $\frac{2m}{r} = \frac{m}{3}$ is a square or a sum of 2 squares (necessarily of numbers $\leqslant m$), (5.11) can be achieved with $\ell = 2$.

If this is not the case and $\frac{3m}{r} = \frac{m}{2}$ is a sum of 3 or fewer squares (necessarily of numbers $\leqslant m$), then (5.11) can be achieved with $\ell = 3$. If this also does not hold, then $\ell = 4$ suffices since then, by Theorem 3.1, the number $\frac{4m}{r} = \frac{2m}{3}$ can be written as a sum of 4 or fewer squares (necessarily of numbers $\leqslant m$).

Note that, since $r = 6$, the number $\frac{m}{2}$ cannot be a multiple of 4. Hence, condition

$$\frac{m}{2} \neq 4^k(8t+7) \quad \text{for all} \quad k, t \in \mathbb{Z}_{\geqslant 0}$$

in Theorem 3.2 is, in this situation, equivalent to

$$\frac{m}{2} \neq 8t+7 \quad \text{for all} \quad t \in \mathbb{Z}_{\geqslant 0}$$

which, in turn, is equivalent to $m \neq 14 \pmod{16}$. Hence, by Theorems 3.2 and 3.3, we conclude that, when $r = 6$, we have $\mathcal{B}(n) = \frac{12}{m} \cdot \frac{m}{6} = 2$ if $\frac{m}{6}$ is a square; otherwise $\mathcal{B}(n) = \frac{12}{m} \cdot \frac{2m}{6} = 4$ if all prime factors of $\frac{m}{3}$ congruent to $3 \pmod 4$

occur with even exponent; if none of these holds then $\mathcal{B}(n) = \frac{12}{m} \cdot \frac{3m}{6} = 6$ if $m \neq 14 \pmod{16}$ and $\mathcal{B}(n) = \frac{12}{m} \cdot \frac{4m}{6} = 8$ otherwise.

When $r = 12$, condition (5.6) becomes

$$\sum_{k=1}^{m} N_{m-k} \leqslant \frac{\ell}{2}. \tag{5.12}$$

Hence, even if $\frac{m}{r}$ were a square, condition (5.12) could never be achieved with $\ell = 1$. If $\frac{2m}{r} = \frac{m}{6}$ is a square, (5.12) can be achieved with $\ell = 2$. If this is not the case and $\frac{3m}{r} = \frac{m}{4}$ is a square (or, equivalently, if m is a square), then (5.12) can be achieved with $\ell = 3$. (Note that if $\frac{m}{4}$ is a square then $\frac{m}{6}$ is not a square.) If this also does not hold and $\frac{4m}{r} = \frac{m}{3}$ is a square or a sum of two squares, then (5.12) can be achieved with $\ell = 4$. In none of the above holds and $\frac{5m}{r} = \frac{5m}{12}$ is a square or a sum of two squares then (5.12) could be achieved with $\ell = 5$. Note, however, that if $\frac{m}{3}$ is not a square nor a sum of two squares then, by Theorem 3.3, at least one prime factor of $\frac{m}{3}$ is congruent to $3 \pmod 4$ and occurs with odd exponent. Then, since $5 \neq 3 \pmod 4$, the number $\frac{5m}{12}$ also has this prime factor occurring with the same odd exponent and so, in this situation, $\frac{5m}{12}$ cannot be written as a sum of 2 or fewer squares, implying that this case is impossible.

If none of the above conditions are true but $\frac{6m}{r} = \frac{m}{2}$ is a sum of 3 or fewer squares, then (5.12) can be achieved with $\ell = 6$. If still $\frac{6m}{r} = \frac{m}{2}$ cannot be written as a sum of 3 or fewer squares then $\frac{7m}{r} = \frac{7m}{12}$ can, and so $\ell = 7$ suffices. Indeed, if $\frac{m}{2}$ cannot be written as a sum of 3 or fewer squares, then

$$\frac{m}{2} = 4^k(8t+7) \quad \text{for some} \quad k, t \in \mathbb{Z}_{\geqslant 0},$$

and $k \geqslant 1$ (since m is multiple of 4); then

$$\frac{7m}{12} = \frac{14}{3} \cdot 4^{k-1}(8t+7),$$

and so $8t + 7 = 0 \pmod 3$, implying that $t = 1 \pmod 3$. Hence,

$$\frac{7m}{12} = \frac{14}{3} \cdot 4^{k-1}(24\,t' + 15) = 14 \cdot 4^{k-1}(8t' + 5) = 4^{k-1}(8t'' + 70) = 4^{k-1}(8t''' + 6)$$

for some $t', t'', t''' \in \mathbb{Z}_{\geqslant 0}$ and so, by Theorem 3.2, the number $\frac{7m}{12}$ can be represented by a sum of 3 or fewer squares (necessarily of numbers $\leqslant m$).

We conclude, by Theorems 3.2 and 3.3 that, when $r = 12$, we have $\mathcal{B}(n) = \frac{12}{m} \cdot \frac{2m}{12} = 2$ if $\frac{m}{6}$ is a square, $\mathcal{B}(n) = \frac{12}{m} \cdot \frac{3m}{12} = 3$ if m is a square, and $\mathcal{B}(n) = \frac{12}{m} \cdot \frac{4m}{12} = 4$ if neither $\frac{m}{6}$ nor m are squares and all prime factors of $\frac{m}{3}$ congruent to $3 \pmod 4$ occur with even exponent. If none of these conditions hold then $\mathcal{B}(n) = \frac{12}{m} \cdot \frac{6m}{12} = 6$, if $\frac{m}{2} \neq 4^k(8t+7)$ (or, equivalently, $m \neq 4^k(16t+14)$) for any $k, t \in \mathbb{Z}_{\geqslant 0}$, and $\mathcal{B}(n) = 7$ otherwise.

□

Remark 5.3 In the Appendix we provide examples that show that all the cases listed in Theorem E are possible. ⊘

Remark 5.4 Note that, in Theorem E, the minimum value of F_1 in $\mathcal{Z}_1$ can always be attained with sums of squares of numbers strictly smaller than m, so that the minimal value can always be obtained with $N_0 = 0$. Hence, if we restrict to symplectic non-Hamiltonian circle actions, the resulting fact that $N_0 = 0$ [MD88] does not give lower bounds for the number of fixed points that are better than in the general case. ⊘

6. A lower bound when n is odd

Here we compute the minimal value $\mathcal{B}(n)$ of the function F_2 restricted to $\mathcal{Z}_2$, obtaining a lower bound for the number of fixed points of the S^1-action when n is *odd*.

Theorem F. *Let $n = 2m+1$ ($m \geqslant 1$) be an odd positive integer and let $\mathcal{B}(n)$ be the minimum of the function F_2 restricted to the set $\mathcal{Z}_2$, where F_2 and $\mathcal{Z}_2$ are respectively defined by (4.5) and (4.8). Then $\mathcal{B}(n)$ can take all values in the set $\{2,4,6,8,12,24\}$. In particular, if $r = \gcd\left(\left\lfloor \frac{n}{2} \right\rfloor - 1, 12\right)$ ($= \gcd(m-1,12)$), we have:*

(i) if $r \leqslant 4$ then $\mathcal{B}(n) = \frac{24}{r}$;
(ii) if $r = 6$ then
- $\mathcal{B}(n) = 4$ *if every prime factor of $\frac{2}{3}(m-1)+1$ congruent to $3 \pmod 4$ occurs with even exponent,*
- $\mathcal{B}(n) = 8$ *otherwise;*

(iii) if $r = 12$ then
- $\mathcal{B}(n) = 2$ *if $\frac{1}{12}(m-1)$ is a triangular number,*
- $\mathcal{B}(n) = 4$ *if $\frac{1}{12}(m-1)$ is not a triangular number and every prime factor of $\frac{2}{3}(m-1)+1$ congruent to $3 \pmod 4$ occurs with even exponent,*
- $\mathcal{B}(n) = 6$ *otherwise.*

Remark 6.1 As usual, we assume that $\gcd(0,12) = 12$. Note also that we consider 0 to be a triangular number. ⊘

Proof. A point $(N_0, \dots, N_m) \in (\mathbb{Z}_{\geqslant 0})^{m+1} \setminus \{0\}$ is in $\mathcal{Z}_2$ if and only if we have

$$G_2 := (1-m)N_m + \sum_{k=1}^{m} \Big(6k(k+1) - (m-1)\Big) N_{m-k} = 0. \tag{6.1}$$

If $m = 1$ this is equivalent to $12N_0 = 0$ and so the minimum of $F_2 := 2N_1$ on $\mathcal{Z}_2$ is $\mathcal{B}(3) = 2$ (attained with $N_0 = 0$ and $N_2 = 1$). Note that here $\frac{m-1}{12} = 0$ is a triangular number and $r := \gcd(m-1,12) = \gcd(0,12) = 12$.

If $m \neq 1$ then (6.1) is equivalent to

$$N_m = \sum_{k=1}^{m} \left(\frac{6k(k+1)}{m-1} - 1 \right) N_{m-k}. \tag{6.2}$$

Hence, to find $\min_{\mathcal{Z}_2} F_2$, we start by substituting (6.2) in (4.5), obtaining

$$F_2 = \frac{24}{m-1} \sum_{k=1}^{m} \frac{k(k+1)}{2} N_{m-k}. \tag{6.3}$$

Since F_2 is even and integer valued and $N_m \in \mathbb{Z}$, we have

$$\frac{12}{m-1}\sum_{k=1}^{m}\frac{k(k+1)}{2}N_{m-k} \in \mathbb{Z}.$$

Since $N_0, \ldots, N_{m-1} \in \mathbb{Z}$, this is equivalently to having

$$\sum_{k=1}^{m-1}\frac{k(k+1)}{2}N_{m-k} \equiv 0 \pmod{\frac{m-1}{r}},$$

with $r := \gcd(m-1, 12) = \gcd\left(\left\lfloor \frac{n}{2}\right\rfloor - 1, 12\right) \in \{1,2,3,4,6,12\}$. This implies that

$$F_2 \equiv 0 \pmod{\frac{24}{r}}. \tag{6.4}$$

Remark 6.2 Condition (6.4) proves Theorem A when n is odd. ⊘

We then want to find the smallest positive value of

$$\sum_{k=1}^{m}\frac{k(k+1)}{2}N_{m-k}$$

which is a multiple of $\frac{m-1}{r}$ and such that

$$\sum_{k=1}^{m}\left(\frac{6k(k+1)}{m-1} - 1\right)N_{m-k} \geqslant 0, \tag{6.5}$$

so that the expression on the right hand side of (6.2) is a non-negative integer. Then, by (6.3), the minimum $\mathcal{B}(n)$ of F_2 on $\mathcal{Z}_2$ is obtained by multiplying this value by $\frac{24}{m-1}$.

Remark 6.3 Note that, when $m \leqslant 13$, condition (6.5) is always satisfied. Hence, the smallest multiple of $\frac{m-1}{r}$ that satisfies all the required conditions is $\frac{m-1}{r}$ itself, leading to

$$\mathcal{B}(n) = \frac{m-1}{r}\cdot\frac{24}{m-1} = \frac{24}{r}, \quad \text{whenever} \quad n = 2m+1 \quad \text{with} \quad m \leqslant 13.$$

⊘

In general, we see that (6.5) is equivalent to

$$\sum_{k=1}^{m}\frac{k(k+1)}{2}N_{m-k} \geqslant \frac{m-1}{12}\sum_{k=1}^{m}N_{m-k},$$

so our goal is to find the smallest positive multiple of $\frac{m-1}{r}$ which can be written as

$$\sum_{k=1}^{m}\frac{k(k+1)}{2}N_{m-k}$$

and is greater or equal to

$$\frac{m-1}{12}\sum_{k=1}^{m}N_{m-k}.$$

In other words, for each m, we want to find the smallest value of $\ell \in \mathbb{Z}_{>0}$ such that

$$\ell \cdot \frac{m-1}{r} = \sum_{k=1}^{m} \frac{k(k+1)}{2} N_{m-k} \geqslant \frac{m-1}{12} \sum_{k=1}^{m} N_{m-k}. \tag{6.6}$$

Note that the first sum in (6.6) is a sum of triangular numbers, possibly with repetitions (whenever one of the N_{m-k}s is greater than 1), and that the sum on the right hand side of (6.6) is precisely the number of triangular numbers used in this representation of $\ell \cdot \frac{m-1}{r}$ as a sum of triangular numbers. We then want to find the smallest value of $\ell \in \mathbb{Z}_{>0}$ such that

$$\sum_{k=1}^{m} N_{m-k} \leqslant \frac{12\,\ell}{r}, \tag{6.7}$$

where $\sum_{k=1}^{m} N_{m-k}$ is the smallest number of triangular numbers $\frac{k(k+1)}{2}$ that is needed to represent the positive integer $\ell \cdot \frac{m-1}{r}$ as a sum of triangular numbers with $k \leqslant m$. We can therefore use the results in Section 3 concerning these numbers.

Since, by Theorem 3.5, we know that every positive integer can be written as a sum of 3 or fewer triangular numbers, condition (6.7) can be achieved with $\ell = 1$ whenever $r \leqslant 4$ and then $\mathcal{B}(n) = \frac{24}{m-1} \cdot \frac{m-1}{r} = \frac{24}{r}$. Note that in all these cases the triangular numbers $\frac{k(k+1)}{2}$ are such that $k \leqslant m$.

When $r = 6$, condition (6.7) becomes

$$\sum_{k=1}^{m} N_{m-k} \leqslant 2\ell. \tag{6.8}$$

Hence, if $\frac{m-1}{r} = \frac{m-1}{6}$ can be written as a sum of 2 or fewer triangular numbers (necessarily $\leqslant m$, yielding $k \leqslant m$), (6.8) can be achieved with $\ell = 1$. Otherwise we need $\ell = 2$, since then, by Theorem 3.5, the number $\frac{2m}{r} = \frac{m}{3}$ can be written as a sum of 3 or fewer triangular numbers (necessarily $\leqslant m$) and so

$$\sum_{k=1}^{m} N_{m-k} \leqslant 3 \leqslant 4 = 2\ell.$$

By Theorem 3.6, we conclude that $\mathcal{B}(n) = \frac{24}{m-1} \cdot \frac{m-1}{6} = 4$ if every prime factor of $4\left(\frac{m-1}{6}\right) + 1$ congruent to 3 (mod 4) occurs with even exponent and $\mathcal{B}(n) = \frac{24}{m-1} \cdot \frac{2(m-1)}{6} = 8$ otherwise.

When $r = 12$, condition (6.7) becomes

$$\sum_{k=1}^{m} N_{m-k} \leqslant \ell. \tag{6.9}$$

Hence, if $\frac{m-1}{r}$ is a triangular number, then (6.9) can be achieved with $\ell = 1$. Otherwise, if $\frac{2(m-1)}{r} = \frac{m-1}{6}$ can be written as a sum of 2 or fewer triangular numbers (necessarily $\leqslant m$), (6.9) can be achieved with $\ell = 2$. If this is not the case, $\ell = 3$ suffices since then, by Theorem 3.5, the number $\frac{3(m-1)}{12} = \frac{m-1}{4}$ can be written as a sum of 3 or fewer triangular numbers (necessarily $\leqslant m$).

By Theorem 3.6, we conclude that $\mathcal{B}(n) = \frac{24}{m-1} \cdot \frac{m-1}{12} = 2$, if $\frac{m-1}{12}$ is a triangular number, $\mathcal{B}(n) = \frac{24}{m-1} \cdot \frac{2(m-1)}{12} = 4$, if every prime factor of $\frac{2}{3}(m-1) + 1$ congruent to 3 (mod 4) occurs with even exponent, and $\mathcal{B}(n) = \frac{24}{m-1} \cdot \frac{3(m-1)}{12} = 6$ in all other cases. $\square$

Remark 6.4 In the Appendix we provide examples that show that all the cases listed in Theorem F are possible. $\oslash$

Remark 6.5 Note that, in Theorem F, the minimum value of F_2 in $\mathcal{Z}_2$ can always be attained with sums of triangular numbers $\frac{k(k+1)}{2}$ with k strictly smaller than m so that the minimal values can always be obtained with $N_0 = 0$. Hence, if we restrict to symplectic non-Hamiltonian circle actions, the resulting fact that $N_0 = 0$ does not give lower bounds for the number of fixed points that are better than in the general case. $\oslash$

7. Proofs of Theorems A, B and C

Proof. (of Theorem A) This result follows immediately from (5.3) and (6.4) in the proofs of Theorems E and F in Sections 5 and 6, since the functions F_1 and F_2 count the total number of fixed points respectively when n is even or odd. Note that, in Theorem A, we write $n = 2m+3$ instead of $n = 2m+1$, when n is odd, to simplify the statement. $\square$

Proof. (of Theorem B) This follows from Theorems E and F in Sections 5 and 6, using the lower bound in Theorem 2.8.

In particular, in Theorem E, if $r \geqslant 4$, then $\dim M \geqslant 16$, and so the number of fixed points must be $\geqslant 4$ by Theorem 2.8. Hence, when $r = 4$, the lower bound of 6 holds even if m is a square since, by Theorem A, we know that $|M^{S^1}|$ is a multiple of 3 (note that, if m is a square, then $m \neq 4^k(16t+14)$ for every $k, t \in \mathbb{Z}_{\geqslant 0}$). When $r = 6$ or $r = 12$, the lower bound of 4 holds even if $\frac{m}{6}$ is a square; note that, if $\frac{m}{6}$ is a square, all prime factors of $\frac{m}{3}$ that are congruent to 3 (mod 4) occur with even exponent. When $r = 12$, the lower bound of 4 also holds if m is a square.

In Theorem F, if $r = 12$ and $m \neq 1$, then $\dim M \geqslant 50$ and so, by Theorem 2.8, the number of fixed points must be $\geqslant 4$. Hence the lower bound of 4 holds even if $\frac{1}{12}(m-1)$ is a triangular number; note that, if this is the case, and k is such that $\frac{1}{12}(m-1) = \frac{k(k+1)}{2}$, then the number $\frac{2}{3}(m-1)+1 = (2k+1)^2$ is a square and so all its prime factors occur with even exponent.

Again we write $n = 2m+3$ to simplify the statement of the theorem. $\square$

Proof. (of Theorem C) The S^1-action is now Hamiltonian, implying that each N_i, the number of fixed points with exactly i negative isotropy weights, coincides with the Betti number $b_{2i}(M)$. Hence, since the classes $[\omega^k] \in H^{2k}(M, \mathbb{R})$ are non trivial, we have $N_i \geqslant 1$ for all $i = 0, \ldots, m$. Moreover, since M is connected and the fixed point set is discrete, there is only one fixed point of index 0 (where the Hamiltonian function is minimal), and so $N_0 = 1$. Consequently, we now want to minimize the functions F_i defined in (4.4) and (4.5), respectively on the sets

$$\widetilde{\mathcal{Z}}_i := \left\{ (N_0, \ldots, N_m) \in (\mathbb{Z}_{\geqslant 1})^{m+1} \mid G_i(N_0, \ldots, N_m) = 0,\, N_0 = 1 \right\}. \tag{7.1}$$

When n is even the proof follows easily from (5.2), knowing that $N_{m-k} \geqslant 1$ for all $k = 1, \dots, m$. Indeed, in this case, the smallest positive integer value of F_1 on $\mathcal{Z}_1$ is attained when $N_0 = \cdots = N_{m-1} = 1$, yielding

$$\sum_{k=1}^{m} k^2 N_{m-k} = \sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6},$$

and

$$F_1 = 2(m+1)(2m+1) = (n+2)(n+1).$$

Note from (5.1), that this value is achieved with

$$N_m = 2\sum_{k=1}^{m}\Big(\frac{6k^2}{m} - 1\Big)N_{m-k} = \frac{12}{m}\sum_{k=1}^{m} k^2 - 2m = 2(2m^2+2m+1) \geqslant 1.$$

When $n = 2m+1 > 3$ is odd, we may no longer be able to take all $N_i = 1$ for $i = 0, \dots, m-1$ as we do in the even case, since the corresponding value of F_2 (given by (6.3)) may not be an integer. Hence we take $\widetilde{N}_{m-k} := N_{m-k} - 1 \in \mathbb{Z}_{\geqslant 0}$ for $k = 0, \dots, m-1$ and then, from (6.2) and (6.3), we have that on $\widetilde{\mathcal{Z}}_2$,

$$\begin{aligned} N_m &= \sum_{k=1}^{m}\left(\frac{6k(k+1)}{m-1} - 1\right) + \sum_{k=1}^{m-1}\left(\frac{6k(k+1)}{m-1} - 1\right)\widetilde{N}_{m-k} \\ &= \frac{2m(m+1)(m+2)}{m-1} - m + \sum_{k=1}^{m-1}\left(\frac{6k(k+1)}{m-1} - 1\right)\widetilde{N}_{m-k} \end{aligned} \tag{7.2}$$

and

$$\begin{aligned} F_2 &= \frac{24}{m-1}\Big(\sum_{k=1}^{m}\frac{k(k+1)}{2} + \sum_{k=1}^{m-1}\frac{k(k+1)}{2}\widetilde{N}_{m-k}\Big) \\ &= \frac{24}{m-1}\Big(\frac{m(m+1)(m+2)}{6} + \sum_{k=1}^{m-1}\frac{k(k+1)}{2}\widetilde{N}_{m-k}\Big). \end{aligned} \tag{7.3}$$

Here we used the fact that the sum of the first m consecutive triangular numbers (starting at 1) is $\frac{m(m+1)(m+2)}{6}$.

Since F_2 is even and integer valued and $\widetilde{N}_{m-k} \in \mathbb{Z}$, we have

$$\frac{m(m+1)(m+2)}{6} + \sum_{k=1}^{m-1}\frac{k(k+1)}{2}\widetilde{N}_{m-k} \equiv 0 \pmod{\frac{m-1}{r}},$$

with $r := \gcd(m-1, 12)$.

We then want to find the smallest multiple of $\frac{m-1}{r}$ greater or equal than $\frac{m(m+1)(m+2)}{6}$, which can be written as

$$\frac{m(m+1)(m+2)}{6} + \sum_{k=1}^{m-1}\frac{k(k+1)}{2}\widetilde{N}_{m-k},$$

and such that

$$\frac{12}{m-1}\Big(\frac{m(m+1)(m+2)}{6} + \sum_{k=1}^{m-1}\frac{k(k+1)}{2}\widetilde{N}_{m-k}\Big) - (1+m) \geqslant \sum_{k=1}^{m-1}\widetilde{N}_{m-k}$$

so that the expression on the right hand side of (7.2) is $\geqslant 1$. Then, by (7.3), the minimum of F_2 on $\widetilde{\mathcal{Z}}_2$ is obtained by multiplying this value by $\frac{24}{m-1}$.

In other words, we want to find the smallest value of $\ell \in \mathbb{Z}_{>0}$ such that

$$\ell \cdot \frac{m-1}{r} = \frac{m(m+1)(m+2)}{6} + \sum_{k=1}^{m-1} \frac{k(k+1)}{2} \widetilde{N}_{m-k} \geqslant \frac{m-1}{12}\Big(\sum_{k=1}^{m-1} \widetilde{N}_{m-k} + (1+m)\Big), \tag{7.4}$$

i.e. we want to find the smallest value of $\ell \in \mathbb{Z}_{>0}$ such that

$$\ell \cdot \frac{m-1}{r} \geqslant \frac{m(m+1)(m+2)}{6} \tag{7.5}$$

and

$$\sum_{k=1}^{m-1} \widetilde{N}_{m-k} \leqslant \frac{12\,\ell}{r} - (1+m), \tag{7.6}$$

where $\sum_{k=1}^{m-1} \widetilde{N}_{m-k}$ is the smallest number of triangular numbers that is needed to represent the nonnegative integer

$$A := \ell \cdot \frac{m-1}{r} - \frac{m(m+1)(m+2)}{6}$$

as a sum of triangular numbers $\frac{k(k+1)}{2}$ with $k \leqslant m-1$.

Now the smallest integer ℓ that verifies (7.5) is

$$\ell = \left\lceil \frac{rm(m+1)(m+2)}{6(m-1)} \right\rceil = \frac{r(m^2+4m+6)}{6} + \left\lceil \frac{r}{m-1} \right\rceil = \frac{r(m^2+4m+6)}{6} + 1.$$

Note that $r(m^2+4m+6) = r\,m(m+4) + 6r \equiv 0 \pmod 6$ since $r\,m$ is always even and $r\,m(m+4) \equiv 0 \pmod 3$ (if $r \not\equiv 0 \pmod 3$ then $m-1 \not\equiv 0 \pmod 3$, implying that either m or $m+4$ is a multiple of 3).

For this value of ℓ we have $A < \frac{m-1}{r} \leqslant m-1$ and so, by Theorem 3.5, A can be represented as a sum of at most three triangular numbers $\frac{k(k+1)}{2}$ with $k \leqslant m-1$. Condition (7.6) can then be achieved with this value of ℓ as

$$\frac{12\,\ell}{r} - (1+m) > 3.$$

Hence the minimum of F_2 on $\widetilde{\mathcal{Z}}_2$ is

$$\frac{24}{m-1} \cdot \frac{\ell(m-1)}{r} = 4(m^2+4m+6) + \frac{24}{r} = n^2 + 6n + 17 + \frac{24}{r}.$$

□

8. Comparing with other bounds

Although our lower bound $\mathcal{B}(n)$ does not, in general, increase with n, there are some values of n for which $\mathcal{B}(n)$ is better than the lower bound $\left\lfloor \frac{n}{2} \right\rfloor + 1$ proposed by Kosniowski [K79] and some for which it is greater than n and we recover the lower bound for Kähler (Hamiltonian) actions. These are listed in the following results which are easy consequences of Theorem B.

Proposition 8.1. *Let $\mathcal{B}(n)$ be the lower bound for the number of fixed points of a J-preserving circle action on a $2n$-dimensional compact connected almost complex manifold (M, J) with $c_1c_{n-1}[M] = 0$ obtained in Theorem B. Then, if*

$$\dim M \in \{4, 6, 8, 10, 12, 14, 18, 20, 22, 26, 28, 34, 44, 46, 50, 58, 74, 82\},$$

we have $\mathcal{B}(n) \geqslant \left\lfloor \frac{n}{2} \right\rfloor + 1$. In particular, the lower bound proposed by Kosniowski is valid for these dimensions, whenever $c_1c_{n-1}[M] = 0$.

Proposition 8.2. *Let $\mathcal{B}(n)$ be the lower bound for the number of fixed points of a J-preserving circle action on a $2n$-dimensional compact connected almost complex manifold (M, J) with $c_1c_{n-1}[M] = 0$ obtained in Theorem B. Then, if*

$$\dim M \in \{4, 8, 10, 14, 20, 26, 34\},$$

we have $\mathcal{B}(n) \geqslant n + 1$.

9. Divisibility results for the number of fixed points

In a letter to V. Gritsenko, Hirzebruch [Hi99] obtains divisibility results for the Chern number $c_n[M]$ (the Euler characteristic of the manifold) under the assumption $c_1c_{n-1}[M] = 0$ (or under the stronger assumption that $c_1 = 0$ in integer cohomology). In particular, he proves the following result.

Theorem 9.1 (Hirzebruch)**.** *Let M be a $2n$-dimensional unitary manifold. If $c_1c_{n-1}[M] = 0$ then*

- *if $n \equiv 1$ or $5 \pmod 8$, the Chern number $c_n[M]$ is divisible by 8;*
- *if $n \equiv 2, 6$ or $7 \pmod 8$, the Chern number $c_n[M]$ is divisible by 4;*
- *if $n \equiv 3$ or $4 \pmod 8$, the Chern number $c_n[M]$ is divisible by 2.*

If an almost complex manifold is equipped with an S^1-action preserving the almost complex structure with a nonempty discrete fixed point set, we know that $c_n[M]$ is equal to the number of fixed points of the action (see for example [GS12, Section 3]). Therefore, we can also obtain divisibility results for $c_n[M]$ from Theorem A.

When $n \equiv 0 \pmod 3$ the divisibility factors of $c_n[M]$ (or $|M^{S^1}|$) obtained from Theorem A are exactly those of Hirzebruch. However, when $n \not\equiv 0 \pmod 3$, Theorem A implies that $c_n[M]$ (or $|M^{S^1}|$) is a multiple of 3, and so we can improve Hirzebruch's result.

Theorem G. *Let (M, J) be a $2n$-dimensional compact connected almost complex manifold equipped with a J-preserving S^1-action with nonempty, discrete fixed point set M^{S^1}. If $c_1c_{n-1}[M] = 0$ and $n \not\equiv 0 \pmod 3$ then*

- *if $n \equiv 0 \pmod 8$, then $|M^{S^1}|$ is divisible by 3;*
- *if $n \equiv 1$ or $5 \pmod 8$, then $|M^{S^1}|$ is divisible by 24;*
- *if $n \equiv 2, 6$ or $7 \pmod 8$, then $|M^{S^1}|$ is divisible by 12;*
- *if $n \equiv 3$ or $4 \pmod 8$, then $|M^{S^1}|$ is divisible by 6.*

Proof. If $n = 2m$ is even, we can write $n \equiv 2k \pmod 8$ with $k \in \{0, 1, 2, 3\}$ and $m \equiv k \pmod 4$. Moreover, $n \not\equiv 0 \pmod 3$ implies that $m \not\equiv 0 \pmod 3$. Hence, if

$r := \gcd(m, 12)$, we have

$$\begin{aligned} r &= 4 \quad \text{if } k = 0, \\ r &= 1 \quad \text{if } m \text{ is odd (i.e. if } k = 1 \text{ or } 3), \\ r &= 2 \quad \text{if } k = 2. \end{aligned}$$

The result for even values of n then follows from Theorem A.

If $n = 2m+3$ is odd, we can write $n \equiv 2k+3 \pmod 8$ with $k \in \{0, 1, 2, 3\}$ and $m \equiv k \pmod 4$. Moreover, $n \not\equiv 0 \pmod 3$ implies that $m \not\equiv 0 \pmod 3$. Hence, if $r := \gcd(m, 12)$, we have

$$\begin{aligned} r &= 4 \quad \text{if } k = 0, \\ r &= 1 \quad \text{if } m \text{ is odd (i.e. if } k = 1 \text{ or } 3), \\ r &= 2 \quad \text{if } k = 2. \end{aligned}$$

The result for odd values of n then follows from Theorem A. □

Remark 9.2 When $n \equiv 0 \pmod 3$, the divisibility factors of $c_n[M]$ (or $|M^{S^1}|$) obtained from Theorem A are exactly those proved by Hirzebruch and listed in Theorem 9.1.

Indeed, if $n = 2m$, then we can write $n \equiv 2k \pmod 8$ with $k \in \{0, 1, 2, 3\}$ and $m \equiv k \pmod 4$. If $n \equiv 0 \pmod 3$, then $m \equiv 0 \pmod 3$ and so, if $r := \gcd(m, 12)$,

$$\begin{aligned} r &= 12 \quad \text{if } k = 0, \\ r &= 3 \quad \text{if } m \text{ is odd}, \\ r &= 6 \quad \text{if } k = 2. \end{aligned} \tag{9.1}$$

If $n = 2m + 3$, we can write $n \equiv 2k + 3 \pmod 8$ with $k \in \{0, 1, 2, 3\}$ and again $m \equiv k \pmod 4$. If $n \equiv 0 \pmod 3$, then $m \equiv 0 \pmod 3$, and we get the same values of $r := \gcd(m, 12)$ as in (9.1). In all cases we recover Hirzebruch's divisibility factors in Theorem 9.1. ⊘

In summary, we obtain the divisibility factors listed in Table 9.1.

$n \pmod 8$	$\lvert M^{S^1}\rvert$ is divisible by	
0	1	if $n \equiv 0 \pmod 3$
	3	otherwise
1	8	if $n \equiv 0 \pmod 3$
	24	otherwise
2	4	if $n \equiv 0 \pmod 3$
	12	otherwise
3	2	if $n \equiv 0 \pmod 3$
	6	otherwise

$n \pmod 8$	$\lvert M^{S^1}\rvert$ is divisible by	
4	2	if $n \equiv 0 \pmod 3$
	6	otherwise
5	8	if $n \equiv 0 \pmod 3$
	24	otherwise
6	4	if $n \equiv 0 \pmod 3$
	12	otherwise
7	4	if $n \equiv 0 \pmod 3$
	12	otherwise

TABLE 9.1. Divisibility factors of $|M^{S^1}|$.

Under the stronger condition that $c_1 = 0$ in integer cohomology, Hirzebruch was able to improve his divisibility factor for $c_n[M]$ in some situations [Hi99].

Proposition 9.3 (Hirzebruch)**.** *If M is a $2n$-dimensional unitary manifold with $c_1 = 0$ and even $n = 2m$ with $m \equiv 1 \pmod 4$, then $c_n[M] \equiv 0 \pmod 8$.*

Knowing this, we are also able to further improve the divisibility factor for $|M^{S^1}|$ under this condition.

Theorem H. *Let (M,J) be a $2n$-dimensional compact connected almost complex manifold equipped with a J-preserving S^1-action with nonempty, discrete fixed point set M^{S^1}, and such that $n \equiv 2 \pmod 8$ and $c_1 = 0$. If $n \not\equiv 0 \pmod 3$, then $|M^{S^1}|$ is divisible by 24.*

Proof. By Theorem G and Proposition 9.3, we have that $|M^{S^1}| \equiv 0 \pmod{12}$ and $|M^{S^1}| \equiv 0 \pmod 8$ so the result follows. □

Using these two results, we can improve, in some situations, the lower bound for the number of fixed points given by $\mathcal{B}(n)$.

Theorem I. *Let (M,J) be a $2n$-dimensional compact connected almost complex manifold equipped with a J-preserving S^1-action with nonempty, discrete fixed point set M^{S^1}, and such that $c_1 = 0$ and $n \equiv 2 \pmod 8$. Then the number of fixed points is at least 24 if $n \not\equiv 0 \pmod 3$ and at least 8 otherwise.*

Remark 9.4 If $n = 2m$, $n \equiv 2 \pmod 8$ and $n \not\equiv 0 \pmod 3$, then, from Theorem B, we always have $\mathcal{B}(n) = 12$, since $\gcd(m, 12) = 1$ (m is odd and is not a multiple of 3); if $n \equiv 0 \pmod 3$, then $\mathcal{B}(n)$ is either 4 or 8, since $\gcd(m, 12) = 3$ (m is odd and a multiple of 3). For example, if $n = 54$, we have $\mathcal{B}(54) = 4$ (since $\frac{m}{3} = 3^2$) but, since $m = 27 \equiv 0 \pmod 3$, we know that, if $c_1 = 0$, then the number of fixed points is at least 8 (c.f. Theorem I). ⊘

10. Examples

We will now show that some of the lower bounds obtained in Theorem B for the number of fixed points are sharp.

Example 10.1 There exists a 4 dimensional almost complex manifold (N^4, J) with $c_1^2[N] = 0$ that admits a J-preserving circle action with 12 fixed points (note that, since $n = 2$, we have $\gcd(\frac{n}{2}, 12) = 1$ and $\mathcal{B}(2) = 12$). Indeed, from (4.3) we can just take

$$N^4 = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2,$$

the 9-point blow-up of $\mathbb{CP}^2$ since

$$b_2(N) = 10 \quad \text{and} \quad b_0(N) = b_4(N) = 1,$$

so that, by (4.3),

$$c_1^2[N] = 10\, b_0(N) - b_2(N) = 0 \quad \text{and} \quad b_0(N) + b_2(N) + b_4(N) = 12.$$

Taking a standard Hamiltonian circle action on $\mathbb{CP}^2$ (with 3 isolated fixed points) and blowing up successively at index 2 fixed points, we can obtain a Hamiltonian circle action on N with exactly 12 fixed points. ⊘

Example 10.2 For $\dim M = 6$ we can take $M = S^6$ with the almost complex structure induced by a vector product in $\mathbb{R}^7$ and equipped with the S^1-action induced by the action on $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$ given by

$$\lambda \cdot (t, z_1, z_2, z_3) = (t, \lambda^n z_1, \lambda^m z_2, \lambda^{-(n+m)} z_3), \quad \lambda \in S^1,$$

with $t \in \mathbb{R}$, $z_1, z_2, z_3 \in \mathbb{C}$, $m, n \in \mathbb{Z} \setminus \{0\}$ and $m + m \neq 0$. This action has exactly 2 fixed points and $N_1 = N_2 = 1$ (note that $\mathcal{B}(3) = 2$). ⊘

Example 10.3 In any dimension, since we can write every even positive integer $2n \geqslant 4$ as

$$2n = 2(2k + 3\ell) = 4k + 6\ell,$$

for some $k, \ell \in \mathbb{Z}_{\geqslant 0}$, we can take

$$M = (N^4)^k \times (S^6)^\ell,$$

where N^4 is the S^1-manifold in Example 10.1 and S^6 has the action in Example 10.2, to obtain an example of dimension $2n$. By Lemma 2.16 this almost complex manifold satisfies

$$c_1 c_{n-1}[M] = 0,$$

and the diagonal circle action preserves the almost complex structure and has $2^\ell \times 12^k$ fixed points.

If $k = \ell = 1$ then $\dim M = 10$ and the action has a minimal number of fixed points. Indeed, it has 24 fixed points and $\mathcal{B}(5) = 24$.

If $k = 0$ and $\ell = 2$ then $\dim M = 12$ and the action has exactly 4 fixed points so it also has a minimal number of fixed points (since $\mathcal{B}(6) = 4$).

If $k = 0$ and $\ell = 3$ then $\dim M = 18$ and the action has 8 fixed points which is also a minimal number ($\mathcal{B}(9) = 8$).

⊘

Remark 10.4 It would be very interesting to find out if there exists an 8 dimensional almost complex manifold (M^8, J) satisfying $c_1 c_3[M] = 0$ and with a J preserving circle action with exactly $\mathcal{B}(4) = 6$ fixed points. If this example could be constructed, then $M^8 \times S^6$ would give us a minimal example with $\mathcal{B}(7) = 12$ fixed points. ⊘

Example 10.5 Returning to Example 10.3, we see that, although the S^1-manifolds $(N^4)^k \times (S^6)^\ell$ do not always have a minimal number of fixed points, $|M^{S^1}| = 2^\ell \times 12^k$, is always consistent with Theorems 9.1 and G in Section 9.

Indeed, if n is even and $k \neq 0$, then $|M^{S^1}|$ is a multiple of 12. If n is even and $k = 0$, then necessarily $n \equiv 0 \pmod 3$. Since $n = 3\ell$ is even, we have $\ell > 1$, and so $|M^{S^1}|$ is a multiple of 4.

If n is odd, then necessarily $\ell > 1$. If $k \neq 0$, then $|M^{S^1}|$ is a multiple of 24. If $k = 0$, then necessarily $n \equiv 0 \pmod 3$. We then have $n = 3\ell$ and 2^ℓ fixed points. If $\ell = 1$, then $n \equiv 3 \pmod 8$ and $|M^{S^1}|$ is divisible by 2; if $\ell = 2$, then $n \equiv 6 \pmod 8$ and $|M^{S^1}|$ is divisible by 4; if $\ell \geqslant 3$, then $|M^{S^1}|$ is a multiple of 8. ⊘

Remark 10.6 If M^{2n} is an almost complex S^1-manifold satisfying $c_1 c_{n-1}[M] = 0$, then $M \times S^6$, where S^6 has the action in Example 10.2, satisfies $c_1 c_{n-1}[M \times S^6] = 0$ (see Lemma 2.16). Then, by Theorem B, we have

$$2\,|M^{S^1}| = \left|\,\left(M \times S^6\right)^{S^1}\right| \geqslant \mathcal{B}(n+3),$$

and so $\mathcal{B}(n+3)/2$ is also a lower bound for the number of fixed points of M. It is easy to check that $\mathcal{B}(n)$, the lower bound for $\left|M^{S^1}\right|$ obtained in Theorem B, is better, i.e.

$$\mathcal{B}(n) \geqslant \frac{\mathcal{B}(n+3)}{2}, \tag{10.1}$$

for every n. In particular, (10.1) trivially holds if $\mathcal{B}(n)$ is equal to 24 or 12 since $\mathcal{B}(n+3) \leqslant 24$ in all cases; the other possibilities are listed in Table 10.1.

$\mathcal{B}(n)$	n	$r = \gcd(m,12)$	$\mathcal{B}(n+3)/2$	
9	$n=2m$	4	3	$n+3=2m+3$
8	$n=2m$	3 or 6	$\leqslant 4$	$n+3=2m+3$
	$n=2m+3$	3 or 6	$\leqslant 4$	$n+3=2(m+3)$ and $\gcd(m+3,12)=3,6$ or 12
7	$n=2m$	12	$\leqslant 3$	$n+3=2m+3$
6	$n=2m$	$2,4,6$ or 12	$\leqslant 6$	$n+3=2m+3$
	$n=2m+3$	4 or 12	$\leqslant 6$	$n+3=2(m+3)$ and $\gcd(m+3,12)=1$ or 3
4	$n=2m$	3, 6 or 12	$\leqslant 4$	$n+3=2m+3$
	$n=2m+3$	6 or 12	$\leqslant 4$	$n+3=2(m+3)$, $m+3$ odd $\gcd(m+3,12)=3$
2	$n=3$	12	2	

TABLE 10.1. Possible values of $\mathcal{B}(n)$ and $\mathcal{B}(n+3)/2$ (non trivial cases).

Similarly, it is also easy to check that

$$\mathcal{B}(n) \geqslant \frac{\mathcal{B}(n+k)}{\mathcal{B}(k)} \tag{10.2}$$

for all $k \geqslant 3$. If $k > 3$, we just need to rule out the possibility of having $\mathcal{B}(n) = \mathcal{B}(k) = 4$ and $\mathcal{B}(n+k) = 24$, since all other cases trivially satisfy (10.2). Note that $n+k$ must be odd if $\mathcal{B}(n+k) = 24$.

In this situation, if $n = 2m$ is even, then k is odd and so $k = 2a+3$ for some $a \geqslant 1$. Then, since $\mathcal{B}(k) = 4$, we have $\gcd(a,12)$ equal to 6 or 12, implying that a is a multiple of 6. But then, since $n+k = 2(m+a)+3$ and $\gcd(m+a,12) = 1$ (since $\mathcal{B}(n+k) = 24$), we have that m is odd and $m \not\equiv 0 \pmod 3$, leading to $\gcd(m,12) = 1$, contradicting the fact that $\mathcal{B}(n) = 4$.

If $n = 2m+3$ is odd, then $k = 2a$ for some $a \geqslant 2$. Then, since $\mathcal{B}(k) = 4$, we have $\gcd(a,12)$ equal to $3,6$ or 12, implying that a is either even or a multiple of 3. Since $\mathcal{B}(n) = 4$, we have that $\gcd(m,12)$ is equal to 6 or 12, and so m must be an even multiple of 3. But then $\gcd(m+a,12) \neq 1$ contradicting the fact that $\mathcal{B}(n+k) = 24$ (since $n+k = 2(m+a)+3$). ⊘

Appendix A. Tables

n	m	$r = \gcd(m, 12)$	$\mathcal{B}(n)$	
26	13	1	12	
20	10	2	6	$10 \not\equiv 14 \pmod{16}$
28	14	2	12	
54	27	3	4	$\frac{m}{3} = 3^2$
18	9	3	8	$\frac{m}{3} = 3$
32	16	4	3	$m = 4^2$
40	20	4	6	m is not a square and $m = 4 \cdot 5 \neq 4^k(16t + 14), \forall k, t \in \mathbb{Z}_{\geqslant 0}$
112	56	4	9	$m = 56$ is not a square and $m = 4 \cdot 14$
108	54	6	2	$\frac{m}{6} = 3^2$
60	30	6	4	$\frac{m}{6} = 5$ is not a square and $\frac{m}{3} = 2 \cdot 5$
180	90	6	6	$\frac{m}{6} = 15$ is not a square, $\frac{m}{3} = 2 \cdot 3 \cdot 5$ and $m = 16 \cdot 5 + 10$
252	126	6	8	$\frac{m}{6} = 21$ is not a square, $\frac{m}{3} = 2 \cdot 3 \cdot 7$ and $m = 16 \cdot 7 + 14$
48	24	12	2	$\frac{m}{6} = 2^2$
72	36	12	3	$\frac{m}{6} = 6$ is not a square and $m = 6^2$
24	12	12	4	$\frac{m}{6}$, m are not squares and $\frac{m}{3} = 2^2$
144	72	12	6	$\frac{m}{6} = 12$ and $m = 72$ are not squares, $\frac{m}{3} = 2^3 \cdot 3$ and $m = 4(16 + 2)$
1008	504	12	7	$\frac{m}{6} = 84$, $m = 504$ are not squares, $\frac{m}{3} = 2^3 \cdot 3 \cdot 7$ and $m = 4(16 \cdot 7 + 14)$

Table A.1. Examples that illustrate all values of $\mathcal{B}(n)$ obtained from Theorem E in Section 5 when $n := \frac{1}{2}\dim M$ is even (by increasing order of r).

$n = 2m+1$	$m-1$	$r = \gcd(m-1, 12)$	$\mathcal{B}(n)$	
39	18	6	4	$\frac{2}{3}(m-1)+1 = 13$
63	30	6	8	$\frac{2}{3}(m-1)+1 = 3 \cdot 7$
75	36	12	2	$\frac{m-1}{12} = \frac{2 \cdot 3}{2}$
51	24	12	4	$\frac{2}{3}(m-1)+1 = 17$
99	48	12	6	$\frac{1}{12}(m-1) = 4$ is not triangular and $\frac{2}{3}(m-1)+1 = 3 \cdot 11$

Table A.2. Examples that illustrate the possible values of $\mathcal{B}(n)$ obtained from the nontrivial cases ($r = 6$ or 12) of Theorem F in Section 6, when $n = \frac{1}{2}\dim M$ is odd.

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