ADDENDUM AND ERRATUM TO "THE FUNDAMENTAL GROUP OF S^1-MANIFOLDS"

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The results of the paper require slight modification in light of a number of inaccuracies in the proof of Lemma 2.5 in Page 1088. The source of the mistakes was failing to account for the possible existence of special **exceptional orbits**[1] on a compact symplectic S^1-manifold M with a nonempty fixed point set, that are not taken to a fixed point by the gradient flow of the generalized moment map. Note that there are no known examples of such actions since all the examples obtained so far of non-Hamiltonian symplectic circle actions on compact manifolds with a non-empty fixed point set are semifree [MD, K]. Note also that examples with exceptional orbits could easily be constructed from these ones by a sequence of blow ups at fixed points but the exceptional orbits obtained in this way would flow to fixed points. In the proof of Lemma 2.5, and consequently in the proofs of our main results, Theorems 2.1 and 3.1, we claim that the existence of a fixed point implies that all orbits are contractible in M which could be wrong if the above exceptional orbits existed.

The main theorems of the paper should therefore be restated as follows.

Theorem 2.1. *Let M be a connected, compact symplectic manifold equipped with a non-Hamiltonian symplectic circle action and let $\phi : M \longrightarrow S^1$ be the corresponding generalized moment map. Let P be a connected component of an arbitrary regular level set of ϕ, let M_{red} be the symplectic quotient P/S^1, and let $N \subset M$ be the union of all connected components of the set of points with non-trivial stabilizer which do not contain a fixed point.*

Then, as fundamental groups of topological spaces, $\pi_1(M)$ is a semidirect product

$$\pi_1(M) = G \rtimes \mathbb{Z}.$$

If the action has no fixed points then $G = \pi_1(P)$. Otherwise,

$$G = \pi_1(M_{\mathrm{red}})$$

if and only if, for every connected component L of N, the fibers of the sphere normal orbibundle of $(L \cap P)/S^1$ inside M_{red} are contractible in $(P \setminus N)/S^1$.

Theorem 3.1. *Let (M, ω) be a connected symplectic manifold (not necessarily compact) equipped with a Hamiltonian S^1-action with proper moment map $\phi : M \longrightarrow \mathbb{R}$. Let P be an arbitrary (compact) regular level set of ϕ, let M_{red} be the symplectic quotient P/S^1 and let $N \subset M$ be the union of all connected components of the set of points with non-trivial stabilizer which do not contain a fixed point.*

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[1] Recall that an exceptional orbit of S^1 is an orbit with a finite non-trivial stabilizer subgroup.

Then, as fundamental groups of topological spaces, the fundamental group of M is $\pi_1(M) = \pi_1(P)$, when the action has no critical points. Otherwise,

$$\pi_1(M) = \pi_1(M_{\text{red}}),$$

if and only if, for every connected component L of N, the fibers of the sphere normal orbibundle of $(L \cap P)/S^1$ inside M_{red} are contractible in $(P \setminus N)/S^1$.

Moreover, if ϕ has a minimum (or a maximum), then $N = \varnothing$ and

$$\pi_1(M) = \pi_1(M_{\text{red}}) = \pi_1(F_{\min}) \ (or = \pi_1(F_{\max})),$$

where $F_{\min}$ and $F_{\max}$ are the sets of minimal and maximal points respectively.

If the action is **semifree**[2] (i.e. if the action is free outside the fixed-point set), then the requirement that the fibers of the sphere normal orbibundle of $(L \cap P)/S^1$ inside M_{red} are contractible in $(P \setminus N)/S^1$ for every connected component L of N is trivially satisfied (since $N = \varnothing$ in this case). Hence we have the following easy corollaries.

Corollary 1. *Let M be a connected, compact symplectic manifold equipped with a non-Hamiltonian semifree symplectic circle action and let $\phi : M \longrightarrow S^1$ be the corresponding generalized moment map. Let P be a connected component of an arbitrary regular level set of ϕ and let M_{red} be the symplectic quotient P/S^1.*

Then, as fundamental groups of topological spaces, $\pi_1(M)$ is a semidirect product

$$\pi_1(M) = G \rtimes \mathbb{Z}.$$

If the action has no fixed points then $G = \pi_1(P)$. Otherwise,

$$G = \pi_1(M_{\text{red}}).$$

Corollary 2. *Let S^1 act semifreely on a connected symplectic manifold (M, ω) (not necessarily compact) with proper moment map $\phi : M \longrightarrow \mathbb{R}$. Let P be an arbitrary (compact) regular level set of ϕ and let M_{red} be the symplectic quotient P/S^1.*

Then, as fundamental groups of topological spaces, the fundamental group of M is $\pi_1(M) = \pi_1(P)$, when the action has no critical points. Otherwise,

$$\pi_1(M) = \pi_1(M_{\text{red}}).$$

Moreover, if ϕ has a minimum (or a maximum),

$$\pi_1(M) = \pi_1(M_{\text{red}}) = \pi_1(F_{\min}) \ (or = \pi_1(F_{\max})),$$

where $F_{\min}$ and $F_{\max}$ are the sets of minimal and maximal points respectively.

In order to account for the possible existence of these special exceptional orbits, the following changes should be made to the paper (where we take the opportunity to correct some misprints):

[2] An action of a group G on a manifold is said to be quasi-free if all the stabilizers are connected. If $G = S^1$ then quasi-free is the same as semifree

1. List of corrections

Page 1088: Lemma 2.5 should be restated as follows.

Lemma 2.5. *Let M be a manifold satisfying the hypotheses of Lemma 2.2, equipped with a circle action with a non-empty fixed point set. Assume that the circle action satisfies the additional condition that every connected component of the set of points with non-trivial stabilizer, contains at least one fixed point. Let $a_0 \in S^1$ be a regular value of the generalized moment map ϕ, and consider the principal circle orbibundle*

$$S^1 \overset{i}{\hookrightarrow} \phi^{-1}(a_0) \overset{p}{\longrightarrow} M_{a_0}, \tag{1.1}$$

where $M_{a_0} := \phi^{-1}(a_0)/S^1$ is the reduced space at a_0. Then, for $y_0 \in \phi^{-1}(a_0)$, the kernel of the map $p_* : \pi_1(\phi^{-1}(a_0), y_0) \longrightarrow \pi_1(M_{a_0}, p(y_0))$ is equal to the kernel of the map

$$j_* : \pi_1(\phi^{-1}(a_0), y_0) \longrightarrow \pi_1(M, y_0),$$

defined in Lemma 2.3.

Page 1088, lines 13-15: replace these lines by the following text which clarifies the need for the additional assumption in the statement of Lemma 2.5:

We first see that $\ker(p_*) \subset \ker(j_*)$. Since $\ker(p_*)$ is generated by the orbits in $\phi^{-1}(a_0)$ we just have to show that these are all contractible in M. If there are no exceptional orbits in $\phi^{-1}(a_0)$ (i.e. orbits with a nontrivial finite stabilizer) then all orbits in each connected component of $\phi^{-1}(a_0)$ are homotopic and then, since at least one of them is contained in the stable or unstable manifold of a critical point in M, all the orbits in $\phi^{-1}(a_0)$ will be contractible in M. If there exist exceptional orbits in $\phi^{-1}(a_0)$ then each orbit is, by assumption, homotopic to a multiple of some exceptional orbit contained in the stable or unstable manifold of a critical point and so, again in this case, all orbits in $\phi^{-1}(a_0)$ will be contractible in M.

Page 1088, line 18: replace "Indeed, if that were not the case, ..., which is impossible." by "Otherwise, there would exist loops $\alpha : S^1 \longrightarrow M$ arbitrarily close to y_0 and homotopic to γ in M with $(\phi \circ \alpha)(S^1) = S^1$, which is impossible."

Page 1089, line 3: replace "$[\gamma] \in N_i$ for one of the groups N_i" by "$[\gamma]$ is in the normal subgroup generated by the union of all the N_i".

Page 1089, line 5: replace " $[\gamma] \in N_i \subset \ker(p_*)$" by " $[\gamma] \in \ker(p_*)$".

Page 1089, line -4: replace this line by the exact sequence

$$\pi_1\left(\left(\phi^c\right)^{-1}(a)\right) \xrightarrow{j_*} \pi_1(M) \underset{\gamma_*}{\overset{(\phi)^c_*}{\rightleftarrows}} \pi_1(S^1).$$

Page 1089, line -2: replace "Moreover," by "If every connected component of the set of points with non-trivial stabilizer contains at least one fixed point (i.e. if $N = \varnothing$) then,"

Page 1090, line 9: add the following text to complete the proof of the theorem:

Let us assume now that $N \neq \varnothing$. Note that each connected component of N is a compact symplectic submanifold of M of codimension at least 2. Since M is compact and $M^{S^1} \neq \varnothing$ all the principal (free) orbits in M are still contractible as well as all exceptional orbits that are not in N. Moreover, N and $M \setminus N$ are both S^1-invariant and the restrictions of the generalized moment map ϕ to these sets are surjective (they are in fact the generalized moment maps of the induced circle actions on N and $M \setminus N$). Consequently, Lemmas 2.2, 2.3, 2.5 and Proposition 2.4 hold for $M \setminus N$.

Indeed, in Lemma 2.2, given a point $y_0 \in M \setminus N$ we can choose a homologically nontrivial loop $\alpha : S^1 \longrightarrow M$ through y_0 contained in $M \setminus N$ since the trajectory of the gradient flow through y_0 is contained in $M \setminus N$.

In Lemma 2.3, we have the exact sequence

$$\pi_1\left(\phi^{-1}(a_0) \setminus N, y_0\right) \xrightarrow{\tilde{j}_*} \pi_1\left(M \setminus N, y_0\right) \xrightarrow{\tilde{\phi}_*} \pi_1(S^1, a_0),$$

where $\tilde{j} : \phi^{-1}(a_0) \setminus N \longrightarrow M \setminus N$ is the inclusion map and $\tilde{\phi} : M \setminus N \longrightarrow S^1$ is the restriction of ϕ to $M \setminus N$. If there are no critical points in $M_{y_0}^{[a,b]} \setminus N = \tilde{\phi}^{-1}([a,b])$ then we still have a deformation retraction to a connected component of a level set $\tilde{\phi}^{-1}(a_0) = \phi^{-1}(a_0) \setminus N$, yielding

$$\pi_1(M_{y_0}^{[a,b]} \setminus N, y_0) = \pi_1(\phi^{-1}(a_0) \setminus N, y_0).$$

If there are critical points in $M_{y_0}^{[a,b]} \setminus N$ then $\pi_1(M_{y_0}^{[a,b]} \setminus N, y_0)$ is obtained from $\pi_1(\phi^{-1}(a_0) \setminus N, y_0)$ in the same way as in the proof of Lemma 2.3 since all the changes involved when passing a critical level of $\tilde{\phi}$ occur in neighborhoods of the critical sets thus avoiding N.

In Proposition 2.4 the Morse theory arguments apply to $M \setminus N$ and so all connected components of all reduced spaces

$$\tilde{\phi}^{-1}(a)/S^1 = M_a \setminus \left(\left(\phi^{-1}(a) \cap N\right)/S^1\right)$$

have the same fundamental group.

The corrected Lemma 2.5 holds now for the principal circle orbibundle

$$\phi^{-1}(a_0) \setminus N \xrightarrow{\tilde{p}} M_{a_0} \setminus \left(\left(\phi^{-1}(a_0) \cap N\right)/S^1\right),$$

yielding $\ker(\tilde{p}_*) = \ker(\tilde{j}_*)$, where $\tilde{p}$ and $\tilde{j}$ are the restrictions of the maps $p : \phi^{-1}(a_0) \longrightarrow M_{a_0}$ and $j : \phi^{-1}(a_0) \longrightarrow M$ to $\phi^{-1}(a_0) \setminus N$. Indeed, $\ker(\tilde{p}_*) \subset \ker(\tilde{j}_*)$ since all the orbits in $\phi^{-1}(a_0) \setminus N$ are contractible in $M \setminus N$ and $\pi_1(M_{y_0}^{[a,b]} \setminus N, y_0)$ can be obtained from $\pi_1(\phi^{-1}(a_0) \setminus N, y_0)$ by a finite number of quotients corresponding to index-2 and index-$(2n-2)$ critical points as in the proof of Lemma 2.5.

Let us assume for simplicity that the level sets of ϕ are connected[3], that N is connected of codimension $2l$ and that all points in N have the same stabilizer. Consider a tubular neighborhood V of N in M. This neighborhood V fibers over N with fiber a $2l$-disk and so it is homotopically equivalent to N. Consequently, $\pi_1(M)$ can be obtained from $\pi_1(M \setminus N)$ by collapsing the fibers of the $(2l-1)$-sphere bundle

$$S\left((M \setminus N) \cap V\right) \longrightarrow N.$$

By the equivariant symplectic neighborhood theorem, V can be equivariantly identified with a neighborhood of the zero section of the symplectic normal bundle of N inside M. A fiber of V over a point $p \in N$ is then identified with the symplectic

[3]Note that the arguments hold locally around every connected component and that there is only a finite number of components.

orthogonal $W_p := (T_pN)^\omega$ of T_pN in T_pM and so, under this identification, ϕ is constant along the fibers of V. Indeed, since the vector field ξ_M generating the S^1-action in M is tangent to N (it is tangent to the S^1-orbits), we have

$$(\phi^* d\theta)(Y) = (\iota(\xi_M)\omega)(Y) = 0$$

for every vector field Y in V tangent to a fiber, and so $\phi_* Y = 0$. Consequently, $V \cap \phi^{-1}(a)$ is a neighborhood of $N \cap \phi^{-1}(a)$ inside $\phi^{-1}(a)$, fibering over $N \cap \phi^{-1}(a)$ with fiber W_p at each $p \in N \cap \phi^{-1}(a)$. This neighborhood, in turn, fibers over N/S^1 and so a neighborhood of $(N \cap \phi^{-1}(a))/S^1$ inside M_{red} can be equivariantly identified with $(V \cap \phi^{-1}(a))/S^1$, fibering over $(N \cap \phi^{-1}(a))/S^1$ with fiber W_p/G_N, where G_N is the stabilizer of any point in N.

Consequently, when $l = 1$, we have the following commutative diagrams with exact rows and columns:

$$\begin{array}{ccccccc}
\mathbb{Z} & \longrightarrow & \pi_1(\phi^{-1}(a) \setminus N) & \xrightarrow{\alpha} & \pi_1(\phi^{-1}(a)) & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle \tilde{j}_*} & & \downarrow{\scriptstyle j_*} & & \\
\mathbb{Z} & \longrightarrow & \pi_1(M \setminus N) & \longrightarrow & \pi_1(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \mathbb{Z} & = & \mathbb{Z} = \pi_1(S^1) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

and

$$\begin{array}{ccccccc}
\mathbb{Z} & \longrightarrow & \pi_1(\phi^{-1}(a) \setminus N) & \xrightarrow{\alpha} & \pi_1(\phi^{-1}(a)) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \times k} & & \downarrow{\scriptstyle \tilde{p}_*} & & \downarrow{\scriptstyle p_*} & & \\
\mathbb{Z} & \xrightarrow{\lambda} & \pi_1\left((\phi^{-1}(a) \setminus N)/S^1\right) & \longrightarrow & \pi_1(M_{\mathrm{red}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

where $k = |G_N| > 1$. Hence, $\ker(j_*) = \alpha(\ker(\tilde{j}_*))$, while $\ker(p_*) = \alpha(\ker(\tilde{p}_*))$ iff $\lambda(1) = 0$. We conclude that $\pi_1(M) = G_1 \rtimes \mathbb{Z}$, where

$$G_1 = \mathrm{im}(j_*) = \pi_1(\phi^{-1}(a))/\ker(j_*) = \pi_1(\phi^{-1}(a))/\alpha(\ker(\tilde{j}_*)) = \pi_1(\phi^{-1}(a))/\alpha(\ker(\tilde{p}_*))$$

is isomorphic to $\pi_1(M_{\mathrm{red}}) = \pi_1(\phi^{-1}(a))/\ker(p_*)$ iff the fibers of the circle normal orbibundle of $(N \cap \phi^{-1}(a))/S^1$ inside M_{red} are contractible in $(\phi^{-1}(a) \setminus N)/S^1$ (i.e. iff $\lambda(1) = 0$).

If $l > 1$ we trivially have $\ker(j_*) = \ker(\tilde{j}_*)$. Moreover, since $\pi_1(S^{2l-1}/G_N) = \mathbb{Z}_k$, we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \pi_1(\phi^{-1}(a) \setminus N) & = & \pi_1(\phi^{-1}(a)) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \tilde{p}_*} & & \downarrow{\scriptstyle p_*} & & \\
\mathbb{Z}_k & \xrightarrow{\lambda} & \pi_1\left((\phi^{-1}(a) \setminus N)/S^1\right) & \longrightarrow & \pi_1(M_{\rm red}) & \longrightarrow & 0 \cdot \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

We conclude that $\ker(p_*) = \ker(\tilde{p}_*)$ iff $\lambda(1) = 0$ and the result follows.

Page 1091, line 18: replace "show that the kernel of the map" by "show that, when $N = \varnothing$, the kernel of the map"

Page 1091, line 25: insert the following paragraph:

If $N \neq \varnothing$ then we proceed as in the proof of Theorem 2.1 above, assuming for simplicity that N is connected and that all points in N have the same stabilizer. We conclude then that $\pi_1(M)$ is isomorphic to $\pi_1(M_{\rm red})$ iff the fibers of the sphere orbibundle associated to the normal orbibundle of $(N \cap \phi^{-1}(a))/S^1$ inside $M_{\rm red}$ are contractible in $(\phi^{-1}(a) \setminus N)/S^1$, and the result follows.

Page 1092, line 5: add reference [M].

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References

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