The Plücker Map

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- $\mathbf{Gr}(k, V)$ is a (compact) topological space.

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- $\mathbf{Gr}(k,V)$ is a complex manifold.

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where $F_i \in \mathbb{C}[x_0, \ldots, x_n]$; $i = 1, \ldots, k$

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- Is $\mathbf{Gr}(k, V)$ a projective variety?

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Examples:

$$v_2 \wedge v_1 = -(v_1 \wedge v_2) \qquad v_2 \wedge v_3 \wedge v_1 = v_1 \wedge v_2 \wedge v_3$$

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- $v_1 \wedge \cdots \wedge v_k = \det(G)(w_1 \wedge \cdots \wedge w_k)$ for some $G \in \mathbf{GL}_k(\mathbb{C})$.

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Plücker map (Julius Plücker, XIXth century)

$$\mathcal{P}\ell : \mathbf{Gr}(k, V) \to \mathbb{P}(\Lambda^k(V))$$
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- $\mathcal{P}\ell$ is injective: If $[v_1 \wedge \cdots \wedge v_k] = [w_1 \wedge \cdots \wedge w_k]$, then

$$v_i \wedge (w_1 \wedge \cdots \wedge w_k) = 0; \ \forall i = 1, \ldots, k$$
$$span_{\mathbb{C}}\{v_1, \ldots, v_k\} = span_{\mathbb{C}}\{w_1, \ldots, w_k\}$$

Primary Goal

Goal: $\mathcal{P}\ell(\mathbf{Gr}(k, V))$ is projective variety; give explicitly the family of corresponding homogeneous polynomials, the so-called Plücker relations.

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$$\mathcal{P}\ell : \mathbf{Gr}(2, V) \to \mathbb{P}(\Lambda^2(V))$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} \mapsto [\sum_{1 \leq i < j \leq 4} p_{ij}(e_i \wedge e_j)]$$

$$p_{ij} = \det \begin{pmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{pmatrix} = a_{1i}a_{2j} - a_{1j}a_{2i}$$

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0 \text{ (Plücker/Klein quadric)}$$