Whitehead torsion

Novos Talentos

Jing Xu
Tutor: Pedro Boavida

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Homotopy

In topology, two continuous functions

$$f : X \to Y \quad \text{and} \quad g : X \to Y$$

from one topological space to another are called homotopic if one can be "continuously deformed" into the other, such a deformation

$$H : X \times [0, 1] \to Y$$

being called a homotopy between the two functions.

Homotopy equivalence

A homotopy equivalence between spaces X and Y is a pair of continuous maps

$$f : X \to Y \quad \text{and} \quad g : Y \to X$$

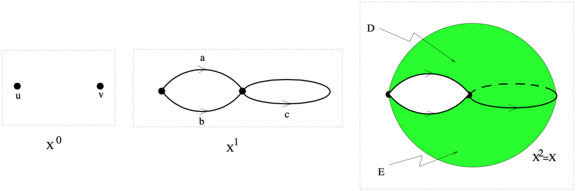
such that $g \circ f$ is homotopic to the identity map id_X and $f \circ g$ is homotopic to id_Y. If such a pair exists, then X and Y are said to be homotopy equivalent, or of the same homotopy type. This relation of homotopy equivalence is often denoted as $X \simeq Y$.

CW complexes

A CW complex is constructed by taking the union of a sequence of topological spaces

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots$$

where X_k is obtained from X_{k-1} by gluing k-dimensional disks along their boundaries.



Simple homotopy equivalence

A map $f : X \to Y$ of finite CW complexes is called a simple homotopy equivalence if it is homotopic to a finite sequence of compositions:

$$X = X_a \xrightarrow{f_a} X_b \xrightarrow{f_b} X_c \cdots X_\omega = Y$$

where each f_i is either an elementary expansion or an elementary collapse. We say that X and Y are simple homotopy equivalent if there exists a simple homotopy equivalence between them.

Whitehead torsion

Theorem: A homotopy equivalence $f : X \to Y$ of finite connected CW complexes is simple if and only if the Whitehead torsion of f is 0.