Master in Mathematics and Applications - Técnico, Lisbon
Numerical Functional Analysis and Optimization - Fall Semester 2016

Midterm Exam – November 10th, 2016 – Solutions

1. Show that

$$\langle x, y\rangle := x_1y_1 + 2x_1y_2 + 2x_2y_1 + 5x_2y_2\,, \qquad x = (x_1, x_2),\; y = (y_1, y_2)\,,$$

defines an inner product on $\mathbb{R}^2$. (Hint: Express $\langle x, y\rangle$ as x^TAy with some $A \in \mathbb{R}^{2\times 2}$.) **[1.5]**

Solution: We can write

$$x_1y_1 + 2x_1y_2 + 2x_2y_1 + 5x_2y_2 = x^T \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} y =: x^TAy = (x, Ay)\,,$$

where $(\cdot,\cdot)$ is the usual inner product in $\mathbb{R}^2$. The eigenvalues, $\lambda_1 = 3 - 2\sqrt{2}$ and $\lambda_2 = 3 - 2\sqrt{2}$, of matrix A are both positive. The matrix A is thus symmetric and positive definite. Therefore

$$\langle x, x\rangle > 0 \;\forall\, x \in \mathbb{R}^2 \setminus \{0\}\,, \qquad \text{and} \qquad \langle x, x\rangle = 0 \quad \Leftrightarrow \quad x = 0\,.$$

The other inner product properties for $\langle\cdot,\cdot\rangle$ follow from the properties of the standard Eucldean inner product $(\cdot,\cdot)$.

2. Let V be a Hilbert space equipped with the inner product $(\cdot,\cdot)$. Let $a \in V, a \neq 0$ be given and consider the functional $f : V \to \mathbb{R}$ defined through $f(u) = (u, a)\,(u, u)$. Show that f is Fréchet differentiable at $u \in V$ and determine the Fréchet derivative $f'(u)$. Compute also $f''(u)$. **[2.0]**

Solution: Note first that both inner products, (u, a) and (u, u), are twice continuously differentiable as continuous bilinear forms from $V \times V$ to $\mathbb{R}$ and well defined since $a, u \in V$. We have

$$f(u + h) - f(u) = (u + h, a)(u + h, u + h) - (u, a)(u, u)$$

$$= (h, a)(u, u) + 2(u, a)(h, u) + (u, a)(h, h) + 2(h, a)(h, u) + (h, a)(h, h)\,,$$

where we have used the symmetry of the inner product $(\cdot,\cdot)$. Defining

$$T(u)h := (h, a)(u, u) + 2(u, a)(h, u)\,,$$

we see that

$$|f(u + h) - f(u) - T(u)h| \leq 3||a||\,||u||\,||h||^2 + ||a||\,||h||^3 = \mathcal{O}(||h||^2) = o(\|h\|)\,,$$

where $\|u\| = \sqrt{(u, u)}$ and where we have used the Cauchy-Schwarz inequality. The operator $T(u) : V \to \mathbf{R}$ thus coincides with the (unique) Fréchet derivative of f at u, i.e.

$$f'(u)h = (h, a)(u, u) + 2(u, a)(h, u)\,.$$

Moreover

$$f'(u + k)h - f'(u)h = (h, a)(u + k, u + k) + 2(u + k, a)(h, u + k) - (h, a)(u, u) - 2(u, a)(h, u)$$

$$= 2(h, a)(k, u) + 2(k, a)(h, u) + 2(u, a)(h, k) + 2(k, a)(h, k) + (h, a)(k, k)\,,$$

where

$$2(k,a)(h,k) + (h,a)(k,k) = \mathcal{O}(||k||^2) = o(\|k\|) \ .$$

It thus follows that the symmetric bilinear operator $f''(u) : V \times V \to \mathbb{R}$ defined by

$$f''(u)hk = 2(h,a)(k,u) + 2(k,a)(h,u) + 2(u,a)(h,k) \, .$$

is the second-order Fréchet derivative of f at u.

3. Consider the initial value problem

$$\begin{cases} u''(t) = -u'(t)\, \sin u(t) + t\,, & 0 < t < 1\,, \\ u(0) = \alpha\,, \qquad u'(0) = \beta\,, & \end{cases} \tag{1}$$

where $\alpha, \beta \in \mathbb{R}$.

a) Show that the initial value problem (1) can be written in the form $u = T(u)$, where

$$T(u)(t) = \int_0^t \cos u(s)\, ds + (\beta - \cos\alpha)\, t + \alpha + \frac{t^3}{6}\,, \qquad 0 \le t < 1\,. \qquad \textbf{[1.5]}$$

Solution: Integrating the differential equation from 0 to t, we get

$$u'(t) = u'(0) - \int_0^t u'(s)\, \sin u(s)\, ds + \frac{t^2}{2} = \beta + \cos u(t) - \cos\alpha + \frac{t^2}{2}\,.$$

Integrating again from 0 to t, we have

$$u(t) = u(0) + \beta\, t + \int_0^t \cos u(s)\, ds - t\, \cos\alpha + \frac{t^3}{6} = \int_0^t \cos u(s)\, ds + (\beta - \cos\alpha)t + \alpha + \frac{t^3}{6}\,,$$

which shows that $u = T(u)$.

b) Let $V = C[0,1]$ and choose $\alpha = \beta = 0$. Show that the operator $T : V \to V$ admits a unique fixed point $u \in V$. **[1.5]**

Solution: With $\alpha = \beta = 0$ we have the integral equation

$$u(t) = \int_0^t \cos u(s)\, ds + \frac{t^3}{6} - t\,.$$

Defining

$$T(u)(t) = \int_0^t \cos u(s)\, ds + \frac{t^3}{6} - t\,, \qquad 0 \le t \le 1\,,$$

we obtain

$$|T(u)(t) - T(v)(t)| = \left| \int_0^t (\cos u(s) - \cos v(s))\, ds \right| \le \int_0^t |\cos u(s) - \cos v(s)|\, ds$$

$$\le \max_{\xi \in \mathbb{R}} |\sin\xi\,| \int_0^t |u(s) - v(s)|\, ds = \int_0^t |u(s) - v(s)|\, ds \le t\, ||u - v||_V\,, \quad \forall t \in [0,1]\,,$$

$$|T^2(u)(t) - T^2(v)(t)| \le \int_0^t |T(u)(s) - T(v)(s)|\, ds \le \int_0^t s\, ds\, ||u - v||_V = \frac{t^2}{2}\, ||u - v||_V\,, \quad \forall t \in [0,1]\,,$$

where $\|u\|_V = \max_{t\in[0,1]} |u(t)|$. Hence,

$$|T^2(u)(t) - T^2(v)(t)| \leq \frac{1}{2}\,||u-v||_V \qquad \forall t \in [0,1] \qquad \Rightarrow \qquad ||T^2(u) - T^2(v)||_V \leq \frac{1}{2}\,||u-v||_V\,,$$

that is, T^2 is a contraction in V and thus admits a unique fixed point $u \in V$. Moreover, if $S = T^2$, it holds

$$u = \lim_{k\to\infty} S^k(u) = \lim_{k\to\infty} S^k(T(u)) = \lim_{k\to\infty} T(S^k(u)) = \lim_{k\to\infty} T(u) = T(u)$$

so $u \in V$ is also the unique fixed point of T.

c) Approximate the solution of the nonlinear Volterra integral equation $u = T(u)$ by Newton's method. Consider $\alpha = \beta = 0$, $u_0(t) = 0$ and compute the first iterate $u_1(t)$. **[1.5]**

Solution: Newton's method, applied to the nonlinear equation $F(u) = u - T(u)$, reads as

$$F'(u_n)\Delta u_n = -F(u_n) \qquad \Leftrightarrow$$

$$\Delta u_n(t) - \int_0^t \sin u_n(s)\,\Delta u_n(s)\,ds = -u_n(t) + \int_0^t \cos u_n(s)\,ds + \frac{t^3}{6} - t\,, \qquad n = 0,1,\dots\,,$$

where $\Delta u_n(t) = u_{n+1}(t) - u_n(t)$. Choosing $u_0(t) = 0$, we obtain

$$\Delta u_0(t) = \int_0^t ds + \frac{t^3}{6} - t = \frac{t^3}{6} \qquad \Rightarrow \qquad u_1(t) = \frac{t^3}{6}\,.$$

4. Consider a Broyden's type update for the inverse of the matrix A_k:

$$A_{k+1}^{-1} = A_k^{-1} + \frac{(s_k - A_k^{-1}y_k)\,s_k^T A_k^{-1}}{s_k^T A_k^{-1} y_k}\,, \qquad k = 0,1,\dots\,,$$

Can we say that

$$\|A_{k+1}^{-1} - A_k^{-1}\|_2 \;\leq\; \|B^{-1} - A_k^{-1}\|_2\,,$$

for all nonsingular $B \in Q(y_k, s_k) = \{B \in \mathbb{R}^{N\times N}\,|\,Bs_k = y_k\}$? Justify your answer. You may assume that $s_k^T A_k^{-1} y_k \neq 0$. **[2.0]**

Solution: Given that $s_k = B^{-1}y_k$, we obtain

$$A_{k+1}^{-1} - A_k^{-1} = \frac{(s_k - A_k^{-1}y_k)\,s_k^T A_k^{-1}}{s_k^T A_k^{-1} y_k} = \left(B^{-1} - A_k^{-1}\right)\frac{y_k s_k^T A_k^{-1}}{s_k^T A_k^{-1} y_k}\,.$$

However, the matrix A_{k+1}^{-1} does not necessarily solve the minimization problem since, although

$$\|A_{k+1}^{-1} - A_k^{-1}\|_2 \leq \|B^{-1} - A_k^{-1}\|_2 \left\| \frac{y_k s_k^T A_k^{-1}}{s_k^T A_k^{-1} y_k} \right\|,$$

where $\|\cdot\|$ can be either the Frobenius or the matrix 2-norm, we have

$$\left\|\frac{xy^T}{x^Ty}\right\|_F = \left\|\frac{xy^T}{x^Ty}\right\|_2 = \frac{\|x\|_2\,\|y\|_2}{x^Ty} \geq 1 \qquad \forall\, x,y \in \mathbb{R}\setminus\{0\}\,,$$

and the equality holds if and only if $x \parallel y$.

The so called "bad" Broyden's method which uses an approximation of the inverse of the Jacobian matrix is formulated as

$$A_{k+1}^{-1} = A_k^{-1} + \frac{(s_k - A_k^{-1} y_k)\, y_k}{y_k^T y_k} \,.$$

In this case, we do have

$$\|A_{k+1}^{-1} - A_k^{-1}\|_2 \;\leq\; \|B^{-1} - A_k^{-1}\|_2 \,,$$

for all nonsingular $B \in Q(y_k, s_k) = \left\{ B \in \mathbb{R}^{N \times N} \,| B s_k = y_k \right\}$ but, as can be shown using Sherman-Morrison formula, this corresponds to a different update formula for A_k, namely

$$A_{k+1} = A_k + \frac{(y_k - A_k s_k)\, y_k^T A_k}{y_k^T A_k s_k} \,.$$