Análise Numérica
LMAC/MMA/MEIC
1º Semestre de 2016/17

Formulário (2º Teste)

i. Normas e condicionamento ($A \in \mathbb{C}^{N\times N}$)

$$\|A\|_1 = \max_{1\le j\le N}\sum_{i=1}^{N}|a_{ij}|, \qquad \|A\|_2 = \sqrt{r_\sigma(A^*A)}, \qquad \|A\|_\infty = \max_{1\le i\le N}\sum_{j=1}^{N}|a_{ij}|, \qquad \|A\|_F = \sqrt{\sum_{i,j=1}^{N}|a_{ij}|^2}$$

$$\text{cond}_p(A) = \|A\|_p\,\|A^{-1}\|_p\,, \quad \text{cond}_*(A) = r_\sigma(A)\,r_\sigma(A^{-1}), \quad r_\sigma(A) = \max_{1\le i\le N}|\lambda_i|$$

ii. Determinação de valores e vectores próprios

Matriz companheira do polinómio $p(x) = x^N + a_{N-1}x^{N-1} + \ldots + a_1 x + a_0$

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots \\ \ldots & & \ldots & & \ldots \\ 0 & \ldots & & 0 & 1 \\ -a_0 & -a_1 & \ldots & \ldots & -a_{N-1} \end{bmatrix} \in \mathbb{R}^{N\times N}$$

Estabilidade do problema de valores e vectores próprios:

$$A(\varepsilon) = A + \varepsilon\, E\,, \quad A, E \in \mathbb{C}^{N\times N}\,, \ \varepsilon > 0\,, \ P^{-1}AP = \text{diag}[\lambda_1, \ldots, \lambda_N]\ , \ P = [u_1, \ldots, u_N]$$

$$Au_i = \lambda_i u_i\,, \ \|u_i\|_2 = 1\,, \quad A^* v_i = \overline{\lambda}_i v_i\,, \ \|v_i\|_2 = 1\,, \quad s_i = v_i^* u_i > 0\,, \ \kappa(\lambda_i) = s_i^{-1}\,, \quad i = 1, \ldots N\,.$$

$$\min_{1\le j\le N}|\lambda(\varepsilon) - \lambda_j| \le \varepsilon\,\|P\|_q\,\|P^{-1}\|_q\,\|E\|_q\,, \quad q = 1, 2, \infty$$

$$|\lambda_k(\varepsilon) - \lambda_k| \le \varepsilon\,\kappa(\lambda_k)\,\|E\|_2 + O(\varepsilon^2)\,, \qquad \|u_k(\varepsilon) - u_k\|_2 \le \frac{\varepsilon}{\min_{j\ne k}|\lambda_k - \lambda_j|} + O(\varepsilon^2)$$

Método das potências: $q^{(0)} \in \mathbb{C}^N$, $\|q^{(0)}\|_2 = 1$

$$z^{(k)} = A\,q^{(k-1)}\,, \qquad q^{(k)} = \frac{z^{(k)}}{\|z^{(k)}\|_2}\,, \qquad \lambda^{(k)} = [q^{(k)}]^* A\,q^{(k)}\,, \qquad k = 1, 2, \ldots$$

Matriz de Householder: $U = I - 2\,w\,w^T$, $w \in \mathbb{R}^N$, $\|w\|_2 = 1$, $U^{-1} = U^T = U$

$$w = \begin{bmatrix} 0_{r-1} \\ u \end{bmatrix}, \quad x = \begin{bmatrix} c \\ d \end{bmatrix}, \quad c \in \mathbb{R}^{r-1}\,, \ u, d \in \mathbb{R}^{N-r+1}\,, \quad 1 \le r \le N-1\,,$$

$$u = \frac{v}{\|v\|_2}\,, \quad v = d \pm \|d\|_2\,e_1\,, \quad Ux = [c_1 \ldots c_{r-1}\ \alpha\ 0 \ldots 0]^T\,.$$

Factorização QR: ($A = QR$, Q ortogonal, R triangular superior)

$$U_r = I - 2\,w^{(r)}\,w^{(r)T}\,, \quad r = 1\ldots, N-1\,, \qquad w = [0, \ldots, 0, w_r, \ldots, w_N]^T\,, \quad R = Q^T A = U_{N-1}\cdots U_1 A$$

iii. Resolução numérica de equações diferenciais ordinárias – problemas de valor inicial

$$\begin{cases} y'(t) = f(t, y(t)), \quad t > t_0 \\ y(t_0) = y_0 \end{cases}$$

Métodos multipasso lineares a p passos:

$$\sum_{j=0}^{p} a_j y_{n+j} = h \sum_{j=0}^{p} b_j f(t_{n+j}, y_{n+j}) \,, \qquad n \geq 0$$

Consistência e ordem:

$$\sum_{j=0}^{p} a_j = 0 \,, \qquad \sum_{j=0}^{p} j^k a_j = k \sum_{j=0}^{p} j^{k-1} b_j \,, \qquad k = 1, \ldots, q$$

Polinómio de estabilidade:

$$\pi(r; \overline{h}) = \rho(r) - \overline{h}\,\sigma(r) \,, \qquad \rho(r) = \sum_{j=0}^{p} a_j r^j \,, \qquad \sigma(r) = \sum_{j=0}^{p} b_j r^j$$

Métodos de Adams:

$$y_{n+p} - y_{n+p-1} = h \sum_{j=0}^{p} b_j f(t_{n+j}, y_{n+j}) \,, \qquad n \geq 0$$

Métodos BDF:

$$\sum_{j=0}^{p} a_j y_{n+j} = h\, b_p\, f(t_{n+p}, y_{n+p}) \,, \qquad n \geq 0$$

Métodos de Runge-Kutta de s etapas:

$$\begin{cases} y_{n+1} = y_n + h \displaystyle\sum_{i=1}^{s} b_i k_i \,, \\ k_i = f\Big(t_n + c_i h,\, y_n + h \displaystyle\sum_{j=1}^{s} a_{ij} k_j\Big) \,, \quad i = 1, \ldots, s \,, \\ c_i = \displaystyle\sum_{j=1}^{s} a_{ij} \,, \quad i = 1, \ldots, s \,. \end{cases}$$

Tabela de Butcher:

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \ldots & a_{1s} \\ c_2 & a_{21} & a_{22} & \ldots & a_{2s} \\ \vdots & \vdots & & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \ldots & a_{ss} \\ \hline & b_1 & b_2 & \ldots & b_s \end{array} \qquad \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

Consistência e ordem (≤ 3):

$$\sum_{i=1}^{s} b_i = 1 \,, \qquad \sum_{i=1}^{s} b_i c_i = \frac{1}{2} \,, \qquad \sum_{i=1}^{s} b_i c_i^2 = \frac{1}{3} \,, \qquad \sum_{i=1}^{s} \sum_{j=1}^{s} b_i a_{ij} c_j = \frac{1}{6}$$

Função de estabilidade:

$$R(\overline{h}) = 1 + \overline{h}\, b^T (I - \overline{h} A)^{-1}\, \mathbf{1} = \frac{\det(I - \overline{h}\, A + \overline{h}\, \mathbf{1}\, b^T)}{\det(I - \overline{h}\, A)} \,, \qquad \mathbf{1} = [1\ 1\ \ldots\ 1]^T \in \mathbb{R}^s$$

iv. Resolução numérica de equações diferenciais ordinárias – problemas com valores na fronteira

$$\begin{cases} y''(x) = f(x, y(x), y'(x))\,, & a \leq x \leq b \\ y(a) = \alpha\,, \qquad y(b) = \beta \end{cases}$$

Método das diferenças finitas:

$$\begin{cases} \dfrac{y_{j+1} - 2y_j + y_{j-1}}{h^2} = f\Big(x_j, y_j, \dfrac{y_{j+1} - y_{j-1}}{2h}\Big), \quad j = 1, \ldots, n-1 \\ y_0 = \alpha\,, \qquad y_n = \beta \end{cases}$$