Análise Numérica
LMAC/MMA/MEIC
1^o Semestre de 2016/17

Formulário (1^{o} Teste)

i. Interpolação de Lagrange

Fórmula interpoladora de Lagrange:

$$p_n(x) = \sum_{j=0}^{n} f_j \, l_j(x)\,, \qquad l_j(x) = \prod_{k=0,k\neq j}^{n} \left(\frac{x - x_k}{x_j - x_k}\right)\,, \qquad f_j = f(x_j)\,, \quad j = 0,\ldots,n$$

Fórmula interpoladora de Newton:

$$p_n(x) = f[x_0] + \sum_{j=1}^{n} f[x_0,\ldots,x_j](x-x_0)\cdots(x-x_{j-1})$$

Fórmula do erro:

$$e_n(x) = f(x) - p_n(x) = f[x_0,\ldots,x_n,x] \prod_{j=0}^{n}(x-x_j) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{j=0}^{n}(x-x_j)\,, \qquad \xi \in \text{int}(x_0,\ldots,x_n,x)$$

ii. Interpolação de Hermite

$$H_{2n+1}(x) = \sum_{j=0}^{n} f_j \phi_j(x) + \sum_{j=0}^{n} f_j' \psi_j(x),$$

$$\phi_j(x) = \Big(1 - 2\,l_j'(x_j)(x-x_j)\Big)\, l_j^2(x), \quad \psi_j(x) = (x-x_j)\, l_j^2(x)\,, \quad f_j = f(x_j)\,, \quad f_j' = f'(x_j)\,, \quad j = 0,\ldots,n$$

Pelas diferenças divididas de Newton:

$$H_{2n+1}(x) = f[x_0] + (x-x_0)\, f[x_0,x_0] + (x-x_0)^2\, f[x_0,x_0,x_1] + (x-x_0)^2(x-x_1)\, f[x_0,x_0,x_1,x_1] +$$

$$+ \ldots + \left(\prod_{j=0}^{n-1}(x-x_j)^2\right)(x-x_n)\, f[x_0,x_0,\ldots,x_n,x_n]$$

Fórmula do erro:

$$e_{2n+1}(x) = f(x) - H_{2n+1}(x) = f[x_0,x_0,\ldots,x_n,x_n,x]\left(\prod_{j=0}^{n}(x-x_j)\right)^2 = \frac{f^{(2n+2)}(\xi)}{(2n+2)!}\left(\prod_{j=0}^{n}(x-x_j)\right)^2, \quad \xi \in \text{int}(x_0,\ldots,x_n,x)$$

iii. Interpolação por splines cúbicos ($M_j = s_j''(x_j)$)

$$s_j(x) = M_j\frac{(x_{j+1}-x)^3}{6h_j} + M_{j+1}\frac{(x-x_j)^3}{6h_j} + A_j(x-x_j) + B_j, \quad x \in [x_j, x_{j+1}], \quad j = 0,\ldots,n-1,$$

$$A_j = \frac{f_{j+1}-f_j}{h_j} - (M_{j+1}-M_j)\frac{h_j}{6}, \qquad B_j = f_j - M_j\frac{h_j^2}{6}, \qquad h_j = x_{j+1}-x_j, \quad j = 0,\ldots,n-1$$

$$\begin{bmatrix} 2 & \lambda_0 & 0 & \ldots & \ldots & 0 \\ \mu_1 & 2 & \lambda_1 & 0 & & \vdots \\ 0 & \mu_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \mu_{n-1} & 2 & \lambda_{n-1} \\ 0 & \ldots & \ldots & 0 & \mu_n & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix},$$

$$\mu_j = \frac{h_{j-1}}{h_{j-1}+h_j}\,, \quad \lambda_j = \frac{h_j}{h_{j-1}+h_j}\,,$$

$$d_j = \frac{6}{h_{j-1}+h_j}\left(\frac{f_{j+1}-f_j}{h_j} - \frac{f_j - f_{j-1}}{h_{j-1}}\right),$$

$$j = 1,\ldots,n-1\,.$$

Condição da primeira derivada nos extremos:

$$\lambda_0 = \mu_n = 1\,, \qquad d_0 = \frac{6}{h_0}\left(\frac{f_1 - f_0}{h_0} - f_0'\right), \quad d_n = \frac{6}{h_{n-1}}\left(f_n' - \frac{f_n - f_{n-1}}{h_{n-1}}\right).$$

Condição livre (natural) nos extremos:

$$\lambda_0 = d_0 = \mu_n = d_n = 0 \quad \Longrightarrow \quad M_0 = M_n = 0$$

Fórmula do erro (com a condição da derivada nos extremos):

$$\max_{x\in[x_0,x_n]} |f(x) - s(x)| \le \frac{5M}{384} \max_{0\le j\le n-1} h_j^4$$

em que

$$M = \max_{x\in[x_0,x_n]} |f^{(4)}(x)|$$

iv. Diferenciação numérica

Diferenças centradas:

$$f'(z) = \frac{f(z+h) - f(z-h)}{2h} - \frac{h^2}{6}\, f^{(3)}(\xi)\,, \quad \xi \in (z-h, z+h)$$

$$f'(z) = \frac{-f(z+2h) + 8f(z+h) - 8f(z-h) + f(z-2h)}{12h} + \mathcal{O}(h^4)$$

$$f''(z) = \frac{f(z+h) - 2f(z) + f(z-h)}{h^2} - \frac{h^4}{12}\, f^{(4)}(\xi)\,, \quad \xi \in (z-h, z+h)$$

Diferenças progreassivas/regressivas:

$$f'(z) = \frac{f(z+h) - f(z)}{h} - \frac{h}{2}\, f''(\xi)\,, \quad \xi \in (z, z+h)$$

$$f'(z) = \frac{f(z) - f(z-h)}{h} + \frac{h}{2}\, f''(\xi)\,, \quad \xi \in (z-h, z)$$

$$f'(z) = \frac{-f(z+2h) + 4f(z+h) - 3f(z)}{2h} + \frac{h^2}{3}\, f^{(3)}(\xi)\,, \quad \xi \in (z, z+2h)$$

v. Polinómios ortogonais com respeito ao produto interno $(f,g)_w = \int_a^b w(x)f(x)g(x)\,dx$

$$[a,b] \ni t \longrightarrow x \in [-1,1]\ : \quad t = \frac{a+b+(b-a)\,x}{2}$$

Polinómios de Legendre: $(a,b) = (-1,1)$, $\quad w(x) = 1\,$, $\quad (f,g)_w = (f,g) = \int_{-1}^1 f(x)g(x)\,dx$

$$P_n(x) = \frac{1}{2^n\, n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big[(x^2-1)^n\Big],\ n = 0,1,\ldots \qquad (P_n, P_n) = \frac{2}{2n+1}$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_{n+1}(x) = \frac{1}{n+1}\Big[(2n+1)\,x\,P_n(x) - n\,P_{n-1}(x)\Big],\ n = 1,2,\ldots$$

Polinómios de Chebyshev: $(a,b) = (-1,1)$, $\quad w(x) = \frac{1}{\sqrt{1-x^2}}\,$, $\quad (f,g)_w = \int_{-1}^1 \frac{f(x)g(x)}{\sqrt{1-x^2}}\,dx$

$$T_n(x) = \cos(n \arccos x)\,, \quad n = 0,1,\ldots \qquad (T_0,T_0)_w = \pi\,, \qquad (T_n,T_n)_w = \frac{\pi}{2}\,,\ n \ge 1$$

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2\,x\,T_n(x) - T_{n-1}(x), \quad n = 1,2,\ldots$$

$$T_{n+1}(t_j) = 0\,, \qquad t_j = \cos\frac{(2j+1)\,\pi}{2n+2}, \qquad j = 0,\ldots,n\,, \qquad T_{n+1}(x) = 2^n \prod_{j=0}^{n}(x - t_j)$$

Polinómios de Laguerre: $(a,b)=(0,\infty)$, $w(x)=e^{-x}$, $(f,g)_w=\int_0^\infty e^{-x} f(x)g(x)\,dx$

$$L_0(x)=1,\quad L_1(x)=1-x,\quad L_{n+1}(x)=\frac{1}{n+1}\Big[(2n+1-x)\,L_n(x)-n\,L_{n-1}(x)\Big],\quad n=1,2,\ldots$$

$$L_n(x)=\frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big[e^{-x}\,x^n\Big],\ n\geq 0\,,\qquad (L_n,L_n)_w=1\,,\ \forall\, n\geq 0$$

Polinómios de Hermite: $(a,b)=(-\infty,\infty)$, $w(x)=e^{-x^2}$, $(f,g)_w=\int_{-\infty}^\infty e^{-x^2} f(x)g(x)\,dx$

$$H_0(x)=1,\quad H_1(x)=2x,\quad H_{n+1}(x)=2x\,H_n(x)-2n\,H_{n-1}(x),\quad n=1,2,\ldots$$

$$H_n(x)=(-1)^n e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big[e^{-x^2}\Big],\ n\geq 0\,,\qquad (H_n,H_n)_w=\sqrt{\pi}\,2^n\,n!\,,\ \forall\, n\geq 0$$

vi. Melhor aproximação mínimos quadrados

$$\phi^*(x)=\sum_{k=1}^{n}\beta_k^*\,\varphi_k(x)\,,\qquad \sum_{k=1}^{n}\beta_k^*\,(\varphi_k,\varphi_j)\;=\;(f,\varphi_j)\,,\quad j=1,\ldots,n$$

vii. Fórmulas de quadratura de Gauss $I_w(f)=\int_a^b w(x)f(x)\,dx\approx I_{n,w}(f)=\sum_{j=0}^n A_j f(x_j)$

Quadraturas de Gauss–Legendre:

$$I_w(f)=I(f)=\int_{-1}^{1} f(x)\,dx\,,\qquad I_n(f)=\sum_{j=0}^{n} A_j f(x_j)$$

$$I(f)-I_n(f)=\frac{2^{2n+3}[(n+1)!]^4}{(2n+3)[(2n+2)!]^2}\;\frac{f^{(2n+2)}(\xi)}{(2n+2)!}\,,\qquad \xi\in\,]-1,1\,[$$

Quadraturas de Gauss–Chebyshev:

$$I_w(f)=\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}\,dx\,,\qquad I_{n,w}(f)=\frac{\pi}{n+1}\sum_{j=0}^{n} f(x_j)$$

$$I_w(f)-I_{n,w}(f)=\frac{\pi}{2^{2n+1}}\;\frac{f^{(2n+2)}(\xi)}{(2n+2)!}\,,\qquad \xi\in\,]-1,1\,[$$