Master in Mathematics and Applications - Técnico, Lisbon
Numerical Functional Analysis and Optimization - Fall Semester 2016

6^{th} Problem set

1. Consider solving the minimization problem

$$\min_{x\in\mathbb{R}^2} \;(x_1-1)^2+(x_2-2)^2 \qquad \text{subject to} \qquad (x_1-1)^2-5x_2=0\,. \tag{1}$$

a) Find all KKT points (x_*,λ^*) for this problem. Is the Linear Independence Constraint Qualification (LICQ) satisfied at the KKT points? Find the local minimizers of problem (6).

b) Show that the (global) solution of the unconstrained minimization problem

$$\min_{x_2\in\mathbb{R}} \;5x_2+(x_2-2)^2$$

is not a solution of the original problem (6). Conclusion?

2. Consider the following constrained minimization problem

$$\min_{x\in\mathbb{R}^N} \;\frac{1}{2}(x_1+1)^2+\frac{1}{2}(x_2+1)^2 \qquad \text{subject to} \qquad \begin{cases} e^{x_1}-x_2-2\geq 0\,,\\[2ex] e^{x_1}+x_2\geq 0\,. \end{cases} \tag{2}$$

a) Give a geometric interpretation of the problem by drawing the constraint functions and a few level curves of the objective function.

b) Find the KKT points of the problem, i.e. points (x_*,λ^*) that satisfy the first-order necessary optimality conditions. Determine the set of feasible directions $F_1(x_*)$ and the critical cone $F_2(x_*,\lambda^*)$ for these points.

c) Find all local solutions of the problem.

3. Consider the following constrained minimization problem

$$\min_{x\in\mathbb{R}^N} f(x) \qquad \text{subject to} \qquad Ax=b\,, \tag{3}$$

where $A\in\mathbb{R}^{M\times N}$, $M<N$, $b\in\mathbb{R}^M$ and $f:\mathbb{R}^N\to\mathbb{R}$ is a twice continuously differentiable function.

a) Write down the first-order necessary optimality conditions. Under which condition LICQ holds true?

b) Write down the second-order necessary and sufficient optimality conditions using a matriz Z whose columns generate the null space of A.

c) Let

$$f(x)=x_1\,x_2+x_2x_3+x_1x_3\,, \qquad A=\begin{bmatrix} 1 & -2 & 3\\ 2 & -4 & 2\end{bmatrix}, \qquad b=\begin{bmatrix} 2\\ 0\end{bmatrix}.$$

Verify that the KKT matrix of problem (3) is non singular. Find the global solution of the problem.

i. Determine a permutation matrix $P\in\mathbb{R}^{3\times 3}$ such that

$$AP=\Big[\,B\,|\,N\,\Big]=\left[\begin{array}{cc|c} 1 & 3 & -2\\ 2 & 2 & -4\end{array}\right].$$

ii. Show that the vector

$$Z=\begin{bmatrix} -B^{-1}N\\ 1\end{bmatrix},$$

generates the null space of AP.

iii. Show that the unknowns x_1 e x_3 can be eliminated using the equation

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = B^{-1}b - B^{-1}Nx_2$$

and that the resulting unconstrained minimization problem is

$$\min_{x_2 \in \mathbb{R}} f\left(P \begin{bmatrix} B^{-1}b - B^{-1}Nx_2 \\ x_2 \end{bmatrix} \right). \tag{4}$$

iv. Solve problem (4) and find the corresponding solution $x_* \in \mathbb{R}^3$.

v. Show that x_* satisfies the first and second-order optimality conditions.

4. Solve the constrained minimization problem

$$\min_{x \in \mathbb{R}^2} (x_1 - 1)^2 + x_2^2 \qquad \text{subject to} \qquad x_1^3 + x_2^2 = 0\,.$$

5. Consider the minimization problem

$$\min_{x \in \mathbb{R}^2} x_1^3 - x_2^3 - 2x_1^2 - x_1 + x_2 \qquad \text{subject to} \qquad \begin{cases} -x_1 - 2x_2 + 2 \geq 0 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}.$$

Find the stationary points (x_*, λ^*) of this problem. Is the LICQ satisfied at these points? Determine the local and global solutions of the problem.

6. Consider the constrained minimization problem

$$\min_{x \in \mathbb{R}^N} f(x) \qquad \text{subject to} \qquad \begin{cases} c_i(x) = 0\,, & i \in \mathcal{E}\,, \\ c_i(x) \geq 0\,, & i \in \mathcal{I}\,, \end{cases} \tag{5}$$

where $f : \mathbb{R}^N \to \mathbb{R}$ is a convex function and the feasible region $\Omega = \{x \mid c_i(x) = 0\,,\ i \in \mathcal{E}\,,\ \ c_i(x) \geq 0\,,\ i \in \mathcal{I}\}$ is a convex set.

a) Show that every local solution of (5) is also a global solution.

b) Show that the set of global solutions is convex.

c) A sufficient condition for the feasible region Ω to be a convex set is that the equality constraint functions $c_i\,,\ i \in \mathcal{E}$, are affine and the inequality constraint functions $c_i\,, i \in \mathcal{I}$ are concave. Show that this condition is not necessary.

7. Consider the minimization problem

$$\min_{x \in \mathbb{R}^2} \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \qquad \text{subject to} \qquad x_1 + x_2 = 1\,. \tag{6}$$

a) Write down the quadratic penalty function $Q(x;\mu)$ for problem (6). Compute the gradient $\nabla_x Q(x;\mu)$ and the Hessian $\nabla_x^2 Q(x,\mu)$ of the quadratic penalty function.

b) Show that

$$x_*(\mu_k) = \begin{bmatrix} \dfrac{1}{\mu_k + 2} & \dfrac{1}{\mu_k + 2} \end{bmatrix}^T$$

is the global minimizer of the unconstrained minimization problem $\min_{x \in \mathbb{R}^2} Q(x, \mu_k)$.

c) Check that the Hessian matrix $\nabla_x^2 Q(x,\mu)$ becomes ill-conditioned when $\mu_k \to 0^+$.

d) Find the global solution of problem (6).

e) Write the augmented Lagrangian $\mathcal{L}_A(x, \lambda; \mu)$ for problem (6). Calculate the gradient $\nabla_x \mathcal{L}_A(x, \lambda; \mu)$ and the Hessian $\nabla_x^2 \mathcal{L}_A(x, \lambda; \mu)$ of the augmented Lagrangian. Show that

$$x_*(\mu_k) = \begin{bmatrix} \dfrac{1 + \mu_k \lambda^k}{\mu_k + 2} & \dfrac{1 + \mu_k \lambda^k}{\mu_k + 2} \end{bmatrix}^T$$

is the global minimizer of the unconstrained minimization problem $\min_{x \in \mathbb{R}^2} \mathcal{L}_A(x, \lambda^k; \mu_k)$.

f) Prove that the update formula for λ^k in the augmented Lagrangian method is given by

$$\lambda^{k+1} = \frac{\mu_k}{\mu_k + 2} \lambda^k + \frac{1}{\mu_k + 2}, \qquad \mu_k > 0\,.$$

Verify also that

$$\lambda^{k+1} - \lambda^* = \frac{\mu_k}{\mu_k + 2} \left(\lambda^k - \lambda^* \right).$$

8. Consider the minimization problem

$$\min_{x \in \mathbb{R}} \; -x^2 \qquad \text{subject to} \qquad 1 - x^2 \geq 0\,. \tag{7}$$

Show that the logarithmic barrier function $P(x, \mu)$ has a minimizer $x = 0$ if $\mu > 1$ and two minimizers $x = \pm\sqrt{1 - \mu}$ if $\mu < 1$.