Master in Mathematics and Applications - Técnico, Lisbon
Numerical Functional Analysis and Optimization - Fall Semester 2016

4^{th} **Problem sheet**

1. Let $u, v \in \mathbb{R}^N$ be column vectors and $A, B \in \mathbb{R}^{N\times N}$ square matrices. Moreover, let $\|A\|_F$ denote the Fröbenius norm and $\|A\|_2$ the (operator) norm induced by the Euclidean vector norm $\|\cdot\|_2$.

a) Show that

$$\|uv^T\|_F = \|uv^T\|_2 = \|u\|_2\,\|v\|_2\,.$$

b) Prove that

$$\|AB\|_F \le \min\{\|A\|_2\,\|B\|_F, \|A\|_F\,\|B\|_2\}\,.$$

2. Let $P \in \mathbb{R}^{N\times N}$ be a ortogonal projection matrix, i.e. P is symmetric and $P^2 = P$.

a) Show that

$$\|x\|_2^2 = \|Px\|_2^2 + \|x - Px\|_2^2 \qquad \forall x \in \mathbb{R}^N\,.$$

b) Let $w \in \mathbb{R}^N$ be a column vector and $L = ww^T/w^Tw$. Show that L and $I - L$ are orthogonal projection matrices. Determine the eigenvalues of L and $I - L$ and show that $\|L\|_2 = \|I - L\|_2$.

3. Let $A \in \mathbb{R}^{N\times N}$, $y, s \in \mathbb{R}^N$, $s \neq 0$ and $Q(y, s) = \{B \in \mathbb{R}^{N\times N}|\, Bs = y\}$. Prove that the minimization problem

$$\min_{B\in Q(s,y)} \|B - A\|_F\,,$$

admits at most one solution.

4. Consider the minimization problem

$$\min_{B\in Q(y,s)} \|B - A\|_2\,,$$

where

$$A = \begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}, \quad s = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad y = \begin{bmatrix} 5 \\ -2 \end{bmatrix}.$$

Prove that every matrix $A(\alpha)$ of the form

$$A(\alpha) = A + \begin{bmatrix} 1 & \alpha \\ -1 & \alpha \end{bmatrix}, \qquad \alpha \in [-1, 1]\,,$$

solves this problem. What would the solution be if the minimization was done with respect to the Fröbenius norm?

5. Let $\{x_k\}_{k\ge 0}$ be a sequence of elements in $\mathbb{R}^N$ and assume that $\lim_{k\to\infty} x_k = x_*$ with a superlinear rate of convergence. Show that

$$\lim_{k\to\infty} \frac{\|x_{k+1} - x_k\|}{\|x_k - x_*\|} = 1\,.$$

6. Let $F(x) = Ax - b$, with $A \in \mathbb{R}^{N\times N}$ non-singular and $b \in \mathbb{R}^N$, and consider the following modified Broyden's update for approximating the solution of the linear system $F(x) = 0$

$$A_{k+1} = A_k + \theta \, \frac{(y_k - A_k s_k) s_k^T}{s_k^T s_k}\,, \qquad \theta \in \mathbb{R}\,,$$

where $y_k = F(x_{k+1}) - F(x_k)$ and $s_k = x_{k+1} - x_k$.

a) Show that $A \in Q(y_k, s_k) = \{B \in \mathbb{R}^{N\times N} \,|\, B s_k = y_k\}$.

b) Show that

$$A_{k+1} - A = \Big(A_k - A\Big)\left(I - \theta \, \frac{s_k s_k^T}{s_k^T s_k}\right), \qquad k = 0, 1, \dots\,,$$

and that

$$\|A_{k+1} - A\|_2 \leq \|A_k - A\|_2\,, \qquad \forall\, \theta \in [0, 2]\,.$$

7. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$F(x) = \begin{bmatrix} x_1 + x_2 - 3 \\ x_1^2 + x_2^2 - 9 \end{bmatrix},$$

where $x = (x_1, x_2)$. The equation $F(x) = 0$ has two roots, $z_* = (3, 0)$ e $y_* = (0, 3)$.

a) Approximate the solution of the nonlinear system $F(x) = 0$ by the Newton's and the Broyden's method. Choose $x_0 = (1, 5)$, $A_0 = J_F(x_0)$ and compute the first five iteratates. Analyse the rate of converegence in both cases.

b) Show that

$$\lim_{k\to\infty} A_k = \begin{bmatrix} 1 & 1 \\ 1.5 & 7.5 \end{bmatrix} \neq J_F(y_*) = \begin{bmatrix} 1 & 1 \\ 0 & 6 \end{bmatrix},$$

where A_k is Broyden's update matrix.

c) Verify that

$$\lim_{k\to\infty} \frac{\|(A_k - J_F(y_*))\,(x_{k+1} - x_k)\,\|}{\|x_{k+1} - x_k\|} = 0\,.$$