Master in Mathematics and Applications - Técnico, Lisbon
Numerical Functional Analysis and Optimization - Fall Semester 2016

3^{rd} Problem sheet

1. Let $V = C[0,1]$ and consider the integral operator $T : V \to V$, defined by

$$T(u)(t) = \int_0^1 k(t,s)\, u^2(s)\, ds + f(t)\,, \qquad 0 \le t \le 1\,,$$

where $k \in C([0,1] \times [0,1])$ and $f \in C[0,1]$.

a) Show that T is Fréchet differentiable and determine the Fréchet derivative $T'(u)$.

b) Derive a bound for the norm $\|T'(u)\|$.

2. Let V be a Hilbert space equipped with the inner product $(\cdot,\cdot)$. Consider the functional $f : V \to \mathbb{R}$ defined by

$$f(u) = (u, Au)\,,$$

where $A \in \mathcal{L}(V)$.

a) Show that f is Fréchet differentiable and compute $f'(u)$.

b) Show that f' is Fréchet differentiable and compute $f''(u)$.

c) Let $V = \mathbb{R}^N$ and assume that $A \in \mathbb{R}^{N\times N}$ is a symmetric matrix. Determine the first two Fréchet derivatives of f.

3. Let $A \in \mathbb{R}^{N\times N}$ be a symmetric matrix and consider the functional $f : \mathbb{R}^N \to \mathbb{R}$ defined by $f(x) = x^T A x + b^T x + c$ where $b \in \mathbb{R}^N$ and $c \in \mathbb{R}$. Show that f is a convex function if and only if A is positive semi-definite and is strictly convex if and only if A is positive definite.

4. Let $f : \mathbb{R}^M \to \mathbb{R}$ be a convex and Fréchet differentiable in $\mathbb{R}^M$ and consider a function $g : \mathbb{R}^N \to \mathbb{R}$ defined by $g(x) = f(Ax + b)$, where $A \in \mathbb{R}^{M\times N}$ and $b \in \mathbb{R}^M$. Determine the first Fréchet derivative of g and show that g is a convex function.

5. Let K be a non-empty, closed, convex subset of a Hilbert space V and assume that $f \in V$.

a) Show that $\phi^* \in K$ is the best approximation of f in K, with respect to the induced norm, if and only if

$$(f - \phi^*, \phi - \phi^*) \le 0 \qquad \forall\, \phi \in K\,. \tag{1}$$

b) Show that, given $f \in V$, there exists a unique $\phi^* \in K$ such that

$$\|f - \phi^*\| = \min_{\phi\in K} \|f - \phi\|\,,$$

and that ϕ^* is also characterized by the inequality (1).

c) Show that if K is a subspace of V then the best approximation ϕ^* is characterized by

$$(f - \phi^*, \phi) = 0 \qquad \forall\, \phi \in K\,.$$

6. Let V be a Banach space and $K \subset V$ a non-empty convex subset. Assume that $f : K \to \mathbb{R}$ is convex and Fréchet differentiable in K.

a) Show that there exists $u \in K$ such that

$$f(u) = \min_{v \in K} f(v)$$

if and only if there exists $u \in K$ such that

$$f'(u)(v-u) \geq 0 \qquad \forall\, v \in K\,. \tag{2}$$

b) Show that when K is a subspace, inequality (2) reduces to the equality

$$f'(u)v = 0 \qquad \forall\, v \in K\,.$$

7. Assume that $F : K \subset \mathbb{R}^N \to \mathbb{R}^M$ be continuously Fréchet differentiable in an open non-empty convex set $K_0 \subset K$.

a) Show that if F' is Lipschitz continuous in K_0, i.e. there exists $L \geq 0$ such that

$$\|F'(x) - F'(y)\| \leq L\, \|x - y\| \quad \forall\, x, y \in K_0\,,$$

then

$$\|F(x+h) - F(x) - F'(x)h\| \leq \frac{L}{2}\, \|h\|^2 \quad \forall\, x, x+h \in K_0\,.$$

b) Show that if F is twice continuously Fréchet differentiable in $K_0 \subset K$ then

$$\|F(x+h) - F(x) - F'(x)h\| \leq \frac{1}{2} \sup_{0 \leq t \leq 1} \|F''(x+th)\|\, \|h\|^2 \quad \forall\, x, x+h \in K_0\,.$$

(Hint: Use the mean value theorem.)

8. Let $F : U \to V$ be continuously Fréchet differentiable, let u^* be a solution to the nonlinear equation $F(u) = 0$ and assume that $[F'(u^*)]^{-1}$ exists as a continuous linear map from V to U. Use the Banach fixed-point theorem to show that the Newton's chord method

$$u_{n+1} = u_n - F'(u_0)\, F(u_n)\,, n = 0, 1, \ldots$$

converges linearly to u^* provided u_0 is chosen sufficiently close to u^*. (Hint: Recall exercise **5** from the 2^{nd} problem sheet and choose $\|F'(u_0) - F'(u^*)\|$ sufficiently small.)