Master in Mathematics and Applications - Técnico, Lisbon
Numerical Functional Analysis and Optimization - Fall Semester 2016

2^{st} **Problem sheet**

1. Consider the following Fredhom integral equation of second kind

$$\lambda\, u(t) - \int_0^1 \frac{u(s)}{1+s^2t^2}\, ds = f(t)\,, \qquad 0 \le t \le 1\,, \tag{1}$$

where $\lambda \in \mathbb{R}\setminus\{0\}$ and $f \in C[0,1]$. Show that, if $|\lambda|$ is chosen sufficiently large, then there exists a unique solution $u \in C[0,1]$ to (8). For those values of λ, bound $\|u\|_\infty$ in terms of $\|f\|_\infty$.

2. Let$f \in C[0,1]$ and consider the boundary value problem

$$\begin{cases} -\,u''(t) = f(t) & 0 < t < 1\,, \\ u(0) = u(1) = 0\,. \end{cases} \tag{2}$$

a) Show that the unique solution u of problem (2) is given by

$$u(t) = \int_0^1 k(t,s)\, f(s)\, ds\,, \tag{3}$$

where

$$k(t,s) = \begin{cases} s\,(1-t)\,, & 0 \le s \le t\,, \\ t\,(1-s)\,, & t \le s \le 1\,. \end{cases}$$

b) Consider the boundary value problem

$$\begin{cases} -\,u''(t) + a(t)\,u(t) = f(t)\,, & 0 < t < 1\,, \\ u(0) = u(1) = 0\,, \end{cases} \tag{4}$$

where $a, f \in C[0,1]$. Show that (4) can be written as a Fredholm integral equation of the second kind.

c) Assume that $\max_{t\in[0,1]} |a(t)| \le a_0$. Show that if $a_0 > 0$ is sufficiently small then problem (4) has a unique solution $u \in C^2[0,1]$.

3. Show that the Fredholm integral equation

$$u(x) - \int_0^1 \sin \pi(x-t) u(t)\, dt = f(x)\,, \qquad 0 \le x \le 1\,,$$

has a unique solution $u \in C[0,1]$ for any given $f \in C[0,1]$. As an approximation of the solution u, use the formula

$$u_n(x) = f(x) + \sum_{j=1}^{n} L^j f(x)$$

to compute u_2.

4. Assume that the conditions of the geometric series theorem are satisfied. Then for any $f \in V$ the equation $(I - L)\,u = f$ has a unique solution $u \in V$. Show that this solution can be approximated by a sequence $\{u_n\}$ defined by

$$u_0 \in V\,, \qquad u_n = f + L\,u_{n-1}\,, \quad n = 1, 2, \ldots$$

by deriving an error bound for $\|u - u_n\|_V$.

5. Let V and W be Banach spaces and assume that $T \in \mathcal{L}(V, W)$ has a bounded inverse $T^{-1} : W \to V$. Show that if $S \in \mathcal{L}(V, W)$ satisfies $\|T - S\| < 1/\|T^{-1}\|$, then $S^{-1} \in \mathcal{L}(W, V)$ exists and

$$\|S^{-1}\| \leq \frac{\|T^{-1}\|}{1 - \|T^{-1}\|\,\|T - S\|}\,.$$

6. Consider the nonlinear Volterra integral equation

$$u(t) = \int_a^t k(t, s, u(s))\,ds + f(t) \qquad a \leq t \leq b\,, \tag{5}$$

where $f \in C[a, b]$ and $k(\cdot, \cdot, \cdot)$ is a continuous function for $a \leq s \leq t \leq b\,,\ u \in \mathbb{R}$. Assume that there exists a constant $M \geq 0$ such that for $a \leq s \leq t \leq b$

$$|k(t, s, u_1) - k(t, s, u_2)| \leq M\,|u_1 - u_2| \qquad \forall\, u_1, u_2 \in \mathbb{R}\,.$$

Let $V = C[a, b]$ and define a nonlinear operator $T : V \to V$ by

$$T(u)(t) = \int_a^t k(t, s, u(s))\,ds + f(t)\,, \qquad a \leq t \leq b\,.$$

a) Show that

$$\|T^m(u) - T^m(v)\|_V \leq \frac{[M(b-a)]^m}{m!}\,\|u - v\|_V \quad \forall u, v \in V\,, \quad m = 0, 1, \ldots\,.$$

b) Prove that the operator T admits a unique fixed point in V, that is, show that the integral equation (5) admits a unique solution $u \in V$.

c) Show that the mapping $|\!|\!|v|\!|\!| := \max_{t\in[0,b]} e^{-\beta t}|v(t)|$, $\beta \in \mathbb{R}$ is anorm in $C[0, b]$, equivalent to the uniform norm $\|\cdot\|_\infty$.

d) Let $a = 0$. Show that $T : C[0, b] \to C[0, b]$ has a unique fixed point in $(C[0, b], |\!|\!|\cdot|\!|\!|)$. Assume that $\beta > M$.

7. Let $V = C[0, 1]$ and consider the following nonlinear Volterra integral equation

$$u(t) = \frac{1}{2}\int_0^t s\,u^2(s)\,ds + f(t)\,, \qquad 0 \leq t \leq 1\,, \tag{6}$$

where $f \in V$ is such that $\|f\|_V \leq \frac{1}{2}$.

a) Define $T : V \to V$ through

$$T(u)(t) = \frac{1}{2}\int_0^t s\, u^2(s)\, ds + f(t)\,, \qquad 0 \le t \le 1$$

and let $K = \{v \in V \mid \|v\|_\infty \le C\}$. Choose $C > 0$ in such a way that the operator T has a unique fixed point $u \in K$ and the iteration

$$\begin{cases} u_{n+1}(t) = \dfrac{1}{2}\displaystyle\int_0^t s\, u_n^2(s)\, ds + f(t)\,, \quad n \ge 0 \\ u_0(t) = 1 \end{cases} \tag{7}$$

converges to u.

b) Let $f(t) = t/2$ and $C = 4/3$. Compute the first two iterates from (7). Derive an upper bound for the error $\|u - u_2\|_V$.

8. Consider the nonlinear Urysohn integral equation

$$u(t) = \mu \int_a^b k(t, s, u(s))\, ds + f(t)\,, \qquad a \le t \le b\,, \tag{8}$$

where $\mu \in \mathbb{R}$, $k \in C([a, b] \times [a, b] \times \mathbb{R})$ and $f \in C[a, b]$. Assume that there exists a constant $L \ge 0$ such that

$$|k(t, s, u_1) - k(t, s, u_2)| \le L\, |u_1 - u_2|\,, \quad \forall u_1, u_2 \in \mathbb{R}\,, \quad a \le t, s \le b\,.$$

a) Show that, if $|\mu|\, L\, (b - a) < 1$, then equation (8) admits a unique solution $u \in C[a, b]$ and the iteration

$$u_{n+1}(t) = \mu \int_a^b k(t, s, u_n(s))\, ds + f(t)\,, \qquad a \le t \le b\,, \quad n = 0, 1, \ldots$$

converges to u, for any choice of $u_0 \in C[a, b]$.

b) Let

$$[a, b] = [0, 1]\,, \qquad f(t) = \frac{7}{8}\, t\,, \qquad k(t, s, u(s)) = \frac{t\, s}{1 + u^2(s)}\,.$$

Prove that, if $|\mu| < 1$, then equation (8) admits a unique solution $u \in C[0, 1]$. Approximate u by computing the first two iterates by the fixed point iteration. Consider $\mu = 1/2$ and $u_0(t) = 1$. Derive an upper bound for the error of the approximation u_2.