Master in Mathematics and Applications - Técnico, Lisbon
Numerical Functional Analysis and Optimization - Fall Semester 2016

1^{st} Problem sheet

1. Consider the following sequence of $C[0,1]$ functions

$$u_n(t) = \begin{cases} 0\,, & 0 \le t \le \dfrac{1}{2}\,, \\ 2nt - n\,, & \dfrac{1}{2} \le t < \dfrac{1}{2} + \dfrac{1}{2n}\,, \\ 1\,, & \dfrac{1}{2} + \dfrac{1}{2n} \le t \le 1\,. \end{cases}$$

Show that $\{u_n\}$ is *not* a Cauchy sequence in the uniform norm $\|\cdot\|_\infty$.

2. Let $a, b \ge 0$ and assume $p, q > 1$ are such that $1/p + 1/q = 1$.

a) Show that

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \qquad \text{(Young's inequality)}\,.$$

b) Let $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ be vectors in $\mathbb{R}^N$. Show that

$$\sum_{j=1}^{N} |x_j\, y_j| \le \|\mathbf{x}\|_p\, \|\mathbf{y}\|_q \qquad \text{(Hölder's inequality)}\,.$$

3. Let $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$. Show that the mappings

$$\|\mathbf{x}\|_p = \Big(\sum_{j=1}^{N} |x_j|^p\Big)^{1/p}, \quad 1 \le p < \infty\,, \qquad \text{and} \qquad \|\mathbf{x}\|_\infty = \max_{1\le j\le N} |x_j|\,,$$

are norms in $\mathbb{R}^N$. Show also that the mapping $\|\cdot\|_p : \mathbb{R}^N \to \mathbb{R}$ is not a norm when $p \in]0,1[$.

4. Let $(V, \|\cdot\|)$ be a normed space. Show that the closed unit ball $B_1(0) = \{u \in V \mid \|u\| \le 1\}$ is a convex set in V.

5. Prove the equivalence of the following norms on $C^1[0,1]$

$$\|f\|_a = |f(0)| + \int_0^1 |f'(x)|\, dx\,, \qquad \|f\|_b = \int_0^1 |f(x)|\, dx + \int_0^1 |f'(x)|\, dx\,.$$

6. Let $A \in \mathbb{R}^{N\times N}$ be a symmetric and positive definite matrix e let $(\cdot,\cdot)$ denote the usual (Euclidean) inner product in $\mathbb{R}^N$. Show that $(Ax, y) = x^T A y$ is an inner product in $\mathbb{R}^N$.

7. Let V be a Hilbert space. Show that the inner product $(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a continuous with respect to its induced norm.

8. A normed space $(V, \|\cdot\|)$ is *strictly convex* if

$$\forall\, u, v \in V \;\text{ such that }\; \|u\| = \|v\| = 1\,, \quad u \neq v \qquad \Longrightarrow \qquad \|u + v\| < 2\,.$$

Let U be a subspace of a strictly convex normed space $(V, \|\cdot\|)$. Show that, given $f \in V$, there exists at most one best approximation of f in U, with respect to the norm $\|\cdot\|$, i.e. a function ϕ^*, say, such that

$$\|f - \phi^*\| = \min_{\phi \in U} \|f - \phi\|\,.$$

9. Let V be an inner product space. A linear operator $P : V \to V$ is called *orthogonal projection* if

$$P^2 = P\,, \qquad (Pu, v) = (u, Pv) \quad \forall\, u, v \in V\,.$$

a) Show that a orthogonal projection is a bounded operator.

b) Assume that $P \neq 0$. Show that

$$\|P\| = \sup_{\substack{v \in V \\ v \neq 0}} \frac{\|Pv\|}{\|v\|} = 1\,.$$

10. Let V, W be normed spaces, let $T : V \to W$ be a continuous linear operator and suppose that there exists a constant $c > 0$ such that

$$\|Tv\|_W \geq c\,\|v\|_V \qquad \forall\, v \in V\,.$$

Show that $T^{-1} : W \to V$ exists as a continuous linear operator.

11. For any fixed $t_0 \in [a, b]$, let $f : V \to \mathbb{R}$ be a functional defined by

$$f(v) = v(t_0) \qquad \forall\, v \in V\,,$$

where $V = C[a, b]$.

Show that f is a continuous linear functional and that

$$\|f\| \;=\; \sup_{v \in V, v \neq 0} \frac{|f(v)|}{\|v\|_\infty} = 1\,.$$