# SATURATED FUSION SYSTEMS OVER 2-GROUPS

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Abstract. We develop methods for listing, for a given 2-group $S$, all nonconstrained centerfree saturated fusion systems over $S$. These are the saturated fusion systems which could, potentially, include minimal examples of exotic fusion systems: fusion systems not arising from any finite group. To test our methods, we carry out this program over four concrete examples: two of order $2^7$ and two of order $2^{10}$. Our long term goal is to make a wider, more systematic search for exotic fusion systems over 2-groups of small order.

For any prime $p$ and any finite $p$-group $S$, a saturated fusion system over $S$ is a category $\mathcal{F}$ whose objects are the subgroups of $S$, whose morphisms are injective group homomorphisms between the objects, and which satisfy certain axioms due to Puig and described here in Section 2. Among the motivating examples are the categories $\mathcal{F} = \mathcal{F}_S(G)$ where $G$ is a finite group with Sylow $p$-subgroup $S$: the morphisms in $\mathcal{F}_S(G)$ are the group homomorphisms between subgroups of $S$ which are induced by conjugation by elements of $G$. A saturated fusion system $\mathcal{F}$ which does not arise in this fashion from a group is called "exotic".

When $p$ is odd, it seems to be fairly easy to construct exotic fusion systems over $p$-groups (see, e.g., [BLO2, §9], [RV], and [Rz]), although we are still very far from having any systematic understanding of how they arise. But when $p = 2$, the only examples we know are those constructed by Levi and Oliver [LO], based on earlier work by Solomon [Sol] and Benson [Bs]. The smallest such example known is over a group of order $2^{10}$, and it is possible that there are no exotic examples over smaller groups. Our goal in this paper is to take a first step towards developing techniques for systematically searching for exotic fusion systems, a search which eventually can be carried out in part using a computer.

A fusion system $\mathcal{F}$ is *constrained* (Definition 2.3) if it contains a normal $p$-subgroup which contains its centralizer. Any constrained fusion system is the fusion system of a unique finite group with analogous properties [BCGLO1, Proposition C]. A fusion system $\mathcal{F}$ over $S$ is *centerfree* (Definition 2.3) if there is no element $1 \neq z \in Z(S)$ such that each morphism in $\mathcal{F}$ extends to a morphism between subgroups containing $z$ which sends $z$ to itself. By [BCGLO2, Corollary 6.14], if there is such a $z$, and if $\mathcal{F}$ is exotic, then there is a smaller exotic fusion system $\mathcal{F}/\langle z\rangle$ over $S/\langle z\rangle$. Thus all *minimal* exotic fusion systems must be nonconstrained and centerfree, and these conditions provide a convenient class of fusion systems to search for and list.

If $\mathcal{F}$ is a saturated fusion system over any $p$-group $S$, then the $\mathcal{F}$-*essential* subgroups of $S$ are the proper subgroups $P \lneqq S$ which "contribute new morphisms" to the category $\mathcal{F}$: it is the smallest set of objects such that each morphism in $S$ is a composite of restrictions of automorphisms of essential subgroups and of $S$ itself. We

2000 *Mathematics Subject Classification.* Primary 20D20. Secondary 20D45, 20D08.

*Key words and phrases.* finite groups, 2-groups, fusion, simple groups.

B. Oliver is partially supported by UMR 7539 of the CNRS.

J. Ventura is partially supported by FCT/POCTI/FEDER and grant PDCT/MAT/58497/2004.

refer to Definition 2.3, Proposition 2.5, and Corollary 2.6 for more details. We define a *critical* subgroup of $S$ to be one which could, potentially, be essential in some fusion system over $\mathcal{F}$. The precise definition (Definition 3.1) is somewhat complicated (and stated without reference to fusion systems), and involves the existence of subgroups of $\mathrm{Out}(P)$ which contain strongly embedded subgroups. Thus Bender's classification of groups with strongly embedded subgroups (at the prime 2) plays a central role in our work. In addition, one important thing about critical subgroups is that the 2-groups we have studied contain very few of them (even those 2-groups which support many "interesting" saturated fusion systems), and we have developed some fairly efficient techniques for listing them.

Thus, the first step when trying to find all saturated fusion systems over a 2-group $S$ is to list its critical subgroups. Afterwards, for each critical $P$ (and for $P = S$), one computes $\mathrm{Out}(P)$, and determines which subgroups of $\mathrm{Out}(P)$ can occur as $\mathrm{Aut}_{\mathcal{F}}(P)$ if $P$ is $\mathcal{F}$-essential. The last step is then to put this all together: to see which combinations of essential subgroups and their automorphism groups can generate a nonconstrained centerfree saturated fusion system $\mathcal{F}$.

To illustrate how this procedure works in practice, we finish by listing all nonconstrained centerfree saturated fusion systems over two groups of order $2^7$ and two groups of order $2^{10}$. We chose them because each is the Sylow subgroup of several "interesting" simple or almost simple groups; in fact, each is the Sylow 2-subgroup of at least one sporadic simple group. The groups we chose are the Sylow 2-subgroups of $M_{22}$, $M_{23}$, and McL; $J_2$ and $J_3$; He, $M_{24}$, and $GL_5(2)$; and $Co_3$. The last case is particularly interesting because it is also the Sylow subgroup of the only known exotic fusion system over a 2-group of order $\leq 2^{10}$.

Not surprisingly, we found no new exotic fusion systems over any of these four groups, and a much wider and more systematic search will be needed to have much hope of finding new exotic examples. For example, over the group $S = UT_5(2)$ of upper triangular $5 \times 5$ matrices over $\mathbb{F}_2$, we show (Theorem 6.10) that the only nonconstrained centerfree saturated fusion systems are those of the simple groups He, $M_{24}$, and $GL_5(2)$. Likewise, over the Sylow 2-subgroup of $Co_3$, we show (Theorem 7.8) that each such fusion system is either the fusion system of $Co_3$, or that of the almost simple group $\mathrm{Aut}(PSp_6(3))$, or the exotic fusion system Sol(3) constructed in [LO]. Thus in these cases, we repeat in part the well known results of Held [He] and Solomon [Sol], except that we classify fusion systems over these 2-groups, and do not try to list all groups which realize a given fusion system. But the techniques we use are somewhat different, and we hope that they can eventually make possible a more systematic search for exotic fusion systems.

This approach also makes it easy to determine all automorphisms of the fusion systems we classify. We don't state it here explicitly, but using the information given about the fusion systems over the four 2-groups we study, one can easily determine their automorphisms, and check they all extend to automorphisms of the associated groups.

The paper is organized as follows. The first section contains background results on finite groups, their automorphism groups, and strongly embedded subgroups, while Section 2 contains background results on fusion systems. Then, in Section 3, critical subgroups are defined, and techniques developed for determining the critical subgroups of a given 2-group. Afterwards, in Sections 4–7, we present our examples, describing the nonconstrained centerfree saturated fusion systems over four different 2-groups.

We would like in particular to thank Kasper Andersen, who helped revive our interest in this program by doing a computer search for some critical subgroups; and Andy Chermak, for (among other things) suggesting we look at the Sylow subgroup of the Janko groups $J_2$ and $J_3$. We also give special thanks to the referee, whose many detailed suggestions helped us to make considerable improvements in the paper.

***Notation :*** Most of the time, when $\alpha \in \mathrm{Aut}(P)$ for some group $P$, we write $[\alpha]$ for the class of $\alpha$ in $\mathrm{Out}(P)$. The one exception to this is the case of automorphisms defined via conjugation: $c_g$ denotes conjugation by $g$, as an element of a group $\mathrm{Aut}(P)$ or $\mathrm{Out}(P)$ (for some $P$ normalized by $g$) which will be specified each time. Occasionally, the same letter will be used to describe a subgroup of $\mathrm{Aut}(P)$ and its image in $\mathrm{Out}(P)$, but that will be stated explicitly in each case.

Since the two authors are topologists, some of our notation clashes with that usually used by group theorists. Commutators are defined $[g,h]=ghg^{-1}h^{-1}$, and $c_g$ denotes conjugation by $g$ in the sense $c_g(x)=gxg^{-1}$. Homomorphisms are written on the left and composed from right to left. When a matrix is used to describe a linear map between vector spaces with respect to given bases, each column contains the coordinates of the image of one basis element. Finally, in what *is* standard notation, we write $Z_n(P)$ for the $n$-th term in the upper central series of a $p$-group $P$; thus $Z_1(P)=Z(P)$ and $Z_n(P)/Z_{n-1}(P)=Z(P/Z_{n-1}(P))$.

## 1. Background results

We collect here some results about groups and their automorphisms which will be needed later. Almost all of them are either well known, or follow from well known constructions.

We first recall some standard notation. For any group $G$ and any prime $p$, $O_p(G)$ denotes the largest normal $p$-subgroup (the intersection of the Sylow $p$-subgroups of $G$), and $O^p(G)$ denotes the smallest normal subgroup of $p$-power index. Also, $O_{p'}(G)$ denotes the largest normal subgroup of order prime to $p$, and $O^{p'}(G)$ denotes the smallest normal subgroup of index prime to $p$.

### 1.1 Automorphisms of $p$-groups and group cohomology

We first consider conditions which can be used to show that certain automorphisms of a $p$-group $P$ lie in $O_p(\mathrm{Aut}(P))$. Recall that the Frattini subgroup $\mathrm{Fr}(P)$ of a $p$-group $P$ is the subgroup generated by commutators and $p$-th powers; i.e., the smallest normal subgroup whose quotient is elementary abelian. It has the property that if $g_1,\dots,g_k\in P$ are elements whose classes generate $P/\mathrm{Fr}(P)$, then they generate $P$.

**Lemma 1.1.** *Fix a prime $p$, a $p$-group $P$, and a sequence of subgroups*

$$P_0 \lhd P_1 \lhd \cdots \lhd P_k = P$$

*such that $P_0 \le \mathrm{Fr}(P)$. Set*

$$\mathcal{A}=\bigl\{\alpha\in\mathrm{Aut}(P)\,\big|\,x^{-1}\alpha(x)\in P_{i-1},\ \text{all } x\in P_i,\ \text{all } i=1,\dots,k\bigr\}\le\mathrm{Aut}(P):$$

*the group of automorphisms which leave each $P_i$ invariant and which induce the identity on each quotient group $P_i/P_{i-1}$. Then $\mathcal{A}$ is a $p$-group. If the $P_i$ are all characteristic in $P$, then $\mathcal{A}\lhd\mathrm{Aut}(P)$, and hence $\mathcal{A}\le O_p(\mathrm{Aut}(P))$.*

*Proof.* To prove that $\mathcal{A}$ is a $p$-group, it suffices to show that each element $\alpha \in \mathcal{A}$ has $p$-power order. This follows, for example, from [G, Theorems 5.1.4 & 5.3.2]. The last statement is then clear. □

We next turn to the problem of determining $\mathrm{Out}(P)$ for a $p$-group $P$. In the next lemma, for any group $G$ and any normal subgroup $H \lhd G$, we let $\mathrm{Aut}(G,H) \le \mathrm{Aut}(G)$ denote the group of automorphisms $\alpha$ of $G$ such that $\alpha(H) = H$, and set $\mathrm{Out}(G,H) = \mathrm{Aut}(G,H)/\mathrm{Inn}(G)$.

**Lemma 1.2.** *Fix a group $G$ and a normal subgroup $H \lhd G$ such that $C_G(H) \le H$ (i.e., $H$ is centric in $G$). Then there is an exact sequence*

$$1 \longrightarrow H^1(G/H; Z(H)) \xrightarrow{\ \eta\ } \mathrm{Out}(G,H) \xrightarrow{\ R\ } N_{\mathrm{Out}(H)}(\mathrm{Out}_G(H))/\mathrm{Out}_G(H) \xrightarrow{\ \chi\ } H^2(G/H; Z(H)), \quad (1)$$

*where $R$ is induced by restriction, and where all maps except (possibly) $\chi$ are homomorphisms. If, furthermore, $H$ is abelian and the extension of $H$ by $G/H$ is split, then $R$ is onto.*

*Proof.* Throughout the proof, $\bar{g} = gH \in G/H$ denotes the class of $g \in G$.

We first prove that there is an exact sequence of the following form:

$$1 \longrightarrow Z^1(G/H; Z(H)) \xrightarrow{\ \widetilde{\eta}\ } \mathrm{Aut}(G,H) \xrightarrow{\ \mathrm{Res}\ } N_{\mathrm{Aut}(H)}(\mathrm{Aut}_G(H)) \xrightarrow{\ \widetilde{\chi}\ } H^2(G/H; Z(H)). \quad (2)$$

Here, $Z^1(G/H; Z(H))$ denotes the group of 1-cocycles for $G/H$ with coefficients in $Z(H)$ ("crossed homomorphisms" in the terminology of [Mc, §IV.4]). Explicitly,

$$Z^1(G/H; Z(H)) = \bigl\{\omega\colon G/H \longrightarrow Z(H) \,\big|\, \omega(\overline{g_1g_2}) = \omega(\overline{g_1}) \cdot g_1\omega(\overline{g_2})g_1^{-1} \ \forall\, g_1, g_2 \in G\bigr\}.$$

For $\omega \in Z^1(G/H; Z(H))$, $\widetilde{\eta}(\omega)$ is defined by setting $\widetilde{\eta}(\omega)(g) = \omega(\bar{g}){\cdot}g$ for $g \in G$. For $g_1, g_2 \in G$,

$$\widetilde{\eta}(\omega)(g_1g_2) = \omega(\overline{g_1g_2}) \cdot g_1g_2 = \omega(\overline{g_1}) \cdot g_1\omega(\overline{g_2})g_1^{-1} \cdot g_1g_2 = \widetilde{\eta}(\omega)(g_1) \cdot \widetilde{\eta}(\omega)(g_2),$$

so $\widetilde{\eta}(\omega) \in \mathrm{Aut}(G,H)$. To see that $\widetilde{\eta}$ is a homomorphism, fix $\omega_1, \omega_2 \in Z^1(G/H; Z(H))$; then for $g_1, g_2 \in G$,

$$\bigl(\widetilde{\eta}(\omega_1) \circ \widetilde{\eta}(\omega_2)\bigr)(g) = \widetilde{\eta}(\omega_1)(\omega_2(\bar{g})g) = \omega_1(\bar{g}) \cdot \omega_2(\bar{g})g = (\omega_1\omega_2)(\bar{g}) \cdot g = \widetilde{\eta}(\omega_1\omega_2)(g)$$

(since $\overline{\omega_2(\bar{g})g} = \bar{g}$). Clearly, $\widetilde{\eta}$ is injective, and $\widetilde{\eta}(\omega)|_H = \mathrm{Id}_H$ for each $\omega$.

The map Res sends $\alpha \in \mathrm{Aut}(G,H)$ to $\alpha|_H \in \mathrm{Aut}(H)$; this lies in the normalizer of $\mathrm{Aut}_G(H)$ since for all $g \in G$, $(\alpha|_H)c_g(\alpha|_H)^{-1} = c_{\alpha(g)} \in \mathrm{Aut}_G(H)$. Assume $\alpha \in \mathrm{Ker}(\mathrm{Res})$; we want to show that $\alpha \in \mathrm{Im}(\widetilde{\eta})$. Since $\alpha \in \mathrm{Aut}(G)$ and $\alpha|_H = \mathrm{Id}_H$, we have $c_g = c_{\alpha(g)} \in \mathrm{Aut}(H)$ for all $g \in G$. Hence $\alpha(g) \equiv g \pmod{Z(H)}$, and the element $\alpha(g){\cdot}g^{-1}$ depends only on $\bar{g} \in G/H$. So we can define $\omega\colon G/H \to Z(H)$ by setting $\omega(\bar{g}) = \alpha(g){\cdot}g^{-1}$ for $g \in G$, and $\alpha$ takes the form $\alpha(g) = \omega(\bar{g}){\cdot}g$. Also, for all $g_1, g_2 \in G$,

$$\omega(\overline{g_1g_2}) = \alpha(g_1g_2) \cdot (g_1g_2)^{-1} = \alpha(g_1)(\alpha(g_2)g_2^{-1})g_1^{-1} = \omega(\overline{g_1}) \cdot g_1\omega(\overline{g_2})g_1^{-1},$$

and hence $\omega \in Z^1(G/H; Z(H))$. This proves the exactness of (2) at $\mathrm{Aut}(G,H)$.

We next define $\widetilde{\chi}$ and prove that $\mathrm{Im}(\mathrm{Res}) = \widetilde{\chi}^{-1}(0)$. Fix some $\varphi \in N_{\mathrm{Aut}(H)}(\mathrm{Aut}_G(H))$, and let $\psi \in \mathrm{Aut}(G/Z(H))$ be defined by $\psi(gZ(H)) = g'Z(H)$ if $\varphi c_g \varphi^{-1} = c_{g'}$ in $\mathrm{Aut}(H)$. This defines $\psi$ uniquely since $C_G(H) = Z(H)$ ($H$ is centric). Intuitively, the obstruction to extending $\varphi$ and $\psi$ to an automorphism of $G$ is the same as the obstruction to two extensions of $H$ by $G/H$ (with the same outer action of $G/H$ on $H$) being isomorphic, and thus lies in $H^2(G/H;Z(H))$ (cf. [Mc, Theorem IV.8.8]).

To show this explicitly, we first choose a map of sets $\widehat{\varphi}\colon G \longrightarrow G$ such that for all $g\in G$ and $h\in H$,

$$\varphi(ghg^{-1})=\widehat{\varphi}(g)\varphi(h)\widehat{\varphi}(g)^{-1},\quad \widehat{\varphi}(hg)=\varphi(h)\widehat{\varphi}(g),\quad \text{and}\quad \widehat{\varphi}(gh)=\widehat{\varphi}(g)\varphi(h) \tag{3}$$

(any two of these imply the third). To construct $\widehat{\varphi}$, let $X=\{g_1,\dots,g_k\}$ be a set of representatives for cosets in $G/H$. For each $i$, let $\widehat{\varphi}(g_i)$ be any element of the coset $\psi(g_iZ(H))\in G/Z(H)$. Then $\varphi(g_ihg_i^{-1})=\widehat{\varphi}(g_i)\varphi(h)\widehat{\varphi}(g_i)^{-1}$ for all $h\in H$ by definition of $\psi$. So if we set $\widehat{\varphi}(hg_i)=\varphi(h)\widehat{\varphi}(g_i)$ for all $h\in H$, then (3) holds for all $g$ and $h$.

For all $g_1,g_2\in G$, $\widehat{\varphi}(g_1g_2)\equiv\widehat{\varphi}(g_1)\widehat{\varphi}(g_2) \pmod{Z(H)}$, since the two elements have the same conjugation action on $H$ by (3). So there is a function $\tau\colon G/H\times G/H \longrightarrow Z(H)$ such that

$$\widehat{\varphi}(g_1g_2)=\varphi(\tau(\overline{g_1},\overline{g_2}))\cdot\widehat{\varphi}(g_1)\widehat{\varphi}(g_2).$$

That $\tau(\overline{g_1},\overline{g_2})$ depends only on the classes of $g_1$ and $g_2$ in $G/H$ follows from the last two identities in (3). For each triple of elements $g_1,g_2,g_3\in G$,

$$\begin{aligned}\widehat{\varphi}(g_1g_2g_3)&=\varphi(\tau(\overline{g_1g_2},\overline{g_3}))\cdot\widehat{\varphi}(g_1g_2)\widehat{\varphi}(g_3)=\varphi(\tau(\overline{g_1g_2},\overline{g_3})\cdot\tau(\overline{g_1},\overline{g_2}))\cdot\widehat{\varphi}(g_1)\widehat{\varphi}(g_2)\widehat{\varphi}(g_3)\\&=\varphi(\tau(\overline{g_1},\overline{g_2g_3}))\cdot\widehat{\varphi}(g_1)\widehat{\varphi}(g_2g_3)=\varphi(\tau(\overline{g_1},\overline{g_2g_3}))\cdot\widehat{\varphi}(g_1)\varphi(\tau(\overline{g_2},\overline{g_3}))\widehat{\varphi}(g_2)\widehat{\varphi}(g_3)\\&=\varphi\big(\tau(\overline{g_1},\overline{g_2g_3})\cdot g_1\tau(\overline{g_2},\overline{g_3})g_1^{-1}\big)\cdot\widehat{\varphi}(g_1)\widehat{\varphi}(g_2)\widehat{\varphi}(g_3);\end{aligned}$$

and hence

$$\tau(\overline{g_1g_2},\overline{g_3})\cdot\tau(\overline{g_1},\overline{g_2})=\tau(\overline{g_1},\overline{g_2g_3})\cdot g_1\tau(\overline{g_2},\overline{g_3})g_1^{-1}. \tag{4}$$

This is precisely the relation which implies $\tau$ is a 2-cocycle (cf. [Mc, §IV.4, (4.8)]).

Assume $\widetilde{\varphi}\colon G\longrightarrow G$ is another map satisfying (3), and let $\widetilde{\tau}$ be the 2-cocycle defined using $\widetilde{\varphi}$. Then $\widehat{\varphi}(g)\equiv\widetilde{\varphi}(g) \pmod{Z(H)}$ for each $g$. So using (3) again, we define $\sigma\colon G/H\longrightarrow Z(H)$ so that $\widetilde{\varphi}(g)=\varphi(\sigma(\bar{g}))\widehat{\varphi}(g)$ for all $g$. Then for $g_1,g_2\in G$,

$$\begin{aligned}\widetilde{\tau}(\overline{g_1},\overline{g_2})&=\varphi^{-1}\big(\widetilde{\varphi}(g_1g_2)\widetilde{\varphi}(g_2)^{-1}\widetilde{\varphi}(g_1)^{-1}\big)\\&=\varphi^{-1}\big(\varphi(\sigma(\overline{g_1g_2}))\widehat{\varphi}(g_1g_2)\widehat{\varphi}(g_2)^{-1}\varphi(\sigma(\overline{g_2})^{-1})\widehat{\varphi}(g_1)^{-1}\varphi(\sigma(\overline{g_1})^{-1})\big)\\&=\sigma(\overline{g_1g_2})\cdot\tau(\overline{g_1},\overline{g_2})\cdot g_1\sigma(\overline{g_2})^{-1}g_1^{-1}\cdot\sigma(\overline{g_1})^{-1}.\end{aligned}$$

In the notation of [Mc, §IV.4], $\widetilde{\tau}=\tau\cdot\delta(\sigma)^{-1}$, and thus $\widetilde{\tau}$ and $\tau$ represent the same element in $H^2(G/H;Z(H))$. We now define $\widetilde{\chi}$ by setting $\widetilde{\chi}(\varphi)=[\tau]\in H^2(G/H;Z(H))$.

If $\varphi\in\operatorname{Im}(\mathrm{Res})$, then $\widehat{\varphi}$ can be chosen to be an automorphism of $G$, so $\tau=1$, and $\varphi\in\operatorname{Ker}(\widetilde{\chi})$. Conversely, if $\varphi\in\operatorname{Ker}(\widetilde{\chi})$, then for any choice of $\widehat{\varphi}$ (with $\tau$ defined as above), there is $\sigma\colon G/H\longrightarrow Z(H)$ such that $\tau=\delta(\sigma)$. So if we define $\widetilde{\varphi}$ by setting $\widetilde{\varphi}(g)=\varphi(\sigma(\bar{g}))\widehat{\varphi}(g)$ for all $g$, the above computations show that $\widetilde{\tau}=1$, and thus that $\widetilde{\varphi}\in\operatorname{Aut}(G,H)$ and $\varphi\in\operatorname{Im}(\mathrm{Res})$.

This proves the exactness of (2). The following sequence is clearly exact:

$$1\longrightarrow\operatorname{Aut}_{Z(H)}(G)\xrightarrow{\ \text{incl}\ }\operatorname{Inn}(G)\xrightarrow{\ \text{restr.}\ }\operatorname{Aut}_G(H)\longrightarrow 1.$$

So if we replace the first three terms in (2) by their quotients modulo these three subgroups, the resulting sequence

$$\begin{aligned}1\longrightarrow Z^1(G/H;Z(H))/\widetilde{\eta}^{-1}(\operatorname{Aut}_{Z(H)}(G))&\xrightarrow{\ \eta\ }\operatorname{Aut}(G,H)/\operatorname{Inn}(G)\\&\xrightarrow{\ R\ }N_{\operatorname{Aut}(H)}(\operatorname{Aut}_G(H))/\operatorname{Aut}_G(H)\xrightarrow{\ \chi\ }H^2(G/H;Z(H)).\end{aligned} \tag{5}$$

is still exact. For $z\in Z(H)$, $c_z(g)=(zgz^{-1}g^{-1})g$ for $g\in G$, and hence $c_z=\widetilde{\eta}(\omega_z)$ where $\omega_z(\bar{g})=(zgz^{-1})g^{-1}$. The group of all such $\omega_z$ is precisely the group $B^1(G/H;Z(H))$

of all 1-coboundaries (cf. [Mc, §IV.2]), and hence the first term in (5) is equal to to $H^1(G/H; Z(H))$. This finishes the proof that (1) is defined and exact.

It remains to prove the last statement. Assume $H$ is abelian, and $G = HK$ where $H \cap K = 1$. Thus $K$ projects isomorphically onto $G/H \cong \mathrm{Out}_G(H)$, and so we can identify these groups. Fix $\beta \in \mathrm{Aut}(H) = \mathrm{Out}(H)$, and assume $\beta \in N_{\mathrm{Aut}(H)}(\mathrm{Aut}_G(H))$. Let $\gamma \in \mathrm{Aut}(K)$ be the automorphism of $K \cong \mathrm{Aut}_G(H)$ induced by conjugation by $\beta$. Then there is an automorphism $\alpha \in \mathrm{Aut}(G)$ such that $\alpha|_H = \beta$ and $\alpha|_K = \gamma$, and thus $R$ is surjective in this case. □

Lemma 1.2 will frequently be applied in the following special case:

**Corollary 1.3.** *Let $G$ be a finite 2-group with centric characteristic subgroup $H \lhd G$. Assume $Z(H)$ has exponent two, and has a basis over $\mathbb{F}_2$ which is permuted freely under the conjugation action of $G/H$. Then there is an isomorphism*

$$R\colon \mathrm{Out}(G) \xrightarrow{\ \cong\ } N_{\mathrm{Out}(H)}(\mathrm{Out}_G(H))/\mathrm{Out}_G(H)$$

*which sends the class of $\alpha \in \mathrm{Aut}(G)$ to the class of $\alpha|_H \in \mathrm{Aut}(H)$.*

*Proof.* The condition on $Z(H)$ implies that $H^i(G/H; Z(H)) = 0$ for all $i > 0$. Also, $\mathrm{Out}(G) = \mathrm{Out}(G, H)$ since $H$ is characteristic in $G$, and so the result follows directly from Lemma 1.2. □

The following slightly related lemma was suggested to us by the referee.

**Lemma 1.4.** *Let $G$ be a finite 2-group, with normal subgroup $H \lhd G$ of index two. Assume there is $x \in G \smallsetminus H$ of order two. Assume also there is a sequence*

$$1 = H_0 \le H_1 \le \cdots \le H_m = H$$

*of subgroups all normal in $G$, such that for each $1 \le i \le m$, $V_i \overset{\text{def}}{=} H_i/H_{i-1}$ is elementary abelian and $C_{V_i}(x) = [x, V_i]$ (equivalently, $|V_i| = |C_{V_i}(x)|^2$). Then all involutions in $xH = G \smallsetminus H$ are conjugate by elements of $H$.*

*Proof.* We show this by induction on $m$. When $m = 1$, all involutions in $xH$ are conjugate since $H^1(\langle x\rangle; H) \cong C_H(x)/[x, H] = 1$ (cf. [A1, 17.7]).

If $m > 1$, then by the induction hypothesis applied to $G/H_1$, all involutions in $xH$ are conjugate modulo $H_1$. Thus if $y \in xH$ is another involution, then $y$ is $H$-conjugate to some $y' \in xH_1$, and $y'$ is $H_1$-conjugate to $x$ by the first part of the proof. □

## 1.2 Strongly embedded subgroups

Strongly embedded subgroups of a finite group play a central role in this paper. We begin with the definition.

**Definition 1.5.** *Fix a prime $p$. For any finite group $G$, a subgroup $G_0 \lneq G$ is called* strongly embedded at $p$ *if $p \big| |G_0|$, and for all $g \in G \smallsetminus G_0$, $G_0 \cap gG_0g^{-1}$ has order prime to $p$. A subgroup $G_0 \lneq G$ is* strongly embedded *if it is strongly embedded at 2.*

The classification of all finite groups with strongly embedded subgroups at 2 is due to Bender.

**Theorem 1.6** (Bender)**.** *Let $G$ be a finite group with strongly embedded subgroup at the prime 2. Fix a Sylow 2-subgroup $S \in \mathrm{Syl}_2(G)$. Then either $S$ is cyclic or quaternion, or $O^{2'}(G/O_{2'}(G))$ is isomorphic to one of the simple groups $PSL_2(2^n)$, $PSU_3(2^n)$, or $\mathrm{Sz}(2^n)$ (where $n \ge 2$, and $n$ is odd in the last case).*

*Proof.* See [Bd]. □

The following lemma about $\mathbb{F}_2$-representations of groups with strongly embedded subgroups (at the prime 2) will be needed in Section 3, and plays a key role in later applications. When $\alpha$ is an automorphism of a vector space $V$, we write

$$[\alpha, V] = \mathrm{Im}[V \xrightarrow{\alpha-\mathrm{Id}} V].$$

**Lemma 1.7.** *Let $G$ be a finite group with strongly embedded subgroup at the prime 2, and let $V$ be an $\mathbb{F}_2$-vector space on which $G$ acts linearly and faithfully. Fix some $S \in \mathrm{Syl}_2(G)$, and let $1 \neq s \in S$ be any nonidentity element. Then the following hold.*

(a) *If $|S| = 2^k$, then $\dim_{\mathbb{F}_2}(V) \geq 2k$.*

(b) *If $Z(S) \cong C_2^n$, then $\dim_{\mathbb{F}_2}([s,V]) \geq n$.*

(c) *If $S$ is cyclic of order 4, then $\dim_{\mathbb{F}_2}([s,V]) \geq 2$. If $S$ is cyclic or quaternion of order $2^k \geq 8$, then $\dim_{\mathbb{F}_2}(V) \geq 3{\cdot}2^{k-2}$ and $\dim_{\mathbb{F}_2}([s,V]) \geq 2^{k-2}$.*

*Proof.* The result is clear when $|S| = 2$ ($\dim(V) \geq 2{\cdot}\dim([s,V]) \geq 2$), so we assume $|S| \geq 4$. If $s \in S$ has order $\geq 4$, then $[s^2,V] \subseteq [s,V]$ ($(s^2-\mathrm{Id})(v) = (s-\mathrm{Id})(v+s(v))$), so it suffices to prove (b) and the statements about $[s,V]$ in (c) when $s$ is an involution. Also, since all involutions in $G$ are conjugate by [Sz2, Lemma 6.4.4], it suffices to prove (b) for just one involution $s$ in $S$. We handle the case where $Z(S)$ is noncyclic in Case 1, and the case where $S$ is cyclic or quaternion in Case 2.

**Case 1:** Assume first that $O^{2'}(G/O_{2'}(G)) \cong L$ where $L$ is simple. There are three subcases to consider. In all cases, we set $|S| = 2^k$ and $|Z(S)| = 2^n$.

**Case 1A:** Assume $L \cong PSL_2(q)$, where $q = 2^k$. Then $S \cong C_2^k$. Also, $L$ contains a dihedral subgroup $\overline{D}$ of order $2(q+1)$. Let $\overline{h_1}, \overline{h_2} \in L$ be a pair of involutions which generate $\overline{D}$, and let $h_1, h_2 \in G$ be a pair of liftings to involutions. Then $\langle h_1, h_2\rangle$ is dihedral of order a multiple of $2(q+1)$, and in particular, $G$ also has a dihedral subgroup $D$ of order $2(q+1)$. Write $D = \langle g, h\rangle$, where $|g| = q+1$ and $|h| = 2$.

By Zsigmondy's theorem (see [Z] or [Ar, p.358]), there is a prime $p$ such that $p|(2^{2k}-1)$ and $p\nmid(2^\ell-1)$ for $\ell < 2k$ — unless $2k = 6$ in which case we take $p = 9$. Thus $p|(q+1)$. Since $g$ acts faithfully and $p$ is a prime power, there is at least one eigenvalue $\lambda \in \overline{\mathbb{F}}_2$ of $g$ with order a multiple of $p$. Since the set of eigenvalues is stable under ($\lambda \mapsto \lambda^2$), the number of eigenvalues (hence the dimension of $V$) is at least equal to the order of 2 modulo $p$; thus $\dim(V) \geq 2k$. Furthermore, the $2k$ eigenvalues $\{\lambda^{2^i}\}$ are permuted in pairs under the action of $h$, and so $\dim([h,V]) \geq k$.

**Case 1B:** Next assume $L \cong \mathrm{Sz}(q)$ for $q = 2^n \geq 8$, where $n$ is odd. Then $|S| = q^2 = 2^{2n}$ and $Z(S) \cong C_2^n$. By Zsigmondy's theorem again, there is a prime $p$ such that 2 has order $4n$ modulo $p$. Since

$$2^{4n} - 1 = q^4 - 1 = (q + \sqrt{2q} + 1)(q - \sqrt{2q} + 1)(q^2 - 1),$$

$p$ divides at least one of the factors $m = q \pm \sqrt{2q} + 1$. By [HB3, Theorem XI.3.10], $L$ contains a dihedral subgroup of order $2m$. Hence by the same argument as that used in Case 1A, $\dim(V) \geq 4n$, and $\dim([h,V]) \leq 2n$ for $h \in G$ of order 2.

**Case 1C:** Now assume $L \cong PSU_3(q)$ for $q = 2^n$ with $n \geq 2$. Then $|S| = q^3 = 2^{3n}$ and $Z(S) \cong C_2^n$. Also, $L$ contains a cyclic subgroup of order $m = (q^2 - q + 1)$ or $m = (q^2 - q + 1)/3$, which comes from regarding $\mathbb{F}_{q^6}$ as a 3-dimensional $\mathbb{F}_{q^2}$-vector space with hermitian product $(x,y) = xy^{q^3}$.

Using Zsigmondy's theorem, choose a prime $p$ such that 2 has order $6n$ modulo $p$ (recall $n \geq 2$). Then $p|m$ since $q^6 - 1 = (q^3 - 1)(q + 1)(q^2 - q + 1)$. So by the same arguments as used in Case 1A, $\dim(V) \geq 6n$.

Let $D \leq L$ be the dihedral subgroup of order $2(q+1)$ generated by diagonal matrices $\mathrm{diag}(u, u^{-1}, 1)$ for $u \in \mathbb{F}_{q^2}$ with $u^{q+1} = 1$, and by the permutation matrix $\left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Since 2 has order $2n$ modulo $q + 1$, $\dim([h, V]) \geq n$ by the arguments used earlier.

**Case 2:** Now assume $S$ is cyclic or quaternion of order $2^k$ with $k \geq 2$, and set $H = O_{2'}(G)$. Let $s \in S$ be the (unique) involution. Since $sH \in Z(G/H)$ by the Brauer-Suzuki theorem [BS, Theorem 2], $[s, H] \neq 1$. For each prime $p\big||H|$, the number of Sylow $p$-subgroups of $H$ is odd, and hence there is at least one subgroup $H_p \in \mathrm{Syl}_p(H)$ which is normalized by $S$. Since $H$ is generated by these $H_p$, at least one of them is not centralized by $s$. So upon replacing $H$ by some appropriate Sylow subgroup $H_p$ and $G$ by $H_pS$, we can assume $G = HS$ where $H$ is a normal $p$-subgroup.

By a theorem of Thompson [G, Theorem 5.3.13], there is a characteristic subgroup $Q \leq H$ such that $S$ still acts faithfully on $Q$, $\mathrm{Fr}(Q) = [Q, Q] \leq Z(Q)$ is elementary abelian, and $Q$ has exponent $p$. Set $Q_0 = [s, Q] \neq 1$. Then $Q_0$ is $S$-invariant since $s \in Z(S)$, and is generated by elements $[s, g]$ ($g \in Q$) which are inverted under conjugation by $s$. Upon replacing $H$ by $Q_0$ and $G$ by $Q_0S$, we are reduced to the case where $H$ has exponent $p$, $Z(H) \geq \mathrm{Fr}(H)$, and $s$ acts on $H/\mathrm{Fr}(H)$ via $-\mathrm{Id}$.

Fix an irreducible $\mathbb{F}_2[H]$-module $W \subseteq V$ with nontrivial $H$-action. Let $K \lhd H$ be the kernel of the action; thus $H/K$ acts faithfully and irreducibly on $W$. Then $Z(H/K)$ is cyclic, since otherwise any faithful representation would split, and hence $|Z(H/K)| = p$ since $H$ has exponent $p$. Set $S_0 = N_S(K)$, and set $W^* = \langle u(W) \,|\, u \in S_0 \rangle \subseteq V$. For each $u \in S$, $u(W^*)$ is a sum of faithful irreducible $H/uKu^{-1}$-modules. Hence the $|S/S_0|$ distinct submodules $u(W^*)$ are linearly independent, and so $\mathrm{rk}(V) \geq |S/S_0|\cdot \mathrm{rk}(W^*)$. Also, either $S_0 = 1$ and $\mathrm{rk}([s, V]) \geq \frac{1}{2}|S|$, or $s \in S_0$ and $\mathrm{rk}([s, V]) \geq |S/S_0|\cdot \mathrm{rk}([s, W^*])$. So we are done if $S_0 = 1$. Otherwise, upon replacing $V$ by $W^*$ and $G$ by $HS_0/K$, we are reduced to the case where $Z(H) \cong C_p$ and $V$ is a sum of faithful, irreducible $\mathbb{F}_2[H]$-modules.

If $H$ is abelian, then $H \cong C_p$, $sgs^{-1} = g^{-1}$ for $g \in H$, and thus $S$ permutes freely the nontrivial irreducible $\bar{\mathbb{F}}_2[H]$-representations. So $\mathrm{rk}(V) = \dim_{\bar{\mathbb{F}}_2}(\bar{\mathbb{F}}_2 \otimes_{\mathbb{F}_2} V) \geq |S|$ in this case, and $\mathrm{rk}([s, V]) \geq \frac{1}{2}|S|$.

If $H$ is nonabelian, then since $\mathrm{Fr}(H) \leq Z(H) \cong C_p$, $H$ must be extraspecial of order $p^{1+2r}$ for some $r \geq 1$. All faithful irreducible $\mathbb{F}_2[H]$-modules have the same rank $ep^r$ for some $e \geq 2$ depending on $p$. By construction, $s$ acts via $-\mathrm{Id}$ on $H/Z(H)$. Hence $S$ acts freely on $(H/\mathrm{Fr}(H)) \smallsetminus 1$, so $|S| = 2^k \big| (p^{2r} - 1)$, and $2^{k-1}$ must divide one of the factors $p^r \pm 1$. So $\mathrm{rk}(V) \geq 2p^r \geq 2(2^{k-1} - 1) \geq 2^k - 2^{k-2} = 3\cdot 2^{k-2}$ when $k \geq 3$. When $k \leq 3$, $\mathrm{rk}(V) \geq 2p \geq 6 \geq 3\cdot 2^{k-2}$, and thus this lower bound on $\mathrm{rk}(V)$ holds for all $k$.

Fix any $g \in H \smallsetminus Z(H)$ such that $sgs^{-1} = g^{-1}$. The eigenvalues (in $\bar{\mathbb{F}}_2$) of the action of $g$ on $V$ include all $p$-th roots of unity with equal multiplicity, and the action of $s$ sends the eigenspace of $\zeta$ to that of $\zeta^{-1}$. Thus $\mathrm{rk}([s, V]) = \frac{p-1}{2p}\cdot \mathrm{rk}(V) \geq \frac{1}{3}\,\mathrm{rk}(V)$. □

## 1.3 General results on groups

The following result is useful when listing subgroups of $\mathrm{Out}(P)$, for a $p$-group $P$, which have a given Sylow $p$-subgroup. The most important case is that where $Q \lhd G$

and $H_0 \le H \le G$ are such that $Q = O_p(G)$, $G = QH$, and $H_0 \in \mathrm{Syl}_p(H)$. But we also have applications which require the more general setting.

**Proposition 1.8.** *Fix a prime $p$, a finite group $G$, and a normal abelian $p$-subgroup $Q \lhd G$. Let $H \le G$ be such that $Q \cap H = 1$, and let $H_0 \le H$ be of index prime to $p$. Consider the set*

$$\mathcal{H} = \bigl\{ H' \le G \bigm| H' \cap Q = 1,\ QH' = QH,\ H_0 \le H' \bigr\}.$$

*Then for each $H' \in \mathcal{H}$, there is $g \in C_Q(H_0)$ such that $H' = gHg^{-1}$.*

*Proof.* Fix $H' \in \mathcal{H}$, and define $\chi\colon H \to Q$ by setting $\chi(h) = h'h^{-1}$, where $h'$ is the unique element in $H' \cap hQ$. By straightforward calculation, $\chi(h_1h_2) = \chi(h_1)\cdot h_1\chi(h_2)h_1^{-1}$ for all $h_1, h_2 \in H$, and thus $\chi \in Z^1(H;Q)$ is a 1-cocycle. Also, $\chi|_{H_0} = 1$. Since $H^1(H;Q)$ injects into $H^1(H_0;Q)$ ($Q$ is abelian and $[H:H_0]$ is prime to $p$), this means that $\chi$ is a coboundary; i.e., $\chi(h) = ghg^{-1}h^{-1}$ for some $g \in Q$. Thus $H' = gHg^{-1}$. Also, $[g, H_0] = 1$, since $ghg^{-1}h^{-1} = \chi(h) = 1$ for each $h \in H_0$. □

As an example of why $Q$ must be assumed abelian in the above proposition, consider the group $G = GL_2(3)$ (and $p = 2$). Set

$$Q = O_2(G) = \left\langle \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right) \right\rangle \cong Q_8.$$

Consider the subgroups

$$H = \left\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \right\rangle \cong \Sigma_3, \quad H' = \left\langle \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \right\rangle, \quad \text{and} \quad H_0 = \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \right\rangle.$$

Then $H$ and $H'$ are both splittings of the surjection $G \to G/Q \cong \Sigma_3$ which contain $H_0$ as Sylow 2-subgroup, but they are not conjugate in $G$. Instead, $H'$ is $G$-conjugate to the subgroup $H'' = \left\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\rangle$. The 1-cocycle $H \to Q$ which sends the subgroup of order three to $I$ and its complement to $-I$ is nontrivial in $H^1(H;Q_8)$, but its restriction is trivial in $H^1(H_0;Q_8)$.

The following very elementary lemma will be used later to list subgroups of a given 2-group which are not normal, and have index two in their normalizers.

**Lemma 1.9.** *Assume $S$ is a 2-group with a normal subgroup $S_0 \lhd S$, such that $S_0$ and $S/S_0$ are both elementary abelian and $|S/S_0| \le 4$. Assume $P \le S$ is such that $P$ is not normal and $|N_S(P)/P| = 2$. Set $P_0 = P \cap S_0$. Let $m$ be the number of cosets $xS_0 \in S/S_0$ such that $xS_0 \ne S_0$ and $[x, S_0] \le P_0$. Then one of the six cases listed in Table 1.1 holds.*

| | $\mathrm{rk}(S_0/P_0)$ | $\lvert P/P_0\rvert$ | $\lvert S/S_0\rvert$ | $m$ | other properties |
|---|---|---|---|---|---|
| **(a)** | 1 | 1 | 2, 4 | 0 | |
| **(b)** | 1 | 2 | 4 | 1 | $P_0 \ntriangleleft S$; $[x, S_0] \le P_0 \Leftrightarrow x \in PS_0$ |
| **(c)** | 1 | 2 | 4 | 3 | $P_0 \lhd S$, $[S, S_0] \lneqq [S,S] \nleq P_0$ |
| **(d)** | 2 | 2 | 2, 4 | 0 | $PS_0/P_0 \cong D_8$ |
| **(e)** | 2 | 4 | 4 | 1 | |
| **(f)** | 3, 4 | 4 | 4 | 0 | $\mathrm{rk}(C_{S_0/P_0}(P/P_0)) = 1$ |

Table 1.1

*Proof.* Since $S_0$ is abelian, $[x, S_0]$ depends only on the class of $x$ in $S/S_0$. So $m$ is well defined, independently of the choice of coset representatives.

By assumption, $P$ is not normal in $S$. Since $S/S_0$ is abelian, this implies $P_0 \lneqq S_0$. For $g \in S_0$, $g \in N_S(P)$ if and only if $[g, P] \le P_0$, or equivalently, $gP_0 \in C_{S_0/P_0}(P/P_0)$. Hence since $|N_S(P)/P| = 2$ and $C_{S_0/P_0}(P/P_0) \ne 1$,

$$\bigl|C_{S_0/P_0}(P/P_0)\bigr| = 2 \qquad \text{and} \qquad N_S(P) \le PS_0. \tag{6}$$

We next claim that if we regard $S_0/P_0$ as an $\mathbb{F}_2[P/P_0]$-module via the conjugation action, then

$$S_0/P_0 \text{ is isomorphic to a submodule of } \mathbb{F}_2[P/P_0] \quad \text{and} \quad \mathrm{rk}(S_0/P_0) \le |P/P_0|. \tag{7}$$

Since $\mathbb{F}_2[P/P_0]$ is injective as a module over itself (since its dual is projective), the (unique) monomorphism from $C_{S_0/P_0}(P/P_0) \cong \mathbb{F}_2$ into the fixed subspace of $\mathbb{F}_2[P/P_0]$ extends to an $\mathbb{F}_2[P/P_0]$-linear homomorphism $\varphi$ from $S_0/P_0$ to $\mathbb{F}_2[P/P_0]$. Also, $\varphi$ must be injective, since otherwise its kernel would have to contain the fixed subgroup $C_{S_0/P_0}(P/P_0)$, and this proves (7).

We are now ready to consider the individual cases. Assume first $\mathrm{rk}(S_0/P_0) = 1$. If $|P/P_0| = 1$, then $P \le S_0$, and $[x, S_0] \le P_0 = P$ only for $x \in N_S(P) = S_0$. Thus $m = 0$, and we are in the situation of (a). If $|P/P_0| > 1$, then $|P/P_0| < |S/S_0|$ (since otherwise $[S : P] = 2$ and $P$ is normal), and thus $|P/P_0| = 2$ and $|S/S_0| = 4$. Also, $N_S(P) = PS_0$ by (6). If $P_0 \lhd S$, then $S_0/P_0 \cong C_2$ is central in $S/P_0$, so $[S, S_0] \le P_0$, and $m = 3$. Also, $[S, S] \nleq P_0$ in this case (otherwise $P$ would be normal in $S$), so $[S, S_0] \lneqq [S, S]$, and we are in case (c). If $P_0 \ntriangleleft S$, then $PS_0 = N_S(P_0)$, $S_0/P_0$ is central in $PS_0/P_0$, so $[x, S_0] \le P_0$ exactly when $x \in PS_0$. Thus $m = 1$, and we are in case (b).

If $\mathrm{rk}(S_0/P_0) = 2$ and $|P/P_0| = 2$, then $S_0/P_0$ is free as an $\mathbb{F}_2[P/P_0]$-module by (7), so $PS_0/P_0$ is nonabelian of order 8 containing $C_2^2$ and hence isomorphic to $D_8$. Every automorphism of $D_8$ which leaves invariant a subgroup isomorphic to $C_2^2$ is inner: this follows as a special case of Lemma 1.2, but also from the well known description of $\mathrm{Out}(D_8) \cong C_2$. So for each $y \in N_S(P_0)$, there is $g \in PS_0$ such that the conjugation action of $yg$ on $PS_0/P_0 \cong D_8$ is the identity. In other words, $c_{yg}|_{PS_0} \equiv \mathrm{Id} \pmod{P_0}$, and in particular, $yg \in N_S(P) \le PS_0$ by (6). This proves that $N_S(P_0) = PS_0$, and thus $[x, P_0] \nleq P_0$ for $x \in S \smallsetminus PS_0$. Also, since $S_0/P_0$ is not fixed by the action of $P/P_0$, $[x, S_0] \nleq P_0$ for $x \in PS_0 \smallsetminus S_0$. Thus $m = 0$, and we are in the situation of (d).

By (7), it remains only to consider the cases where $|P/P_0| = |S/S_0| = 4$ and $2 \le \mathrm{rk}(S_0/P_0) \le 4$. If $\mathrm{rk}(S_0/P_0) = 2$, then since $P/P_0$ acts on it nontrivially by (6), it must be a free $\mathbb{F}_2[P/P_1]$ module for some $P_1 \le P$ of index two containing $P_0$. Hence $[x, S_0] \le P_0$ for $x \in P_1$ ($x$ centralizes $S_0/P_0$), $[x, S_0] \nleq P_0$ for $x \in S \smallsetminus P_1$, so $m = 1$, and we are in case (e). If $\mathrm{rk}(S_0/P_0) = 3, 4$, then $S_0/P_0$ is isomorphic to a submodule of index one or two in $\mathbb{F}_2[P/O_0]$ by (7), hence each $x \in S \smallsetminus S_0$ acts nontrivially on $S_0/P_0$, and so $m = 0$. Together with (6), this proves we are in case (f). □

## 2. Fusion systems

We first recall the definition of an (abstract) saturated fusion system. For any group $G$ and any $x \in G$, $c_x$ denotes conjugation by $x$ ($c_x(g) = xgx^{-1}$). For $H, K \le G$, we write

$$\mathrm{Hom}_G(H, K) = \bigl\{\varphi \in \mathrm{Hom}(H, K) \bigm| \varphi = c_x \text{ some } x \in G\bigr\}.$$

We also set $\mathrm{Aut}_G(H) = \mathrm{Hom}_G(H, H) \cong N_G(H)/C_G(H)$.

**Definition 2.1** ([Pg], [BLO2, Definition 1.1]). *A fusion system over a finite $p$-group $S$ is a category $\mathcal{F}$, with $\mathrm{Ob}(\mathcal{F})$ the set of all subgroups of $S$, which satisfies the following two properties for all $P, Q \le S$:*

- $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$*; and*
- *each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ is the composite of an isomorphism in $\mathcal{F}$ followed by an inclusion.*

When $\mathcal{F}$ is a fusion system over $S$, two subgroups $P, Q \le S$ are said to be *$\mathcal{F}$-conjugate* if they are isomorphic as objects of the category $\mathcal{F}$. A subgroup $P \le S$ is called *fully centralized* in $\mathcal{F}$ (*fully normalized* in $\mathcal{F}$) if $|C_S(P)| \ge |C_S(P')|$ ($|N_S(P)| \ge |N_S(P')|$) for all $P' \le S$ which is $\mathcal{F}$-conjugate to $P$.

**Definition 2.2** ([Pg], [BLO2, Definition 1.2]). *A fusion system $\mathcal{F}$ over a finite $p$-group $S$ is* saturated *if the following two conditions hold:*

(I) (Sylow axiom) *For all $P \le S$ which is fully normalized in $\mathcal{F}$, $P$ is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.*

(II) (Extension axiom) *If $P \le S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ are such that $\varphi(P)$ is fully centralized, and if we set*

$$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\},$$

*then there is $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar\varphi|_P = \varphi$.*

For any finite group $G$ and any Sylow subgroup $S \in \mathrm{Syl}_p(G)$, the fusion system of $G$ (at $p$) is the category $\mathcal{F}_S(G)$, whose objects are the subgroups of $S$, and with morphism sets $\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)$. This is easily shown to be saturated using the Sylow theorems (cf. [BLO2, Proposition 1.3]). A saturated fusion system is *exotic* if it is not the fusion system of any finite group.

The following definitions play a central role in this paper. In general, when $\mathcal{F}$ is a fusion system over $S$ and $P \le S$, we write $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ and $\mathrm{Out}_S(P) = \mathrm{Aut}_S(P)/\mathrm{Inn}(P)$.

**Definition 2.3.** *Fix a prime $p$, a $p$-group $S$, and a saturated fusion system $\mathcal{F}$ over $S$. Let $P \le S$ be any subgroup.*

- *$P$ is* $\mathcal{F}$-centric *if $C_S(P') = Z(P')$ for all $P'$ which is $\mathcal{F}$-conjugate to $P$.*
- *$P$ is* $\mathcal{F}$-radical *if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$; i.e., if $\mathrm{Out}_{\mathcal{F}}(P)$ contains no nontrivial normal $p$-subgroup.*
- *$P$ is* $\mathcal{F}$-essential *if $P$ is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly embedded subgroup at $p$.*
- *$P$ is* central in $\mathcal{F}$ *if every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ in $\mathcal{F}$ extends to a morphism $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ,PR)$ such that $\bar\varphi|_P = \mathrm{Id}_P$.*
- *$P$ is* normal in $\mathcal{F}$ *if $P \lhd S$, and every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ in $\mathcal{F}$ extends to a morphism $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ,PR)$ such that $\bar\varphi(P) = P$.*
- *The fusion system $\mathcal{F}$ is* nonconstrained *if there is no subgroup $P \le S$ which is $\mathcal{F}$-centric and normal in $\mathcal{F}$.*
- *For any $\varphi \in \mathrm{Aut}(S)$, $\varphi\mathcal{F}\varphi^{-1}$ denotes the fusion system over $S$ defined by*

$$\mathrm{Hom}_{\varphi\mathcal{F}\varphi^{-1}}(P,Q) = \varphi{\cdot}\mathrm{Hom}_{\mathcal{F}}(\varphi^{-1}(P), \varphi^{-1}(Q)){\cdot}\varphi^{-1}$$

*for all $P, Q \leq S$.*

When $\mathcal{F} = \mathcal{F}_S(G)$ for a finite group $G$ with $S \in \mathrm{Syl}_p(G)$, then $P \leq S$ is $\mathcal{F}$-centric if and only if it is *$p$-centric* in $G$: that is, $Z(P) \in \mathrm{Syl}_p(C_G(P))$, or equivalently, $C_G(P) = Z(P) \times C'_G(P)$ for some (unique) subgroup $C'_G(P)$ of order prime to $p$. The subgroup $P$ is $\mathcal{F}$-essential if and only if it is $p$-centric in $G$, $N_S(P) \in \mathrm{Syl}_p(N_G(P))$, and $N_G(P)/(P{\cdot}C_G(P))$ has a strongly embedded subgroup at $p$.

We say that a fusion system is "centerfree" if it contains no nontrivial central subgroup. Our main goal in this paper is to develop techniques for listing, for a given 2-group $S$, all centerfree nonconstrained saturated fusion systems over $S$ (up to isomorphism). This restriction is motivated in part by the two results stated in the following theorem: they imply that any minimal exotic fusion system is centerfree and nonconstrained.

**Theorem 2.4.** *Let $\mathcal{F}$ be a saturated fusion system over a finite $p$-group $S$.*

(a) *If $\mathcal{F}$ is constrained, then there is up to isomorphism a unique $p'$-reduced $p$-constrained finite group $G$ (i.e., $O_{p'}(G) = 1$ and $C_G(O_p(G)) \leq O_p(G)$) such that $\mathcal{F} \cong \mathcal{F}_S(G)$.*

(b) *If $A \lhd S$ is central in $\mathcal{F}$, then $\mathcal{F}$ is exotic if and only if $\mathcal{F}/A$ is exotic. Here, $\mathcal{F}/A$ is the fusion system over $S/A$ such that for all $P, Q \leq S$ containing $A$, $\mathrm{Hom}_{\mathcal{F}/A}(P/A, Q/A)$ is the image of $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ under projection.*

*Proof.* See [BCGLO1, Proposition C] and [BCGLO2, Corollary 6.14]. In both cases, much more precise results are shown. In (a), one can choose $G$ with normal $p$-subgroup $Q$ such that $Q \cong O_p(\mathcal{F})$ (the maximal normal $p$-subgroup of $\mathcal{F}$) and $G/Q \cong \mathrm{Aut}_{\mathcal{F}}(O_p(\mathcal{F}))$. Under the hypotheses of (b), if $\mathcal{F}/A$ is the fusion system of a finite group $G$, then $\mathcal{F}$ is the fusion system of a central extension of $G$ by $A$. □

One of the key problems when constructing fusion systems over a $p$-group $S$ is to determine which subgroups of $S$ can contribute automorphisms; i.e., for which $P \leq S$ the group $\mathrm{Aut}_{\mathcal{F}}(P)$ need not be generated by restrictions of automorphisms of larger subgroups. This is what motivates the definition of $\mathcal{F}$-essential subgroups. The following proposition and corollary are due to Puig [Pg, Theorem 5.8], and were originally pointed out to us by Grodal.

**Proposition 2.5.** *Let $\mathcal{F}$ be a saturated fusion system over a $p$-group $S$, and let $P \lneqq S$ be an $\mathcal{F}$-centric subgroup which is fully normalized in $\mathcal{F}$. Then $P$ is $\mathcal{F}$-essential if and only if $\mathrm{Aut}_{\mathcal{F}}(P)$ is* not *generated by restrictions of morphisms between strictly larger subgroups of $S$.*

*Proof.* Since $P$ is fully normalized, $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$. Since $P \lneqq S$, we have $N_S(P) \gneqq P$, and so $\mathrm{Aut}_S(P) \gneqq \mathrm{Inn}(P)$ since $P$ is $\mathcal{F}$-centric.

Let $G_0 \leq \mathrm{Aut}_{\mathcal{F}}(P)$ be the subgroup generated by those $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$ which extend to morphisms between strictly larger subgroups of $S$. We first claim that

$$G_0 = \big\langle \varphi \in \mathrm{Aut}_{\mathcal{F}}(P) \,\big|\, \varphi^{-1}\mathrm{Aut}_S(P)\varphi \cap \mathrm{Aut}_S(P) \gneqq \mathrm{Inn}(P) \big\rangle. \tag{1}$$

To see this, fix $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$ such that $\varphi^{-1}\mathrm{Aut}_S(P)\varphi \cap \mathrm{Aut}_S(P) \gneqq \mathrm{Inn}(P)$, and consider the group

$$N_\varphi \stackrel{\text{def}}{=} \{g \in N_S(P) \,|\, \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(P)\}.$$

Then $\mathrm{Aut}_{N_\varphi}(P) = \varphi^{-1}\mathrm{Aut}_S(P)\varphi \cap \mathrm{Aut}_S(P) \gneqq \mathrm{Inn}(P)$, so $N_\varphi \gneqq P$. By the extension axiom, $\varphi$ extends to a morphism in $\mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$, and this proves that $\varphi \in G_0$.

Conversely, if $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$ extends to $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(Q,S)$ for some $Q \gneqq P$, then $\varphi \mathrm{Aut}_Q(P)\varphi^{-1} \le \mathrm{Aut}_S(P)$, and

$$\varphi^{-1}\mathrm{Aut}_S(P)\varphi \cap \mathrm{Aut}_S(P) \ge \mathrm{Aut}_Q(P) \gneqq \mathrm{Inn}(P).$$

This proves (1). For all $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \smallsetminus G_0$ and all $\beta \in G_0$,

$$\alpha^{-1}\beta^{-1}\mathrm{Aut}_S(P)(\beta\alpha) \cap \mathrm{Aut}_S(P) = \mathrm{Inn}(P)$$

by (1), and thus the intersection of each Sylow subgroup of $\alpha^{-1}G_0\alpha$ with $\mathrm{Aut}_S(P)$ is $\mathrm{Inn}(P)$. In other words, $\mathrm{Inn}(P) \in \mathrm{Syl}_p(\alpha G_0 \alpha^{-1} \cap G_0)$ for all $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \smallsetminus G_0$, which implies that $G_0/\mathrm{Inn}(P)$ is strongly embedded in $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$. Conversely, if $\mathrm{Out}_{\mathcal{F}}(P)$ contains any strongly embedded subgroup at $p$, then there is a strongly embedded subgroup $H$ which contains the Sylow $p$-subgroup $\mathrm{Out}_S(P)$, and $G_0/\mathrm{Inn}(P) \le H \lneqq \mathrm{Out}_{\mathcal{F}}(P)$ by (1) and the definition of a strongly embedded subgroup. □

As a corollary, we get Alperin's fusion theorem stated for restriction to essential subgroups. Roughly, it says that every saturated fusion system is generated by automorphisms of $S$ and of essential subgroups, and their restrictions.

**Corollary 2.6.** *Fix a saturated fusion system $\mathcal{F}$ over a $p$-group $S$. Then for each $P, P' \le S$ and each $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P,P')$, there are subgroups $P = P_0, P_1, \ldots, P_k = P'$, subgroups $Q_i \ge \langle P_{i-1}, P_i\rangle$ $(i = 1,\ldots,k)$ which are $\mathcal{F}$-essential or equal to $S$, and automorphisms $\varphi_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$, such that $\varphi_i(P_{i-1}) = P_i$ for all $i$ and $\varphi = (\varphi_k|_{P_{k-1}}) \circ \cdots \circ (\varphi_1|_{P_0})$.*

*Proof.* By Alperin's fusion theorem in the form shown in [BLO2, Theorem A.10], this holds if we allow the $Q_i$ to be any $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups of $S$ which are fully normalized in $\mathcal{F}$. So the corollary follows immediately from that together with Proposition 2.5. □

## 3. Semicritical and critical subgroups

The following definition gives necessary conditions for subgroups of a $p$-group to possibly be $\mathcal{F}$-radical or $\mathcal{F}$-essential in some fusion system.

**Definition 3.1.** *Let $S$ be a finite $p$-group. A subgroup $P \lneqq S$ will be called* semicritical *if the following two conditions hold:*

(a) *$P$ is ($p$-)centric in $S$; and*

(b) $\mathrm{Out}_S(P) \cap O_p(\mathrm{Out}(P)) = 1$.

*A subgroup $P \le S$ will be called* critical *if it is semicritical, and if*

(c) *there are subgroups*

$$\mathrm{Out}_S(P) \le G_0 \lneqq G \le \mathrm{Out}(P)$$

*such that $G_0$ is strongly embedded in $G$ at $p$ and $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(G)$.*

The importance of (semi)critical subgroups lies in the following proposition.

**Proposition 3.2.** *Fix a $p$-group $S$, a saturated fusion system $\mathcal{F}$ over $S$, and a subgroup $P \lneqq S$. If $P$ is $\mathcal{F}$-centric and $\mathcal{F}$-radical, then it is a semicritical subgroup of $S$. If $P$ is $\mathcal{F}$-essential, then $P$ is a critical subgroup of $S$.*

*Proof.* Set $G = \mathrm{Out}_{\mathcal{F}}(P)$. If $P$ is $\mathcal{F}$-centric and $\mathcal{F}$-radical, then

$$\mathrm{Out}_S(P) \cap O_p(\mathrm{Out}(P)) \le G \cap O_p(\mathrm{Out}(P)) \le O_p(G) = 1,$$

and so $P$ is a semicritical subgroup of $S$.

If $P$ is $\mathcal{F}$-essential, then by definition, $P$ is $\mathcal{F}$-centric (hence centric in $S$), fully normalized in $\mathcal{F}$, and $G \stackrel{\text{def}}{=} \mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly embedded subgroup $G_0 \lneqq G$ at $p$. Since any strongly embedded subgroup at $p$ contains a Sylow $p$-subgroup, we can assume (after replacing $G_0$ by a conjugate subgroup if necessary) that $G_0 \ge \mathrm{Out}_S(P) \in \mathrm{Syl}_p(G)$. Since $O_p(G) \le gG_0g^{-1} \cap G_0$ for all $g \in G$, this shows that $O_p(G) = 1$, hence that $P$ is $\mathcal{F}$-radical, and thus a semicritical subgroup of $S$. This proves that $P$ is critical in $S$. □

For the above definition to be useful, simple criteria are needed which allow us to eliminate most subgroups as not being critical. This works best when $p = 2$. The following proposition gives some criteria for doing this; criteria which are useful mostly when $P$ has index $\ge 4$ in its normalizer. For example, point (a) implies that $P$ is not critical in $S$ if $\mathrm{Out}_S(P)$ contains a subgroup isomorphic to $D_8$ — since $D_8$ contains noncentral involutions.

Recall that when $V$ is a vector space and $\alpha$ is a linear automorphism of $V$, we write $[\alpha, V] = \mathrm{Im}[V \xrightarrow{\alpha-\mathrm{Id}} V]$.

**Proposition 3.3.** *Fix a critical subgroup $P$ of a 2-group $S$, and set $S_0 = N_S(P)/P \cong \mathrm{Out}_S(P)$. Then the following hold.*

(a) *Either $S_0$ is cyclic, or $Z(S_0) = \{g \in S_0 \mid g^2 = 1\}$. If $\mathrm{rk}(Z(S_0)) > 1$, then $|S_0| = |Z(S_0)|^m$ for $m = 1$, 2, or 3.*

(b) *All involutions in $S_0$ are conjugate in $\mathrm{Out}(P)$, and hence in $\mathrm{Aut}(P/\mathrm{Fr}(P))$. In fact, there is a subgroup $R \le \mathrm{Out}(P)$ (or $R \le \mathrm{Aut}(P/\mathrm{Fr}(P))$) of odd order, which normalizes $S_0$ and permutes its involutions transitively.*

(c) *Set $|S_0| = 2^k$. Then $\mathrm{rk}(P/\mathrm{Fr}(P)) \ge 2k$. If $k \ge 2$, then $\mathrm{rk}([s, P/\mathrm{Fr}(P)]) \ge 2$ for all $1 \ne s \in S_0$.*

(d) *Assume $Z(S_0) \cong C_2^n$ with $n \ge 2$, and fix $1 \ne s \in Z(S_0)$. Then $\mathrm{rk}([s, P/\mathrm{Fr}(P)]) \ge n$.*

*Proof.* Fix subgroups

$$\mathrm{Out}_S(P) \le G_0 \lneqq G \le \mathrm{Out}(P)$$

such that $G_0$ is strongly embedded in $G$ and $\mathrm{Out}_S(P) \in \mathrm{Syl}_2(G)$. In particular, $O_2(G) = 1$. Since the kernel of the natural map from $\mathrm{Out}(P)$ to $\mathrm{Out}(P/\mathrm{Fr}(P))$ is a 2-group by Lemma 1.1, the induced action of $G$ on $P/\mathrm{Fr}(P)$ is still faithful.

By Bender's theorem ([Bd] or Theorem 1.6), either $S_0 \cong \mathrm{Out}_S(P)$ is cyclic or quaternion, or $O^{2'}(G/O_{2'}(G))$ is isomorphic to one of the simple groups $PSL_2(2^n)$, $PSU_3(2^n)$, or $\mathrm{Sz}(2^n)$ (where $n \ge 2$, and $n$ is odd in the last case).

**(a)** This is clear if $S_0$ is cyclic or quaternion. If not, let $L$ be the simple group $L = O^{2'}(G/O_{2'}(G))$. If $L \cong PSL_2(2^n)$, then $S_0 \cong C_2^n$. If $L \cong \mathrm{Sz}(q)$, where $q = 2^n$ for odd $n \ge 3$, then by Suzuki's description of the Sylow 2-subgroups [Sz, §4, Lemma 1] (see also [Sz, §9]), $|S_0| = q^2$, $Z(S_0) \cong C_2^n$, and all involutions in $S_0$ are in $Z(S_0)$. So (a) holds in both of these cases.

If $L \cong PSU_3(q)$, where $q = 2^n$ for odd $n \geq 3$, then we can identify

$$S_0 = \{V(r,s) \,|\, r,s \in \mathbb{F}_{q^2} \,|\, r+\bar{r} = s\bar{s}\} \qquad \text{where} \qquad \bar{r} = r^q \quad \text{and} \quad V(r,s) = \left(\begin{smallmatrix} 1 & s & r \\ 0 & 1 & \bar{s} \\ 0 & 0 & 1 \end{smallmatrix}\right).$$

Also, $V(r,s){\cdot}V(u,v) = V(r+u+s\bar{v}, s+v)$. Thus $|S_0| = 2^{3n}$, $Z(S_0) = \{V(r,0) \,|\, r \in \mathbb{F}_q\} \cong C_2^n$, and $V(r,s)^2 = V(s\bar{s},0) = 1$ only if $s=0$. So (a) holds in this case, also.

**(b)** By [Sz2, Lemma 6.4.4], all involutions in $G_0$ are conjugate to each other. Since the involutions in $S_0$ are all in $Z(S_0)$, they must be conjugate to each other by elements in $N_{G_0}(S_0)$; and we can write $N_{G_0}(S_0) = S_0 \rtimes R$ where $|R|$ is odd.

**(c,d)** These follow immediately from Lemma 1.7, applied with $V = P/\mathrm{Fr}(P)$. □

Proposition 3.3 will be our main tool when identifying those critical subgroups which have index $\geq 4$ in their normalizer. The following lemma is an easy consequence of Lemma 1.1, and will be useful in many situations in the index two case. It will often be applied with $\Theta = 1$, or with $\Theta = Z_2(P)$ (the subgroup such that $Z_2(P)/Z(P) = Z(P/Z(P))$).

**Lemma 3.4.** *Fix a prime $p$, a $p$-group $S$, a subgroup $P \leq S$, and a subgroup $\Theta \leq P$ characteristic in $P$. Assume there is $g \in N_S(P){\smallsetminus}P$ such that*

(a) $[g,P] \leq \Theta{\cdot}\mathrm{Fr}(P)$, *and*

(b) $[g,\Theta] \leq \mathrm{Fr}(P)$.

*Then $c_g \in O_p(\mathrm{Aut}(P))$, and hence $P$ is not semicritical in $S$.*

*Proof.* Point (a) implies that $c_g$ is the identity on $P/\Theta{\cdot}\mathrm{Fr}(P)$, and (b) implies it is the identity on $\Theta{\cdot}\mathrm{Fr}(P)/\mathrm{Fr}(P)$. Hence $c_g \in O_p(\mathrm{Aut}(P))$ by Lemma 1.1, and so $P$ is not semicritical in $S$. □

Lemma 3.4 will be our main tool when looking for critical subgroups of index two in their normalizer. But there are two more, closely related, lemmas which will also be useful in certain cases. The following one can be thought of as a refinement of Lemma 3.4, at least when $p = 2$.

**Lemma 3.5.** *Fix a 2-group $S$, a semicritical subgroup $P \leq S$, and $g \in N_S(P){\smallsetminus}P$ such that $c_g$ has order two in $\mathrm{Out}_S(P)$. Then there is $\alpha \in \mathrm{Aut}(P)$ of odd order, and $x \in [g,P]$, such that $x \notin \mathrm{Fr}(P)$ and $[g,\alpha(x)] \notin \mathrm{Fr}(P)$.*

*Proof.* Since $P$ is semicritical, $c_g \notin O_2(\mathrm{Out}(P))$. Hence by the Baer-Suzuki theorem (cf. [A1, Theorem 39.6]), there is $\beta \in \mathrm{Out}(P)$ such that $\Delta = \langle c_g, \beta c_g \beta^{-1}\rangle \leq \mathrm{Out}(P)$ is not a 2-group. Thus $\Delta$ contains a dihedral group of order $2m$ for $m$ odd, and we can assume that $\beta$ is chosen so that $|\Delta| = 2m$. Let $\widehat{\gamma} \in \Delta \leq \mathrm{Out}(P)$ be an element of order $m$ inverted by $c_g$, and let $\gamma \in \mathrm{Aut}(P)$ be an automorphism of odd order whose class in $\mathrm{Out}(P)$ is $\widehat{\gamma}$.

Set $V = P/\mathrm{Fr}(P)$, regarded as an $\mathbb{F}_2[\Delta]$-module. Since $\widehat{\gamma}$ has odd order, $V$ splits as a product $V = C_V(\widehat{\gamma}) \times V'$, where $\widehat{\gamma}$ acts on $V' = [\widehat{\gamma}, V]$ without fixed component. Let $V_0 \subseteq V'$ be an irreducible $\mathbb{F}_2[\Delta]$-submodule. Since $\widehat{\gamma}$ acts nontrivially on $V_0$ (and the subgroup of elements of $\Delta$ which act trivially is normal), $c_g$ acts nontrivially on $V_0$, and hence $[g, V_0] \neq 1$.

Fix $1 \neq \widehat{x} \in [g,V_0] \leq C_{V_0}(g)$. The $\widehat{\gamma}$-orbit of $\widehat{x}$ is $\Delta$-invariant (since $c_g(\widehat{x}) = \widehat{x}$ and $\Delta = \langle \widehat{\gamma}, c_g\rangle$), and hence it generates $V_0$ since $V_0$ is irreducible. Thus there is $i$ such that

$\widehat{\gamma}^i(\widehat{x}) \notin C_{V_0}(g)$. Now choose $x \in [g,P]$ such that $x\mathrm{Fr}(P) = \widehat{x}$; then $[g, \gamma^i(x)] \notin \mathrm{Fr}(P)$. Set $\alpha = \gamma^i$; then $\alpha$ and $x$ satisfy the conclusion of the lemma. □

In the special case of Lemma 3.5 where $[g,S]$ has order two, one can take this much farther. Recall $Z_2(S) \lhd S$ is the subgroup such that $Z_2(S)/Z(S) = Z(S/Z(S))$.

**Lemma 3.6.** *Let $S$ be a 2-group, and fix elements $z \in Z(S)$ and $g \in Z_2(S)$ such that $z^2 = 1$ and $[g,S] \le \langle z \rangle$. Assume $P$ is a critical subgroup of $S$ such that $g \notin P$. Then the following hold.*

(a) *$|N_S(P)/P| = 2$, $z \notin \mathrm{Fr}(P)$, and $P = C_S(h)$ for some $h \in S$ such that $h^2 = 1$ and $[g,h] = z$.*

(b) *If $y \in S \smallsetminus \langle g, z \rangle$ is such that $[y,S] \le \langle z \rangle$, then either $y \in Z(P)$ and $h$ is not $S$-conjugate to $yh$, or $gy \in Z(P)$ and $h$ is not $S$-conjugate to $gyh$.*

*Proof.* **(a)** By assumption (and since $P$ is centric), $[g,P] \le [g,S] = \langle z \rangle \le Z(S) \le P$. In particular, $g \in N_S(P)$. By Lemma 3.4 (applied with $\Theta = 1$), $[g,P] \nleq \mathrm{Fr}(P)$, and thus $[g,P] = \langle z \rangle$ and $z \notin \mathrm{Fr}(P)$. Since $\mathrm{rk}([g, P/\mathrm{Fr}(P)]) = 1$, $|N_S(P)/P| = 2$ by Proposition 3.3(c).

Set $\Theta = \Omega_1(Z(P))$: the 2-torsion subgroup of $Z(P)$. Then $[g,P] \le \langle z \rangle \le \Theta$, so $[g,\Theta] \nleq \mathrm{Fr}(P)$ by Lemma 3.4 again, and thus $z \in [g,\Theta]$. Fix $h \in \Theta$ such that $[g,h] = z$. In particular, $h^2 = 1$ and $P \le C_S(h)$.

Now $|N_S(P)/P| = 2$, $g \in N_S(P)$, and $g \notin C_S(h)$ imply that $N_{C_S(h)}(P) = P$. Hence $P = C_S(h)$ since otherwise its normalizer in $C_S(h)$ would be strictly larger.

**(b)** Since $[y,P] \le [y,S] \le \langle z \rangle$, $y \in N_S(P)$. If neither $y$ nor $gy$ is in $P$, then $yP$ and $gP$ are distinct nonidentity elements of $N(P)/P$, which is impossible since $|N(P)/P| = 2$. Thus one of them is in $P$; say $y \in P$. If $y \notin Z(P)$, then $z \in [y,P] \le \mathrm{Fr}(P)$, which again contradicts (a). Thus $y \in Z(P)$.

It remains to show $h$ is not $S$-conjugate to $yh$. Assume otherwise: let $a$ be such that $aha^{-1} = yh$. Then $aPa^{-1} = C_S(aha^{-1}) = C_S(yh) \ge P$, so $a \in N_S(P)$. Thus $h, zh, yh \in Z(P)$ are all $N_S(P)$-conjugate to $h$, so $|N_S(P)/P| > 2$, which again contradicts (a). □

We can now outline the general procedure which will be used to determine all of the critical subgroups of a given 2-group $S$. We first try to find a normal subgroup $S_0 \lhd S$, as large as possible, which we can show is contained in all critical subgroups. For example, in many cases, we do this for $S_0 = Z_2(S)$, using Lemmas 3.5 and 3.6. We then search for critical subgroups $P \le S$ such that $|N_S(P)/P| = 2$, by first applying Lemma 1.9 (when possible) to list all subgroups of index 2 in their normalizer, and then applying Lemma 3.4 to eliminate most of them. Afterwards, we search for subgroups $P \le S$ such that $|N_S(P)/P| = 2^k \ge 4$, $\mathrm{rk}(P/\mathrm{Fr}(P)) \ge 2k$, and $\mathrm{rk}([s, P/\mathrm{Fr}(P)]) \ge 2$ for all $s \in N_S(P) \smallsetminus P$, and check (using Proposition 3.3) which of them could be critical. In practice, this seems to work surprisingly well on groups of order $\le 2^{10}$, at least on those where we have tested it.

## 4. Fusion systems over the Sylow 2-subgroup of $J_2$ and $J_3$

We are now ready to begin working with some concrete examples. In the next four sections, we list all nonconstrained centerfree saturated fusion systems over each of four different 2-groups $S$. In each case, this procedure can be broken up into three steps: first determine the critical subgroups of $S$ (or a list of subgroups which includes all critical subgroups), then determine the automorphism group of each critical subgroup, and finally work out all possible combinations of which critical subgroups can be $\mathcal{F}$-essential for any given $\mathcal{F}$ and what their $\mathcal{F}$-automorphism groups can be. The last step is carried out only up to isomorphism, in the sense that we make a list of fusion systems over $S$ and show that for each $\mathcal{F}$, there is some $\varphi \in \mathrm{Aut}(S)$ such that $\varphi\mathcal{F}\varphi^{-1}$ is in the list (see Definition 2.3). If we did find a candidate for a new exotic fusion system, then there would be the additional step of proving that it is saturated; but otherwise this is done by identifying it (by elimination) with the fusion system of some finite group.

In this section and the next, $S_0 = UT_3(4)$ denotes the group of $3\times 3$ upper triangular matrices over $\mathbb{F}_4$ with 1's in all diagonal entries. Let $e_{ij}^a \in S_0$ (for $i<j$) be the elementary matrix with entry $a \in \mathbb{F}_4$ in the $(i,j)$ position, and set $E_{ij} = \{e_{ij}^a \,|\, a \in \mathbb{F}_4\}$. Thus, for example,

$$Z(S_0) = [S_0, S_0] = E_{13} = \{e_{13}^a \,|\, a \in \mathbb{F}_4\}.$$

We note here for reference throughout this section the relations

$$(e_{12}^a e_{23}^b)^2 = [e_{12}^a, e_{23}^b] = e_{13}^{ab} \quad \text{for all } a,b \in \mathbb{F}_4 \tag{1}$$

We also let $c_{ij}^a$ denote conjugation by $e_{ij}^a$, as an automorphism of $S_0$ and also as a homomorphism between subgroups of $S_0$ or groups containing $S_0$, and write $\langle c_{ij}^* \rangle = \{c_{ij}^a \,|\, a \in \mathbb{F}_4\}$.

Let $a \mapsto \bar{a} = a^2$ denote the field automorphism on $\mathbb{F}_4$, and let $M \mapsto \overline{M}$ denote the induced field automorphism on $S_0$. Let $\tau \in \mathrm{Aut}(S_0)$ be the automorphism "transpose inverse" which sends $e_{ij}^a$ to $e_{4-j,4-i}^a$. Consider the semidirect product

$$S_{\phi\theta} = UT_3(4) \rtimes \langle \phi, \theta \rangle,$$

where for $M \in S_0 = UT_3(4)$, $\phi M \phi^{-1} = \overline{M}$ and $\theta M \theta^{-1} = \tau(\overline{M})$ (and $\phi^2 = \theta^2 = [\phi,\theta] = 1$). Thus

$$\tau\left(\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & c & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & c & b{+}ac \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad c_\theta\left(\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & \bar{c} & \overline{b{+}ac} \\ 0 & 1 & \bar{a} \\ 0 & 0 & 1 \end{pmatrix} \;; \tag{2}$$

and $S_{\phi\theta}$ is a Sylow 2-subgroup of the full automorphism group $\mathrm{Aut}(PSL_3(4)) = PGL_3(4) \rtimes \langle \phi, \theta \rangle$. In this section, we determine the nonconstrained saturated fusion systems over the group

$$S_\theta \overset{\mathrm{def}}{=} UT_3(4) \rtimes \langle \theta \rangle,$$

while in the next section we work with the group $S_\phi \overset{\mathrm{def}}{=} UT_3(4) \rtimes \langle \phi \rangle$.

The following subgroups will play an important role throughout this section:

$$A_1 = \langle E_{12}, E_{13} \rangle = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a,b \in \mathbb{F}_4 \right\} \qquad Q_0 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & \bar{a} \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a,b \in \mathbb{F}_4 \right\}$$

$$A_2 = \langle E_{13}, E_{23} \rangle = \left\{ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a,b \in \mathbb{F}_4 \right\} \qquad Q = \langle Q_0, \theta \rangle\,.$$

Thus $A_1$ and $A_2$ are the "rectangular subgroups", both isomorphic to $C_2^4$; while $Q_0 \cong C_2 \times Q_8$ and $Q \cong Q_8 \times_{C_2} D_8$. Also,

$$Q_0/E_{13} = [\theta, S_0/E_{13}] = C_{S_0/E_{13}}(\theta)\,, \quad Q_0 = [S_\theta, S_\theta] \lhd S_\theta\,, \quad \text{and} \quad Q \lhd S_\theta\,. \tag{3}$$

We start with some elementary facts about $S_\theta$ and its subgroups. Throughout this section and the next, $\omega$ denotes an element in $\mathbb{F}_4 \smallsetminus \mathbb{F}_2$, so that $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$.

**Lemma 4.1.** (a) *All involutions in $S_\theta \smallsetminus S_0$ are $S_0$-conjugate to $\theta$. For each involution $g \in S_\theta \smallsetminus S_0$, $C_{S_0}(g) \cong Q_8$, $C_{S_0}(g) \le Q_0$, and $e_{13}^1 \in \mathrm{Fr}(C_{S_0}(g))$.*

(b) *All involutions in $S_0$ are in $A_1$ or in $A_2$, and all involutions in $S_\theta \smallsetminus S_0$ are in $Q$.*

(c) *$A_1$ and $A_2$ are the only subgroups of $S_\theta$ isomorphic to $C_2^4$.*

(d) *The subgroups $S_0$ and $Q$ are both characteristic in $S_\theta$.*

*Proof.* **(a)** Conjugation by $\theta$ acts on $E_{13}$ with fixed subgroup $\langle e_{13}^1 \rangle$, and on $S_0/E_{13}$ by exchanging the complementary subgroups $A_1/E_{13}$ and $A_2/E_{13}$. The hypotheses of Lemma 1.4 thus hold, and all involutions in the coset $\theta S_0$ are $S_0$-conjugate to $\theta$.

By (2), $C_{S_0}(\theta)$ is generated by $e_{13}^1$, together with matrices $\left(\begin{smallmatrix} 1 & a & \omega \\ 0 & 1 & \bar{a} \\ 0 & 0 & 1 \end{smallmatrix}\right)$ for $0 \ne a \in \mathbb{F}_4$. Thus $C_{S_0}(\theta) \le Q_0$ (this also follows from (3)), has order 8, and is isomorphic to $Q_8$ since it is not cyclic and its only involution is $e_{13}^1$ (by (1)). Since each involution $g \in S_\theta \smallsetminus S_0$ is conjugate to $\theta$, and since $Q_0 \lhd S_\theta$, the same holds for $C_{S_0}(g)$.

**(b)** The first statement holds by (1). Since all involutions in $S_\theta \smallsetminus S_0$ are $S_0$-conjugate to $\theta$, and since $\theta \in Q$ and $Q \lhd S_\theta$, all such involutions are in $Q$.

**(c)** Assume $A \le S_\theta$ and $A \cong C_2^4$. If $A \le S_0$, then $A \subseteq (A_1 \cup A_2)$ by (b), and $A = A_1$ or $A_2$ since no element of $A_1 \smallsetminus E_{13}$ commutes with any element of $A_2 \smallsetminus E_{13}$. If $A \nleq S_0$ and $g \in A \smallsetminus S_0$, then $A \cap S_0 \cong C_2^3$, $A \cap S_0 \le C_{S_0}(g) \cong Q_8$ by (a), and this is impossible.

**(d)** The subgroup $S_0 = \langle A_1, A_2 \rangle$ is characteristic by (c). By (b), $Q$ is generated by the centralizers of all involutions in $S_\theta \smallsetminus S_0$, and so it is also characteristic. □

## 4.1 Candidates for critical subgroups

The following proposition is the main result of this subsection.

**Proposition 4.2.** *If $P$ is a critical subgroup of $S_\theta$, then $P$ is one of the subgroups $Q$, $S_0 = UT_3(4)$, $A_1$, or $A_2$.*

Proposition 4.2 follows immediately from Lemmas 4.3 and 4.4. We first deal with the normal critical subgroups.

**Lemma 4.3.** *If $P \lhd S_\theta$ is a normal critical subgroup of $S_\theta$, then $P = Q$ or $P = S_0 = UT_3(4)$.*

*Proof.* By Proposition 3.3(c), $\mathrm{rk}(P/\mathrm{Fr}(P)) \ge 2k$ if $|S_\theta/P| = 2^k$. Thus $2^7 \ge |S_\theta| \ge 2^k \cdot 2^{2k} = 2^{3k}$, so $k \le 2$ and $|S_\theta/P| \le 4$. In particular, $S_\theta/P$ is abelian, and so $P \ge [S_\theta, S_\theta] = Q_0$ by (3).

Assume first that $|S_\theta/P| = 4$. Then $|P| = 2^5$ and $\mathrm{rk}(P/\mathrm{Fr}(P)) \ge 4$, so $|\mathrm{Fr}(P)| \le 2$. It follows that $\mathrm{Fr}(P) = \mathrm{Fr}(Q_0) = \langle e_{13}^1 \rangle$.

If $P \le S_0$, then $P = \langle Q_0, e_{12}^a \rangle$ for some $a \ne 0$, and hence $\mathrm{Fr}(P) \ge E_{13}$, which we saw is impossible. Thus $P = \langle Q_0, h\theta \rangle$ for some $h \in S_0$; and since $S_0 = Q_0 E_{12}$, we can assume $h = e_{12}^a$ for some $a \in \mathbb{F}_4$. Also, $(h\theta)^2 = [e_{12}^a, \theta] \in \mathrm{Fr}(P) = \langle e_{13}^1 \rangle$, and this is possible only if $a = 0$. Thus $P = \langle Q_0, \theta \rangle = Q$ in this case.

Now assume $|S_\theta/P| = 2$, and fix $g \in S_\theta \smallsetminus P$. Since $S_\theta/\mathrm{Fr}(S_\theta) \cong C_2^3$, there are seven subgroups of index 2 in $S_\theta$. Assume $P \ne S_0$. Then $P = \langle Q_0, e_{12}^a, e_{12}^b\theta \rangle$ for some $a, b \in \mathbb{F}_4$ where $a \ne 0$. Also, $[Q_0, e_{12}^a] = E_{13}$. If $b \notin \{0, a\}$, then

$$\mathrm{Fr}(P) = \langle E_{13}, [e_{12}^a, \theta], [e_{12}^b, \theta] \rangle = Q_0 = \mathrm{Fr}(S_\theta),$$

so $[g,P] \le \mathrm{Fr}(P)$ for $g \in S_\theta \smallsetminus P$, and $P$ is not critical by Lemma 3.4 (applied with $\Theta = 1$).

We are left with the case $b \in \{0,a\}$, and thus $P = \langle Q_0, e_{12}^a, \theta\rangle$. Then $\mathrm{Fr}(P) = \langle E_{13}, [e_{12}^a, \theta]\rangle \cong C_2 \times C_4$, and so $E_{13}$ is characteristic in $P$ since it is the 2-torsion subgroup of $\mathrm{Fr}(P)$. Thus $C_P(E_{13}) = P_0 \stackrel{\mathrm{def}}{=} P \cap S_0$ is characteristic in $P$. For $g \in S_0 \smallsetminus P$, $[g,P] \le P_0$ and $[g,P_0] \le E_{13} \le \mathrm{Fr}(P)$; and thus $P$ is not critical by Lemma 3.4 applied with $\Theta = P_0$. □

It now remains to show:

**Lemma 4.4.** *If $P \le S_\theta$ is a critical subgroup and not normal, then $P = A_1$ or $P = A_2$.*

*Proof.* Fix such a $P$. Assume first that $|N(P)/P| \ge 4$. Since $S_\theta$ has order $2^7$, $|N(P)| \le 2^6$, and so $|P| \le 2^4$. Since $P$ is critical, we must have $\mathrm{rk}(P/Fr(P)) \ge 4$ by Proposition 3.3(c). This can only happen if $P \cong C_2^4$, and by Lemma 4.1(c), $P = A_1$ or $P = A_2$.

Now assume $|N(P)/P| = 2$. Then $Q_0 = \mathrm{Fr}(S_\theta) \not\le P$ because $P$ not normal in $S_\theta$. Also, $e_{13}^1 \in Z(S_\theta) \le P$ since $P$ is centric. If $e_{13}^\omega \notin P$, then by Lemma 3.6, there is some $h \in S_\theta \smallsetminus S_0$ ($S_0 = C_{S_\theta}(e_{13}^\omega)$) such that $h^2 = 1$, $P = C_{S_\theta}(h)$, and $e_{13}^1 \notin \mathrm{Fr}(P)$. But this contradicts Lemma 4.1(a), and thus $E_{13} \le P$. Also, $Q_0 \le N(P)$ since $[Q_0, S_\theta] = E_{13} \le P$.

Thus $[Q_0 : P \cap Q_0] = 2$. Since $|Q_0| = 2^4$, $E_{13} \lneqq P \cap Q_0 \lneqq Q_0$, and there is a unique $a \in \mathbb{F}_4 \smallsetminus 0$ such that $e_{12}^a e_{23}^{\bar a} \in P$. Hence $(e_{12}^a e_{23}^{\bar a})^2 = e_{13}^1 \in \mathrm{Fr}(P)$ by (1). Fix $b \in \mathbb{F}_4 \smallsetminus \{a, 0\}$ and set $g = e_{12}^b e_{23}^{\bar b} \in N(P) \smallsetminus P$. By (3), $[g,P] \le [Q_0, S_\theta] = E_{13}$.

By Lemma 3.5, there is some $\alpha \in \mathrm{Aut}(P)$, and elements $x \in [g,P] \le E_{13}$ and $y = \alpha(x)$, such that $x \notin \mathrm{Fr}(P)$ and $[g,y] \notin \mathrm{Fr}(P)$. Since $e_{13}^1 \in \mathrm{Fr}(P)$, this means $x = e_{13}^c$ for some $c \in \{\omega, \bar\omega\}$. Also, $y^2 = \alpha(x^2) = 1$.

If $y \notin S_0$, then by Lemma 4.1(a), $y = h\theta h^{-1}$ for some $h \in S_0$, and so

$$[g,y] = [g, h\theta h^{-1}] = h[h^{-1}gh, \theta]h^{-1} \in h[Q_0, \theta]h^{-1} = \langle e_{13}^1\rangle \le \mathrm{Fr}(P)$$

(recall $Q_0 \lhd S_\theta$ by (3)). But $[g,y] \notin \mathrm{Fr}(P)$ by assumption, so we conclude $y \in S_0$.

Since $y^2 = 1$, $y \in A_1$ or $A_2$ by Lemma 4.1(b). Also, $y \notin E_{13}$ since $[g,y] \in [S_0, y] \ne 1$. We can assume (upon replacing $P$ by $\theta P \theta^{-1}$ if necessary) that $y \in A_1 \smallsetminus E_{13}$. Fix $d \in \mathbb{F}_4 \smallsetminus 0$ such that $y \in e_{12}^d E_{13}$. By (1),

$$[y, e_{12}^a e_{23}^{\bar a}] = [e_{12}^d, e_{12}^a e_{23}^{\bar a}] = e_{13}^{d\bar a} \in [y,P] \cap E_{13} \le \mathrm{Fr}(P). \tag{4}$$

Now, $[y,P] = \alpha([x,P]) \le \alpha(\langle e_{13}^1\rangle)$ has order at most two, while $[y,P] \cap E_{13} \ne 1$ by (4). Since $e_{13}^1 \in \mathrm{Fr}(P)$ but $x = e_{13}^c \notin \mathrm{Fr}(P)$ (and $[y,P] \le \mathrm{Fr}(P)$), this implies $[y,P] = \langle e_{13}\rangle$. It also implies $[x,P] = [e_{13}^c, P] \ne 1$, and hence $P \not\le S_0$. Fix $r \in S_0$ such that $r\theta \in P$; then $(r\theta)y(r\theta)^{-1} \in A_2 \smallsetminus E_{13}$, so $[y, r\theta] \notin E_{13}$, which is impossible since $[y,P] = \langle e_{13}^1\rangle$. Thus there are no critical subgroups of this form. □

## 4.2 Automorphisms of critical subgroups

Before describing the automorphism group of $S_0$, we need to give names to some automorphisms. For each $f \in \mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_4, \mathbb{F}_4)$, define $\rho_1^f, \rho_2^f \in \mathrm{Aut}(S_0)$ by setting $\rho_i|_{A_i} = \mathrm{Id}$, and

$$\rho_1^f(e_{23}^x) = e_{23}^x e_{13}^{f(x)} \qquad \text{and} \qquad \rho_2^f(e_{12}^x) = e_{12}^x e_{13}^{f(x)}.$$

Note that $\rho_i^f \circ \rho_i^{f'} = \rho_i^{f+f'}$, and hence $R_i \stackrel{\mathrm{def}}{=} \{\rho_i^f \mid f \in \mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_4, \mathbb{F}_4)\}$ is a subgroup of $\mathrm{Aut}(S_0)$ isomorphic to $C_2^4$. One easily sees that $R_1$ and $R_2$ commute in $\mathrm{Aut}(S_0)$, and that they generate the group of all automorphisms of $S_0$ which induce the identity on $E_{13}$ and on $S_0/E_{13}$. Thus $R_1 \times R_2$ is a normal subgroup of $\mathrm{Aut}(S_0)$, and is contained in $O_2(\mathrm{Aut}(S_0))$.

Next define $\gamma_0, \gamma_1 \in \mathrm{Aut}(S_0)$ by letting $\gamma_0$ be conjugation by $\mathrm{diag}(\omega, 1, \bar{\omega})$, and letting $\gamma_1$ be conjugation by $\mathrm{diag}(\omega, 1, \omega)$. Thus

$$\gamma_0\left(\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 & \omega a & \bar{\omega} b \\ 0 & 1 & \omega c \\ 0 & 0 & 1 \end{smallmatrix}\right) \qquad \text{and} \qquad \gamma_1\left(\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 & \omega a & b \\ 0 & 1 & \bar{\omega} c \\ 0 & 0 & 1 \end{smallmatrix}\right) .$$

Also, $\gamma_0$ and $\gamma_1$ both have order 3,

$$\Gamma_0 \stackrel{\text{def}}{=} \langle \gamma_0, c_\theta \rangle \cong \Sigma_3, \qquad \Gamma_1 \stackrel{\text{def}}{=} \langle \gamma_1, \tau \rangle \cong \Sigma_3,$$

and $[\Gamma_0, \Gamma_1] = 1$ in $\mathrm{Aut}(S_0)$.

**Lemma 4.5.** (a) $\mathrm{Aut}(S_0) = (R_1 \times R_2){\cdot}(\Gamma_0 \times \Gamma_1) \cong C_2^8 \rtimes (\Sigma_3 \times \Sigma_3)$, *and hence*

$$\mathrm{Out}(S_0) = \big((R_1/\langle c_{12}^* \rangle) \times (R_2/\langle c_{23}^* \rangle)\big){\cdot}(\Gamma_0 \times \Gamma_1) \cong C_2^4 \rtimes (\Sigma_3 \times \Sigma_3).$$

(b) *Restriction induces an isomorphism*

$$\mathrm{Out}(S_\theta) \xrightarrow[\cong]{\text{Res}} C_{\mathrm{Out}(S_0)}(c_\theta)/\langle c_\theta \rangle.$$

(c) *Set* $H = \langle \mathrm{Aut}_{S_0}(A_1), \gamma_0|_{A_1} \rangle \cong A_4$. *Then for any pair of subgroups* $U, U_0 \leq C_{\mathrm{Aut}(A_1)}(H)$ *of order three, there is* $\psi \in \mathrm{Aut}(S_\theta)$ *such that* $\psi|_{S_0}$ *commutes with* $\gamma_0$ *in* $\mathrm{Aut}(S_0)$, *and such that* $(\psi|_{A_1})U(\psi|_{A_1})^{-1} = U_0$.

*Proof.* **(a)** The elements we have defined clearly generate subgroups of $\mathrm{Aut}(S_0)$ and $\mathrm{Out}(S_0)$ of the form described in (a). It remains to show that

$$\mathrm{Aut}(S_0) = \langle R_1, R_2, \Gamma_0, \Gamma_1 \rangle. \tag{5}$$

Let $\alpha \in \mathrm{Aut}(S_0)$ be arbitrary. By Lemma 4.1(c), $\alpha$ either sends each subgroup $A_i$ to itself or switches them. Hence there is $\alpha_1 \in \{\alpha, \tau\alpha\}$ such that $\alpha_1(A_i) = A_i$ for $i = 1, 2$.

Next, we can choose $r, s \in \{0, 1, 2\}$ such that if we set $\alpha_2 = \gamma_1^r \gamma_2^s \alpha_1$, then $\alpha_2(e_{12}^1) \equiv e_{12}^1$ and $\alpha_2(e_{23}^1) \equiv e_{23}^1 \pmod{E_{13}}$. Finally, there is $\alpha_3 \in \{c_\phi \alpha_2, \alpha_2\}$ such that $\alpha_3(e_{12}^a) \equiv e_{12}^a \pmod{E_{13}}$ for all $a \in \mathbb{F}_4$. By (1) (and since $E_{13} \leq Z(S_0)$), for all $a \in \mathbb{F}_4$,

$$\alpha_3(e_{13}^a) = [\alpha_3(e_{12}^a), \alpha_3(e_{23}^1)] = [e_{12}^a, e_{23}^1] = e_{13}^a$$

and hence

$$[e_{12}^1, e_{23}^a] = e_{13}^a = [\alpha_3(e_{12}^1), \alpha_3(e_{23}^a)] = [e_{12}^1, \alpha_3(e_{23}^a)].$$

Since $\alpha_3(e_{23}^a) \in E_{13}E_{23}$, this implies $\alpha_3(e_{23}^a) \equiv e_{23}^a \pmod{E_{13}}$ for all $a$. Thus $\alpha_3$ induces the identity on $S_0/E_{13}$ and on $E_{13}$.

Let $\varphi\colon S_0/E_{13} \longrightarrow E_{13}$ be the function such that for all $g \in S_0$, $\alpha_3(g) = g{\cdot}\varphi(gE_{13})$. Since $E_{13} = Z(S_0)$ and $\alpha$ is a homomorphism, $\varphi$ is also a homomorphism. So there is a pair of functions $f, f' \in \mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_4, \mathbb{F}_4)$ such that $\varphi(e_{12}^a e_{23}^b E_{13}) = e_{13}^{f(a)+f'(b)}$, and $\alpha_3 = \rho_2^f \circ \rho_1^{f'} \in R_1 \times R_2$. Since $\alpha \in (\Gamma_0 \times \Gamma_1) \circ \alpha_3$, this proves (5).

**(b)** By Lemma 4.1(d), $S_0$ is characteristic in $S_\theta$. Also, $Z(S_0) = E_{13}$ is free as an $\mathbb{F}_2[\langle c_\theta \rangle]$-module. So by Corollary 1.3, the map induced by restriction

$$\mathrm{Out}(S_\theta) \xrightarrow[\cong]{\text{Res}} N_{\mathrm{Out}(S_0)}(\mathrm{Out}_{S_\theta}(S_0))/\mathrm{Out}_{S_\theta}(S_0) = C_{\mathrm{Out}(S_0)}(\langle c_\theta \rangle)/\langle c_\theta \rangle$$

is an isomorphism.

**(c)** For each $\alpha \in \mathrm{Aut}(A_1)$, let $M(\alpha) \in GL_4(2)$ be the matrix of $\alpha$ with respect to the ordered basis $\{e_{13}^1, e_{13}^\omega, e_{12}^1, e_{12}^\omega\}$ for $A_1$ as a vector space over $\mathbb{F}_2$. Thus $M(H)$ is generated by

$$M(c_{23}^1) = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \qquad M(c_{23}^\omega) = \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix} \qquad \text{and} \qquad M(\gamma_0|_{A_1}) = \begin{pmatrix} Z^{-1} & 0 \\ 0 & Z \end{pmatrix},$$

where matrices are written in $2 \times 2$ blocks and $Z = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. From this, it follows that

$$M\big(C_{\mathrm{Aut}(A_1)}(H)\big) = \left\{ \begin{pmatrix} B & C \\ 0 & B \end{pmatrix} \,\middle|\, B \in \langle Z \rangle,\ C \in M_2(\mathbb{F}_2),\ CZ = Z^{-1}C \right\}.$$

In particular, $O_2(C_{\mathrm{Aut}(A_1)}(H)) = C_{R_2^*}(H)$, where $R_2^* = \{\alpha|_{A_1} \,|\, \alpha \in R_2\}$; and also $C_{\mathrm{Aut}(A_1)}(H)/O_2(C_{\mathrm{Aut}(A_1)}(H)) \cong C_3$. So for any pair of subgroups $U, U_0 \le C_{\mathrm{Aut}(A_1)}(H)$ of order three, there is $\eta \in R_2$ such that $\eta|_{A_1} \in C_{R_2^*}(H)$ and $(\eta|_{A_1})U(\eta|_{A_1})^{-1} = U_0$.

Now, $c_\theta \eta c_\theta^{-1} \in R_1$ commutes with $\eta \in R_2$, so $\eta \circ (c_\theta \eta c_\theta^{-1})$ commutes with $c_\theta$ in $\mathrm{Aut}(S_0)$. Hence this can be extended to $\psi \in \mathrm{Aut}(S_\theta)$ by setting $\psi(\theta) = \theta$. Also, $\psi|_{A_1} = \eta|_{A_1}$ since elements of $R_1$ are the identity on $A_1$. Since $\gamma_0 \eta \gamma_0^{-1}$ is equal to $\eta$ after restriction to $A_1$ and both are the identity on $A_2$, they are equal as elements of $R_2 \le \mathrm{Aut}(S_0)$. So $\gamma_0 = c_\theta \gamma_0^{-1} c_\theta$ also commutes with $c_\theta \eta c_\theta^{-1} \in R_1$, and thus commutes with their composite $\psi|_{S_0}$. □

Now set

$$\Gamma = \langle \gamma_1, \gamma_0, c_\theta \rangle \le \mathrm{Aut}(S_0) \qquad \text{so that} \qquad \Gamma \cong C_3 \times \Sigma_3.$$

To simplify notation, we also write $\Gamma_0 \le \Gamma$ to denote their images in $\mathrm{Out}(S_0)$. Finally, define $\dot\gamma_1 \in \mathrm{Aut}(S_\theta)$ by setting $\dot\gamma_1|_{S_0} = \gamma_1$ (conjugation by $\mathrm{diag}(\omega, 1, \omega)$) and $\dot\gamma_1(\theta) = \theta$.

**Lemma 4.6.** *Let $\mathcal{F}$ be any saturated fusion system over $S_\theta$ for which $S_0$ is $\mathcal{F}$-essential. Then there is some $\varphi \in \mathrm{Aut}(S_\theta)$ such that either*

- $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(S_0) = \Gamma_0 \cong \Sigma_3$ *and* $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(S_\theta) = 1$*; or*
- $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(S_0) = \Gamma \cong C_3 \times \Sigma_3$ *and* $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(S_\theta) = \langle [\dot\gamma_1] \rangle$.

*Proof.* By Lemma 4.5(a,b),

$$\mathrm{Out}(S_\theta)/O_2(\mathrm{Out}(S_\theta)) \cong C_{\mathrm{Out}(S_0)}(\langle c_\theta \rangle)/O_2(C_{\mathrm{Out}(S_0)}(\langle c_\theta \rangle)) \cong \Sigma_3$$

(represented by $\Gamma_1$). Thus $|\mathrm{Out}_{\mathcal{F}}(S_\theta)| = 1$ or 3, since it contains $\mathrm{Out}_{S_\theta}(S_\theta) = 1$ as a Sylow 2-subgroup.

Set $Q = O_2(\mathrm{Out}(S_0))$ for short. Then $\mathrm{Out}_{\mathcal{F}}(S_0) \cap Q = 1$, and by Lemma 4.5(a),

$$\mathrm{Out}(S_0)/Q = \langle \gamma_0, c_\theta \rangle \times \langle \gamma_1, \tau \rangle \cong \Sigma_3 \times \Sigma_3.$$

Also, $\mathrm{Out}_{\mathcal{F}}(S_0) \cdot Q/Q$ contains $\langle c_\theta \rangle$ as a Sylow 2-subgroup not in the center (since $O_2(\mathrm{Out}_{\mathcal{F}}(S_0)) = 1$), and this is possible only if $\mathrm{Out}_{\mathcal{F}}(S_0) \cdot Q = \Gamma_0 \cdot Q$ or $\Gamma \cdot Q$. By Lemma 1.8, there is some $\varphi_0 \in O_2(\mathrm{Aut}(S_0))$ such that $[[\varphi_0], c_\theta] = 1$ in $\mathrm{Out}(S_0)$ and $\varphi_0 \mathrm{Out}_{\mathcal{F}}(S_0) \varphi_0^{-1}$ is equal to $\Gamma_0$ or $\Gamma$. By Lemma 4.5(b), $\varphi_0$ extends to some $\varphi \in \mathrm{Aut}(S_\theta)$, and thus $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(S_0) = \Gamma_0$ or $\Gamma$.

If $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(S_0) = \Gamma$, then by the extension axiom, $\gamma_1$ extends to an element of $\mathrm{Aut}_{\varphi\mathcal{F}\varphi^{-1}}(S_\theta)$. Hence $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(S_\theta) = \langle [\dot\gamma_1] \rangle$ since this extension is unique (mod $\mathrm{Inn}(S_\theta)$) by Lemma 4.5(b) again.

Conversely, if $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(S_\theta)$ has order 3, then a generator of this group restricts to an element of order three in $\mathrm{Aut}_{\mathcal{F}}(S_0)$ (since $S_0$ is characteristic in $S_\theta$). Since no element of order three in $\Gamma_0$ extends to $S_\theta$, we conclude that $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(S_0) = \Gamma$. □

We now check the possibilities for $\mathrm{Aut}_{\mathcal{F}}(A_i)$ when the $A_i$ are essential. Consider the following subgroups of $\mathrm{Aut}(A_i)$:

$$\Lambda_i \stackrel{\text{def}}{=} \mathrm{Aut}_{GL_3(4)}(A_i) \cong GL_2(4) \qquad \text{and} \qquad \Lambda_i^0 \stackrel{\text{def}}{=} [\Lambda_i, \Lambda_i] \cong SL_2(4).$$

Thus $\Lambda^i$ is the group of those automorphisms of $A_i$ induced by conjugation by elements of $GL_3(4) \geq S_0$. Note that we can regard $A_i$ as a vector space over $\mathbb{F}_4$, where scalar multiplication is given by $u{\cdot}e_{ij}^x = e_{ij}^{ux}$ for $u, x \in \mathbb{F}_4$; and then $\Lambda_i = \mathrm{Aut}_{\mathbb{F}_4}(A_i)$ is the group of $\mathbb{F}_4$-linear automorphisms. Since $A_1$ and $A_2$ are $S_\theta$-conjugate, $\mathrm{Aut}_{\mathcal{F}}(A_1) = \Lambda_1$ if and only if $\mathrm{Aut}_{\mathcal{F}}(A_2) = \Lambda_2$, and similarly for the $\Lambda_i^0$.

**Lemma 4.7.** *Let $\mathcal{F}$ be any saturated fusion system over $S_\theta$, and assume $A_1$ and $A_2$ are $\mathcal{F}$-essential. Then $S_0$ is also $\mathcal{F}$-essential. There is an automorphism $\varphi \in \mathrm{Aut}(S_\theta)$ such that either*

- $\mathrm{Aut}_{\varphi\mathcal{F}\varphi^{-1}}(A_i) = \Lambda_i^0$, $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(S_0) = \Gamma_0 \cong \Sigma_3$, *and* $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(S_\theta) = 1$*; or*
- $\mathrm{Aut}_{\varphi\mathcal{F}\varphi^{-1}}(A_i) = \Lambda_i$, $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(S_0) = \Gamma \cong C_3 \times \Sigma_3$, *and* $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(S_\theta) = \langle [\dot{\gamma}_1] \rangle$.

*Proof.* Set $\Delta = \mathrm{Aut}_{\mathcal{F}}(A_1)$. Thus $\Delta$ is a subgroup of $\mathrm{Aut}(A_1) \cong GL_4(2) \cong A_8$ which has $\mathrm{Aut}_{S_\theta}(A_1) \cong C_2^2$ as Sylow 2-subgroup, and which contains a strongly embedded subgroup. By Bender's theorem (Theorem 1.6), $O^{2'}(\Delta/O_{2'}(\Delta))$ is isomorphic to $SL_2(4) \cong A_5$. The only nontrivial odd order subgroup of $GL_4(2)$ which has $A_5$ in its normalizer is $C_3$, with normalizer isomorphic to $SL_2(4) \rtimes \langle \phi \rangle \cong (C_3 \times A_5) \rtimes C_2$. If $H \leq GL_4(2) \cong A_8$ and $H \cong A_5$, then since the only proper subgroups of $A_5$ of index $\leq 8$ have index 5 and 6, each orbit of $H$ acting on $\{1, \ldots, 8\}$ has length 1, 5, or 6. Thus $H$ is in one of two conjugacy classes: either it acts as $SL_2(4)$ with respect to some $\mathbb{F}_4$-vector space structure, or it acts via the permutation action on $\mathbb{F}_2^5/\mathrm{diag}$. Since the fixed set of $\mathrm{Aut}_{S_\theta}(A_1) = \langle c_{23}^* \rangle$ acting on $A_1$ is 2-dimensional, this last action cannot occur. We conclude that $\Delta$ must be $\mathrm{Aut}(A_1)$-conjugate to $\Lambda_1^0$ or $\Lambda_1$.

Let $\Delta^0 \lhd \Delta$ be the subgroup isomorphic to $\Lambda_1^0 \cong SL_2(4)$. Thus $\Delta^0$ has odd index in $\Delta$, and $\langle c_{23}^* \rangle \in \mathrm{Syl}_2(\Delta^0) = \mathrm{Syl}_2(\Delta)$. Hence there is an element of order three in $N_{\Delta^0}(\langle c_{23}^* \rangle)$, which by the extension axiom extends to some $\xi \in \mathrm{Aut}_{\mathcal{F}}(S_0)$. Since $\Delta^0$ is $\mathrm{Aut}(A_1)$-conjugate to $\Lambda_1^0$, $\xi|_{A_1}$ acts without fixed component, and in particular acts nontrivially on $E_{13}$. Hence $[\xi]$ does not commute with $c_\theta$ in $\mathrm{Out}_{\mathcal{F}}(S_0)$, and so $S_0$ is also $\mathcal{F}$-essential. By Lemma 4.6, we can assume (after replacing $\mathcal{F}$ by $\psi\mathcal{F}\psi^{-1}$ for some appropriate $\psi \in \mathrm{Aut}(S_\theta)$) that $\mathrm{Aut}_{\mathcal{F}}(S_0) = \Gamma_0$ or $\mathrm{Aut}_{\mathcal{F}}(S_0) = \Gamma$. In either case, $\gamma_0|_{A_1} \in \mathrm{Aut}_{\mathcal{F}}(A_1)$.

Since $\Delta$ is $\mathrm{Aut}(A_1)$-conjugate to $\Lambda_1^0$ or $\Lambda_1$, it is contained in the centralizer of some subgroup $U \leq \mathrm{Aut}(A_1)$ of order three which acts on $A_1$ without fixed component, thus defining a $\mathbb{F}_4$-vector space structure. Similarly, the "standard" subgroups $\Lambda_1^0 \leq \Lambda_1$ are centralized by $U_0 = \langle \gamma_0\gamma_1|_{A_1} \rangle$; i.e., by conjugation by $\mathrm{diag}(\omega, 1, 1)$.

If $\mathrm{Out}_{\mathcal{F}}(S_0) = \Gamma$, then $\Delta \geq \langle \mathrm{Aut}_{S_0}(A_1), \gamma_0|_{A_1}, \gamma_1|_{A_1} \rangle$, and so $\langle \gamma_0|_{A_1}, \gamma_1|_{A_1} \rangle \in \mathrm{Syl}_3(\Delta)$. Thus $\Delta \cong GL_2(4)$, $U = Z(\Delta) \leq \langle \gamma_0|_{A_1}, \gamma_1|_{A_1} \rangle$, and hence

$$U \leq C_{\langle \gamma_0|_{A_1}, \gamma_1|_{A_1} \rangle}(\mathrm{Aut}_{S_0}(A_1)) = U_0.$$

This proves that $U = U_0$, and hence $\Delta = \Lambda_1$ in this case.

Now assume $\mathrm{Aut}_{\mathcal{F}}(S_0) = \Gamma_0$. Set $H = \langle \mathrm{Aut}_{S_0}(A_1), \gamma_0|_{A_1} \rangle \cong A_4$. Then $H \leq \Delta$, and so $U$ and $U_0$ both centralize $H$. By Lemma 4.5(c), there is $\varphi \in \mathrm{Aut}(S_\theta)$ such that $[\varphi|_{S_0}, \gamma_0] = 1$ in $\mathrm{Out}(S_0)$ and $(\varphi|_{A_1})U(\varphi|_{A_1})^{-1} = U_0$. Thus

$$(\varphi|_{A_1})\Delta(\varphi|_{A_1})^{-1} \leq C_{\mathrm{Aut}(A_1)}(U_0) = \Lambda_1,$$

and so $\mathrm{Aut}_{\varphi\mathcal{F}\varphi^{-1}}(A_1) = \varphi\Delta\varphi^{-1} = \Lambda_1^0$ or $\Lambda_1$. Also, since $\varphi$ commutes with $\gamma_0$, we still have $\mathrm{Aut}_{\varphi\mathcal{F}\varphi^{-1}}(S_0) = \Gamma_0$. Since $\gamma_1|_{A_1}$ is not the restriction of an element of $\Gamma_0$, it cannot be in $\mathrm{Aut}_{\varphi\mathcal{F}\varphi^{-1}}(A_1)$, and thus $\mathrm{Aut}_{\varphi\mathcal{F}\varphi^{-1}}(A_1) = \Lambda_1^0$.

In either case, $\mathrm{Aut}_{\varphi\mathcal{F}\varphi^{-1}}(A_2) = c_\theta \mathrm{Aut}_{\varphi\mathcal{F}\varphi^{-1}}(A_1)c_\theta^{-1}$ is equal to $\Lambda_2^0$ or $\Lambda_2$. □

## 4.3 Fusion systems over $S_\theta$

**Theorem 4.8.** *Let $\mathcal{F}$ be any nonconstrained saturated fusion system over the group $S_\theta = UT_3(4) \rtimes \langle\theta\rangle$, where $\theta$ acts on $UT_3(4) \le PGL_3(4)$ by sending a matrix $M$ to $\tau(\overline{M})$. Then $\mathcal{F}$ is isomorphic to the fusion system of one of the groups $PSL_3(4) \rtimes \langle\theta\rangle$, $PGL_3(4) \rtimes \langle\theta\rangle$, $J_2$, or $J_3$.*

*Proof.* By Proposition 4.2, the only possible $\mathcal{F}$-essential subgroups are $S_0$, $Q$, $A_1$, and $A_2$. If $S_0$ is not $\mathcal{F}$-essential, then by Lemma 4.7, neither $A_1$ nor $A_2$ is $\mathcal{F}$-essential. Hence $Q$ is the only $\mathcal{F}$-essential subgroup, and $\mathcal{F}$ is generated by automorphisms of $Q$ and $S_\theta$. Since $Q$ is characteristic in $S_\theta$ (Lemma 4.1(d)), this implies $Q \lhd \mathcal{F}$, which contradicts the assumption that $\mathcal{F}$ is nonconstrained. Thus $S_0$ must be $\mathcal{F}$-essential.

If $S_0$ is the only $\mathcal{F}$-essential subgroup, then since it is also characteristic in $S_\theta$ (Lemma 4.1(d) again), it would be normal in $\mathcal{F}$, again contradicting the assumption that $\mathcal{F}$ is nonconstrained. Thus either $Q$ is $\mathcal{F}$-essential, or $A_1$ and $A_2$ are $\mathcal{F}$-essential, or all of them are.

Since $Q \cong D_8 \times_{C_2} Q_8$, $\mathrm{Inn}(Q)$ is the group of automorphisms which induce the identity on $Q/Z(Q)$, and $\mathrm{Out}(Q) \cong \Sigma_5$ is the group which permutes the five involutions in $Q/Z(Q)$ which lift to involutions in $Q$. Hence if $Q$ is $\mathcal{F}$-essential, then $\mathrm{Out}_{\mathcal{F}}(Q) = A_5$: this is the only subgroup which contains $\mathrm{Out}_{S_\theta}(Q)$ as Sylow 2-subgroup and which has a strongly embedded subgroup.

**Case 1:** Assume first that $Q$ is not $\mathcal{F}$-essential, and hence that $S_0$ and the $A_i$ are $\mathcal{F}$-essential. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the fusion systems over $S_\theta$ generated by the following automorphism groups and their restrictions:

$$\begin{array}{lll} \mathrm{Out}_{\mathcal{F}_1}(S_\theta) = 1 & \mathrm{Out}_{\mathcal{F}_1}(S_0) = \Gamma_0 \cong \Sigma_3 & \mathrm{Aut}_{\mathcal{F}_1}(A_i) = \Lambda_i^0 \\ \mathrm{Out}_{\mathcal{F}_2}(S_\theta) = \langle[\dot{\gamma}_1]\rangle & \mathrm{Out}_{\mathcal{F}_2}(S_0) = \Gamma \cong C_3 \times \Sigma_3 & \mathrm{Aut}_{\mathcal{F}_2}(A_i) = \Lambda_i\ . \end{array}$$

Here, the subgroups $\Lambda_i^0 \le \Lambda_i \le \mathrm{Aut}(A_i)$ are defined as before: $\Lambda_i = \mathrm{Aut}_{GL_3(4)}(A_i) \cong GL_2(4)$ and $\Lambda_i^0 \cong SL_2(4)$ is its commutator subgroup.

By Lemma 4.7, we can assume (after replacing $\mathcal{F}$ by $\varphi\mathcal{F}\varphi^{-1}$ for appropriate $\varphi$) that either $\mathrm{Aut}_{\mathcal{F}}(A_i) = \Lambda_i^0$ (for $i = 1, 2$) and $\mathrm{Out}_{\mathcal{F}}(S_0) = \Gamma_0$, or $\mathrm{Aut}_{\mathcal{F}}(A_i) = \Lambda_i$ and $\mathrm{Out}_{\mathcal{F}}(S_0) = \Gamma$. Furthermore, by Lemma 4.6, $\mathrm{Out}_{\mathcal{F}}(S_\theta)$ is determined (exactly) by $\mathrm{Out}_{\mathcal{F}}(S_0)$. Since $\mathcal{F}$ is generated by automorphisms of $S_\theta$, $S_0$, and the $A_i$ and their restrictions, this proves that $\mathcal{F} = \mathcal{F}_1$ or $\mathcal{F} = \mathcal{F}_2$. In other words, these are the only possible isomorphism classes of saturated fusion systems over $S_\theta$ satisfying these conditions.

If $G$ is one of the groups $PSL_3(4) \rtimes \langle\theta\rangle$ or $PGL_3(4) \rtimes \langle\theta\rangle$, then any $S \in \mathrm{Syl}_2(G)$ is isomorphic to $S_\theta$, and $S_0$ is strongly closed in $\mathcal{F}_S(G)$ under any identification $S = S_\theta$. Hence $Q \cap S_0 \cong C_2 \times Q_8$ is invariant under the action of $\mathrm{Aut}_G(Q)$, which is impossible if $\mathrm{Out}_{\mathcal{F}}(Q) \cong A_5$. Thus $Q$ is not $\mathcal{F}$-essential. Since $\mathcal{F}_S(G)$ is nonconstrained and centerfree, it must be isomorphic to $\mathcal{F}_1$ or $\mathcal{F}_2$; and by comparing automorphism groups of the $A_i$, one sees that $\mathcal{F}_{S_\theta}(PSL_3(4) \rtimes \langle\theta\rangle) \cong \mathcal{F}_1$ and $\mathcal{F}_{S_\theta}(PGL_3(4) \rtimes \langle\theta\rangle) \cong \mathcal{F}_2$.

**Case 2:** Now assume $Q$ and $S_0$ are both $\mathcal{F}$-essential. Let $\mathcal{F}_3$ and $\mathcal{F}_4$ be the fusion systems over $S_\theta$ generated by the following automorphism groups and their restrictions:

$$\mathrm{Out}_{\mathcal{F}_3}(S_\theta) = \langle[\dot\gamma_1]\rangle \qquad \mathrm{Out}_{\mathcal{F}_3}(S_0) = \Gamma \qquad \mathrm{Out}_{\mathcal{F}_3}(Q) = A_5$$
$$\mathrm{Out}_{\mathcal{F}_4}(S_\theta) = \langle[\dot\gamma_1]\rangle \qquad \mathrm{Out}_{\mathcal{F}_4}(S_0) = \Gamma \qquad \mathrm{Out}_{\mathcal{F}_4}(Q) = A_5 \qquad \mathrm{Aut}_{\mathcal{F}_4}(A_i) = \Lambda_i\ .$$

For $\mathcal{F} = \mathcal{F}_3$ or $\mathcal{F}_4$, all involutions in $E_{13}$, and all involutions in $S_0 \smallsetminus E_{13}$, are $\mathcal{F}$-conjugate via automorphisms of $S_0$. Also, all noncentral involutions in $Q$ are $\mathcal{F}$-conjugate via automorphisms of $Q$, and hence (by Lemma 4.1(a)) all involutions in $S_\theta \smallsetminus S_0$ are $\mathcal{F}$-conjugate to the involutions in $E_{13}$. Thus there are exactly two $\mathcal{F}$-conjugacy classes of involutions when $\mathcal{F} = \mathcal{F}_3$ (those in $S_0 \smallsetminus E_{13}$ and the others); while these form a single class if $\mathcal{F} = \mathcal{F}_4$ (since the $A_i$ are $\mathcal{F}$-essential).

For arbitrary $\mathcal{F}$ of this type, $\mathrm{Out}_{\mathcal{F}}(Q) = A_5$ has index 2 in $\mathrm{Out}(Q)$, and so $\mathrm{Aut}_{\mathcal{F}}(Q)$ contains all automorphisms of $Q$ of odd order. Since $\dot\gamma_1|_Q$ has order 3 in $\mathrm{Aut}_{\mathcal{F}}(Q)$, it must extend (by the extension axiom) to some automorphism in $\mathrm{Aut}_{\mathcal{F}}(S_\theta)$. Thus $\mathrm{Out}_{\mathcal{F}}(S_\theta)$ has order 3 by Lemma 4.6. By Lemmas 4.6 and 4.7, we can assume (after replacing $\mathcal{F}$ by $\varphi\mathcal{F}\varphi^{-1}$ for appropriate $\varphi$) that $\mathrm{Out}_{\mathcal{F}}(S_0) = \Gamma$ and $\mathrm{Out}_{\mathcal{F}}(S_\theta) = \langle[\dot\gamma_1]\rangle$, and also that $\mathrm{Aut}_{\mathcal{F}}(A_i) = \Lambda_i$ if the $A_i$ are $\mathcal{F}$-essential. Thus $\mathcal{F} = \mathcal{F}_3$ or $\mathcal{F} = \mathcal{F}_4$.

By Janko's original characterization of the sporadic simple groups $J_2$ and $J_3$ [J], both contain involution centralizers of odd index isomorphic to $(D_8 \times_{C_2} Q_8) \rtimes A_5$, and $J_2$ has two conjugacy classes of involutions while $J_3$ has only one class. Also, $S_\theta$ is isomorphic to the Sylow 2-subgroups of these groups; this is shown explicitly in [GH, p.331], and also follows since $S_\theta$ is a Sylow 2-subgroup of $(D_8 \times_{C_2} Q_8) \rtimes A_5$. Thus $\mathcal{F}_{S_\theta}(J_2) \cong \mathcal{F}_3$ and $\mathcal{F}_{S_\theta}(J_3) \cong \mathcal{F}_4$. □

In fact, the main result of [GH] is that if $G$ is a finite group with Sylow 2-subgroup isomorphic to $S_\theta$, then either $G/O_{2'}(G)$ is isomorphic to one of the groups $PSL_3(4) \rtimes \langle\theta\rangle$, $PGL_3(4) \rtimes \langle\theta\rangle$, $J_2$, or $J_3$, or $G/O_{2'}(G) \cong C_G(x)$ for some involution $x$.

## 5. Fusion systems over the Sylow 2-subgroup of $M_{22}$

Again in this section, $S_0 = UT_3(4)$ denotes the group of $3\times 3$ upper triangular matrices over $\mathbb{F}_4$ with 1 in all diagonal entries, $x \mapsto \bar{x} = x^2$ denotes the field automorphism on $\mathbb{F}_4$, and $M \mapsto \overline{M}$ denotes the induced field automorphism on $S_0$. Set $S_\phi = S_0 \rtimes \langle\phi\rangle$, where $\phi M \phi^{-1} = \overline{M}$ for all $M \in S_0$ and $\phi^2 = 1$. We want to list all nonconstrained centerfree saturated fusion systems over $S_\phi$, up to isomorphism.

As before, $e_{ij}^a \in S_0$ (for $i < j$) denotes the elementary matrix with entry $a \in \mathbb{F}_4$ in the $(i,j)$ position, satisfying the relations

$$(e_{12}^a e_{23}^b)^2 = [e_{12}^a, e_{23}^b] = e_{13}^{ab} \quad \text{for all } a,b \in \mathbb{F}_4. \tag{1}$$

Also, $E_{ij} = \{e_{ij}^a \,|\, a \in \mathbb{F}_4\}$, $c_{ij}^a$ denotes conjugation by $e_{ij}^a$, and $\omega$ denotes an element in $\mathbb{F}_4 \smallsetminus \mathbb{F}_2$. Thus $\mathbb{F}_4 = \{0, 1, \omega, \bar\omega\}$. Finally,

$$Z(S_0) = E_{13} = \langle e_{13}^1, e_{13}^\omega\rangle, \qquad Z(S_\phi) = \langle e_{13}^1\rangle, \qquad \text{and} \qquad [S_\phi, S_\phi] = \langle E_{13}, e_{12}^1, e_{23}^1\rangle.$$

The following subgroups will play an important role in this section:

$$A_1 = \langle E_{12}, E_{13}\rangle \cong C_2^4 \qquad H_1 = \langle A_1, \phi\rangle \qquad N_1 = \langle A_1, e_{23}^1, \phi\rangle$$
$$A_2 = \langle E_{13}, E_{23}\rangle \cong C_2^4 \qquad H_2 = \langle A_2, \phi\rangle \qquad N_2 = \langle A_2, e_{12}^1, \phi\rangle\ .$$

Note that $N_i = N_{S_\phi}(H_i)$.

**Lemma 5.1.** (a) *For any involution $g \in S_\phi \smallsetminus S_0$, $g$ is $S_0$-conjugate to $\phi$, $C_{S_0}(g) \le \langle E_{13}, e_{12}^1, e_{23}^1 \rangle$, $C_{S_0}(g) \cap E_{13} = \langle e_{13}^1 \rangle$, and $e_{13}^1 \in \mathrm{Fr}(C_{S_0}(g))$.*

(b) *$A_1$ and $A_2$ are the only subgroups of $S_\phi$ isomorphic to $C_2^4$.*

*Proof.* **(a)** Since $C_{E_{13}}(\phi) = [\phi, E_{13}]$ and $C_{S_0/E_{13}}(\phi) = [\phi, S_0/E_{13}]$, the hypotheses of Lemma 1.4 apply to the pair $S_0 \lhd S_\phi$. Hence each involution $g \in S_\phi \smallsetminus S_0$ is conjugate to $\phi$, and $C_{S_0}(g)$ is $S_0$-conjugate to $C_{S_0}(\phi) = \langle e_{13}^1, e_{12}^1, e_{23}^1 \rangle \cong D_8$ for such $g$. Since the subgroups $\langle E_{13}, e_{12}^1, e_{23}^1 \rangle$, $E_{13}$, and $\langle e_{13}^1 \rangle$ are all normal in $S_\phi$, and since $C_{S_0}(\phi)$ satisfies all of the above conditions, so does $C_{S_0}(g)$.

**(b)** By Lemma 4.1, $A_1$ and $A_2$ are the only subgroups of $S_0$ isomorphic to $C_2^4$. So assume $P \not\le S_0$ and $P \cong C_2^4$. Set $P_0 = P \cap S_0$, and fix $g \in P \smallsetminus P_0$. Then $C_2^3 \cong P_0 \le C_{S_0}(g)$, and we just showed in the proof of (a) that $C_{S_0}(g) \cong D_8$. So this situation is impossible. □

## 5.1 Candidates for critical subgroups

Our main result here is the following:

**Proposition 5.2.** *If $P$ is a critical subgroup of $S_\phi$ then $P$ is equal to one of the subgroups $S_0 = UT_3(4)$, $N_1$, or $N_2$; or is conjugate to $H_1$ or $H_2$.*

*Proof.* In Lemma 5.3, we show that if $P$ is normal, then $P$ is one of the subgroups $S_0$, $N_1$, or $N_2$. In Lemma 5.4, we show that if $P$ is not normal and has index 2 in its normalizer, then $P$ is conjugate to $H_1$ or $H_2$.

Now assume $P$ is not normal and $|N(P)/P| \ge 4$. Since $S_\phi$ has order $2^7$, $|N(P)| \le 2^6$, and so $|P| \le 2^4$. Since $P$ is critical, $\mathrm{rk}(P/\mathrm{Fr}(P)) \ge 4$ by Proposition 3.3(c). This implies $P \cong C_2^4$, so $P = A_1$ or $A_2$ by Lemma 5.1(b), and these subgroups are normal. □

For use in the proofs of Lemmas 5.3 and 5.4, we define the subgroup

$$Q_0 = \langle E_{13}, e_{12}^1, e_{23}^1 \rangle \cong C_2 \times D_8.$$

Then

$$Q_0/E_{13} = [\phi, S_0/E_{13}] = C_{S_0/E_{13}}(\phi) \quad \text{and} \quad Q_0 = [S_\phi, S_\phi] \lhd S_\phi\,. \tag{2}$$

**Lemma 5.3.** *If $P \lhd S_\phi$ is a normal critical subgroup of $S_\phi$, then $P$ is one of the three subgroups $S_0$, $N_1$, or $N_2$.*

*Proof.* By Proposition 3.3(c), $\mathrm{rk}(P/\mathrm{Fr}(P)) \ge 2k$ if $|S_\phi/P| = 2^k$. Thus $|S_\phi| \ge 2^{3k}$, so $k \le 2$, and $|S_\phi/P| \le 4$. Hence $S_\phi/P$ is abelian, so $P \ge Q_0 = [S_\phi, S_\phi]$ by (2), and $\mathrm{Fr}(P) \ge \mathrm{Fr}(Q_0) = \langle e_{13}^1 \rangle$.

**Case 1 :** If $|S_\phi/P| = 4$, then $|P| = 2^5$ and $|P/Q_0| = 2$. Since $\mathrm{rk}(P/\mathrm{Fr}(P)) \ge 4$ and $\mathrm{Fr}(P) \ne 1$, we have $\mathrm{Fr}(P) = \mathrm{Fr}(Q_0) = \langle e_{13}^1 \rangle$.

Set $\mathfrak{X} = \{e_{12}^\omega, e_{23}^\omega, e_{12}^\omega e_{23}^\omega\}$ (as a *set* of elements of $S_0$). If $P = \langle Q_0, x \rangle$ for some $x \in \mathfrak{X}$, then $[x, Q_0] \not\le \langle e_{13}^1 \rangle$ by the relations in (1), so $\mathrm{Fr}(P) \not\le \langle e_{13}^1 \rangle$. If $P = \langle Q_0, x\phi \rangle$ for some $x \in \mathfrak{X}$, then $(x\phi)^2 = [x, \phi] \notin \langle e_{13}^1 \rangle$, and again $\mathrm{Fr}(P) \not\le \langle e_{13}^1 \rangle$. So these cases cannot occur.

This leaves only the possibility

$$P = \langle Q_0, \phi \rangle = \langle e_{13}^\omega, \phi \rangle \times_{\langle e_{13}^1 \rangle} \langle e_{12}^1, e_{23}^1 \rangle \cong D_8 \times_{C_2} D_8 \cong Q_8 \times_{C_2} Q_8.$$

Then $\mathrm{Out}(P) \cong \Sigma_3 \wr C_2$. If $P$ were critical, then by Proposition 3.3(b), there would be an odd order subgroup of $\mathrm{Out}(P)$ which normalizes $\mathrm{Out}_{S_\phi}(P) = \langle e_{12}^\omega, e_{23}^\omega \rangle \cong C_2^2$ and permutes its involutions transitively, and this is not the case. Thus this group $P$ is not critical; and we conclude that $S_\phi$ contains no normal critical subgroups of index 4.

**Case 2 :** Assume $|S_\phi/P| = 2$, and fix $g \in S_\phi \smallsetminus P$. Since $S_\phi/\mathrm{Fr}(S_\phi) \cong C_2^3$, there are seven subgroups of index 2 in $S_\phi$. If $\mathrm{Fr}(P) = Q_0 = [S_\phi, S_\phi]$, then $[g, P] \le \mathrm{Fr}(P)$, and so $P$ is not critical by Lemma 3.4 (applied with $\Theta = 1$). This is the case for three of the seven subgroups

$$\langle Q_0, e_{12}^\omega\phi, e_{23}^\omega \rangle, \qquad \langle Q_0, e_{12}^\omega, e_{23}^\omega\phi \rangle, \qquad \text{and} \qquad \langle Q_0, e_{12}^\omega\phi, e_{23}^\omega\phi \rangle;$$

which leaves only $S_0$, $N_1$, $N_2$, and $N_3 = \langle Q_0, e_{12}^\omega e_{23}^\omega, \phi \rangle$ to consider. So it remains to check that $N_3$ is not critical.

Now, $\mathrm{Fr}(N_3) = \langle E_{13}, e_{12}^1 e_{23}^1 \rangle \cong C_2 \times C_4$, and hence its 2-torsion subgroup $E_{13}$ is characteristic in $N_3$. Let $\Theta \le N_3$ be such that $\Theta/E_{13} = Z(N_3/E_{13})$. Then $\Theta = [S_\phi, S_\phi]$ is characteristic in $N_3$, and for $g \in S_\phi \smallsetminus N_3$ $[g, N_3] \le \Theta$, and $[g, \Theta] \le E_{13} \le \mathrm{Fr}(N_3)$. So also in this case, $P$ is not critical by Lemma 3.4. □

It remains to handle the critical subgroups which are not normal.

**Lemma 5.4.** *Let $P \le S_\phi$ be a critical subgroup with index 2 in $N_{S_\phi}(P)$ and not normal in $S_\phi$. Then $P$ is $S_\phi$-conjugate to $H_1$ or $H_2$.*

*Proof.* We have $e_{13}^1 \in P$ since $P$ is centric. If $e_{13}^\omega \notin P$, then by Lemma 3.6, there is $h \in S_\phi \smallsetminus S_0$ ($S_0 = C_{S_\phi}(e_{13}^\omega)$) such that $h^2 = 1$, $P = C_{S_\phi}(h)$, and $e_{13}^1 \notin \mathrm{Fr}(P)$. This is impossible by Lemma 5.1(a), and hence $E_{13} \le P$.

Now, $Q_0 = \langle E_{13}, e_{12}^1, e_{23}^1 \rangle \not\le P$ because $P$ is not normal in $S_\phi$ (see (2)). Also, $[P, Q_0] \le [S_\phi, Q_0] = E_{13} \le P$, so $N_{S_\phi}(P) \ge Q_0$. Thus $[Q_0 : P \cap Q_0] = 2$: it cannot be larger because $|N_{S_\phi}(P)/P| = 2$. So exactly one of the matrices $e_{12}^1$, $e_{23}^1$ or $e_{12}^1 e_{23}^1$ is in $P$. By symmetry, we can assume that $g \stackrel{\text{def}}{=} e_{23}^1 \notin P$ (hence $g$ generates $N(P)/P$), and that $P$ contains $e_{12}^1$ or $e_{12}^1 e_{23}^1$.

If $P \le S_0$, then $[P, S_0] \le E_{13} \le P$, and $S_0 \le N(P)$. Thus $N(P) = S_0$ (since $P$ is not normal in $S_\phi$), and $[S_0 : P] = 2$. It follows that $P = \langle E_{13}, e_{12}^1 h_1, e_{12}^\omega h_2, e_{23}^\omega h_3 \rangle$ for some $h_i \in \langle g \rangle = \langle e_{23}^1 \rangle$. Then $e_{23}^a \in P$ for some $a \in \{\omega, \bar\omega\}$, and $\mathrm{Fr}(P)$ contains the elements

$$[e_{12}^1 h_1, e_{23}^a] = e_{13}^a \qquad \text{and} \qquad [e_{12}^\omega h_2, e_{23}^a] = e_{13}^{a\omega}$$

(using (1) again). Thus $\mathrm{Fr}(P) = E_{13} \ge [g, P]$, and so $P$ is not critical by Lemma 3.4 applied with $\Theta = 1$.

Now assume $P \not\le S_0$, and set $P_0 = P \cap S_0$. Then $|P_0| \le 2^4$, since $|P| \le \frac{1}{4}|S| = 2^5$. If $e_{12}^1 \in P$, then since $[\langle e_{23}^1, e_{12}^\omega \rangle, S_\phi] \le \langle E_{13}, e_{12}^1 \rangle \le P$, $\langle e_{23}^1, e_{12}^\omega \rangle \le N(P)$, and so $e_{12}^\omega h \in P$ for some $h \in \langle g \rangle$. Thus $P_0 = \langle E_{13}, e_{12}^1, e_{12}^\omega h \rangle$. Furthermore, $S_\phi/P_0 \cong D_8$, and $D_8$ contains exactly two conjugacy classes of subgroups which are not normal. Since $P \not\le S_0$, this proves that up to conjugacy, $P = \langle E_{13}, e_{12}^1, e_{12}^\omega h, \phi \rangle$ for some $h \in \langle g \rangle$. If $h = 1$, then $P = H_1$. If $h = g = e_{23}^1$, then $(e_{12}^\omega e_{23}^1)^2 = e_{13}^\omega \in \mathrm{Fr}(P)$ by (1), so $\mathrm{Fr}(P) \ge E_{13} = [g, P]$, and again $P$ is not critical by Lemma 3.4.

By a similar argument, if $e_{12}^1 e_{23}^1 \in P$, then $e_{12}^\omega e_{23}^\omega h \in P$ for some $h \in \langle g \rangle$, and (again up to conjugacy) $P = \langle E_{13}, e_{12}^1 e_{23}^1, e_{12}^\omega e_{23}^\omega h, \phi \rangle$. If $h = 1$, then by (1),

$$\mathrm{Fr}(P) \ge \big\langle (e_{12}^1 e_{23}^1)^2 = e_{13}^1,\ (e_{12}^\omega e_{23}^\omega)^2 = e_{13}^{\bar\omega} \big\rangle = E_{13},$$

so $\mathrm{Fr}(P) = \langle E_{13}, [\phi, e_{12}^{\omega}e_{23}^{\omega}]\rangle = \langle E_{13}, e_{12}^{1}e_{23}^{1}\rangle$. Thus $[g,P] \le \mathrm{Fr}(P)$ in this case, and $P$ is not critical by Lemma 3.4. If $h = e_{23}^{1}$, then

$$P = \langle E_{13}, e_{12}^{1}e_{23}^{1}, e_{12}^{\omega}e_{23}^{\bar\omega}, \phi\rangle, \qquad Z(P) = \langle e_{13}^{1}\rangle, \qquad Z_2(P) = \langle E_{13}, e_{12}^{1}e_{23}^{1}\rangle;$$

so $[g,P] \le Z_2(P)$ and $[g, Z_2(P)] \le \mathrm{Fr}(P)$, and $P$ is not critical by Lemma 3.4 applied with $\Theta = Z_2(P)$. □

## 5.2 Automorphisms of critical subgroups

By Proposition 5.2, the only critical subgroups of $S_\phi$, and hence the only essential subgroups in a saturated fusion system over $S_\phi$, are $S_0$, $N_1$, $N_2$, and subgroups conjugate to $H_1$ and $H_2$. The automorphism group of $S_0$ was computed in Lemma 4.5(a). In this subsection, we first compute $\mathrm{Out}(H_1)$ and $\mathrm{Out}(N_1)$, and then determine all possibilities for $\mathrm{Out}_{\mathcal{F}}(S_0)$, $\mathrm{Out}_{\mathcal{F}}(H_i)$, and $\mathrm{Out}_{\mathcal{F}}(N_i)$ when $\mathcal{F}$ is a saturated fusion system over $S_\phi$.

We first recall some of the notation used for automorphisms of $S_0$. For each $f \in \mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_4, \mathbb{F}_4)$, we defined $\rho_1^f, \rho_2^f \in \mathrm{Aut}(S_0)$ by setting

$$\rho_1^f\left(\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 & a & b+f(a) \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right) \qquad \text{and} \qquad \rho_2^f\left(\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 & a & b+f(c) \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right);$$

and set $R_i = \{\rho_i^f \mid f \in \mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_4, \mathbb{F}_4)\} \cong C_2^4$. Also, we defined $\gamma_0, \gamma_1, \tau \in \mathrm{Aut}(S_0)$ by setting

$$\gamma_0\left(\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 & \omega a & \bar\omega b \\ 0 & 1 & \omega c \\ 0 & 0 & 1 \end{smallmatrix}\right), \quad \gamma_1\left(\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 & \omega a & b \\ 0 & 1 & \bar\omega c \\ 0 & 0 & 1 \end{smallmatrix}\right), \quad \tau\left(\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 & c & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{smallmatrix}\right)^{-1};$$

and set $\Gamma_0 = \langle \gamma_0, c_\phi\tau\rangle$ and $\Gamma_1 = \langle \gamma_1, \tau\rangle$. By Lemma 4.5(a),

$$\mathrm{Out}(S_0) = \big((R_1/\langle c_{23}^*\rangle) \times (R_2/\langle c_{12}^*\rangle)\big)\cdot(\Gamma_0 \times \Gamma_1) \cong C_2^4 \rtimes (\Sigma_3 \times \Sigma_3).$$

**Lemma 5.5.** *The group* $\mathrm{Out}(S_\phi)$ *is a 2-group. If* $\alpha \in \mathrm{Aut}(S_0)$ *commutes with* $c_\phi$ *as elements of* $\mathrm{Out}(S_0)$*, then* $\alpha$ *extends to an automorphism of* $S_\phi$*.*

*Proof.* Since $c_\phi$ acts freely on the basis $\{e_{13}^{\omega}, e_{13}^{\bar\omega}\}$ of $Z(S_0)$, and since $S_0$ is a characteristic subgroup of $S_\phi$, the map induced by restriction

$$\mathrm{Out}(S_\phi) \xrightarrow{\ \cong\ } N_{\mathrm{Out}(S_0)}(\langle c_\phi\rangle)/\langle c_\phi\rangle = C_{\mathrm{Out}(S_0)}(c_\phi)/\langle c_\phi\rangle$$

is an isomorphism by Corollary 1.3. This proves the last statement. Since the centralizer of $c_\phi$ in

$$\mathrm{Out}(S_0)/O_2(\mathrm{Out}(S_0)) \cong \Sigma_3 \times \Sigma_3$$

has order 4, $C_{\mathrm{Out}(S_0)}(c_\phi)$ is a 2-group, and hence $\mathrm{Out}(S_\phi)$ is a 2-group. □

We next check the possibilities for $\mathrm{Out}_{\mathcal{F}}(S_0)$ when $\mathcal{F}$ is a saturated fusion system.

**Lemma 5.6.** *If* $\mathcal{F}$ *is a saturated fusion system over* $S_\phi$*, then there is an automorphism* $\varphi \in \mathrm{Aut}(S_\phi)$ *such that*

$$\mathrm{Aut}_{\varphi\mathcal{F}\varphi^{-1}}(S_0) \le \langle \gamma_0, \gamma_1, \mathrm{Aut}_{S_\phi}(S_0)\rangle.$$

*Proof.* Set $\Delta = \mathrm{Out}_{\mathcal{F}}(S_0)$ and $Q = O_2(\mathrm{Out}(S_0))$ for short. Then $\Delta \cap Q = 1$ since $\mathrm{Out}_{S_\phi}(S_0) = \langle c_\phi\rangle \in \mathrm{Syl}_2(\Delta)$ ($S_0$ is fully normalized since it is normal). So there is a unique subgroup $\Delta' \le \langle [\gamma_0], [\gamma_1], c_\phi\rangle$ such that $Q\Delta = Q\Delta'$ in $\mathrm{Out}(S_0)$.

By Proposition 1.8, there is $\alpha \in \mathrm{Aut}(S_\phi)$ such that $[\alpha] \in C_Q(c_\phi)$ and $\Delta' = [\alpha]\Delta[\alpha]^{-1}$. Then $\alpha$ extends to an automorphism $\varphi \in \mathrm{Aut}(S_\phi)$ by Lemma 5.5, and

$$\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(S_0) = [\alpha]\Delta[\alpha]^{-1} = \Delta' \le \langle [\gamma_0], [\gamma_1], c_\phi\rangle.$$ □

We next describe $\mathrm{Out}(P)$ for $P = H_i$ and $N_i$, and list the possibilities for $\mathrm{Out}_{\mathcal{F}}(P)$ when $\mathcal{F}$ is a saturated fusion system over $S_\phi$. When doing this, it will be helpful to translate automorphisms of $A_1$ to matrices.

Define $\rho_i^* \in \mathrm{Aut}(S_0)$ by setting

$$\rho_1^*\Bigl(\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\Bigr) = \begin{pmatrix} 1 & a & b+\bar{a} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \rho_2^*\Bigl(\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\Bigr) = \begin{pmatrix} 1 & a & b+\bar{c} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} .$$

Thus $\rho_i^* \in R_i$ is the identity on $A_{3-i}$, and $\rho_2^* = \tau\rho_1^*\tau^{-1}$. The $\rho_i^*$ commute with $c_\phi$ in $\mathrm{Aut}(S_0)$, and hence extend to automorphisms $\dot{\rho}_i^* \in \mathrm{Aut}(S_\phi)$ by sending $\phi$ to itself. Similarly, we let $\dot{\tau} \in \mathrm{Aut}(S_\phi)$ be the extension of $\tau$ which sends $\phi$ to itself.

Let $\eta_1 \in \mathrm{Aut}(H_1)$ be the automorphism such that

$$\eta_1(\phi) = \phi, \quad \eta_1(e_{13}^a) = e_{12}^a, \quad \text{and} \quad \eta_1(e_{12}^a) = e_{12}^a e_{13}^a \quad (\text{for all } a \in \mathbb{F}_4).$$

Define $\eta_1' \in \mathrm{Aut}(H_1)$ by setting $\eta_1' = \dot{\rho}_1^*\eta_1\dot{\rho}_1^{*-1}$. Finally, let $\eta_2, \eta_2' \in \mathrm{Aut}(H_2)$ be the automorphisms $\eta_2 = \dot{\tau}\eta_1\dot{\tau}^{-1}$ and $\eta_2' = \dot{\tau}\eta_1'\dot{\tau}^{-1}$.

**Lemma 5.7.** *The following hold for any saturated fusion system $\mathcal{F}$ over $S_\phi$.*

(a) *If $H_i$ is $\mathcal{F}$-essential for $i = 1$ or $2$, then*

$$\mathrm{Out}_{\mathcal{F}}(H_i) = \langle[\eta_i], \mathrm{Out}_{N_i}(H_i)\rangle \cong \Sigma_3 \qquad \text{or} \qquad \mathrm{Out}_{\mathcal{F}}(H_i) = \langle[\eta_i'], \mathrm{Out}_{N_i}(H_i)\rangle \cong \Sigma_3 .$$

(b) *If $\mathrm{Out}_{\mathcal{F}}(S_0) \le \langle[\gamma_0], c_\phi\rangle$ and $H_1$ is $\mathcal{F}$-essential, then there is $\varphi \in \mathrm{Aut}(S_\phi)$ such that $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(S_0) = \mathrm{Out}_{\mathcal{F}}(S_0)$ and $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(H_1) = \langle[\eta_1], c_{23}^1\rangle$. If in addition, $H_2$ is $\mathcal{F}$-essential, then $\varphi$ can be chosen such that we also have $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(H_2) = \langle[\eta_2], c_{12}^1\rangle$.*

*Proof.* Since $c_\phi$ acts freely on the basis $\{e_{13}^\omega, e_{13}^{\bar{\omega}}, e_{12}^\omega, e_{12}^{\bar{\omega}}\}$ of $A_1$, and since $A_1$ is a characteristic subgroup of $H_1$, the map induced by restriction

$$\mathrm{Out}(H_1) \xrightarrow[\cong]{\mathrm{Res}_{A_1}} N_{\mathrm{Aut}(A_1)}(\langle c_\phi\rangle)/\langle c_\phi\rangle = C_{\mathrm{Aut}(A_1)}(c_\phi)/\langle c_\phi\rangle$$

is an isomorphism by Corollary 1.3.

For each $\alpha \in \mathrm{Aut}(A_1)$, let $M(\alpha)$ denote the matrix for $\alpha$ with respect to the ordered basis $\{e_{13}^1, e_{12}^1, e_{13}^\omega, e_{12}^\omega\}$. Matrices will be written as $2\times 2$ blocks, where

$$I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad J = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \quad Z = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right), \quad \text{and} \quad Y = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right).$$

For example, $M(c_\phi) = \left(\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right)$ and $M(c_{23}^1) = \left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right)$. By direct computation,

$$C_{GL_4(2)}\bigl(\left(\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right)\bigr) = \bigl\{\left(\begin{smallmatrix} B & C \\ 0 & B \end{smallmatrix}\right) \bigm| B \in GL_2(2),\ C \in M_2(\mathbb{F}_2)\bigr\} \cong C_2^4 \rtimes GL_2(2). \tag{3}$$

Hence $\mathrm{Out}(H_1) \cong C_2^3 \rtimes GL_2(2) \cong C_2^3 \rtimes \Sigma_3$. Also, since $M(\eta_1|_{A_1}) = \left(\begin{smallmatrix} Z & 0 \\ 0 & Z \end{smallmatrix}\right)$ and $M(c_{23}^1|_{A_1}) = \left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right)$ (and $\langle Z, J\rangle = GL_2(2)$),

$$\mathrm{Out}(H_1) = O_2(\mathrm{Out}(H_1))\cdot\langle[\eta_1], c_{23}^1\rangle.$$

**(a)** We prove this for $H_1$; the case $H_2$ then follows by symmetry. Assume $H_1$ is $\mathcal{F}$-essential for some saturated fusion system $\mathcal{F}$. Set $\Delta = \mathrm{Out}_{\mathcal{F}}(H_1)$ and $Q = O_2(\mathrm{Out}(H_1))$ for short. Then $\Delta \cap Q = 1$, $Q\Delta = \mathrm{Out}(H_1)$, and $c_{23}^1 \in \Delta$. By Proposition 1.8, $\Delta = \langle[\alpha\eta_1\alpha^{-1}], c_{23}^1\rangle$ for some $\alpha \in O_2(\mathrm{Aut}(H_1))$ (thus $[\alpha] \in Q$) which centralizes $c_{23}^1$ in $\mathrm{Out}(H_1)$. Translated to matrices, and since we are working modulo $\langle M(c_\phi)\rangle = \langle\left(\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right)\rangle$, this means that $M(\alpha|_{A_1}) = \left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)$ for some $C \in M_2(\mathbb{F}_2)$ (by (3)), and that $JCJ^{-1} = C$ or $C + I$. Since $JZJ^{-1} = I + Z$, we get

$$C \in \langle C_{M_2(\mathbb{F}_2)}(J), Z\rangle = \langle I, Y, Z\rangle$$

(as an *additive* subgroup of $M_2(\mathbb{F}_2)$). Also, since $\left(\begin{smallmatrix} I & Z \\ 0 & I \end{smallmatrix}\right)$ commutes with $\left(\begin{smallmatrix} Z & 0 \\ 0 & Z \end{smallmatrix}\right)$ (the matrix of $\eta_1|_{A_1}$), and since we are working modulo $\langle M(c_\phi)\rangle = \langle\left(\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right)\rangle$, we can always choose $C = 0$ or $C = Y$. Since $\left(\begin{smallmatrix} I & Y \\ 0 & I \end{smallmatrix}\right) = \left(\begin{smallmatrix} J & Y \\ 0 & J \end{smallmatrix}\right)\left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right)$ where $\left(\begin{smallmatrix} J & Y \\ 0 & J \end{smallmatrix}\right) = M(\rho_1^*|_{A_1})$, this shows that we can take $\alpha = \mathrm{Id}$ or $\alpha = (\dot\rho_1^* c_{23}^1)|_{H_1}$. Also,

$$(\dot\rho_1^* c_{23}^1)\eta_1(\dot\rho_1^* c_{23}^1)^{-1} = \dot\rho_1^*\big(c_{23}^1\eta_1 {c_{23}^1}^{-1}\big)\dot\rho_1^{*-1} = \dot\rho_1^*\eta_1^{-1}\dot\rho_1^{*-1} = \eta_1'^{-1},$$

and thus $\Delta$ must be one of the two groups $\langle[\eta_1], c_{23}^1\rangle$ or $\langle[\eta_1'], c_{23}^1\rangle$.

**(b)** Now assume $\mathrm{Out}_{\mathcal{F}}(S_0) \le \langle\gamma_0, c_\phi\rangle$, and $H_1$ is $\mathcal{F}$-essential. Set $\varphi = \mathrm{Id}_{S_\phi}$ if $\mathrm{Out}_{\mathcal{F}}(H_1) = \langle[\eta_1], c_{23}^1\rangle$, and $\varphi = \dot\rho_1^* \in \mathrm{Aut}(S_\phi)$ if $\mathrm{Out}_{\mathcal{F}}(H_1) = \langle[\eta_1'], c_{23}^1\rangle$. In either case, $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(H_1) = \langle[\eta_1], c_{23}^1\rangle$. Also, $\varphi|_{H_2} = \mathrm{Id}$ and $\varphi|_{S_0}$ commutes with $\gamma_0$, and thus $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(P) = \mathrm{Out}_{\mathcal{F}}(P)$ for $P = S_0$ and $H_2$.

Similarly, if $H_2$ is also $\mathcal{F}$-essential, we can set $\psi = \mathrm{Id}_{S_\phi}$ if $\mathrm{Out}_{\mathcal{F}}(H_2) = \langle[\eta_2], c_{12}^1\rangle$, and $\psi = \dot\rho_2^* \in \mathrm{Aut}(S_\phi)$ if $\mathrm{Out}_{\mathcal{F}}(H_2) = \langle[\eta_2'], c_{23}^1\rangle$. Then $\mathrm{Out}_{\psi\mathcal{F}\psi^{-1}}(H_2) = \langle[\eta_2], c_{12}^1\rangle$, and $\mathrm{Out}_{\psi\mathcal{F}\psi^{-1}}(P) = \mathrm{Out}_{\mathcal{F}}(P)$ for $P = S_0$ and $H_1$. □

We now turn our attention to $N_1$ and $N_2$. Consider the basis

$$\mathbf{b}_1 = \{e_{12}^\omega, e_{12}^\omega e_{13}^\omega, e_{12}^{\bar\omega}, e_{12}^{\bar\omega}e_{13}^{\bar\omega}\}$$

of $A_1$, which $\mathrm{Aut}_{N_1}(A_1) = \langle c_{23}^1, c_\phi\rangle$ permutes freely. Let $\nu_1 \in \mathrm{Aut}(N_1)$ be the automorphism such that

$$\nu_1(e_{12}^\omega) = e_{12}^\omega e_{13}^\omega, \quad \nu_1(e_{12}^\omega e_{13}^\omega) = e_{12}^{\bar\omega}, \quad \nu_1(e_{12}^{\bar\omega}) = e_{12}^\omega,$$
$$\nu_1(e_{12}^{\bar\omega}e_{13}^{\bar\omega}) = e_{12}^{\bar\omega}e_{13}^{\bar\omega}, \quad \nu_1(e_{23}^1) = e_{23}^1\phi\,, \quad \text{and} \quad \nu_1(\phi) = e_{23}^1\ .$$

Thus $\nu_1$ permutes cyclically the first three elements in $\mathbf{b}_1$ and fixes the fourth, and from this it is easily seen to be an automorphism of $N_1 = A_1 \rtimes \langle e_{23}^1, \phi\rangle$. Set $\nu_2 = \dot\tau\nu_1\dot\tau^{-1}$.

**Lemma 5.8.** *If $\mathcal{F}$ is a saturated fusion system over $S_\phi$, and $N_i$ is $\mathcal{F}$-essential for $i = 1$ or $2$, then $\mathrm{Out}_{\mathcal{F}}(N_i) = \langle[\nu_i], \mathrm{Out}_{S_\phi}(N_i)\rangle \cong \Sigma_3$. If $N_i$ is not $\mathcal{F}$-essential, then $\mathrm{Out}_{\mathcal{F}}(N_i) = \mathrm{Out}_{S_\phi}(N_i)$.*

*Proof.* We prove this for $N_1$. Since $N_1/A_1$ acts freely on the basis $\mathbf{b}_1$, and since $A_1$ is characteristic in $N_1$, the map induced by restriction

$$\mathrm{Out}(N_1) \xrightarrow{\ \cong\ } N_{\mathrm{Aut}(A_1)}(\langle c_\phi, c_{23}^1\rangle)/\langle c_\phi, c_{23}^1\rangle$$

is an isomorphism by Corollary 1.3.

The action of $\langle c_\phi, c_{23}^1\rangle \cong C_2^2$ on $A_1$ permutes the elements of $\langle E_{13}, e_{12}^1\rangle$ in orbits of order one or two, and permutes the remaining eight elements in two orbits of order four:

$$\mathbf{b}_1 = \{e_{12}^\omega, e_{12}^\omega e_{13}^\omega, e_{12}^{\bar\omega}, e_{12}^{\bar\omega}e_{13}^{\bar\omega}\} \qquad \text{and} \qquad \mathbf{b}_2 = \{e_{12}^\omega e_{13}^1, e_{12}^\omega e_{13}^{\bar\omega}, e_{12}^{\bar\omega}e_{13}^1, e_{12}^{\bar\omega}e_{13}^\omega\}\ ,$$

each of which is a basis. Hence each element of the normalizer of $\langle c_\phi, c_{23}^1\rangle$ either sends each of these bases to itself or exchanges them. Clearly, each permutation of the basis $\mathbf{b}_1$ defines an element of $N_{\mathrm{Aut}(A_1)}(\langle c_\phi, c_{23}^1\rangle)$ (and determines a permutation of $\mathbf{b}_2$), and so these define a subgroup isomorphic to $\Sigma_4$ and of index at most two in this normalizer. The automorphism which sends each element of $\mathbf{b}_1$ to the product of the other three elements centralizes $\langle c_\phi, c_{23}^1\rangle$ and exchanges the two bases.

This proves that $N_{\mathrm{Aut}(A_1)}(\langle c_\phi, c_{23}^1\rangle) \cong C_2 \times \Sigma_4$, and hence

$$\mathrm{Out}(N_1) \cong N_{\mathrm{Aut}(A_1)}(\langle c_\phi, c_{23}^1\rangle)/\langle c_\phi, c_{23}^1\rangle \cong C_2 \times \Sigma_3.$$

The class of $\nu_1$ in $\mathrm{Out}(N_1)$ thus generates the unique subgroup of order three. So either $\mathrm{Out}_{\mathcal{F}}(N_1) = \langle[\nu_1], \mathrm{Out}_{S_\phi}(N_1)\rangle \cong \Sigma_3$, in which case $N_1$ is $\mathcal{F}$-essential, or $\mathrm{Out}_{\mathcal{F}}(N_1) = \mathrm{Out}_{S_\phi}(N_1)$ and $N_1$ is not $\mathcal{F}$-essential. □

We now describe some restrictions on which combinations of subgroups can be essential in a centerfree nonconstrained saturated fusion system.

**Lemma 5.9.** *Let $\mathcal{F}$ be any centerfree nonconstrained saturated fusion system over $S_\phi$. Then for each of $i = 1$ and $2$, either $H_i$ or $N_i$ is $\mathcal{F}$-essential, but not both. If $N_1$ and $N_2$ are both $\mathcal{F}$-essential, then $\mathrm{Out}_{\mathcal{F}}(S_0) \nleq \langle \gamma_1, c_\phi \rangle$.*

*Proof.* By Proposition 5.2 (and since $\mathrm{Out}(S_\phi)$ is a 2-group), $\mathcal{F}$ is generated by automorphisms in $\mathrm{Inn}(S_\phi)$, $\mathrm{Aut}_{\mathcal{F}}(S_0)$, $\mathrm{Aut}_{\mathcal{F}}(H_i)$, and $\mathrm{Aut}_{\mathcal{F}}(N_i)$ (for $i = 1, 2$), and their restrictions. Since $\langle c_\phi \rangle \in \mathrm{Syl}_2(\mathrm{Out}_{\mathcal{F}}(S_0))$, each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S_0)$ must send $A_1$ and $A_2$ to themselves.

If neither $H_1$ nor $N_1$ is $\mathcal{F}$-essential, then all morphisms in $\mathcal{F}$ are composites of restrictions of automorphisms of $S_\phi$, $S_0$, $H_2$, and $N_2$, all of which send $A_2$ to itself. Hence $A_2$ is normal in $\mathcal{F}$, which contradicts the assumption that $\mathcal{F}$ is nonconstrained. Similarly, if neither $H_2$ nor $N_2$ is $\mathcal{F}$-essential, then $A_1 \lhd \mathcal{F}$, which again contradicts our assumption.

Thus at least one subgroup in each pair $(H_1, N_1)$ and $(H_2, N_2)$ must be $\mathcal{F}$-essential. If $N_1$ is $\mathcal{F}$-essential, then $\nu_1 \in \mathrm{Aut}_{\mathcal{F}}(N_1)$ by Lemma 5.8, and $\nu_1(H_1) = \langle A_1, e_{23}^1 \rangle$. This last subgroup is normal in $S_\phi$, while $N(H_1) = N_1$. Hence $H_1$ is not fully normalized in $\mathcal{F}$, and so cannot be $\mathcal{F}$-essential. Similarly, if $N_2$ is $\mathcal{F}$-essential, then $H_2$ is not.

It remains to prove the last statement. Assume otherwise: assume $N_1$ and $N_2$ are $\mathcal{F}$-essential, and $\mathrm{Out}_{\mathcal{F}}(S_0) \leq \langle \gamma_1, c_\phi \rangle$. Then neither $H_1$ nor $H_2$ is $\mathcal{F}$-essential, so $\mathcal{F}$ is generated by automorphisms of $S_\phi$, $N_1$, and $N_2$; together with $\gamma_1, c_\phi \in \mathrm{Aut}(S_0)$. All of these automorphisms fix $e_{13}^1$ (since $S_\phi$, $N_1$, and $N_2$ all have center $e_{13}^1$). Thus $e_{13}^1$ is in the center of $\mathcal{F}$, and this contradicts the assumption that $\mathcal{F}$ is centerfree. □

## 5.3 Fusion systems over $S_\phi$

In order to better describe the subgroups generated by certain sets of elements of the $\mathrm{Aut}(A_i)$, we define an explicit isomorphism from $\mathrm{Aut}(A_1)$ to the alternating group $A_8$. We first describe this on an abstract 4-dimensional $\mathbb{F}_2$-vector space $V$ with ordered basis $\{v_1, v_2, v_3, v_4\}$.

Let $\Lambda^2(V) = (V \otimes V)/\langle v \otimes v \mid v \in V \rangle$ be the second exterior power of $V$, let $[v \otimes w] \in \Lambda^2(V)$ be the class of $v \otimes w$, and set $v_{ij} = [v_i \otimes v_j]$. Thus $\{v_{ij} \mid i < j\}$ is a basis for $\Lambda^2(V)$. Define $\mathfrak{q}\colon \Lambda^2(V) \longrightarrow \mathbb{F}_2$ by setting $\mathfrak{q}(x) = 0$ if $x = [v \otimes w]$ for some $v, w \in V$, and $\mathfrak{q}(x) = 1$ otherwise. Let $\mathfrak{b}\colon V \times V \longrightarrow \mathbb{F}_2$ be the associated form $\mathfrak{b}(x, y) = \mathfrak{q}(x + y) + \mathfrak{q}(x) + \mathfrak{q}(y)$. Thus $\mathfrak{q}(v_{ij}) = 0$ for all $i, j$, and $\mathfrak{b}(v_{ij}, v_{kl}) = 1$ if $i, j, k, l$ are distinct and is zero otherwise. One can show that $\mathfrak{b}$ is bilinear and hence $\mathfrak{q}$ is quadratic by comparing them with the bilinear and quadratic forms which take the same values on the $v_{ij}$. Hence this defines an explicit isomorphism from $\mathrm{Aut}(V) \cong GL_4(2)$ to $\Omega(\Lambda^2(V), \mathfrak{q}) \cong \Omega_6^+(2)$ (the commutator subgroup of the orthogonal group $O(\Lambda^2(V), \mathfrak{q})$), by sending $\alpha$ to $\Lambda^2(\alpha)$.

We next construct an explicit isomorphism $\Omega(\Lambda^2(V), \mathfrak{q}) \cong A_8$. Let $P_e(\underline{\mathbf{8}})$ be the group of subsets of even order in $\underline{\mathbf{8}} = \{1, 2, \ldots, 8\}$, regarded as an $\mathbb{F}_2$-vector space with addition given by symmetric difference $X + Y = ((X \smallsetminus Y) \cup (Y \smallsetminus X))$. Let $\mathfrak{q}$ be the quadratic form on $P_e(\underline{\mathbf{8}})/\langle \underline{\mathbf{8}} \rangle$ defined by $\mathfrak{q}(X) = \frac{1}{2}|X|$, associated to the bilinear form $\mathfrak{b}(X, Y) = |X \cap Y|$. The symmetric group $\Sigma_8$ acts on $P_e(\underline{\mathbf{8}})/\langle \underline{\mathbf{8}} \rangle$ preserving the form, and this defines isomorphisms $\Sigma_8 \xrightarrow{\;\cong\;} SO(P_e(\underline{\mathbf{8}})/\langle \underline{\mathbf{8}} \rangle, \mathfrak{q})$ and $A_8 \xrightarrow{\;\cong\;} \Omega(P_e(\underline{\mathbf{8}})/\langle \underline{\mathbf{8}} \rangle, \mathfrak{q})$.

Define $\kappa\colon \Lambda^2(V) \xrightarrow{\cong} P_e(\underline{\mathbf{8}})/\langle\underline{\mathbf{8}}\rangle$ by setting

$$\begin{array}{lll} \kappa(v_{12}) = \{1234\} & \kappa(v_{13}) = \{1256\} & \kappa(v_{14}) = \{1357\} \\ \kappa(v_{34}) = \{1238\} & \kappa(v_{24}) = \{2356\} & \kappa(v_{23}) = \{1367\} \end{array}$$

This clearly preserves the quadratic forms on the two spaces. Let

$$\chi_V\colon \mathrm{Aut}(V) \xrightarrow[\cong]{\Lambda^2(-)} \Omega(\Lambda^2(V),\mathfrak{q}) \xrightarrow[\cong]{\kappa_*} \Omega(P_e(\underline{\mathbf{8}})/\langle\underline{\mathbf{8}}\rangle,\mathfrak{q}) \xleftarrow{\cong} A_8$$

denote the isomorphism induced by $\Lambda^2(-)$ and $\kappa$.

We apply this here with $V = A_1$, and with the ordered basis $\{v_1, v_2, v_3, v_4\} = \{e^1_{13}, e^\omega_{13}, e^1_{12}, e^\omega_{12}\}$. We first give an explicit example of how $\chi_{A_1}(\alpha)$ can be determined in practice for $\alpha \in \mathrm{Aut}(A_1)$.

Consider the case $\alpha = c^1_{23}$. By (1), $c^1_{23}(e^a_{13}) = e^a_{13}$ and $c^1_{23}(e^a_{12}) = e^a_{12}e^a_{13}$, so that (upon writing elements additively)

$$c^1_{23}(v_1) = v_1, \quad c^1_{23}(v_2) = v_2, \quad c^1_{23}(v_3) = v_1 + v_3, \quad c^1_{23}(v_4) = v_2 + v_4.$$

Hence $\Lambda^2(c_\phi)$ and $\kappa_*(\Lambda^2(c_\phi))$ make the following assignments:

$$\begin{array}{ccc} v_{12} \mapsto v_{12} & v_{13} \mapsto v_{13} & v_{14} \mapsto v_{12} + v_{14} \\ \{1234\} \mapsto \{1234\} & \{1256\} \mapsto \{1256\} & \{1357\} \mapsto \{2457\} \end{array}$$

and

$$\begin{array}{ccc} v_{12} + v_{34} \mapsto v_{14} + v_{23} + v_{34} & v_{13} + v_{24} \mapsto v_{13} + v_{24} & v_{14} + v_{23} \mapsto v_{14} + v_{23} \\ \{48\} \mapsto \{47\} & \{13\} \mapsto \{13\} & \{56\} \mapsto \{56\}. \end{array}$$

Note that by taking sums of complementary pairs in the second row, we got information on how $\kappa_*(\Lambda^2(c_\phi))$ acts on certain sets of order two. Recall that in the quotient group $P_e(\underline{\mathbf{8}})/\langle\underline{\mathbf{8}}\rangle$, each subset of $\underline{\mathbf{8}}$ is identified with its complement. So we also get that $\{57\} = \{13\} + \{1357\}$ is sent to $\{13\} + \{2457\} = \{68\}$. Since $\{56\}$ is left invariant, the permutation which induces $\kappa_*(\Lambda^2(c_\phi))$ must exchange 5 and 6 and send 7 to 8. Upon continuing with arguments of this type, we eventually show that $\kappa_*(\Lambda^2(c_\phi))$ is induced by the permutation $(5\,6)(7\,8)$, and hence that $\chi_{A_1}(c^1_{23}) = (5\,6)(7\,8)$. In fact, if one just wants to check that $(5\,6)(7\,8)$ is indeed the right answer, the procedure is much simpler: it suffices to check that this permutation does indeed induce $\kappa_*(\Lambda^2(c_\phi))$ on the six basis elements as listed above.

We now list images under $\chi_{A_1}$ of several of the automorphisms we need to consider. In each case, $M(\alpha)$ denotes the matrix of $\alpha$ with respect to the ordered basis $\{e^1_{13}, e^\omega_{13}, e^1_{12}, e^\omega_{12}\}$:

$$\begin{array}{rcccc} \alpha = & c_\phi & c^1_{23} & c^\omega_{23} & \rho^*_1|_{A_1} \\ M(\alpha) = & \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} & \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} & \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix} & \begin{pmatrix} I & J \\ 0 & I \end{pmatrix} \\ \chi_{A_1}(\alpha) = & (1\,2)(5\,6) & (5\,6)(7\,8) & (5\,8)(6\,7) & (1\,2)(3\,4)\ . \end{array} \tag{4}$$

Here, $J = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $Z = \left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$. We also get the following values for $\chi(\alpha|_{A_1})$, for certain automorphisms $\alpha \in \mathrm{Aut}(P)$ of order 3 which can occur in $\mathrm{Aut}_{\mathcal{F}}(P)$:

$$\begin{array}{rccccc}
(\alpha,P) = & (\gamma_0,S_0) & (\gamma_1,S_0) & (\nu_1,N_1) & (\eta_1,H_1) & (\eta_1',H_1) \\
M(\alpha|_{A_1}) = & \begin{pmatrix} Z^{-1} & 0 \\ 0 & Z \end{pmatrix} & \begin{pmatrix} I & 0 \\ 0 & Z \end{pmatrix} & \left(\begin{smallmatrix} 1&0&0&0\\0&1&1&1\\0&1&0&0\\0&0&0&1 \end{smallmatrix}\right) & \begin{pmatrix} 0 & I \\ I & I \end{pmatrix} & \begin{pmatrix} J & J \\ I & I{+}J \end{pmatrix} \\
\chi_{A_1}(\alpha|_{A_1}) = & (5\,6\,7) & (1\,3\,2)(5\,7\,6) & (2\,5\,8)(1\,6\,7) & (4\,8\,7) & (3\,8\,7)\ .
\end{array} \tag{5}$$

This is now applied in the following lemma, which identifies certain groups of automorphisms of $A_1$.

**Lemma 5.10.** (a) $\langle \mathrm{Aut}_{S_\phi}(A_1), \eta_1\rangle \cong \langle \mathrm{Aut}_{S_\phi}(A_1), \eta_1'\rangle \cong \Sigma_5$ *and* $\gamma_0|_{A_1}$ *belongs to both of these groups of automorphisms;*

(b) $\langle \mathrm{Aut}_{S_\phi}(A_1), \eta_1, \gamma_1\rangle = \langle \mathrm{Aut}_{S_\phi}(A_1), \eta_1, \gamma_0, \gamma_1\rangle \cong (C_3 \times A_5) \rtimes C_2 \cong \Gamma L_2(4)$*;*

(c) $\langle \mathrm{Aut}_{S_\phi}(A_1), \eta_1', \gamma_1\rangle = \langle \mathrm{Aut}_{S_\phi}(A_1), \nu_1, \eta_1', \gamma_0, \gamma_1\rangle \cong A_7$*;*

(d) $\langle \mathrm{Aut}_{S_\phi}(A_1), \nu_1, \gamma_0\rangle \cong A_6$*;*

(e) $\langle \mathrm{Aut}_{S_\phi}(A_1), \nu_1, \gamma_0\gamma_1\rangle = \langle \mathrm{Aut}_{S_\phi}(A_1), \nu_1, \gamma_0, \gamma_1\rangle \cong A_7$.

*Here, we write* $\nu_1$*,* $\eta_1$*,* $\eta_1'$*, and* $\gamma_i$*, but mean their restrictions to* $A_1$*.*

*Proof.* The proof will be based on the isomorphism $\chi = \chi_{A_1}\colon \mathrm{Aut}(A_1) \xrightarrow{\ \cong\ } A_8$ constructed above. To simplify notation, we identify these two groups, and omit "$\chi(-)$" where it would be appropriate.

Whenever $I$ and $J$ are disjoint subsets of $\underline{\mathbf{8}} = \{1,\ldots,8\}$ ($m \geq 1$), we let $A_{I,J} \leq A_8$ ($A_I \leq A_8$) denote the subgroups of permutations which leave $I$ and $J$ invariant (leave $I$ invariant), and fix all other elements in $\underline{\mathbf{8}}$. Elements of the subsets are listed without brackets or commas. Thus, for example, $A_{125678}$ ($\cong A_6$) is the subgroup of even permutations which fix 3 and 4, while $A_{12;5678}$ contains those permutations which fix 3 and 4 and leave the subset $\{1,2\}$ invariant.

We refer to (4) and (5) for the images in $A_8$ of certain elements of $\mathrm{Aut}(A_1)$.

**(a)** Consider first

$$H_a \stackrel{\text{def}}{=} \langle \mathrm{Aut}_{S_\phi}(A_1), \eta_1\rangle = \langle c_{23}^{\omega}, c_{23}^{1}, c_\phi, \eta_1\rangle = \langle (5\,8)(6\,7), (5\,6)(7\,8), (1\,2)(5\,6), (4\,8\,7)\rangle\ .$$

Then $H_a \leq A_{12;45678}$. Also, the image of $H_a$ under projection to $\Sigma_5$ (permutations of $\{4,5,6,7,8\}$) contains the 2-cycle $(5\,6)$ and the 5-cycle $c_{23}^{\omega}\eta_1 = (5\,8\,6\,7\,4)$ (where we compose from right to left). Thus the projection is surjective, and this proves that

$$H_a = A_{12;45678} \cong \Sigma_5. \tag{6}$$

In particular, $\gamma_0 = (5\,6\,7) \in H_a$.

Simiarly, if we set

$$H_a' \stackrel{\text{def}}{=} \langle \mathrm{Aut}_{S_\phi}(A_1), \eta_1'\rangle = \langle c_{23}^{\omega}, c_{23}^{1}, c_\phi, \eta_1'\rangle = \langle (5\,8)(6\,7), (5\,6)(7\,8), (1\,2)(5\,6), (3\,8\,7)\rangle\ ,$$

then

$$H_a' = A_{12;35678} \cong \Sigma_5 \quad \text{and hence} \quad \gamma_0 = (5\,6\,7) \in H_a'. \tag{7}$$

**(b)** By (6),

$$H_b \stackrel{\text{def}}{=} \langle \mathrm{Aut}_{S_\phi}(A_1), \eta_1, \gamma_1\rangle = \langle H_a, \gamma_1\rangle = \langle A_{12;45678}, (1\,3\,2)(5\,7\,6)\rangle = A_{123;45678}.$$

Thus $H_b \cong (C_3 \times A_5) \rtimes C_2 \cong \Gamma L_2(4)$ and $\gamma_0 = (5\,6\,7) \in H_b$.

**(c)** By (7),

$$H_c \stackrel{\text{def}}{=} \langle \mathrm{Aut}_{S_\phi}(A_1), \eta_1', \gamma_1 \rangle = \langle A_{12;35678}, (1\,3\,2)(5\,7\,6) \rangle = A_{1235678} \cong A_7.$$

In particular, $\nu_1 = (2\,5\,8)(1\,6\,7)$ and $\gamma_0 = (5\,6\,7)$ are both in $H_c$.

**(d)** We have

$$\begin{aligned} H_d &\stackrel{\text{def}}{=} \langle \mathrm{Aut}_{S_\phi}(A_1), \gamma_0, \nu_1 \rangle = \langle (1\,2)(5\,6), (5\,8)(6\,7), (5\,6)(7\,8), (5\,6\,7), (2\,5\,8)(1\,6\,7) \rangle \\ &= \langle A_{12;5678}, (2\,5\,8)(1\,6\,7) \rangle = A_{125678} \cong A_6. \end{aligned}$$

**(e)** Consider the subgroup

$$H_e \stackrel{\text{def}}{=} \langle \mathrm{Aut}_{S_\phi}(A_1), \nu_1, \gamma_0\gamma_1 \rangle = \langle \mathrm{Aut}_{S_\phi}(A_1), (2\,5\,8)(1\,6\,7), (1\,3\,2) \rangle.$$

Then $\nu_1^{-1}(1\,3\,2)\nu_1 = (7\,3\,8) = \eta_1' \in H_e$, and so

$$H_e = \langle H_a', (2\,5\,8)(1\,6\,7), (1\,3\,2) \rangle = \langle A_{12;35678}, (2\,5\,8)(1\,6\,7), (1\,3\,2) \rangle = A_{1235678}.$$

Thus $H_e \cong A_7$, and $\gamma_0, \gamma_1 \in H_e$. □

We are now ready to list fusion systems over $S_\phi$. In the statement and the proof of the following theorem, we follow the usual notation by writing $P\Gamma L_n(q) = PGL_n(q) \rtimes \langle \phi \rangle$ and $P\Sigma L_n(q) = PSL_n(q) \rtimes \langle \phi \rangle$, where $\phi$ is a generator of $\mathrm{Aut}(\mathbb{F}_q)$ (extended to an automorphism on matrix groups).

**Theorem 5.11.** *If $\mathcal{F}$ is a nonconstrained centerfree saturated fusion system over $S_\phi$, then it is isomorphic to the fusion system of one of the following groups: $M_{22}$, $M_{23}$,* McL*, $P\Sigma L_3(4)$, $P\Gamma L_3(4)$, or $PSL_4(5) \cong P\Omega_6^+(5)$.*

*Proof.* Let $\mathcal{F}$ be a saturated fusion system over $S_\phi$. Assume $\mathcal{F}$ is nonconstrained and centerfree. By Lemma 5.6, upon replacing $\mathcal{F}$ by $\varphi\mathcal{F}\varphi^{-1}$ for some $\varphi \in \mathrm{Aut}(S_\phi)$, we can assume that

$$\mathrm{Out}_{\mathcal{F}}(S_0) \le \langle [\gamma_0], [\gamma_1], c_\phi \rangle. \tag{8}$$

We first list the different choices for the set of $\mathcal{F}$-essential subgroups, then we list the different combinations for $\mathrm{Aut}_{\mathcal{F}}(P)$ (or $\mathrm{Out}_{\mathcal{F}}(P)$) for each $\mathcal{F}$-essential subgroup $P$. Using that, we show that $\mathcal{F}$ is isomorphic to one of a list of six explicitly defined fusion systems over $S_\phi$, which we then compare with those in the statement of the theorem.

The following are some conditions which must hold for $\mathcal{F}$:

(a) $\mathrm{Out}_{\mathcal{F}}(S_\phi) = 1$. This holds since $\mathrm{Out}(S_\phi)$ is a 2-group (Lemma 5.5).

(b) The only possible $\mathcal{F}$-essential subgroups are $S_0$, $N_1$, $N_2$, $H_1$, $H_2$, and their conjugates (Proposition 5.2).

(c) Exactly one of the subgroups $H_1$ or $N_1$ is essential, and exactly one of the subgroups $H_2$ or $N_2$ is essential (Lemma 5.9).

(d) If $H_i$ is $\mathcal{F}$-essential ($i = 1$ or 2), then $\gamma_0 \in \mathrm{Aut}_{\mathcal{F}}(S_0)$. If $H_1$ is $\mathcal{F}$-essential, then by Lemma 5.7(a), $\eta_1$ or $\eta_1'$ is in $\mathrm{Aut}_{\mathcal{F}}(H_1)$, so $\gamma_0|_{A_1} \in \mathrm{Aut}_{\mathcal{F}}(A_1)$ by Lemma 5.10(a). This is the restriction of an automorphism in $\mathrm{Aut}_{\mathcal{F}}(S_0)$ by the extension axiom, and thus $\gamma_0 \in \mathrm{Aut}_{\mathcal{F}}(S_0)$ by (8). Similarly, if $H_2$ is $\mathcal{F}$-essential, then $\tau\gamma_0\tau^{-1} = \gamma_0 \in \mathrm{Aut}_{\mathcal{F}}(S_0)$.

(e) If $\gamma_0, \gamma_1 \in \mathrm{Aut}_{\mathcal{F}}(S_0)$, and $H_i$ is essential for $i = 1$ or 2, then $\mathrm{Out}_{\mathcal{F}}(H_i) = \langle [\eta_i], \mathrm{Out}_{S_\phi}(H_i)\rangle$. To see this when $i = 1$, assume otherwise: thus $\eta_1' \in \mathrm{Aut}_{\mathcal{F}}(H_1)$ by Lemma 5.7(a). Then $\nu_1|_{A_1} \in \mathrm{Aut}_{\mathcal{F}}(A_1)$ by Lemma 5.10(c), which implies by the extension axiom that $\nu_1|_{A_1}$ extends to an automorphism in $\mathrm{Aut}_{\mathcal{F}}(N_1)$. Thus $N_1$ is $\mathcal{F}$-essential by Lemma 5.8, so $H_1$ is not $\mathcal{F}$-essential by Lemma 5.9, which is a contradiction.

(f) If $N_1$ and $N_2$ are both $\mathcal{F}$-essential, then at least one of the automorphisms $\gamma_0$, $\gamma_0\gamma_1$, or $\gamma_0\gamma_1^{-1}$ must be in $\mathrm{Aut}_{\mathcal{F}}(S_0)$. To see this, note first that by (8), $\mathrm{Out}_{\mathcal{F}}(S_0) = \langle \Delta, c_\phi\rangle$ for some $\Delta \le \langle [\gamma_0], [\gamma_1]\rangle \cong C_3^2$. Also, by Lemma 5.9, $\Delta \not\le \langle [\gamma_1]\rangle$. Thus for some $i$, $[\gamma_0\gamma_1^i] \in \Delta \le \mathrm{Out}_{\mathcal{F}}(S_0)$.

If $H_1$ or $H_2$ is $\mathcal{F}$-essential, then by (d), $\gamma_0 \in \mathrm{Aut}_{\mathcal{F}}(S_0)$. If neither of these groups is $\mathcal{F}$-essential, then $N_1$ and $N_2$ are both $\mathcal{F}$-essential by (c), and so $\gamma_0\gamma_1^i \in \mathrm{Aut}_{\mathcal{F}}(S_0)$ (some $i = 0, \pm 1$) by (f). Since $\gamma_0, \gamma_1 \in \mathrm{Aut}(S_0)$ are both inverted by $c_\phi$, $\mathrm{Out}_{\mathcal{F}}(S_0)$ contains a subgroup $\Sigma_3$ in all cases, and hence $S_0$ is $\mathcal{F}$-essential for any $\mathcal{F}$.

Together with points (b) and (c), this shows that the choices for the set of $\mathcal{F}$-essential subgroups (up to conjugacy) are among the following:

$$\{H_1, H_2, S_0\}\ , \quad \{H_1, N_2, S_0\}\ , \quad \{N_1, H_2, S_0\} \quad \text{and} \quad \{N_1, N_2, S_0\}\ . \tag{9}$$

If $N_1$ and $H_2$ are $\mathcal{F}$-essential, then $H_1$ and $N_2$ are essential in the fusion system $\dot\tau\mathcal{F}\dot\tau^{-1}$ which is isomorphic to $\mathcal{F}$. We claim that upon combining (9) with the restrictions on the automorphism groups $\mathrm{Out}_{\mathcal{F}}(-)$ imposed by points (a) and (d)–(f), we are reduced to the following list of eleven candidates for fusion systems $\mathcal{F}$, up to isomorphism:

$$\begin{aligned} &\{H_1; H_2; S_0\}: \quad \{\eta_1; \eta_2; \gamma_0\}\ , \ \{\eta_1; \eta_2; \gamma_0, \gamma_1\}\ , \ \{\eta_1'; \eta_2; \gamma_0\}\ , \ \{\eta_1'; \eta_2'; \gamma_0\}\ ; \\ &\{H_1; N_2; S_0\}: \quad \{\eta_1; \nu_2; \gamma_0\}\ , \ \{\eta_1; \nu_2; \gamma_0, \gamma_1\}\ , \ \{\eta_1'; \nu_2; \gamma_0\}\ ; \\ &\{N_1; N_2; S_0\}: \quad \{\nu_1; \nu_2; \gamma_0\}\ , \ \{\nu_1; \nu_2; \gamma_0, \gamma_1\}\ , \ \{\nu_1; \nu_2; \gamma_0\gamma_1\}\ , \ \{\nu_1; \nu_2; \gamma_0\gamma_1^{-1}\}\ . \end{aligned} \tag{10}$$

The first entry in each row of (10) gives the $\mathcal{F}$-essential subgroups. The later entries list, for each $\mathcal{F}$-essential subgroup $P$, generators of $\mathrm{Aut}_{\mathcal{F}}(P)$ in addition to $\mathrm{Aut}_{S_\phi}(P)$. In each case, $\mathcal{F}$ is the fusion system generated by the given automorphism groups of the given essential subgroups and $\mathrm{Inn}(S_\phi)$; i.e., the fusion system generated by $\mathrm{Inn}(S_\phi)$, $\gamma_k \in \mathrm{Aut}(S_0)$, $\eta_i$ or $\eta_i'$ in $\mathrm{Aut}(H_i)$, and $\nu_j \in \mathrm{Aut}(N_j)$, where $k$, $i$, and $j$ are as listed. Thus, for example, the last entry in the first row describes the fusion system generated by $\mathrm{Inn}(S_\phi)$, $\gamma_0 \in \mathrm{Aut}(S_0)$, $\eta_1' \in \mathrm{Aut}(H_1)$, and $\eta_2' \in \mathrm{Aut}(H_2)$.

We next justify the claim. When $H_1$ or $H_2$ is $\mathcal{F}$-essential, then $\mathrm{Out}_{\mathcal{F}}(S_0) = \langle [\gamma_0], c_\phi\rangle$ or $\langle [\gamma_0], [\gamma_1], c_\phi\rangle$ by (d). By (e), the second is possible only if $\eta_i \in \mathrm{Aut}_{\mathcal{F}}(H_i)$ for all $H_i$ which are $\mathcal{F}$-essential. Thus the seven fusion systems listed in the first two rows are the only possible ones for which $H_1$ or $H_2$ is $\mathcal{F}$-essential (up to replacing $\mathcal{F}$ by $\dot\tau\mathcal{F}\dot\tau^{-1}$). If $N_1$ and $N_2$ are $\mathcal{F}$-essential, then by (f), $\mathcal{F}$ must be one of the four fusion systems listed in the third row of (10).

By Lemma 5.10(e), if $N_1$ is $\mathcal{F}$-essential (so $\nu_1 \in \mathrm{Aut}_{\mathcal{F}}(N_1)$ by Lemma 5.8) and $\gamma_0\gamma_1 \in \mathrm{Aut}_{\mathcal{F}}(S_0)$, then $\gamma_0|_{A_1}, \gamma_1|_{A_1} \in \mathrm{Aut}_{\mathcal{F}}(A_1)$. So by the extension axiom (and (8)), $\gamma_0, \gamma_1 \in \mathrm{Aut}_{\mathcal{F}}(S_0)$ in this case. Likewise, if $\gamma_0\gamma_1^{-1} = \tau(\gamma_0\gamma_1)\tau^{-1} \in \mathrm{Aut}_{\mathcal{F}}(S_0)$ and $N_2$ is $\mathcal{F}$-essential, then $\gamma_0, \gamma_1 \in \mathrm{Aut}_{\mathcal{F}}(S_0)$. In other words, $\mathcal{F}$ cannot have the form corresponding to either of the last two entries in the last row of (10).

By Lemma 5.7(b), if $\mathrm{Out}_{\mathcal{F}}(S_0) = \langle [\gamma_0], c_\phi\rangle$, $H_1$ is $\mathcal{F}$-essential, and $\mathrm{Out}_{\mathcal{F}}(H_1) = \langle [\eta_1'], c_{23}^1\rangle$, then there is an automorphism $\varphi \in \mathrm{Aut}(S_\phi)$ such that $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(S_0) = \mathrm{Out}_{\mathcal{F}}(S_0)$ and $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(H_1) = \langle [\eta_1], c_{23}^1\rangle$. If, furthermore, $H_2$ is also $\mathcal{F}$-essential, then $\varphi$ can be chosen such that $\mathrm{Out}_{\varphi\mathcal{F}\varphi^{-1}}(H_2) = \langle [\eta_2], c_{12}^1\rangle$. In other words, we can eliminate all of the cases in the first two rows of (10) which involve $\eta_1'$ or $\eta_2'$, since the corresponding fusion systems are isomorphic to others in the list.

| $\mathrm{Out}_{\mathcal{F}}(S_0)$ | $\mathcal{F}$-essential | $\mathrm{Aut}_{\mathcal{F}}(A_1)$ | $\mathrm{Aut}_{\mathcal{F}}(A_2)$ | $G$ |
|---|---|---|---|---|
| $\langle[\gamma_0], c_\phi\rangle$ | $H_1, H_2, S_0$ | $\Sigma_5$ | $\Sigma_5$ | $P\Sigma L_3(4)$ |
| $\langle[\gamma_0], [\gamma_1], c_\phi\rangle$ | $H_1, H_2, S_0$ | $(C_3 \times A_5) \rtimes C_2$ | $(C_3 \times A_5) \rtimes C_2$ | $P\Gamma L_3(4)$ |
| $\langle[\gamma_0], c_\phi\rangle$ | $H_1, N_2, S_0$ | $\Sigma_5$ | $A_6$ | $M_{22}$ |
| $\langle[\gamma_0], [\gamma_1], c_\phi\rangle$ | $H_1, N_2, S_0$ | $(C_3 \times A_5) \rtimes C_2$ | $A_7$ | $M_{23}$ |
| $\langle[\gamma_0], c_\phi\rangle$ | $N_1, N_2, S_0$ | $A_6$ | $A_6$ | $PSL_4(5) \cong P\Omega_6^+(5)$ |
| $\langle[\gamma_0], [\gamma_1], c_\phi\rangle$ | $N_1, N_2, S_0$ | $A_7$ | $A_7$ | McL |

TABLE 5.2

We have now shown that $\mathcal{F}$ is isomorphic to one of six fusion systems: the first two in each row of (10). These six are described in more detail in Table 5.2, where in all cases, $\mathrm{Out}_{\mathcal{F}}(H_i) = \langle[\eta_i], \mathrm{Out}_{S_\phi}(H_i)\rangle$ if $H_i$ is $\mathcal{F}$-essential. If $H_i$ is $\mathcal{F}$-essential, then $\mathrm{Aut}_{\mathcal{F}}(A_i) \cong \Sigma_5$ if $\mathrm{Out}_{\mathcal{F}}(S_0) = \langle[\gamma_0], c_\phi\rangle$ by Lemma 5.10(a), while $\mathrm{Aut}_{\mathcal{F}}(A_i) \cong (C_3 \times A_5) \rtimes C_2$ if $\mathrm{Out}_{\mathcal{F}}(S_0) = \langle[\gamma_0], c_\phi\rangle$ by Lemma 5.10(b). The descriptions of $\mathrm{Aut}_{\mathcal{F}}(A_i)$ when $N_i$ is $\mathcal{F}$-essential follow in a similar way from Lemma 5.10(d,e). By inspection, these six fusion systems are distinguished by the groups $\mathrm{Aut}_{\mathcal{F}}(A_1)$ and $\mathrm{Aut}_{\mathcal{F}}(A_2)$ as described in the table.

It remains to prove that the groups $G$ listed in the table do realize these fusion systems: that they all have Sylow 2-subgroups isomorphic to $S_\phi$, and have automorphism groups $\mathrm{Aut}_G(A_i)$ as described. This is clear for the groups $P\Sigma L_3(4)$ and $P\Gamma L_3(4)$ using the well-known isomorphisms $\Sigma L_2(4) \cong \Sigma_5$ and $\Gamma L_2(4) \cong (C_3 \times A_5) \rtimes C_2$ (or by directly determining $\mathrm{Aut}_G(H_i)$ and $\mathrm{Aut}_G(S_0)$).

Since $A_6$ has no subgroup of index 7 or 8, and $A_7$ no subgroup of index 8, the group $GL_4(2) \cong A_8$ contains unique conjugacy classes of subgroups isomorphic to $A_6$ and $A_7$. Since $A_6$ and $A_7$ are simple, this implies that up to isomorphism, there are unique semidirect products $C_2^4 \rtimes A_6$ and $C_2^4 \rtimes A_7$ which are not direct products. By Lemma 5.10, $\mathrm{Aut}_{S_\phi}(A_i)$ is contained in a subgroup isomorphic to $A_7$, and hence $S_\phi$ is a Sylow 2-subgroup of the (of any) semidirect product $C_2^4 \rtimes A_6$ or $C_2^4 \rtimes A_7$ which is not a direct product.

When $q \equiv \pm 5 \pmod 8$, then $P\Omega_6^\pm(q)$ is the commutator subgroup of the projective orthogonal group of a quadratic form on $V = \mathbb{F}_q^6$ with orthonormal basis $\{v_1, \dots, v_6\}$. This group contains two conjugacy classes of subgroups $C_2^4 \rtimes A_6$: the groups of automorphisms which preserve up to sign one of the two bases $\{v_i\}$ or $\{v_1 \pm v_2, v_3 \pm v_4, v_5 \pm v_6\}$. (These two orthogonal bases are inequivalent, since 2 is always a nonsquare for such $q$.) Since these are subgroups of odd index, $P\Omega_6^\pm(q)$ has Sylow 2-subgroups isomorphic to $S_\phi$, and its fusion system is the one with these automorphism groups (and is independent of $q$).

As for the other groups, $M_{22}$ contains subgroups $C_2^4 \rtimes \Sigma_5$ (the quintet subgroup) and $C_2^4 \rtimes A_6$ (the hexad subgroup); while $M_{23}$ contains $(C_2^4 \rtimes C_3) \rtimes \Sigma_5$ (the quintet subgroup) and $C_2^4 \rtimes A_7$ (the heptad subgroup). See [Co, Table 3] for more detail. By

[Fi, Theorem 1], McLaughlin's group McL contains two conjugacy classes of subgroups $C_2^4 \rtimes A_7$. So all three of these groups have the fusion systems described in Table 5.2. □

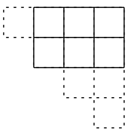
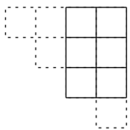
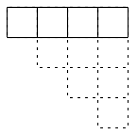
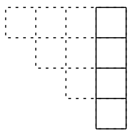
Note also that McL contains $M_{22}$, $P\Omega_6^-(3)$, and $P\Sigma L_3(4)$ as subgroups of odd index, while $M_{23}$ contains $M_{22}$ and $P\Sigma L_3(4)$ as subgroups of odd index.

## 6. Fusion systems over $UT_5(2)$

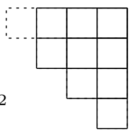
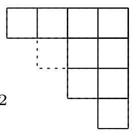
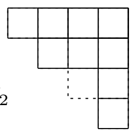
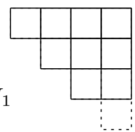
Throughout this section, $T = UT_5(2)$ denotes the group of $5 \times 5$ upper triangular matrices over $\mathbb{F}_2$. We let $e_{ij} \in T$ (for $i < j$) be the elementary matrix with nontrivial entry in the $(i,j)$ position. Also, $c_{ij}$ denotes conjugation by $e_{ij}$, regarded as an automorphism of $T$ or as a homomorphism between subgroups of $T$. For later reference, we note here the following relations among the $e_{ij}$:

$$(e_{ij}e_{k\ell})^2 = [e_{ij}, e_{k\ell}] = \begin{cases} e_{i\ell} & \text{if } j = k \\ e_{kj} & \text{if } i = \ell \\ 1 & \text{if } i \neq \ell \text{ and } j \neq k. \end{cases} \tag{1}$$

For any pair of sets of indices $I, J \subseteq \{1,2,3,4,5\}$, let $E_{I;J} \leq T$ denote the subgroup generated by all $e_{ij}$ for $i \in I$ and $j \in J$ (and $i < j$). In particular, we focus attention on the "rectangular" subgroups $A_1 = E_{12;345}$, $A_2 = E_{123;45}$, $U_1 = E_{1;2345}$, and $U_2 = E_{1234;5}$. These can be described pictorally as follows:

$A_1 =$  $A_2 =$  $U_1 =$  $U_2 =$ 

We also need to consider the following index two subgroups $Q_i$:

$Q_1 = {}$ $=A_1A_2U_2$  $Q_2 = {}$ $=A_2U_1U_2$  $Q_3 = {}$ $=A_1U_1U_2$  $Q_4 = {}$ $=A_1A_2U_1$ .

We will show in Proposition 6.5 that the $Q_i$ are the only critical subgroups of $T$.

The following lemma is very elementary and well known; we include it here for the sake of completeness.

**Lemma 6.1.** *The only elementary abelian subgroups of rank* 6 *in* $T$ *are* $A_1$ *and* $A_2$.

*Proof.* Set $R = E_{123;345} = A_1A_2$ and $R_0 = Z(R) = E_{12;45}$ for short. By (1), all involutions in $R$ are in $A_1 \cup A_2$, and no element of $A_1 \smallsetminus R_0$ commutes with any element of $A_2 \smallsetminus R_0$. Hence each elementary abelian subgroup of $R$ is contained in $A_1$ or in $A_2$.

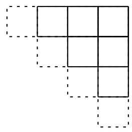
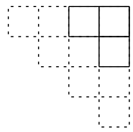
Assume $A \leq T$ is elementary abelian of rank six, and set $B = A \cap R$. We just saw that $B$ is contained in $A_1$ or $A_2$; it suffices to handle the case $B \leq A_1$. Since $T/R \cong C_2^2$, $\mathrm{rk}(B) \geq 4$. If $\mathrm{rk}(B) = 4$, then $AR = T$, so there are elements $g, h \in A$ such that $g \in e_{12}R$ and $h \in e_{45}R$. Then

$$B \cap R_0 \leq C_{R_0}(\langle g, h\rangle) = C_{R_0}(\langle e_{12}, e_{45}\rangle) = \langle e_{15}\rangle$$

(since $R = C_T(R_0)$), so $\mathrm{rk}(B) \leq 1 + \mathrm{rk}(A_1/R_0) = 3$, a contradiction. Thus $\mathrm{rk}(B) = 5$, $A = \langle B, g\rangle$ for some $g \in T \smallsetminus R$, $B \cap R_0 \leq C_{R_0}(g)$ has rank at least three, and this is impossible since $\mathrm{rk}(C_{R_0}(a)) = 2$ for $a = e_{12}$, $e_{45}$, or $e_{12}e_{45}$. □

## 6.1 Determining the critical subgroups

Throughout this subsection, we write

$T' = [T,T] =$  and $Z_2 = \langle e_{15}, e_{14}, e_{25}\rangle =$ 

for short. These subgroups will appear repeatedly. Using (1), they are seen to be terms in the upper and lower central series for $T$:

$$Z_2 = [T,T'] = Z_2(T) \qquad \text{and} \qquad T' = Z_3(T). \tag{2}$$

Also, $\tau \in \mathrm{Aut}(T)$ is the automorphism $\tau(e_{ij}) = e_{6-j,6-i}$. We first show:

**Lemma 6.2.** *All critical subgroups of $T$ contain $Z_2$.*

*Proof.* Fix a critical subgroup $P \le T$, and assume first that $e_{14} \notin P$. We apply Lemma 3.6 with $z = e_{15}$, $g = e_{14}$, and $y = e_{25}$. By the proposition, $P = C_T(h)$ for some $h$ such that $[e_{14}, h] = e_{15}$. Also, either $e_{25} \in Z(P)$ and $h$ is not $T$-conjugate to $e_{25}h$, or $e_{14}e_{25} \in Z(P)$ and $h$ is not $T$-conjugate to $e_{14}e_{25}h$.

If $e_{25} \in Z(P)$, then since $[h, e_{14}] = e_{15} \ne 1$,

$$h \in C_T(e_{25}) \smallsetminus C_T(e_{14}) = \langle e_{45}, A_1A_2\rangle \smallsetminus \langle e_{12}, A_1A_2\rangle = e_{45}{\cdot}A_1A_2.$$

Since $[A_1A_2, e_{24}] = 1$, $[h, e_{24}] = [e_{45}, e_{24}] = e_{25}$, contradicting the condition that $h$ not be $T$-conjugate to $e_{25}h$. Similarly, if $e_{14}e_{25} \in Z(P)$, then

$$h \in C_T(e_{14}e_{25}) \smallsetminus C_T(e_{14}) = \langle e_{12}e_{45}, A_1A_2\rangle \smallsetminus \langle e_{12}, A_1A_2\rangle = e_{12}e_{45}{\cdot}A_1A_2,$$

so $[h, e_{24}] = [e_{12}e_{45}, e_{24}] = e_{14}e_{25}$, contradicting the condition that $h$ not be $T$-conjugate to $e_{14}e_{25}h$.

This proves that $e_{14} \in P$, and a similar argument shows that $e_{25} \in P$. □

We next reduce to the case of subgroups having index 2 in their normalizers.

**Lemma 6.3.** *If $P$ is a critical subgroup of $T$, then $|N_T(P)/P| = 2$.*

*Proof.* Assume otherwise: let $P$ be a critical subgroup of $T$ with $|N(P)/P| \ge 4$. By Proposition 3.3(c),

$$\mathrm{rk}([g, P/\mathrm{Fr}(P)]) \ge 2 \quad \text{for each} \quad g \in N(P){\smallsetminus}P \tag{3}$$

and

$$|N(P)/P| = 2^k \quad \Longrightarrow \quad \mathrm{rk}(P/\mathrm{Fr}(P)) \ge 2k. \tag{4}$$

By Lemma 6.2, $P \ge Z_2$. Hence $[T', P] \le [T', T] = Z_2 \le P$ by (2), so $N(P) \ge T'$. We now consider separately the cases where $e_{15} \in \mathrm{Fr}(P)$ or $e_{15} \notin \mathrm{Fr}(P)$. We will frequently be using (1) for commutator and squaring relations, without referring to it each time.

**Case 1:** Assume first that $e_{15} \in \mathrm{Fr}(P)$. Since $[e_{13}, P] \le [e_{13}, T] = \langle e_{14}, e_{15}\rangle$, this implies $\mathrm{rk}([e_{13}, P/\mathrm{Fr}(P)]) \le 1$. Hence $e_{13} \in P$ by (3), and $e_{35} \in P$ by symmetry.

We claim that

(a) $e_{14} \notin \mathrm{Fr}(P)$ implies $P \le \langle T', e_{12}, e_{23}, e_{45}\rangle$; and

(b) $e_{25} \notin \mathrm{Fr}(P)$ implies $P \le \langle T', e_{12}, e_{34}, e_{45}\rangle$.

If $e_{14} \notin \mathrm{Fr}(P)$, then since $e_{13} \in P$ and $e_{15} \in \mathrm{Fr}(P)$, this implies $e_{14}, e_{14}e_{15} \notin [e_{13}, P]$. Also, $[e_{13}, T] = \langle e_{14}, e_{15}\rangle$, and hence

$$P \le \bigl\{g \in T \bigm| [e_{13}, g] \in \langle e_{15}\rangle\bigr\} = \langle e_{35}, C_T(e_{13})\rangle = \langle T', e_{12}, e_{23}, e_{45}\rangle.$$

Point (b) follows by symmetry with respect to $\tau \in \mathrm{Aut}(T)$.

**Case 1a:** Assume $e_{24} \notin P$. Thus $e_{24} \in N_T(P) \smallsetminus P$, and $\mathrm{rk}([e_{24}, P/\mathrm{Fr}(P)]) \ge 2$ by (3). Since $[e_{24}, T] = \langle e_{14}, e_{25}\rangle$, this implies that

$$e_{14}, e_{25} \in [e_{24}, P] \qquad \text{and} \qquad \mathrm{Fr}(P) \cap Z_2 = \langle e_{15}\rangle. \tag{5}$$

The second point, together with (a) and (b), implies $P \le \langle T', e_{12}, e_{45}\rangle$.

Set $P_0 = \langle Z_2, e_{13}, e_{15}\rangle$. We have now shown that $P_0 \le P \le \langle P_0, e_{24}, e_{12}, e_{45}\rangle$, and that $e_{24} \notin P$. Also, since $[e_{24}, P_0] = 1$ and $|[e_{24}, P]| \ge 4$ by (5), $[P{:}P_0] \ge 4$. We conclude that $P = \langle P_0, e_{12}x, e_{45}y\rangle$ for some $x, y \in \langle e_{24}\rangle$.

By (5) again, $(e_{12}e_{24})^2 = e_{14} \notin \mathrm{Fr}(P)$ and $(e_{45}e_{24})^2 = e_{25} \notin \mathrm{Fr}(P)$. Hence $x = y = 1$, and $P = \langle Z_2, e_{13}, e_{35}, e_{12}, e_{45}\rangle$. But then $\mathrm{Out}_T(P) \cong T/P \cong D_8$ has noncentral involutions, while by Proposition 3.3(a), this is impossible for a critical subgroup $P \le T$.

**Case 1b:** Now assume $e_{24} \in P$. Thus $T' \le P$, and so $P$ is normal in $T$. Also, $[P{:}T'] = 16\big/[T{:}P] \le 4$ since $[T{:}P] \ge 4$.

Assume first $e_{12}, e_{45} \in P$. Then $P = \langle T', e_{12}, e_{45}\rangle$ (since it cannot be larger), and

$$\mathrm{Fr}(P) = \bigl\langle \mathrm{Fr}(T'), [e_{12}, T'], [e_{45}, T']\bigr\rangle = \bigl\langle e_{15}, [e_{12}, e_{24}], [e_{45}, e_{24}]\bigr\rangle = Z_2.$$

So $[e_{23}, P/\mathrm{Fr}(P)] = \langle e_{13}\rangle$ has rank one, which contradicts (3).

Thus either $e_{12} \notin P$ or $e_{45} \notin P$. By symmetry (with respect to $\tau \in \mathrm{Aut}(T)$), we can assume $e_{12} \notin P$. Then $\mathrm{rk}([e_{12}, P/\mathrm{Fr}(P)]) \ge 2$ by (3). Since $[e_{12}, P] \le [e_{12}, T] = \langle e_{13}, e_{14}, e_{15}\rangle$ and $e_{15} \in \mathrm{Fr}(P)$, this implies $e_{13}, e_{14} \notin \mathrm{Fr}(P)$. Hence by (a), $P \le \langle T', e_{12}, e_{23}, e_{45}\rangle$. If $[P{:}T'] \le 2$, then $T/P \cong C_2^k$ for $k \ge 3$, hence $\mathrm{rk}([e_{12}, P/\mathrm{Fr}(P)]) \ge 3$ by Proposition 3.3(d), and we just saw this is impossible.

Thus $[P{:}T'] = 4$, and $e_{12} \notin P \le \langle T', e_{12}, e_{23}, e_{45}\rangle$. It follows that $P = \langle T', e_{23}x, e_{45}y\rangle$ for some $x, y \in \langle e_{12}\rangle$. If $y \ne 1$, then $e_{45} \in N_T(P) \smallsetminus P$, $\mathrm{rk}([e_{45}, P/\mathrm{Fr}(P)]) \ge 2$, which is impossible since $[e_{45}, P] \le \langle e_{15}, e_{25}, e_{35}\rangle$ and $e_{25} = [e_{23}x, e_{35}] \in \mathrm{Fr}(P)$. Thus $y = 1$. If $x \ne 1$, then $(e_{12}e_{23})^2 = e_{13} \in \mathrm{Fr}(P)$, while we already showed that $e_{13} \notin \mathrm{Fr}(P)$.

We are thus left with the case $P = \langle T', e_{23}, e_{45}\rangle$. Then $\mathrm{Fr}(P) = \langle e_{15}, [e_{23}, e_{35}]\rangle = \langle e_{15}, e_{25}\rangle$. So

$$\begin{aligned} \mathrm{rk}([e_{12}, P/\mathrm{Fr}(P)]) &= \mathrm{rk}(\langle e_{13}, e_{14}, e_{15}\rangle/\langle e_{15}\rangle) = 2 \\ \mathrm{rk}([e_{34}, P/\mathrm{Fr}(P)]) &= \mathrm{rk}(\langle e_{14}, e_{24}, e_{35}\rangle) = 3. \end{aligned}$$

But this contradicts Proposition 3.3(b), which says that all involutions in $\mathrm{Out}_T(P)$ are conjugate in $\mathrm{Out}(P)$, and hence that $[e_{12}, P/\mathrm{Fr}(P)]$ and $[e_{34}, P/\mathrm{Fr}(P)]$ have the same rank. So this subgroup is not critical.

**Case 2:** Now assume $e_{15} \notin \mathrm{Fr}(P)$. Since $e_{14} \in P$ and $[e_{14}, T] = \langle e_{15}\rangle$, this implies $e_{14} \in Z(P)$, and similarly $e_{25} \in Z(P)$. So $P \le C_T(Z_2) = A_1A_2$. Since $P$ is centric in $T$, this also implies $P \ge Z(A_1A_2) = A_1 \cap A_2 = \langle Z_2, e_{24}\rangle$. Set $R_0 = \langle Z_2, e_{24}\rangle$ for short.

If $|P/R_0| \le 2$, then $|P| \le 2^5$, so $\mathrm{rk}(P/\mathrm{Fr}(P)) \le 5$ and $|N(P)/P| \ge |A_1A_2/P| \ge 2^3$. This contradicts (4), and we conclude $|P/R_0| \ge 4$.

Now, $e_{13}e_{35} \notin P$ since $(e_{13}e_{35})^2 = e_{15} \notin \mathrm{Fr}(P)$. Assume neither $e_{13}$ nor $e_{35}$ is in $P$. Since $|P/R_0| \geq 4$, this implies $P = \langle R_0, e_{23}x, e_{34}y \rangle$ for some $x, y \in \langle e_{13}, e_{35} \rangle$. Also,

$$e_{12}Pe_{12}^{-1} = \langle R_0, e_{23}e_{13}x, e_{34}y \rangle \quad \text{and} \quad e_{45}Pe_{45}^{-1} = \langle R_0, e_{23}x, e_{34}e_{35}y \rangle,$$

so up to conjugacy, we can assume $x \in \langle e_{35} \rangle$ and $y \in \langle e_{13} \rangle$. Since $\mathrm{rk}([e_{13}, P/\mathrm{Fr}(P)]) \geq 2$ by (3), $e_{14} = (e_{13}e_{34})^2 \notin \mathrm{Fr}(P)$, so $y = 1$. By a similar argument, $x = 1$, and thus $P = \langle R_0, e_{23}, e_{34} \rangle$. But then $[e_{13}, P] = \langle e_{14} \rangle$, contradicting (3) again.

Thus either $e_{13} \in P$ or $e_{35} \in P$, and they cannot both be in $P$ since $[e_{13}, e_{35}] = e_{15} \notin \mathrm{Fr}(P)$. By symmetry (with respect to $\tau \in \mathrm{Aut}(T)$), it suffices to consider the case $e_{35} \in P$ and $e_{13} \notin P$. Then $e_{45} \in N(P)$ since $[e_{45}, T] \leq P$, and thus $N(P) \geq \langle A_1A_2, e_{45} \rangle$ has order $\geq 2^9$. If $|P| \leq 2^6$, then $|N(P)/P| \geq 2^3$, so $\mathrm{rk}(P/\mathrm{Fr}(P)) \geq 6$ by (4), $P$ is elementary abelian of rank 6, and $P = A_2$ by Lemma 6.1. But since $N(A_2)/A_2 = T/A_2$ has order 16, $A_2$ is not critical by (4).

Thus $|P| = 2^7$, $P$ has index 2 in $A_1A_2$, and hence $P = \langle R_0, e_{23}x, e_{34}y, e_{35} \rangle$ for some $x, y \in \langle e_{13} \rangle$. Also, since $\mathrm{rk}([e_{13}, P/\mathrm{Fr}(P)]) \geq 2$ by (3) again, $[e_{13}, P] = [e_{13}, T] = \langle e_{14}, e_{15} \rangle$ and $\langle e_{14}, e_{15} \rangle \cap \mathrm{Fr}(P) = 1$. Thus $(e_{13}e_{34})^2 = e_{14} \notin \mathrm{Fr}(P)$, implying $y = 1$. Since $e_{12}Pe_{12}^{-1} = \langle R_0, e_{23}e_{13}x, e_{34}, e_{35} \rangle$, we can now assume up to conjugacy that $P = \langle R_0, e_{23}, e_{34}, e_{35} \rangle$. In this case, $Z(P) = R_0$, $[e_{13}, P] \leq Z(P)$ and $[e_{13}, Z(P)] = 1$, and $P$ is not critical by Lemma 3.4 applied with $\Theta = Z(P)$. □

The following lemma will be used when determining the normal critical subgroups of index two in $T$. We formulate it here in a more general form, so it can also be applied in the next section.

**Lemma 6.4.** *Assume $S = \langle g_1, g_2, g_3, g_4 \rangle$ is a group of order $2^7$, with center $Z \overset{\text{def}}{=} Z(S) = \mathrm{Fr}(S) = \langle z_1, z_2, z_3 \rangle \cong C_2^3$, satisfying the relations $g_i^2 = 1$ ($i = 1, 2, 3, 4$), $[g_i, g_{i+1}] = z_i$ ($i = 1, 2, 3$), and $[g_i, g_j] = 1$ when $|i - j| \geq 2$. Consider the subgroups*

$$U_i = \langle Z, g_j \mid j \neq i \rangle \ (1 \leq i \leq 4), \quad U_{13} = \langle Z, g_1g_3, g_2, g_4 \rangle, \quad U_{24} = \langle Z, g_1, g_3, g_2g_4 \rangle.$$

*Let $P \lhd S$ be a subgroup of index two, not equal to $U_i$ for any $i = 1, 2, 3, 4$. Then either $\mathrm{Fr}(P) = Z$; or $P = U_{13}$ or $U_{24}$, $\mathrm{Fr}(P) = \langle z_1x, z_3y \rangle$ for some $x, y \in \langle z_2 \rangle$, and $Z(P) = Z$.*

*Proof.* If $g_1 \in P$, then $g_2a \in P$ for some $a \in \langle g_3, g_4 \rangle$ (since $P \neq U_2$), and $[g_1, g_2a] = z_1 \in \mathrm{Fr}(P)$. If $g_2 \in P$, then $g_1a \in P$ for some $a \in \langle g_3, g_4 \rangle$ ($P \neq U_1$), and $[g_2, g_1a] \in \{z_1, z_1z_2\}$ is in $\mathrm{Fr}(P)$. If $g_1g_2 \in P$, then $(g_1g_2)^2 = z_1 \in \mathrm{Fr}(P)$. Since $[S{:}P] = 2$, one of the elements $g_1, g_2, g_1g_2$ is in $P$, so in all cases, $z_1x \in \mathrm{Fr}(P)$ for some $x \in \langle z_2 \rangle$. By a similar argument, $z_3y \in \mathrm{Fr}(P)$ for some $y \in \langle z_2 \rangle$.

Thus $\mathrm{Fr}(P) = Z$ whenever $z_2 \in \mathrm{Fr}(P)$. If neither $g_2$ nor $g_3$ is in $P$, then $g_2g_3 \in P$, and $z_2 = (g_2g_3)^2 \in \mathrm{Fr}(P)$. If $g_2 \in P$ and $g_3x \in P$ for $x \in \langle g_4 \rangle$, then $z_2 = [g_2, g_3x] \in \mathrm{Fr}(P)$. If $g_2 \in P$ and neither $g_3$ nor $g_3g_4$ is in $P$, then $g_4 \in P$, and hence $P = \langle Z, g_2, g_4, g_1g_3 \rangle = U_{13}$ (since $P \neq U_3$). By a similar argument, if $g_3 \in P$, then either $z_2 \in \mathrm{Fr}(P)$ or $P = U_{24}$. Thus $\mathrm{Fr}(P) = Z$ with these two exceptions.

If $P = U_{13} = \langle Z, g_2, g_4, g_1g_3 \rangle$, then clearly $Z(P) \geq Z$. If $g = g_1^i g_2^j g_3^k g_4^\ell x \in Z(P)$, where $i, j, k, \ell = 0, 1$ and $x \in Z$, then $i = k = 0$ since $[g, g_2] = 1$, and $j = \ell = 0$ since $[g, g_1g_3] = 1$. Thus $g = x \in Z$, and so $Z(P) = Z$. The proof that $Z(U_{24}) = Z$ is similar. □

We are now ready to handle the subgroups of $T$ which contain $Z_2$ and have index 2 in their normalizer. This requires some detailed case-by-case checks.

**Proposition 6.5.** *The only possible critical subgroups of* $T = UT_5(2)$ *are the subgroups* $Q_i$ $(i = 1, 2, 3, 4)$ *of index 2.*

*Proof.* Let $P$ be a critical subgroup of $T$. By Lemma 6.3, $|N(P)/P| = 2$. By Lemma 3.4,

$$g \in N(P)\smallsetminus P, \quad \Theta \text{ char } P \quad \Longrightarrow \quad [g,P] \not\leq \Theta{\cdot}\mathrm{Fr}(P) \quad \text{or} \quad [g,\Theta] \not\leq \mathrm{Fr}(P). \tag{6}$$

**Case 1:** Assume $P \lhd T$. Thus $P$ has index 2 in $T$, and $P \geq [T,T] = T'$. Also, $e_{15} = [e_{13}, e_{35}] \in \mathrm{Fr}(P)$,

$$e_{14} = [e_{34}, e_{13}] = [e_{12}, e_{24}] = [e_{12}e_{34}, e_{24}] \in \mathrm{Fr}(P)$$

since one of the elements $e_{12}$, $e_{34}$, or $e_{12}e_{34}$ is in $P$, and similarly $e_{25} \in \mathrm{Fr}(P)$. Thus $\mathrm{Fr}(P) \geq Z_2$. For any $g \in T\smallsetminus P$, $[g,P] \leq T'$ and $[g,T'] \leq [T,T'] = Z_2 \leq \mathrm{Fr}(P)$ (2). Hence by (6), $T'$ is not characteristic in $P$.

We must show $P = Q_i$ for some $i = 1, 2, 3, 4$. Assume otherwise: assume $P$ is not one of the $Q_i$. Consider the group $S$ of Lemma 6.4. There is an epimorphism $\varphi\colon T \longrightarrow S$, defined by $\varphi(e_{i,i+1}) = g_i$ and $\varphi(e_{i,i+2}) = z_i$, with $\mathrm{Ker}(\varphi) = Z_2$. Since $P \neq Q_i$ for each $i$, $\varphi(P)$ satisfies the hypotheses of the lemma. So either $\mathrm{Fr}(P) = \varphi^{-1}(Z(S)) = T'$ and hence $T'$ is characteristic in $P$; or $P = \langle e_{12}, e_{34}, e_{23}e_{45}\rangle$ or $\langle e_{12}e_{34}, e_{23}, e_{45}\rangle$.

By Lemma 6.4 again, in both of these last two cases, $\mathrm{Fr}(P) = \langle Z_2, e_{13}x, e_{35}y\rangle$ for some $x, y \in \langle e_{24}\rangle$, and $Z(P/Z_2) = \varphi^{-1}(Z(S))/Z_2 = T'/Z_2$. Thus $Z(\mathrm{Fr}(P)) = Z_2$, and hence $Z_2$ and $T'$ are both characteristic in $P$. But we have already seen that this implies $P$ cannot be critical.

**Case 2:** Now assume $P \ntriangleleft T$. Thus $P \ngeq T' = [T,T]$, while $P \geq Z_2$ by Lemma 6.2. Since $[T,T'] = Z_2 \leq P$ by (2), $T' \leq N_T(P)$. So we can always choose $g \in N_{T'}(P)\smallsetminus P$, in which case $[g,P] \leq [T',T] = Z_2$. By (6), applied with $\Theta = 1$ or $\Theta = Z_2(P) \geq Z_2$,

$$\mathrm{Fr}(P) \ngeq Z_2, \qquad \text{and} \qquad [g, Z_2(P)] \not\leq \mathrm{Fr}(P) \quad \text{for } g \in N_{T'}(P)\smallsetminus P \tag{7}$$

We next claim that

$$\{e_{13}, e_{35}\} \cap P \neq \varnothing. \tag{8}$$

Assume otherwise: assume $e_{13}, e_{35} \notin P$. Then both of these are in $N(P)$, and since $|N(P)/P| = 2$, $e_{13}e_{35} \in P$. By Lemma 3.5, there is $\alpha \in \mathrm{Aut}(P)$ of odd order, and $x \in [e_{13}, P]$, such that $x \notin \mathrm{Fr}(P)$ and $[e_{13}, \alpha(x)] \notin \mathrm{Fr}(P)$. Set $y = \alpha(x)$. Since $x \in \{e_{14}, e_{14}e_{15}\}$ has order two, $y^2 = 1$. Also, $[e_{13}, y] \in \{e_{14}, e_{14}e_{15}\}$, and $[e_{13}e_{35}, y] \notin \{e_{14}, e_{14}e_{15}\}$ since $e_{14} \notin \mathrm{Fr}(P)$.

Set $Q = U_1U_2 = \langle e_{12}, e_{13}, e_{14}, e_{15}, e_{25}, e_{35}, e_{45}\rangle$. By the commutator relations (1), $[e_{13}, y] \in \{e_{14}, e_{14}e_{15}\}$ implies $y \equiv e_{34} \pmod{\langle Q, e_{23}, e_{24}\rangle}$. Combined with the condition $[e_{13}e_{35}, y] \notin \{e_{14}, e_{14}e_{15}\}$, we have $y \equiv e_{23}e_{34}$ (mod $\langle Q, e_{24}\rangle$. But then the class $yQ$ has order four in $T/Q \cong D_8$, which contradicts the assumption $y^2 = 1$. This finishes the proof of (8).

Set $T_0 = \langle T', e_{12}, e_{45}\rangle$. We want to apply Lemma 1.9 to identify subgroups of $S = T/Z_2$ of index two in their normalizer. To do this, we regard $T/Z_2$ as an extension

$$1 \longrightarrow \underset{=\langle e_{13}, e_{24}, e_{35}, e_{12}, e_{45}\rangle}{T_0/Z_2} \longrightarrow T/Z_2 \longrightarrow \underset{=\langle e_{23}, e_{34}\rangle}{T/T_0} \longrightarrow 1,$$

where $S_0 = T_0/Z_2 \cong C_2^5$ and $S/S_0 \cong C_2^2$. Using the notation of Lemma 1.9 (but with $P$ a subgroup of $T$ and not of $S = T/Z_2$), we set $P_0 = P \cap T_0$.

Recall, in the notation of Lemma 1.9, that $m$ is the number of classes $xT_0 \in T/T_0$ such that $xT_0 \neq T_0$ and $[x, T_0] \leq P_0$. Since $[e_{23}, T_0/Z_2] = \langle e_{13}Z_2\rangle$, $[e_{34}, T_0/Z_2] = \langle e_{35}Z_2\rangle$, and $[e_{23}e_{34}, T_0/Z_2] = \langle e_{13}Z_2, e_{35}Z_2\rangle$, we see that

$$m = 2^k - 1 \qquad \text{where} \qquad k = \bigl|\{e_{13}, e_{35}\} \cap P\bigr|. \tag{9}$$

Thus (8) implies $m \geq 1$.

By Lemma 1.9, we must consider the following cases, where we omit those where $m = 0$. In all cases, since $N_T(P) \geq T'$ and $P \ngeq T'$, $[T'{:}P \cap T'] = 2$. Recall that we always choose $g \in N_{T'}(P){\smallsetminus}P$.

**(b) $\mathbf{rk}(T_0/P_0) = 1$, $|P/P_0| = 2$, $m = 1$, and $P_0 \ntriangleleft S$.** Then $\{e_{13}, e_{35}\} \nsubseteq P$ by (9), and we can choose $g \in \{e_{13}, e_{35}\}$ in $N_T(P){\smallsetminus}P$. Since $P_0$ has index two in $T_0$ and does not contain $g$, there are elements $x, y, z \in \langle g\rangle$ such that $e_{24}x, e_{12}y, e_{45}z \in P$. Thus $e_{14} = [e_{12}y, e_{24}x]$ and $e_{25} = [e_{24}x, e_{45}z]$ are both in $\mathrm{Fr}(P)$, so $Z_2 \leq \mathrm{Fr}(P)$, which contradicts (7).

**(c) $\mathbf{rk}(T_0/P_0) = 1$, $|P/P_0| = 2$, $m = 3$, and $P_0 \triangleleft S$.** Then $P_0 \geq \langle Z_2, e_{13}, e_{35}\rangle$ by (9). Hence $e_{24} \notin P$ ($P \ngeq T'$), and we take $g = e_{24}$. For $x \in T_0$, $(e_{23}e_{34}x)^2 \equiv (e_{23}e_{34})^2 = e_{24} \pmod{[T, T_0] \leq P_0}$, so $(e_{23}e_{34}x)^2 \notin P_0$, and $e_{23}e_{34}x \notin P$. So up to symmetry, we can assume $PT_0 = \langle T_0, e_{23}\rangle$. Thus $P = \langle Z_2, e_{13}, e_{35}, e_{12}x, e_{45}y, e_{23}z\rangle$ for some $x, y, z \in \langle g\rangle$. In all cases, $Z(P) = \langle e_{15}\rangle$, $Z_2(P) \leq \langle T', e_{45}\rangle$, and $[e_{24}, Z_2(P)] \leq \langle e_{15}, e_{25}\rangle \leq \mathrm{Fr}(P)$. So this case is impossible by (7).

**(e) $\mathbf{rk}(T_0/P_0) = 2$, $|P/P_0| = 4$, and $m = 1$.** By (9), exactly one of the elements $e_{13}$ or $e_{35}$ is in $P_0$. Up to symmetry, we can assume $e_{13} \in P_0$ while $e_{35} \notin P_0$. Set $g = e_{35}$. Since $[e_{34}, T_0/Z_2] = \langle e_{35}Z_2\rangle$ is not in $P/Z_2$, and since $P_0/Z_2$ is invariant under the conjugation action of $e_{34}$ on $T_0/Z_2$ (since $|P/P_0| = 4$), $P_0/Z_2 \leq C_{T_0/Z_2}(e_{34}) = \langle e_{12}Z_2, T'/Z_2\rangle$. Also, $|P_0/Z_2| = 8$ since $|T_0/P_0| = 4$ and $|T_0/Z_2| = 32$, and so $P_0 = \langle Z_2, e_{13}, e_{24}x, e_{12}y\rangle$ for some $x, y \in \langle g\rangle$.

Now, $e_{15} = [e_{12}y, e_{25}] \in \mathrm{Fr}(P)$. By Lemma 3.5, there is $\alpha \in \mathrm{Aut}(P)$ and $r \in [e_{35}, P]$ such that $r \notin \mathrm{Fr}(P)$ and $[e_{35}, \alpha(r)] \notin \mathrm{Fr}(P)$. Set $s = \alpha(r)$. Since $[e_{35}, T] = \langle e_{25}, e_{15}\rangle$, these conditions imply $r, [e_{35}, s] \in \{e_{25}, e_{25}e_{15}\}$. Also, $s^2 = 1$ since $r^2 = 1$.

Set $H = E_{1234;45} = \langle A_2, e_{45}\rangle \leq C_T(e_{35})$. The condition on $[e_{35}, s]$ (together with (1)) implies $s = e_{23}v$ for $v \in \langle H, e_{12}, e_{13}\rangle$; and $v \in \langle H, e_{13}\rangle$ since $s^2 = 1$.

Set $K = \langle Z_2, e_{35}\rangle$. Since $|P/P_0| = 4$, and since $\langle P_0, e_{35}, e_{45}\rangle = T_0$, there is $w \in \langle e_{35}, e_{45}\rangle$ such that $e_{34}w \in P$. Then $[s, P]$ contains the elements

$$[s, e_{12}y] \in [e_{23}{\cdot}\langle e_{13}, H\rangle, e_{12}H] \in e_{13}H \qquad \text{and} \qquad [s, e_{34}w] = [e_{23}v, e_{34}w] \in e_{24}K,$$

where the last inclusion holds since $[v, e_{34}w] \in [\langle e_{13}, H\rangle, H] \leq K$ and $[T, w] \leq K$. Thus $|[s, P]| \geq 4$, which is impossible since $[s, P] = \alpha([r, P]) = \langle \alpha(e_{15})\rangle$.

This finishes the proof that $P$ is not critical when it is not normal. □

## 6.2 Automorphisms of critical subgroups

Recall that $\tau \in \mathrm{Aut}(T)$ denotes the transpose along the "back" diagonal composed with $A \mapsto A^{-1}$; i.e., the automorphism $\tau(e_{ij}) = e_{6-j,6-i}$. We claim there are also automorphisms $\theta_1, \psi_1 \in \mathrm{Aut}(T)$ such that

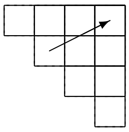
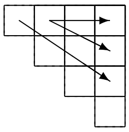
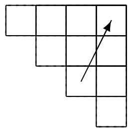
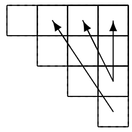
$$\theta_1(e_{23}) = e_{23}e_{15}\,, \qquad \psi_1(e_{12}) = e_{12}e_{35}\,, \quad \psi_1(e_{13}) = e_{13}e_{15}e_{25}\,;$$

and which send all other generators $e_{ij}$ to themselves. This is clear for $\theta_1$, since it has the form $\theta_1(g) = g{\cdot}\varphi(g)$ for some $\varphi \in \mathrm{Hom}(T, Z(T))$. By a similar argument, $\psi_1|_{Q_1}$

is an automorphism of $Q_1$, and it extends to an automorphism of $T$ if $\psi_1([e_{12}, g]) = [e_{12}e_{35}, \psi_1(g)]$ for all $g \in Q_1$. This is clear when $g = e_{ij}$ for $j \geq 4$ ($g = \psi_1(g)$ commutes with $e_{12}$ and $e_{35}$), and holds for the other two generators by direct calculation:

$$\psi_1([e_{12}, e_{13}]) = 1 = [e_{12}e_{35}, e_{13}e_{15}e_{25}] \quad \text{and} \quad \psi_1([e_{12}, e_{23}]) = e_{13}e_{15}e_{25} = [e_{12}e_{35}, e_{23}].$$

We also define $\theta_2, \psi_2 \in \mathrm{Aut}(T)$ by setting $\theta_2 = \tau\theta_1\tau^{-1}$ and $\psi_2 = \tau\psi_1\tau^{-1}$. It is helpful to visualize these automorphisms pictorially as follows:

$\theta_1 =$ , $\psi_1 =$ , $\theta_2 =$ , $\psi_2 =$ .

For each $\varphi \in \{\theta_i, \psi_i\}$ and each $i < j$, the arrows in the diagram for $\varphi$ starting in position $(i, j)$ point to the positions of the basis elements which occur in $e_{ij}^{-1}\varphi(e_{ij})$.

Let $\mathrm{Aut}^0(T) \leq \mathrm{Aut}(T)$ be the subgroup of automorphisms which send $A_1$ to itself, and set $\mathrm{Out}^0(T) = \mathrm{Aut}^0(T)/\mathrm{Inn}(T)$. By Lemma 6.1, each automorphism of $T$ either sends the $A_i$ to themselves or exchanges them, so $\mathrm{Aut}(T) = \mathrm{Aut}^0(T) \rtimes \langle\tau\rangle$.

For each $i = 1, 2, 3, 4$, $N_{GL_5(2)}(Q_i)$ is the group of all $A = (a_{jk}) \in GL_5(2)$ such that $a_{jk} = 0$ for all $j > k$ such that $(j, k) \neq (i+1, i)$. Thus $N_{GL_5(2)}(Q_i)/Q_i \cong GL_2(2) \cong \Sigma_3$, and is generated by the classes (mod $Q_i$) of $e_{i,i+1}$ and the permutation matrix for the transposition $(i\ i{+}1)$. We now define

$$\Delta_i = \mathrm{Out}_{GL_5(2)}(Q_i) = \langle c_{i,i+1}, [\sigma_{i,i+1}]\rangle \cong \Sigma_3\,, \tag{10}$$

where $\sigma_{i,i+1} \in \mathrm{Aut}(Q_i)$ is conjugation by that permutation matrix; i.e., the automorphism which exchanges the $i$-th and $(i{+}1)$-st rows and columns.

**Proposition 6.6.** (a) $\mathrm{Out}^0(T) = \langle[\theta_1], [\theta_2], [\psi_1], [\psi_2]\rangle \cong C_2^4$. *Hence* $|\mathrm{Out}(T)| = 2^5$.

(b) *For each* $i = 1, 2, 3, 4$, $\mathrm{Out}(Q_i) = O_2(\mathrm{Out}(Q_i)){\cdot}\Delta_i$, *and* $O_2(\mathrm{Out}(Q_i))$ *is elementary abelian. If* $\mathcal{F}$ *is a saturated fusion system over* $T$ *and* $Q_i$ *is* $\mathcal{F}$*-essential, then* $\mathrm{Out}_{\mathcal{F}}(Q_i) = [\varphi]\Delta_i[\varphi]^{-1}$ *for some* $\varphi \in O_2(\mathrm{Aut}(Q_i))$ *which extends to an automorphism of* $T$.

*Proof.* Set

$$R = E_{123;345} = A_1A_2 \qquad \text{and} \qquad R_0 = E_{12;45} = A_1 \cap A_2 = Z(R) = \mathrm{Fr}(R).$$

Let $\mathrm{Out}^0(R) \leq \mathrm{Out}(R)$ be the subgroup of classes of automorphisms which send $A_1$ to itself. In Steps 1 and 2, we describe $\mathrm{Out}^0(T)$, $\mathrm{Out}(Q_1)$, and $\mathrm{Out}(Q_4)$ by comparison with $\mathrm{Out}^0(R)$. Then in Step 3, we prove (b) for $Q_2$ and $Q_3$.

In Steps 2 and 3, it will be helpful to represent automorphisms of $A_1$ by matrices. So for each $\alpha \in \mathcal{A}_1$, $M(\alpha)$ will denote the matrix for $\alpha|_{A_1}$ with respect to the ordered basis

$$\mathbf{b} = \{e_{15}, e_{25}, e_{14}, e_{24}, e_{13}, e_{23}\}.$$

**Step 1:** Let $\widetilde{\kappa}$ be the homomorphism from $\mathrm{Aut}^0(R)$ to $\mathrm{Aut}(R/A_1) \times \mathrm{Aut}(R/A_2)$ induced by the projections of $R$ onto $R/A_2$ and $R/A_1$. Then $\mathrm{Inn}(R) \leq \mathrm{Ker}(\widetilde{\kappa})$, so $\widetilde{\kappa}$ induces a homomorphism $\kappa$ on $\mathrm{Out}^0(R)$. We claim the sequence

$$1 \longrightarrow O_2(\mathrm{Out}(R)) \xrightarrow{\text{incl}} \mathrm{Out}^0(R) \xrightarrow{\kappa} \mathrm{Aut}(R/A_1) \times \mathrm{Aut}(R/A_2) \longrightarrow 1 \tag{11}$$

is exact. Here, $\mathrm{Aut}(R/A_i) \cong \Sigma_3$ since $R/A_i \cong C_2^2$, and $\kappa$ is onto since its restriction to the subgroup $\mathrm{Out}_{GL_5(2)}(R) \cong \Sigma_3{\times}\Sigma_3$ is onto. So $O_2(\mathrm{Out}(R)) \leq \mathrm{Ker}(\kappa)$. Conversely, for each $\alpha \in \mathrm{Aut}(R)$ such that $[\alpha] \in \mathrm{Ker}(\kappa)$, $\alpha$ induces the identity on $R/(A_1 \cap A_2) = R/R_0$ where $R_0 = \mathrm{Fr}(R)$, and hence $\alpha \in O_2(\mathrm{Aut}(R))$ by Lemma 1.1. Thus (11) is exact.

If $\alpha \in \mathrm{Aut}(R)$ induces the identity on $R/R_0$, then it also restricts to the identity on $R_0$ — since $[e_{i3}, e_{3j}] = e_{ij}$ for $i = 1, 2$ and $j = 4, 5$ by (1). Hence each such $\alpha$ has the form $\alpha(g) = g{\cdot}\widehat{\alpha}(gR_0)$ for some map $\widehat{\alpha}$ from $R/R_0$ to $R_0$, and $\widehat{\alpha}$ is a homomorphism since $R_0 = Z(R)$. Thus $O_2(\mathrm{Aut}(R)) \cong \mathrm{Hom}(R/R_0, R_0) \cong C_2^{16}$. Since $\mathrm{Inn}(R) \cong R/R_0$ has rank 4, $O_2(\mathrm{Out}(R)) \cong C_2^{12}$.

Now, $R_0 = Z(R)$ is free as a module over $\mathbb{F}_2[T/R] = \mathbb{F}_2[\langle c_{12}, c_{45}\rangle]$. Also, $R$ is generated by the only subgroups of $T$ isomorphic to $C_2^6$ (Lemma 6.1), and hence is characteristic in any subgroup of $T$ which contains it. So if $P$ is any of the groups $T$, $Q_1$, or $Q_4$, then restriction to $R$ induces an isomorphism

$$\mathrm{Out}(P) \xrightarrow[\cong]{\mathrm{Res}_R} N_{\mathrm{Out}(R)}(\mathrm{Out}_P(R))/\mathrm{Out}_P(R). \tag{12}$$

by Corollary 1.3. When $P = Q_i$ for $i = 1$ or 4, then $\kappa$ sends $\mathrm{Out}_P(R) \cong C_2$ nontrivially to one of the factors $\mathrm{Aut}(R/A_1)$ or $\mathrm{Aut}(R/A_2)$ in the extension (11), and sends $\Delta_i$ isomorphically to the other factor. Thus

$$\kappa\bigl(N_{\mathrm{Out}(R)}(\mathrm{Out}_P(R))\bigr) = \kappa(\mathrm{Out}_P(R){\cdot}\Delta_i) \cong C_2 \times \Sigma_3,$$

and hence

$$\mathrm{Out}(Q_i) = O_2(\mathrm{Out}(Q_i)){\cdot}\Delta_i \qquad \text{where} \qquad O_2(\mathrm{Out}(Q_i)) \cong C_{O_2(\mathrm{Out}(R))}(\mathrm{Out}_P(R)).$$

In particular, $O_2(\mathrm{Out}(Q_i))$ is elementary abelian.

Assume $Q_i$ is $\mathcal{F}$-essential, and set $\Delta_i' = \mathrm{Out}_{\mathcal{F}}(Q_i)$. Then $\Delta_i' \cap O_2(\mathrm{Out}(Q_i)) = 1$ since $O_2(\Delta_i') = 1$, and $O_2(\mathrm{Out}(Q_i)){\cdot}\Delta_i' = \mathrm{Out}(Q_i)$ since otherwise $\Delta_i'$ would have order two. Hence by Proposition 1.8, $\Delta_i' = \varphi\Delta_i\varphi^{-1}$ for some $\varphi \in \mathrm{Out}(Q_i)$ which centralizes $\mathrm{Out}_T(Q_i)$. Since $Z(Q_i) \cong C_2^2$ is a free $\mathbb{F}_2[T/Q_i]$-module, $H^2(T/Q_i; Z(Q_i)) = 0$, and Lemma 1.2 implies that $\varphi$ extends to an automorphism of $T$.

**Step 2:** When $P = T$, (12) restricts to an isomorphism

$$\mathrm{Out}^0(T) \xrightarrow[\cong]{\mathrm{Res}_R} N_{\mathrm{Out}^0(R)}(\langle c_{12}, c_{45}\rangle)/\langle c_{12}, c_{45}\rangle \cong C_{O_2(\mathrm{Out}(R))}(\langle c_{12}, c_{45}\rangle),$$

where the last isomorphism follows using (11). We now prove point (a) by describing this centralizer explicitly. Write $O_2(\mathrm{Aut}(R)) = \mathcal{A}_1 \times \mathcal{A}_2$, where $\mathcal{A}_1 \cong \mathrm{Hom}(R/A_2, R_0)$ is the subgroup of automorphisms which are the identity on $A_2$ and on $R/A_2$, and $\mathcal{A}_2 \cong \mathrm{Hom}(R/A_1, R_0)$ is the subgroup of automorphisms which are the identity on $A_1$ and on $R/A_1$. Set

$$\widehat{\mathcal{A}}_1 = \mathcal{A}_1/\langle c_{34}, c_{35}\rangle \qquad \text{and} \qquad \widehat{\mathcal{A}}_2 = \mathcal{A}_2/\langle c_{13}, c_{23}\rangle;$$

thus $O_2(\mathrm{Out}(R)) = \widehat{\mathcal{A}}_1 \times \widehat{\mathcal{A}}_2$. The actions of $c_{12}$ and $c_{45}$ clearly preserve this decomposition, and hence $\mathrm{Res}_R$ induces an isomorphism

$$\mathrm{Out}^0(T) \cong C_{O_2(\mathrm{Out}(R))}(\langle c_{12}, c_{45}\rangle) = C_{\widehat{\mathcal{A}}_1}(\langle c_{12}, c_{45}\rangle) \times C_{\widehat{\mathcal{A}}_2}(\langle c_{12}, c_{45}\rangle). \tag{13}$$

Recall that $M(-)$ is the matrix for an automorphism of $A_1$ with respect to the basis $\mathbf{b}$ defined above. Thus

$$\bigl\{M(\alpha|_{A_1}) \bigm| \alpha \in \mathcal{A}_1\bigr\} = \left\{\left(\begin{smallmatrix} I & 0 & B \\ 0 & I & C \\ 0 & 0 & I \end{smallmatrix}\right) \,\middle|\, B, C \in M_2(\mathbb{F}_2)\right\}.$$

Write $\lambda(B, C) = \left(\begin{smallmatrix} I & 0 & B \\ 0 & I & C \\ 0 & 0 & I \end{smallmatrix}\right)$ for short; then $M(c_{34}) = \lambda(0, I)$ and $M(c_{35}) = \lambda(I, 0)$.

Now, $c_{45}$ and $c_{12}$ act on these matrices via conjugation by $\left(\begin{smallmatrix} I & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{smallmatrix}\right)$ and by $\left(\begin{smallmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{smallmatrix}\right)$, respectively, where $J = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Hence

$$[c_{45}, \lambda(B,C)] = \lambda(C,0) \qquad \text{and} \qquad [c_{12}, \lambda(B,C)] = \lambda(JBJ^{-1}{+}B, JCJ^{-1}{+}C).$$

From this it follows that $M$ induces an isomorphism

$$C_{\widehat{\mathcal{A}}_1}(\langle c_{12}, c_{45}\rangle) \xrightarrow{\ \cong\ } \bigl\{\lambda(B,0) \bigm| JBJ^{-1}{+}B \in \{0,I\}\bigr\} \big/ \bigl\langle \lambda(I,0)\bigr\rangle.$$

Also, $J\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)J^{-1}{+}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} c & a+c+d \\ 0 & c \end{smallmatrix}\right) \in \{0,I\}$ if and only if $a+c+d=0$. Since $M(\theta_1|_R) = \lambda(\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), 0)$ and $M(\psi_1|_R) = \lambda(\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right), 0)$, this proves that

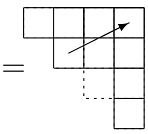
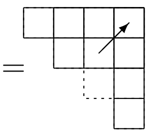
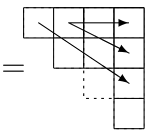
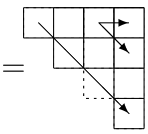
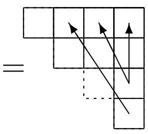
$$C_{\widehat{\mathcal{A}}_1}(\langle c_{12}, c_{45}\rangle) = \langle \theta_1|_R, \psi_1|_R\rangle \cong C_2^2.$$

After combining this with the corresponding argument for $\widehat{\mathcal{A}}_2$, and with (13), we have now proven that $\mathrm{Out}^0(T) \cong C_2^4$ with basis $\{[\theta_1], [\psi_1], [\theta_2], [\psi_2]\}$.

**Step 3:** It remains to prove (b) for $Q_2$ and $Q_3$. We do this for $Q_3$, and the result for $Q_2$ then follows via conjugation by $\tau$. Recall that $\sigma_{34} \in \mathrm{Aut}(Q_3)$ is the automorphism which switches the third and fourth rows and columns. We define automorphisms $\beta_1, \ldots, \beta_5 \in \mathrm{Aut}(Q_3)$ as follows:

$$\beta_1 = \theta_1|_{Q_3}, \quad \beta_2 = \sigma_{34}\beta_1\sigma_{34}, \quad \beta_3 = \psi_1|_{Q_3}, \quad \beta_4 = \sigma_{34}\beta_3\sigma_{34}, \quad \text{and} \quad \beta_5 = \psi_2|_{Q_3}.$$

These can be described pictorially as follows:

$\beta_1 =$  $\beta_2 =$  $\beta_3 =$  $\beta_4 =$  $\beta_5 =$  .

Thus, for example, $\beta_4(e_{12}) = e_{12}e_{45}$, $\beta_4(e_{14}) = e_{14}e_{15}e_{25}$, and $\beta_4$ sends all of the other generators $e_{ij}$ to themselves. We will show that $O_2(\mathrm{Out}(Q_3)) \cong C_2^5$ with the classes of these elements as basis.

By Lemma 1.2, there is a short exact sequence

$$1 \to H^1(Q_3/A_1; A_1) \longrightarrow \mathrm{Out}(Q_3) \xrightarrow{\ \mathrm{Res}_{A_1}\ } N_{\mathrm{Aut}(A_1)}(\mathrm{Aut}_{Q_3}(A_1))/\mathrm{Aut}_{Q_3}(A_1) \to 1. \tag{14}$$

In terms of matrices, we are looking for the centralizer in $GL_6(2)$ of

$$M(c_{12}) = \left(\begin{smallmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{smallmatrix}\right), \qquad M(c_{35}) = \left(\begin{smallmatrix} I & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{smallmatrix}\right), \quad \text{and} \quad M(c_{45}) = \left(\begin{smallmatrix} I & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{smallmatrix}\right),$$

where $J = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ as before. The centralizer in $\mathrm{Aut}(A_1)$ of $\langle c_{35}, c_{45}\rangle$ is the group of those $\alpha$ such that $M(\alpha) = \left(\begin{smallmatrix} A & B & C \\ 0 & A & 0 \\ 0 & 0 & A \end{smallmatrix}\right)$ for some $A, B, C \in M_2(\mathbb{F}_2)$ with $A$ invertible; and such a matrix commutes with $M(c_{12})$ exactly when $A$, $B$, and $C$ all commute with $J$. Since a matrix in $M_2(\mathbb{F}_2)$ commutes with $J$ if and only if it has the form $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$ for some $a, b \in \mathbb{F}_2$, this proves that

$$M\bigl(C_{\mathrm{Aut}(A_1)}(\mathrm{Aut}_{Q_3}(A_1))\bigr) = \Bigl\langle M(c_{12}), M(c_{35}), M(c_{45}), \left(\begin{smallmatrix} I & Y & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{smallmatrix}\right), \left(\begin{smallmatrix} I & 0 & Y \\ 0 & I & 0 \\ 0 & 0 & I \end{smallmatrix}\right) \Bigr\rangle$$

where $Y = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Hence

$$C_{\mathrm{Aut}(A_1)}(\mathrm{Aut}_{Q_3}(A_1)) = \langle c_{12}, c_{35}, c_{45}, \beta_1|_{A_1}, \beta_2|_{A_1}\rangle \cong C_2^5. \tag{15}$$

So $C_{\mathrm{Aut}(A_1)}(\mathrm{Aut}_{Q_3}(A_1))/\mathrm{Aut}_{Q_3}(A_1)$ is a group of order 4 generated by the classes of $\beta_1|_{A_1}$ and $\beta_2|_{A_1}$.

By (1), for $g \in Q_3 \smallsetminus A_1$, $[g, A_1] = \langle e_{15}, e_{25}\rangle \cong C_2^2$ if $g \in \langle A_1, e_{35}, e_{45}\rangle$, while $[g, A_1] \cong C_2^3$ otherwise. Hence each $\beta \in \mathrm{Aut}(Q_3)$ leaves the subgroup $\langle A_1, e_{35}, e_{45}\rangle$ invariant.

The group of automorphisms of $Q_3/A_1 \cong C_2^3$ which leave $\langle e_{35}A_1, e_{45}A_1\rangle$ invariant is isomorphic to $\Sigma_4$, and is generated by the actions of $\beta_4$, $\beta_3$, and $\Delta_3$ on $Q_3/A_1$. This, together with (15) shows that

$$N_{\mathrm{Aut}(A_1)}(\mathrm{Aut}_{Q_3}(A_1))/\mathrm{Aut}_{Q_3}(A_1) = \mathrm{Res}_{A_1}\big(\langle[\beta_1],[\beta_2],[\beta_3],[\beta_4],\Delta_3\rangle\big) \cong C_2^4 \rtimes \Sigma_3, \quad (16)$$

where the $\beta_i|_{A_1}$ generate an elementary abelian subgroup since their matrices all have the form $\left(\begin{smallmatrix} I & B & C \\ 0 & I & 0 \\ 0 & 0 & I \end{smallmatrix}\right)$ for some $B, C \in M_2(\mathbb{F}_2)$.

We next claim that $H^1(Q_3/A_1; A_1) \cong C_2$. To see this, set $V = A_1 \times \langle \widehat{e}_1, \widehat{e}_2\rangle \cong C_2^8$, regarded as an $\mathbb{F}_2[Q_3/A_1]$-module (with $A_1$ as submodule) by setting $e_{35}(\widehat{e}_i) = \widehat{e}_i{\cdot}e_{i4}$, $e_{45}(\widehat{e}_i) = \widehat{e}_i{\cdot}e_{i3}$, $e_{12}(\widehat{e}_1) = \widehat{e}_1$, and $e_{12}(\widehat{e}_2) = \widehat{e}_1\widehat{e}_2$. This module is free (the $Q_3/A_1$-orbit of $\widehat{e}_2$ is a basis), and hence is cohomologically trivial. So the exact sequence in cohomology for the extension $1 \to A_1 \to V \to V/A_1 \to 1$ takes the form

$$\underset{=\langle e_{15}\rangle \cong C_2}{H^0(Q_3/A_1; V)} \longrightarrow \underset{=\langle \widehat{e}_1A_1\rangle \cong C_2}{H^0(Q_3/A_1; V/A_1)} \xrightarrow{\ \delta\ } H^1(Q_3/A_1; A_1) \longrightarrow \underset{=0}{H^1(Q_3/A_1; V)}.$$

Thus $H^1(Q_3/A_1; A_1) \cong C_2$ is generated by $\delta(\widehat{e}_1A_1)$, which is represented by the cocycle which sends $g \in Q_3/A_1$ to $g(\widehat{e}_1)\widehat{e}_1^{-1}$. This induces an automorphism $\beta \in \mathrm{Aut}(Q_3)$ such that $\beta|_{\langle A_1, e_{12}\rangle} = \mathrm{Id}$, $\beta(e_{35}) = e_{35}e_{14}$, and $\beta(e_{45}) = e_{45}e_{13}$. By inspection, $\beta = \beta_5 c_{13}$, and this finishes the proof that $\mathrm{Ker}(\mathrm{Res}_{A_1}) = \langle[\beta_5]\rangle$.

Upon combining this with (14) and (16), we have now shown that $\mathrm{Out}(Q_3)$ is generated by the $[\beta_i]$ and $\Delta_3$. Also, $\langle[\beta_5]\rangle = \mathrm{Ker}(\mathrm{Res}_{A_1})$ is normal in $\mathrm{Out}(Q_3)$, and hence central. The subgroup of elements in $\mathrm{Out}(Q_3)$ which leave invariant the subgroup $U_2$ contains $[\beta_1]$, $[\beta_2]$, $[\beta_4]$, $[\beta_3]$, and $\Delta_3$, but not $[\beta_5]$. Thus $\langle[\beta_5]\rangle$ splits off as a direct factor in $\mathrm{Out}(Q_3)$. So by (16), $O_2(\mathrm{Out}(Q_3)) \cong C_2^5$ with the $[\beta_i]$ as basis.

Assume $Q_3$ is $\mathcal{F}$-essential, and set $\Delta_3' = \mathrm{Out}_{\mathcal{F}}(Q_3)$. Then $|\Delta_3'| = 2n$ for some odd $n > 1$ by the Sylow axiom, and $n = 3$ since it divides $|\mathrm{Out}(Q_3)|$. Also, $\Delta_3' \cap O_2(\mathrm{Out}(Q_3)) = 1$, and hence $O_2(\mathrm{Out}(Q_3)){\cdot}\Delta_3' = \mathrm{Out}(Q_3) = O_2(\mathrm{Out}(Q_3)){\cdot}\Delta_3$. By Proposition 1.8, $\Delta_3' = [\beta]\Delta_3[\beta]^{-1}$ for some $\beta \in \mathrm{Aut}_{\mathcal{F}}(Q_3)$ which commutes with $c_{34}$ in $\mathrm{Out}_{\mathcal{F}}(Q_3)$. Since $c_{34}\beta_2c_{34}^{-1} \equiv \beta_1\beta_2$ and $c_{34}\beta_4c_{34}^{-1} \equiv \beta_4\beta_3 \pmod{\mathrm{Inn}(Q_3)}$, we have

$$[\beta] \in C_{\mathrm{Out}(Q_3)}(\langle c_{34}\rangle) = \langle[\beta_1],[\beta_3],[\beta_5],c_{34}\rangle.$$

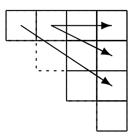
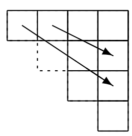
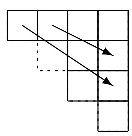
All of these extend to automorphisms of $T$ by the definitions at the beginning of Step 3, and this finishes the proof of (b) for $Q_3$. □

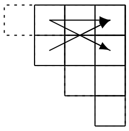
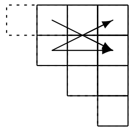
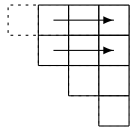
The following computations will also be needed later.

**Lemma 6.7.** *The following commutativity relations hold:*

$$[\psi_1|_{Q_2}, \Delta_2] = 1 \text{ in } \mathrm{Out}(Q_2) \qquad [(\theta_1\psi_1)|_{Q_1}, \Delta_1] = 1 \text{ in } \mathrm{Out}(Q_1)$$
$$[\psi_2|_{Q_3}, \Delta_3] = 1 \text{ in } \mathrm{Out}(Q_3) \qquad [(\theta_2\psi_2)|_{Q_4}, \Delta_4] = 1 \text{ in } \mathrm{Out}(Q_4)\ .$$

*Proof.* When $\varphi \in \{\theta_1, \theta_2, \psi_1, \psi_2\}$, $\varphi c_g \varphi^{-1} = c_{\varphi(g)}$ and $\varphi(g)g^{-1} \in Q_i$ for each $g \in T$ and each $i = 1, 2, 3, 4$, and thus $[\varphi|_{Q_i}, c_{i,i+1}] = 1$ in $\mathrm{Out}(Q_i)$. So we need only check the commutators with $\sigma_{i,i+1}$ (see (10)). This can be done by direct computation, but can also be seen using the pictorial description of these automorphisms. For example,

$\psi_1|_{Q_2} =$  $=$  $\circ\, c_{35}$, $\qquad \sigma_{23}(\psi_1|_{Q_2})\sigma_{23}^{-1} =$  $\circ\, c_{25}$, and so $[\psi_1, \sigma_{23}] = c_{35}c_{25} \in \mathrm{Inn}(Q_2)$. Similarly, in $\mathrm{Aut}(Q_1)$,

$\theta_1\psi_1|_{Q_1} =$  , $\sigma_{12}(\theta_1\psi_1|_{Q_1})\sigma_{12}^{-1} =$  , $c_{35} =$  ,

and so $[\theta_1\psi_1|_{Q_1}, \sigma_{12}] = c_{35}$. The remaining cases follow via conjugation with the automorphism $\tau$. □

## 6.3 Fusion systems over $T = UT_5(2)$

We are now ready to describe the nonconstrained saturated fusion systems over $T$. We begin by looking at automorphisms of $Q_2$ and $Q_3$ in such a fusion system.

**Proposition 6.8.** *Let $\mathcal{F}$ be a nonconstrained saturated fusion system over $T$. Then $Q_2$ and $Q_3$ are both $\mathcal{F}$-essential. Also, $\mathcal{F}$ is isomorphic to a fusion system $\mathcal{F}^*$ over $T$ such that $\mathrm{Out}_{\mathcal{F}^*}(Q_2) = \Delta_2$ and $\mathrm{Out}_{\mathcal{F}^*}(Q_3) = \Delta_3$.*

*Proof.* By Proposition 6.6(a), $\mathrm{Out}(T)$ is a 2-group, and hence $\mathrm{Out}_{\mathcal{F}}(T) = 1$. So if $Q_3$ is not $\mathcal{F}$-essential, then by Proposition 6.5, $\mathcal{F}$ is generated by restrictions of automorphisms of $Q_1, Q_2, Q_4$, all of which send $A_2$ to itself. Hence each morphism in $\mathcal{F}$ extends to a morphism between subgroups containing $A_2$ which sends $A_2$ to itself, and so $A_2$ is normal in $\mathcal{F}$. But $A_2$ is centric in $T$, and so this contradicts the assumption that $\mathcal{F}$ is nonconstrained. Thus $Q_3$ is $\mathcal{F}$-essential; and by a similar argument, $Q_2$ is also $\mathcal{F}$-essential.

By Proposition 6.6(b), $\mathrm{Out}_{\mathcal{F}}(Q_3) = (\varphi|_{Q_3})\Delta_3(\varphi|_{Q_3})^{-1}$ for some $\varphi \in \mathrm{Aut}(T)$, and $\varphi \in \mathrm{Aut}^0(T)$ since it leaves $Q_3$ invariant. So upon replacing $\mathcal{F}$ by $\varphi^{-1}\mathcal{F}\varphi$, we can assume $\mathrm{Out}_{\mathcal{F}}(Q_3) = \Delta_3$. Then, by Proposition 6.6(b) again, $\mathrm{Out}_{\mathcal{F}}(Q_2) = (\psi|_{Q_2})\Delta_2(\psi|_{Q_2})^{-1}$ for some $\psi \in \mathrm{Aut}^0(T)$. Since $\theta_1|_{Q_2} = \mathrm{Id}$ and $\psi_1|_{Q_2}$ centralizes $\Delta_2$ (Lemma 6.7), we can assume $\psi \in \langle\theta_2, \psi_2\rangle$. In particular, $(\psi|_{Q_3})\Delta_3(\psi|_{Q_3})^{-1} = \Delta_3$ by Lemma 6.7 again (and since $\theta_2|_{Q_3} = \mathrm{Id}$). So if we set $\mathcal{F}^* = \psi^{-1}\mathcal{F}\psi$, then $\mathrm{Out}_{\mathcal{F}^*}(Q_2) = \Delta_2$ and $\mathrm{Out}_{\mathcal{F}^*}(Q_3) = \Delta_3$. □

We now study how the automorphisms of $Q_1$ and $Q_4$ fit with those of $Q_2$ and $Q_3$. In the following proposition, $3\Sigma_6$ denotes a nonsplit extension with kernel of order 3 and quotient group $\Sigma_6$.

**Proposition 6.9.** *Fix a nonconstrained saturated fusion system $\mathcal{F}$ over $T$, and assume $\mathrm{Out}_{\mathcal{F}}(Q_2) = \Delta_2$ and $\mathrm{Out}_{\mathcal{F}}(Q_3) = \Delta_3$. Then $Q_1$ and $Q_4$ are both $\mathcal{F}$-essential. Also, for each pair $(i,j) = (1,2)$ or $(4,1)$, either*

- $\mathrm{Out}_{\mathcal{F}}(Q_i) = \Delta_i$ *and* $\mathrm{Aut}_{\mathcal{F}}(A_j) \cong \Sigma_3 \times GL_3(2)$*; or*
- $\mathrm{Out}_{\mathcal{F}}(Q_i) = (\theta_j\psi_j)\Delta_i(\theta_j\psi_j)^{-1}$ *and* $\mathrm{Aut}_{\mathcal{F}}(A_j) \cong 3\Sigma_6$.

*Proof.* By Proposition 6.6(a), $\mathrm{Out}(T)$ is a 2-group, and hence $\mathrm{Out}_{\mathcal{F}}(T) = 1$. So by Proposition 6.5, $\mathcal{F}$ is generated by $\mathrm{Inn}(T)$ together with $\mathrm{Aut}_{\mathcal{F}}(Q_i)$ for $i = 1, 2, 3, 4$.

If neither $Q_1$ nor $Q_4$ is $\mathcal{F}$-essential, then $\mathcal{F}$ is generated by $\mathrm{Inn}(T)$ together with $\Delta_2$, and $\Delta_3$, all of which leave $U_1$ and $U_2$ invariant. Thus $U_1$ and $U_2$ would both be normal in $\mathcal{F}$, which contradicts our assumption that $\mathcal{F}$ is nonconstrained. So $Q_1$ or $Q_4$ is $\mathcal{F}$-essential.

For each $i = 1, 4$, if $Q_i$ is $\mathcal{F}$-essential, then by Proposition 6.6(b), $\mathrm{Aut}_{\mathcal{F}}(Q_i) = \varphi_i \Delta_i \varphi_i^{-1}$ for some $\varphi_i \in \mathrm{Aut}^0(T)$. (We drop "restricted to $Q_i$" to simplify the notation.) Since $\theta_1\psi_1$ commutes with $\Delta_1$ in $\mathrm{Out}(Q_1)$ by Lemma 6.7, we can assume $\varphi_1 \in \langle \theta_1, \theta_2, \psi_2 \rangle$. Similarly, we can assume $\varphi_4 \in \langle \theta_1, \theta_2, \psi_1 \rangle$. Set

$$\sigma_{12}^* = \varphi_1 \sigma_{12} \varphi_1^{-1} \qquad \text{and} \qquad \sigma_{45}^* = \varphi_4 \sigma_{45} \varphi_4^{-1},$$

so that $\mathrm{Out}_{\mathcal{F}}(Q_1) = \langle c_{12}, \sigma_{12}^* \rangle$ and $\mathrm{Out}_{\mathcal{F}}(Q_4) = \langle c_{45}, \sigma_{45}^* \rangle$.

In Steps 1 and 2, when $\alpha \in \mathrm{Aut}(A_1)$, we again let $M(\alpha) \in GL_6(2)$ be its matrix with respect to the ordered basis $\{e_{15}, e_{25}, e_{14}, e_{24}, e_{13}, e_{23}\}$.

**Step 1** We first prove that if $Q_1$ is $\mathcal{F}$-essential, then $\varphi_1 \in \langle \theta_2, \psi_2 \rangle$; while if $Q_4$ is $\mathcal{F}$-essential, then $\varphi_4 \in \langle \theta_1, \psi_1 \rangle$.

Assume $Q_1$ is $\mathcal{F}$-essential. Fix $X \in M_2(\mathbb{F}_2)$ such that $M(\varphi_1|_{A_1}) = \left(\begin{smallmatrix} I & 0 & X \\ 0 & I & 0 \\ 0 & 0 & I \end{smallmatrix}\right)$. Thus $X = 0$ if $\varphi_1 \in \langle \theta_2, \psi_2 \rangle$, and $X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ otherwise. Set $W = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then

$$M(\sigma_{34}|_{A_1}) = \left(\begin{smallmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{smallmatrix}\right) \quad \text{and} \quad M(\sigma_{12}^*|_{A_1}) = \left(\begin{smallmatrix} I & 0 & X \\ 0 & I & 0 \\ 0 & 0 & I \end{smallmatrix}\right) \left(\begin{smallmatrix} W & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & W \end{smallmatrix}\right) \left(\begin{smallmatrix} I & 0 & X \\ 0 & I & 0 \\ 0 & 0 & I \end{smallmatrix}\right) = \left(\begin{smallmatrix} W & 0 & Y \\ 0 & W & 0 \\ 0 & 0 & W \end{smallmatrix}\right),$$

where $\sigma_{34} \in \mathrm{Aut}_{\mathcal{F}}(Q_3)$ and $\sigma_{12}^* \in \mathrm{Aut}_{\mathcal{F}}(Q_1)$, and $Y = XW + WX = 0$ or $I$. So if we set $Q_{13} = Q_1 \cap Q_3$ and $\alpha = [\sigma_{12}^*|_{Q_{13}}, \sigma_{34}|_{Q_{13}}] \in \mathrm{Aut}_{\mathcal{F}}(Q_{13})$, then

$$M(\alpha|_{A_1}) = \left[ \left(\begin{smallmatrix} W & 0 & Y \\ 0 & W & 0 \\ 0 & 0 & W \end{smallmatrix}\right), \left(\begin{smallmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{smallmatrix}\right) \right] = \left(\begin{smallmatrix} I & YW & YW \\ 0 & I & 0 \\ 0 & 0 & I \end{smallmatrix}\right).$$

Thus $\alpha$ induces the identity on $\mathrm{Fr}(Q_{13}) = \langle e_{15}, e_{25} \rangle$ and on $A_1/\mathrm{Fr}(Q_{13})$, and induces the identity on $Q_{13}/A_1$ since $\varphi_1$ (and hence $\sigma_{12}^*$) does. Since these are characteristic subgroups of $Q_{13}$, $\alpha \in O_2(\mathrm{Aut}_{\mathcal{F}}(Q_{13})) \le \mathrm{Out}_T(Q_{13})$ by Lemma 1.1. Hence $YW = 0$ or $YW = I$. Since $Y \in \{0, I\}$ and $W = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, we conclude that $Y = 0$, and thus $X = 0$. This proves that $\varphi_1 \in \langle \theta_2, \psi_2 \rangle$ if $Q_1$ is $\mathcal{F}$-essential; and also (via conjugation by $\tau$) that $\varphi_4 \in \langle \theta_1, \psi_1 \rangle$ if $Q_4$ is $\mathcal{F}$-essential.

**Step 2** We now strengthen the conclusion of Step 1, by proving that $Q_1$ $\mathcal{F}$-essential implies $\varphi_1 \in \langle \theta_2 \psi_2 \rangle$, and $Q_4$ $\mathcal{F}$-essential implies $\varphi_4 \in \langle \theta_1 \psi_1 \rangle$.

Assume first $Q_1$ and $Q_4$ are both $\mathcal{F}$-essential; we show $\varphi_4 \in \langle \theta_1 \psi_1 \rangle$. Set $Q_{14} = Q_1 \cap Q_4 = A_1 A_2$. Let $X \in M_2(\mathbb{F}_2)$ be such that $M(\varphi_4|_{A_1}) = \left(\begin{smallmatrix} I & 0 & X \\ 0 & I & 0 \\ 0 & 0 & I \end{smallmatrix}\right)$. Thus $X = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, or $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, depending on whether $\varphi_4 = \mathrm{Id}$, $\theta_1$, $\psi_1$, or $\theta_1\psi_1$. Set $W = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ as before. Since $\varphi_1 \in \langle \theta_2, \psi_2 \rangle$ and $\theta_2|_{A_1} = \psi_2|_{A_1} = \mathrm{Id}$, $\sigma_{12}^*|_{A_1} = \sigma_{12}|_{A_1}$. Hence

$$M(\sigma_{12}^*|_{A_1}) = \left(\begin{smallmatrix} W & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & W \end{smallmatrix}\right) \quad \text{and} \quad M(\sigma_{45}^*|_{A_1}) = \left(\begin{smallmatrix} I & 0 & X \\ 0 & I & 0 \\ 0 & 0 & I \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{smallmatrix}\right) \left(\begin{smallmatrix} I & 0 & X \\ 0 & I & 0 \\ 0 & 0 & I \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & I & X \\ I & 0 & X \\ 0 & 0 & I \end{smallmatrix}\right),$$

where $\sigma_{12}^* \in \mathrm{Aut}_{\mathcal{F}}(Q_1)$ and $\sigma_{45}^* \in \mathrm{Aut}_{\mathcal{F}}(Q_4)$. Set $\beta = [\sigma_{12}^*, \sigma_{45}^*] \in \mathrm{Aut}_{\mathcal{F}}(Q_{14})$. Then

$$M(\beta|_{A_1}) = \left[ \left(\begin{smallmatrix} W & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & W \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & I & X \\ I & 0 & X \\ 0 & 0 & I \end{smallmatrix}\right) \right] = \left(\begin{smallmatrix} I & 0 & X+WXW^{-1} \\ 0 & I & X+WXW^{-1} \\ 0 & 0 & I \end{smallmatrix}\right).$$

Set $R_0 = A_1 \cap A_2$. Thus $\beta$ induces the identity on $A_1/R_0$, and also on $A_2/R_0$ since $\varphi_1$ (and hence $\sigma_{12}^*$) induces the identity on $A_2/R_0$. Since $R_0 = \mathrm{Fr}(Q_{14})$, $\beta \in O_2(\mathrm{Aut}_{\mathcal{F}}(Q_{14}))$ by Lemma 1.1, so $\beta \in \mathrm{Aut}_T(Q_{14})$ by the Sylow axiom, and thus $X + WXW^{-1} \in \{I, 0\}$. If $X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, then $X + WXW^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, which is impossible. It follows that $\varphi_4 \in \langle \theta_1 \psi_1 \rangle$ in this situation.

Now assume $Q_4$ is $\mathcal{F}$-essential and $Q_1$ is not. If $\varphi_4 = \theta_1$, then $\mathcal{F}$ is generated by $\mathrm{Inn}(T)$ and $\mathrm{Aut}_{\mathcal{F}}(Q_i)$ for $i = 2, 3, 4$, and all of these automorphism groups leave $U_1$ invariant. Thus $U_1$ is normal in $\mathcal{F}$ in this case, which contradicts the assumption that $\mathcal{F}$ is nonconstrained.

Assume $\varphi_4 = \psi_1$. Set $V = \langle e_{23}, e_{24}, e_{25}\rangle \leq A_1$, and let $\mathcal{A} \leq \mathrm{Aut}_{\mathcal{F}}(A_1)$ be the subgroup of those elements which leave $V$ invariant. Consider the homomorphism

$$\Psi\colon \mathcal{A} \xrightarrow{\text{(res,proj)}} \mathrm{Aut}(V)\times \mathrm{Aut}(A_1/V) \xrightarrow{\widehat{M}} GL_3(2)\times GL_3(2)$$

where $\widehat{M}$ sends a pair of automorphisms to their matrices with respect to the bases $\{e_{i5}, e_{i4}, e_{i3}\}$ for $i = 2$ or 1, respectively. Then $\Psi(\mathrm{Aut}_{Q_1}(A_1)) = \{(X,X)\}$ for $X \in GL_3(2)$ upper triangular, while $\Psi(\sigma_{34}|_{A_1}) = \left(\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right)\right)$. Thus $\mathrm{Im}(\Psi)$ contains all matrices $(M,M)$ for $M \in H$, where $H \leq GL_3(2)$ is the subgroup of matrices with first column $\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$. The above computation of $M(\sigma^*_{45}|_{A_1})$ when $\varphi_4 = \psi_1$ (hence $X = \left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)$) shows that $\Psi(\sigma^*_{45}) = \left(\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1&1\\1&0&1\\0&0&1\end{smallmatrix}\right)\right)$.

Now, $\Psi$ is not onto, since otherwise $2^6\big||\mathrm{Aut}_{\mathcal{F}}(A_1)|$, contradicting the Sylow axiom. Since $H$ is a maximal subgroup in $GL_3(2)$ (it has prime index), the above computations show that $\mathrm{Im}(\Psi)$ surjects onto each factor $GL_3(2)$. Hence the subgroup $K$ of all $g \in GL_3(2)$ such that $(1,g) \in \mathrm{Im}(\Psi)$ is normal, and $K \neq GL_3(2)$ since $\Psi$ is not onto. Thus $K = 1$ since $GL_3(2)$ is simple, and $\mathrm{Im}(\Psi)$ has the form $\{(g,\alpha(g))\}$ for some $\alpha \in \mathrm{Aut}(GL_3(2))$. By the above computations, $\alpha|_H = \mathrm{Id}$, and

$$\alpha\left(\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right)\right) = \left(\begin{smallmatrix}0&1&1\\1&0&1\\0&0&1\end{smallmatrix}\right) \quad \text{implies} \quad \alpha\left(\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right)\right) = \left(\begin{smallmatrix}0&1&1\\1&0&1\\0&0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&1&1\\1&1&0\\0&1&0\end{smallmatrix}\right).$$

Thus $\alpha$ sends an element of order three to one of order four, which is impossible.

This finishes the proof that $\varphi_4 \in \langle\theta_1\psi_1\rangle$ in both cases ($Q_1$ $\mathcal{F}$-essential or not). As usual, it then follows by symmetry that $\varphi_1 \in \langle\theta_2\psi_2\rangle$ if $Q_1$ is $\mathcal{F}$-essential.

**Step 3** Assume $Q_4$ is $\mathcal{F}$-essential, and $\mathrm{Aut}_{\mathcal{F}}(Q_4) = \Delta_4$. If $Q_1$ is not $\mathcal{F}$-essential, then $\mathcal{F}$ is generated by automorphisms of $Q_2$, $Q_3$, and $Q_4$, all of which leave $U_1$ invariant. Hence $U_1$ is normal in $\mathcal{F}$, which contradicts the assumption that $\mathcal{F}$ is nonconstrained.

Thus $Q_1$ is $\mathcal{F}$-essential. Since $\varphi_1|_{A_1} = \mathrm{Id}$, the restriction to $A_1$ of $\mathrm{Aut}_{\mathcal{F}}(Q_1)$ is equal to that of $\Delta_1$. So $\mathrm{Aut}_{\mathcal{F}}(A_1)$ is generated by restrictions of automorphisms of $\Delta_i$ for $i = 1,3,4$. This is the product of the actions of $\Delta_1$ on each of the three columns $\langle e_{1i}, e_{2i}\rangle$ in $A_1$ ($i = 3,4,5$) with the actions of $\langle\Delta_3,\Delta_4\rangle$ on each of the two rows. The actions of $\Delta_3$ and $\Delta_4$ generate the full $GL_3(2)$-action on each row (any $3\times 3$ matrix can be diagonalized by row and column operations on its first two and last two rows and columns), and thus $\mathrm{Aut}_{\mathcal{F}}(A_1) \cong \Sigma_3 \times GL_3(2)$.

Similarly, if $Q_1$ is $\mathcal{F}$-essential and $\mathrm{Aut}_{\mathcal{F}}(Q_1) = \Delta_1$, then $Q_4$ is also $\mathcal{F}$-essential and $\mathrm{Aut}_{\mathcal{F}}(A_2) \cong \Sigma_3 \times GL_3(2)$.

**Step 4** Now assume $Q_4$ is $\mathcal{F}$-essential and $\mathrm{Aut}_{\mathcal{F}}(Q_4) = (\theta_1\psi_1)\Delta_4(\theta_1\psi_1)^{-1}$. We will show that $\mathrm{Aut}_{\mathcal{F}}(A_1) \cong 3\Sigma_6$, and that $Q_1$ is also $\mathcal{F}$-essential. The corresponding result when $\mathrm{Aut}_{\mathcal{F}}(Q_1) = (\theta_2\psi_2)\Delta_1(\theta_2\psi_2)^{-1}$ then follows by symmetry.

Consider the subgroup

$$\mathrm{Aut}^0_{\mathcal{F}}(A_1) \overset{\text{def}}{=} \left\langle \mathrm{Aut}_T(A_1), \sigma_{34}|_{A_1}, \sigma^*_{45}|_{A_1}\right\rangle \leq \mathrm{Aut}_{\mathcal{F}}(A_1) \ :$$

the subgroup generated by restrictions of elements in $\mathrm{Aut}_{\mathcal{F}}(Q_i)$ for $i = 2,3,4$. This time, we identify $A_1$ with $\mathbb{F}_4^3$. Fix $\omega \in \mathbb{F}_4 \smallsetminus \mathbb{F}_2$, and give $A_1$ the structure of a $\mathbb{F}_4$-vector space by setting $\omega e_{1j} = e_{2j}$ and $\omega e_{2j} = e_{1j}e_{2j}$. For $\alpha \in \mathrm{Aut}_{\mathbb{F}_4}(A_1)$, let $M^*(\alpha) \in GL_3(4)$ be the matrix for $\alpha$ with respect to the $\mathbb{F}_4$-basis $\{e_{15}, e_{14}, e_{13}\}$.

Write $\bar{\omega} = \omega^2 = \omega + 1 \in \mathbb{F}_4$. Then

$$M^*(c_{34}) = \left(\begin{smallmatrix} 1&0&0\\0&1&1\\0&0&1 \end{smallmatrix}\right), \qquad M^*(\sigma_{34}) = \left(\begin{smallmatrix} 1&0&0\\0&0&1\\0&1&0 \end{smallmatrix}\right), \qquad M^*(c_{45}) = \left(\begin{smallmatrix} 1&1&0\\0&1&0\\0&0&1 \end{smallmatrix}\right);$$

and

$$M^*(\sigma^*_{45}) = M^*\big((\theta_1\psi_1)\sigma_{45}(\theta_1\psi_1)^{-1}\big) = \left(\begin{smallmatrix} 1&0&\bar{\omega}\\0&1&0\\0&0&1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0&1&0\\1&0&0\\0&0&1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1&0&\bar{\omega}\\0&1&0\\0&0&1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0&1&\bar{\omega}\\1&0&\bar{\omega}\\0&0&1 \end{smallmatrix}\right).$$

Also, $c_{12}$ acts on $A_1$ as the field automorphism $\phi_2\colon (a,b,c) \mapsto (\bar{a},\bar{b},\bar{c})$, with respect to the given basis.

Consider the following six points in the projective plane $P(\mathbb{F}_4^3)$: $\lambda_1 = \langle(1,\omega,0)\rangle$, $\lambda_2 = \langle(1,\bar{\omega},0)\rangle$, $\lambda_3 = \langle(\omega,0,1)\rangle$, $\lambda_4 = \langle(\omega,1,1)\rangle$, $\lambda_5 = \langle(\bar{\omega},0,1)\rangle$, $\lambda_6 = \langle(\bar{\omega},1,1)\rangle$. These form an "oval", in the sense that no three of them lie in a projective line. By a direct check, the above generators permute these points in the following way:

| | | | | |
|---|---|---|---|---|
| $\phi_2$ | $\left(\begin{smallmatrix} 1&0&0\\0&1&1\\0&0&1 \end{smallmatrix}\right)$ | $\left(\begin{smallmatrix} 1&0&0\\0&0&1\\0&1&0 \end{smallmatrix}\right)$ | $\left(\begin{smallmatrix} 1&1&0\\0&1&0\\0&0&1 \end{smallmatrix}\right)$ | $\left(\begin{smallmatrix} 0&1&\bar{\omega}\\1&0&\bar{\omega}\\0&0&1 \end{smallmatrix}\right)$ |
| $(1\,2)(3\,5)(4\,6)$ | $(3\,4)(5\,6)$ | $(1\,5)(2\,3)$ | $(1\,2)(4\,6)$ | $(1\,2)(3\,6)$ |

The first two and last two of these permutations generate the subgroup of elements of $\Sigma_6$ which leave $\{1,2\}$ invariant, and hence this set of five permutations generates $\Sigma_6$. Since this extension of $C_3$ by $\Sigma_6$ is not split, this proves that $\mathrm{Aut}^0_{\mathcal{F}}(A_1) \cong 3\Sigma_6$.

Let $\zeta \in \mathrm{Aut}(A_1)$ be such that $M^*(\zeta) = \omega{\cdot}I = \mathrm{diag}(\omega,\omega,\omega)$. Then $\zeta \in \mathrm{Aut}^0_{\mathcal{F}}(A_1)$ by the above computation. Also, $\zeta$ commutes with all elements of $\mathrm{Aut}_{Q_1}(A_1)$, so it extends to an element $\bar{\zeta} \in \mathrm{Aut}_{\mathcal{F}}(Q_1)$ by the extension axiom. Thus $Q_1$ is $\mathcal{F}$-essential since $\mathrm{Aut}_{\mathcal{F}}(Q_1)$ is not a 2-group. Also, the restriction to $A_1$ of each element of $\mathrm{Aut}_{\mathcal{F}}(Q_1) = \langle \mathrm{Aut}_T(Q_1), \bar{\zeta}\rangle$ lies in $\mathrm{Aut}^0_{\mathcal{F}}(A_1)$, and hence $\mathrm{Aut}_{\mathcal{F}}(A_1) = \mathrm{Aut}^0_{\mathcal{F}}(A_1) \cong 3\Sigma_6$. □

We can now summarize these results in the following theorem. The much more difficult classification of simple groups with Sylow 2-subgroup $UT_5(2)$ is due to Held [He], and is also shown in [A2, Chapter 14].

**Theorem 6.10.** *Every nonconstrained saturated fusion system over $T = UT_5(2)$ is isomorphic to the fusion system of one of the simple groups $GL_5(2)$, $M_{24}$, or* He.

*Proof.* By Proposition 6.5, $\mathcal{F}$ is generated by $\mathrm{Aut}_{\mathcal{F}}(T)$ and the $\mathrm{Aut}_{\mathcal{F}}(Q_i)$ for $i = 1,2,3,4$. Also, $\mathrm{Aut}_{\mathcal{F}}(T) = \mathrm{Inn}(T)$ since $\mathrm{Aut}(T)$ is a 2-group (Proposition 6.6(a)). By Proposition 6.8, we can assume $\mathrm{Out}_{\mathcal{F}}(Q_i) = \Delta_i$ for $i = 2,3$. Then by Proposition 6.9, there are at most four possibilities for $\mathcal{F}$, of which two are isomorphic via $\tau$.

We refer to [He], and to [A2, §40], for a description of the groups $\mathrm{Aut}_G(A_i)$ when $G = GL_5(2)$, $M_{24}$, or Held's group. Each of these groups contains Sylow 2-subgroups $S \cong UT_5(2)$. Also, $\mathcal{F}_S(G)$ is nonconstrained and centerfree in each case, and hence must be isomorphic to one of the three distinct fusion systems which we found. Thus $\mathcal{F}$ is isomorphic to the fusion system of $GL_5(2)$ if $\mathrm{Aut}_{\mathcal{F}}(A_1) \cong \mathrm{Aut}_{\mathcal{F}}(A_2) \cong \Sigma_3 \times GL_3(2)$ (if $\mathrm{Aut}_{\mathcal{F}}(Q_i) = \Delta_i$ for $i = 1,4$); $\mathcal{F}$ is isomorphic to the fusion system of $M_{24}$ if $\mathrm{Aut}_{\mathcal{F}}(A_1) \cong \Sigma_3 \times GL_3(2)$ and $\mathrm{Aut}_{\mathcal{F}}(A_2) \cong 3\Sigma_6$ or vice versa (if $\mathrm{Aut}_{\mathcal{F}}(Q_i) = \Delta_i$ for $i = 1$ or $i = 4$ but not both); and $\mathcal{F}$ is isomorphic to the fusion system of Held's group if $\mathrm{Aut}_{\mathcal{F}}(A_1) \cong \mathrm{Aut}_{\mathcal{F}}(A_2) \cong 3\Sigma_6$ (if $\mathrm{Aut}_{\mathcal{F}}(Q_i) = (\theta_j\psi_j)\Delta_i(\theta_j\psi_j)^{-1}$ for $(i,j) = (1,2)$ and $(4,1)$). □

## 7. Fusion systems over the Sylow subgroup of $Co_3$

Our notation here for elements in a Sylow 2-subgroup of $\mathrm{Spin}_7(3)$ is based on that used in [LO]. Fix $Y, B \in SL_2(9)$ such that $Y$ has order 8 and $\langle Y, B\rangle \cong Q_{16}$, and set $A = Y^2$. In particular, $Y^4 = B^2 = -I$, and $\langle A, B\rangle \cong Q_8$. Consider the groups

$$\mathbb{S}_0 \overset{\text{def}}{=} \langle Y, B\rangle^3/\langle(-I,-I,-I)\rangle \qquad \text{and} \qquad \mathbb{S} \overset{\text{def}}{=} \mathbb{S}_0 \overset{(1\,2)}{\rtimes} \underset{\boldsymbol{\tau}}{C_2},$$

and let $[\![X_1, X_2, X_3]\!]$ denote the class of $(X_1, X_2, X_3)$ in $\mathbb{S}_0$. Thus

$$\boldsymbol{\tau}^2 = 1 \quad \text{and} \quad \boldsymbol{\tau}[\![X_1, X_2, X_3]\!]\boldsymbol{\tau}^{-1} = [\![X_2, X_1, X_3]\!].$$

Write $\mathbf{a}_1 = [\![A, I, I]\!]$, $\mathbf{a}_2 = [\![I, A, I]\!]$, $\mathbf{a}_3 = [\![I, I, A]\!]$, $\mathbf{b}_1 = [\![B, I, I]\!]$, $\mathbf{b}_2 = [\![I, B, I]\!]$, $\mathbf{b}_3 = [\![I, I, B]\!]$, $\mathbf{c} = [\![Y, Y, Y]\!]$, and $\mathbf{z}_i = \mathbf{a}_i^2$. Finally, set

$$T^* = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{c}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \boldsymbol{\tau}\rangle \le \mathbb{S} :$$

a group of order $2^{10}$. For later reference, we list the following relations in $T^*$ (for all $i \neq j$), which in fact form a complete presentation for this group:

$$\begin{gathered} \mathbf{a}_i^2 = \mathbf{b}_i^2 = [\mathbf{a}_i, \mathbf{b}_i] = \mathbf{z}_i, \qquad \mathbf{z}_i^2 = 1 = \mathbf{z}_1\mathbf{z}_2\mathbf{z}_3, \qquad [\mathbf{a}_i, \mathbf{b}_j] = 1 = [\mathbf{a}_i, \mathbf{a}_j] = [\mathbf{b}_i, \mathbf{b}_j]; \\ \mathbf{c}^2 = \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3, \qquad [\mathbf{c}, \mathbf{a}_i] = 1, \qquad \mathbf{c}\mathbf{b}_i\mathbf{c}^{-1} = \mathbf{a}_i\mathbf{b}_i, \qquad \mathbf{b}_i\mathbf{c}\mathbf{b}_i^{-1} = \mathbf{a}_i^{-1}\mathbf{c}; \qquad (1) \\ \boldsymbol{\tau}^2 = 1, \quad \boldsymbol{\tau}\mathbf{c}\boldsymbol{\tau}^{-1} = \mathbf{c}, \quad \boldsymbol{\tau}\mathbf{a}_i\boldsymbol{\tau}^{-1} = \mathbf{a}_{\sigma(i)}, \quad \boldsymbol{\tau}\mathbf{b}_i\boldsymbol{\tau}^{-1} = \mathbf{b}_{\sigma(i)} \quad (\text{where } \sigma = (1\,2) \in \Sigma_3). \end{gathered}$$

The embedding of $T^*$ as a Sylow 2-subgroup of $\mathrm{Spin}_7(3)$ is described in detail in [LO, § 2]. For example, the subgroup $\langle \mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2, \mathbf{a}_3, \mathbf{b}_3\rangle$ is a Sylow subgroup of $\mathrm{Spin}_3(3) \times_{C_2} \mathrm{Spin}_4^+(3) \le \mathrm{Spin}_7(3)$, via the identifications $\mathrm{Spin}_3(3) \cong SL_2(3)$, $\mathrm{Spin}_4^+(3) \cong SL_2(3) \times SL_2(3)$, and $Q_8 \in \mathrm{Syl}_2(SL_2(3))$. Instead of repeating that argument here, we give an explicit homomorphism $\rho\colon T^* \longrightarrow \Omega_7(3)$ to help motivate some of our constructions. Let $\delta_i$ be the diagonal matrix with entry $-1$ in $i$-th position and 1 elsewhere, and set $\delta_{ij} = \delta_i\delta_j$, etc. Let $\pi_\sigma$ be the permutation matrix for $\sigma \in \Sigma_7$. Thus $\pi_\sigma\delta_i\pi_\sigma^{-1} = \delta_{\sigma(i)}$. Define $\rho$ by setting

$$\begin{array}{lll} \rho(\mathbf{a}_1) = \delta_{14}\pi_{(12)(34)} & \rho(\mathbf{a}_2) = \delta_{24}\pi_{(12)(34)} & \rho(\mathbf{a}_3) = \delta_{56} \\ \rho(\mathbf{b}_1) = \delta_{12}\pi_{(13)(24)} & \rho(\mathbf{b}_2) = \delta_{23}\pi_{(13)(24)} & \rho(\mathbf{b}_3) = \delta_{57} \\ \rho(\mathbf{c}) = \delta_{46}\pi_{(34)(56)} & \rho(\boldsymbol{\tau}) = \delta_{1567}. & \end{array}$$

It is straightforward to check that the relations in $T^*$ listed above all hold, and hence that this defines a homomorphism with kernel $\langle \mathbf{z}_3\rangle$.

Two families of subgroups of $T^*$ will play an important role in what follows. First define

$$R_0 = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\rangle \qquad R_1 = \langle R_0, \mathbf{c}\rangle \qquad R_2 = \langle R_0, \boldsymbol{\tau}\rangle \qquad R_3 = \langle R_0, \mathbf{c}\boldsymbol{\tau}\rangle\ .$$

Thus $T^*/R_0 \cong C_2^2$, and $R_1$, $R_2$, and $R_3$ are the three subgroups of index two in $T^*$ which contain $R_0$. Also, $Z(R_0) = Z(R_1) = \langle \mathbf{z}_1, \mathbf{z}_2\rangle$, while $Z(R_i) = \langle \mathbf{z}_3\rangle = Z(T^*)$ for $i = 2, 3$.

Next consider the following subgroups:

$$\begin{gathered} \mathbf{Q} = \langle \mathbf{z}_1, \mathbf{a}_1\mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1\mathbf{b}_2, \mathbf{b}_3, \boldsymbol{\tau}\rangle = \langle \mathbf{a}_1\mathbf{a}_2, \mathbf{b}_1\mathbf{b}_2\rangle \times_{C_2} \langle \mathbf{a}_3, \mathbf{b}_3\rangle \times_{C_2} \langle \mathbf{z}_1, \boldsymbol{\tau}\rangle \\ R_4 = \langle \mathbf{Q}, \mathbf{a}_1, \mathbf{c}\rangle\ ; \qquad H_1 = \langle \mathbf{Q}, \mathbf{c}\rangle \qquad H_2 = \langle \mathbf{Q}, \mathbf{a}_1\mathbf{c}\rangle \qquad H_3 = \langle \mathbf{Q}, \mathbf{a}_1\rangle\ . \end{gathered}$$

Thus $\mathbf{Q}$ is extraspecial of order $2^7$ (a central product of three $D_8$'s), $R_4/\mathbf{Q} \cong C_2^2$, and $H_1$, $H_2$, and $H_3$ are the three subgroups of index two in $R_4$ which contain $\mathbf{Q}$. Also,

$H_3 \lhd T^*$, while $H_2 = \mathbf{b}_1 H_1 \mathbf{b}_1^{-1}$ and $N_{T^*}(H_1) = N_{T^*}(H_2) = R_4$. These three subgroups will be seen to be permuted transitively by $\mathrm{Out}(R_4)$.

Consider again the homomorphism $\rho\colon T^* \longrightarrow \Omega_7(3)$ defined above, and also the induced action of $T^*$ on $V \cong \mathbb{F}_2^7$ with canonical (orthonormal) basis $\{e_1, \dots, e_7\}$. By inspection, $R_1$ is the subgroup of those elements which act on each of the factors $\langle e_1, e_2, e_3, e_4\rangle$ and $\langle e_5, e_6, e_7\rangle$ with determinant one, and $R_0$ is the subgroup of elements whose action on each factor lies in the spinor group. Also, $R_4$ is the subgroup of elements which leave invariant each of the summands $\langle e_1, e_2\rangle$, $\langle e_3, e_4\rangle$, and $\langle e_5, e_6\rangle$, while $\mathbf{Q}$ is the group of elements which sends each of the $\langle e_i\rangle$ to itself.

Before we begin looking at the critical subgroups of $T^*$, we prove the following lemma about $\mathbf{Q}$, and about another subgroup $\mathbf{A} \cong C_4^3$ which we will need to work with.

**Lemma 7.1.** (a) *Set* $\mathbf{A} = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{c}\rangle$. *Then* $\mathbf{A} \cong C_4^3$, $\mathbf{A} \lhd T^*$, *and* $T^*/\mathbf{A} \cong C_2 \times D_8$.

(b) *If* $P \le T^*$ *is such that* $|P| = 2^7$ *and* $|\mathrm{Fr}(P)| = 2$, *then* $P = \mathbf{Q}$.

*Proof.* **(a)** Since $\mathbf{c}^2 = \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ and $(\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3)^2 = 1$, $\mathbf{A} = \langle \mathbf{a}_1\rangle \times \langle \mathbf{a}_2\rangle \times \langle \mathbf{c}\rangle \cong C_4^3$. By the relations (1), $\mathbf{A}$ is normal in $T^*$, and $T^*/\mathbf{A} = \langle \mathbf{b}_1\mathbf{A}, \mathbf{b}_2\mathbf{A}, \mathbf{b}_3\mathbf{A}, \boldsymbol{\tau}\mathbf{A}\rangle \cong D_8 \times C_2$.

**(b)** Let $\mathbf{A}_0 = \langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\rangle$ be the 2-torsion subgroup of $\mathbf{A}$. Since $|T^*/\mathbf{A}| = 2^4$, $P\cap\mathbf{A}$ is a normal subgroup of $P$ of order at least $2^3$. If $|P\cap\mathbf{A}| = 2^3$, then $P/(P\cap\mathbf{A}) \cong T^*/\mathbf{A} \cong C_2 \times D_8$, so $\mathrm{Fr}(P) \cap \mathbf{A} = 1$, and $P \cap \mathbf{A} = \mathbf{A}_0$ since it cannot have 4-torsion. Since $\mathrm{Fr}(T^*/\mathbf{A}) = \langle \mathbf{b}_1\mathbf{b}_2\mathbf{A}\rangle$, $\mathrm{Fr}(P) = \langle \mathbf{b}_1\mathbf{b}_2 g\rangle$ for some $g \in \mathbf{A}$, which is impossible since $[\mathbf{b}_1\mathbf{b}_2 g, \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3] = \mathbf{z}_3 \ne 1$.

It follows that $|P\cap\mathbf{A}| \ge 2^4$. In particular, $\mathrm{Fr}(P) \le \mathbf{A}$ since $P\cap\mathbf{A}$ is not elementary abelian, and $P \ge \mathbf{A}_0$ and $|P \cap \mathbf{A}| = 2^4$ since $P$ contains no subgroup $C_4^2$. So $P\mathbf{A}/\mathbf{A}$ is an elementary abelian subgroup of order $2^3$ in $T^*/\mathbf{A} \cong D_8 \times C_2$. Hence either $P\mathbf{A}/\mathbf{A} = \langle \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\rangle$ and thus $P\mathbf{A} = R_1$, or $P\mathbf{A}/\mathbf{A} = \langle \mathbf{b}_1\mathbf{b}_2, \mathbf{b}_3, \boldsymbol{\tau}\rangle$ and $P\mathbf{A} = R_4$. In either case, $\mathbf{b}_3 g \in P$ for some $g \in \mathbf{A}$, and so $[\mathbf{b}_3 g, \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3] = \mathbf{z}_3 \in \mathrm{Fr}(P)$. Thus $\mathrm{Fr}(P) = \langle \mathbf{z}_3\rangle$.

If $P\mathbf{A} = R_1$, then $\mathbf{b}_1 g \in P$ for some $g \in \mathbf{A}$, so $[\mathbf{b}_1 g, \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3] = \mathbf{z}_1 \in \mathrm{Fr}(P)$, and we just saw this is impossible. Hence $P\mathbf{A} = R_4$.

Consider the quotient group

$$R_4/\mathbf{A}_0 = R_4/\langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\rangle = \langle \mathbf{a}_1, \mathbf{a}_2, \boldsymbol{\tau}\rangle \times_{\langle \mathbf{a}_3\rangle} \langle \mathbf{c}, \mathbf{b}_3\rangle \times \langle \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\rangle \cong D_8 \times_{C_2} D_8 \times C_2.$$

Since $\mathrm{Fr}(P) \le \mathbf{A}_0$, $P/\mathbf{A}_0 \cong C_2^4$. Hence $Z(R_4/\mathbf{A}_0) \le P/\mathbf{A}_0$, since every abelian subgroup of rank four in $R_4/\mathbf{A}_0$ contains the center. In particular, $\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3 \in P$, so $P/\langle \mathbf{z}_3\rangle \le C_{R_4/\langle \mathbf{z}_3\rangle}(\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3) = \mathbf{Q}/\langle \mathbf{z}_3\rangle$, and hence $P = \mathbf{Q}$. □

In fact, $\mathbf{A}$ is the unique abelian subgroup of order $2^6$ in $T^*$, but we will not need to use that.

### 7.1 Determining the critical subgroups

We start as usual by reducing to the case of subgroups of index 2 in their normalizers.

**Lemma 7.2.** *If* $P$ *is a critical subgroup of* $T^*$, *then* $|N_{T^*}(P)/P| = 2$.

*Proof.* Assume otherwise: let $P$ be a critical subgroup of $T^*$ with $|N_{T^*}(P)/P| \ge 4$. By Proposition 3.3(c),

$$g \notin P, \quad [g, P] \le P \quad \Longrightarrow \quad \mathrm{rk}([g, P/\mathrm{Fr}(P)]) \ge 2. \tag{2}$$

Since $P$ is centric in $T^*$, $Z(T^*) = \langle \mathbf{z}_3 \rangle \le P$. Since $[x,P] \le [x,T^*] \le \langle \mathbf{z}_3 \rangle$ for $x \in \langle \mathbf{z}_1, \mathbf{a}_3 \rangle$, $\mathbf{z}_1, \mathbf{a}_3 \in P$ by (2). In particular, $\mathbf{z}_3 = \mathbf{a}_3^2 \in \mathrm{Fr}(P)$.

Since $[\mathbf{a}_1\mathbf{a}_2, P] \le [\mathbf{a}_1\mathbf{a}_2, T^*] = \langle \mathbf{z}_1, \mathbf{z}_3 \rangle$, $\mathrm{rk}([\mathbf{a}_1\mathbf{a}_2, P/\mathrm{Fr}(P)]) \le 1$, and hence $\mathbf{a}_1\mathbf{a}_2 \in P$ by (2). Similarly, $[\mathbf{b}_3, P] \le [\mathbf{b}_3, T^*] = \langle \mathbf{a}_3 \rangle$ implies $\mathbf{b}_3 \in P$. Set $T_0 = \langle \mathbf{z}_1, \mathbf{a}_1\mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_3 \rangle \le P$. Then $|T_0| = 2^5$, $T_0 \lhd T^*$, and

$$T^*/T_0 = R_4/T_0 \rtimes \langle \mathbf{b}_1 \rangle \cong \underset{\langle \mathbf{a}_1, \mathbf{b}_1\mathbf{b}_2, \mathbf{c}, \boldsymbol{\tau} \rangle}{C_2^4} \rtimes \underset{\mathbf{b}_1}{C_2}.$$

Now, $[\mathbf{a}_1, T^*] = \langle \mathbf{z}_1, \mathbf{z}_3, \mathbf{a}_1\mathbf{a}_2 \rangle = [\mathbf{b}_1\mathbf{b}_2, T^*]$. So by (2), either $\mathrm{Fr}(P) \cap \langle \mathbf{z}_1, \mathbf{a}_1\mathbf{a}_2 \rangle = \langle \mathbf{z}_3 \rangle$, or $\mathbf{a}_1, \mathbf{b}_1\mathbf{b}_2 \in P$. If $\mathbf{a}_1 \in P$, then $\mathbf{z}_1 = \mathbf{a}_1^2 \in \mathrm{Fr}(P)$, and so $\mathbf{b}_1\mathbf{b}_2 \in P$, $\mathrm{Fr}(T^*) \le P$, and thus $P \lhd T^*$. Set $T_1 = \langle T_0, \mathbf{a}_1, \mathbf{b}_1\mathbf{b}_2 \rangle$; thus $|T_1| = 2^7$ and so $[P{:}T_1] \le 2$. If $\mathbf{b}_1 \notin P$, then since $[T_1, \mathbf{b}_1] \le \langle \mathbf{z}_1, \mathbf{z}_3 \rangle \le \mathrm{Fr}(P)$ and $|P/T_1| \le 2$, $\mathrm{rk}([\mathbf{b}_1, P/\mathrm{Fr}(P)]) \le 1$, contradicting (2) again. Thus $P = \langle T_1, \mathbf{b}_1 \rangle = R_0$, $\mathrm{Fr}(P) = \langle \mathbf{z}_1, \mathbf{z}_3 \rangle$, $\mathrm{rk}([\boldsymbol{\tau}, P/\mathrm{Fr}(P)]) = 2$, and $\mathrm{rk}([\mathbf{c}, P/\mathrm{Fr}(P)]) = 3$. This contradicts Proposition 3.3(b) (all involutions in $\mathrm{Out}_{T^*}(P)$ are conjugate in $\mathrm{Out}(P)$). We now conclude that

$$\mathbf{a}_1 \notin P, \qquad \text{and} \qquad \mathrm{Fr}(P) \cap \langle \mathbf{z}_1, \mathbf{a}_1\mathbf{a}_2 \rangle = \langle \mathbf{z}_3 \rangle. \tag{3}$$

Since $[\mathbf{a}_1\mathbf{a}_2, R_4] = \langle \mathbf{z}_3 \rangle \le \mathrm{Fr}(P)$ and $[\mathbf{a}_1\mathbf{a}_2, \mathbf{b}_1] = \mathbf{z}_1$, (3) implies $P \le R_4$. Also, $P \ge \mathrm{Fr}(R_4)$, and so $N_{T^*}(P) \ge R_4$. By Proposition 3.3(a), all involutions in $\mathrm{Out}_{T^*}(P) \cong N_{T^*}(P)/P$ are central. Hence if $P \lhd T^*$, then all elements of $R_4/P$ are central in $T^*/P$, which is impossible since $\mathbf{a}_1 \notin P$ and $[\mathbf{c}, \mathbf{b}_1] = \mathbf{a}_1$ ($\mathbf{c} \notin P$ since $P$ is normal, and hence $\mathbf{c}P \notin Z(T^*/P)$). Thus $N_{T^*}(P) = R_4$. Also, $R_4/P \cong C_2^k$ for $k \ge 2$. If $k \ge 3$, then $\mathrm{rk}(P/\mathrm{Fr}(P)) \ge 6$ by Proposition 3.3(c), so $2^7 \le |P| = 2^{9-k}$, a contradiction. Thus

$$N_{T^*}(P) = R_4, \qquad [P : T_0] = 4, \qquad \text{and} \qquad [R_4 : P] = 4. \tag{4}$$

If $x\mathbf{b}_1\mathbf{b}_2 \in P$ for some $x \in \langle \mathbf{a}_1 \rangle$, then since $[x\mathbf{b}_1\mathbf{b}_2, T^*] \le \langle \mathbf{z}_1, \mathbf{a}_1\mathbf{a}_2 \rangle$ in both cases, $[x\mathbf{b}_1\mathbf{b}_2, P] = \langle \mathbf{z}_3 \rangle$ by (3). Hence

$$P \le \bigl\{ g \in R_4 \bigm| [x\mathbf{b}_1\mathbf{b}_2, g] \in \langle \mathbf{z}_3 \rangle \bigr\} = \begin{cases} \langle T_0, \mathbf{b}_1\mathbf{b}_2, \boldsymbol{\tau} \rangle & \text{if } x = 1 \\ \langle T_0, \mathbf{a}_1\mathbf{b}_1\mathbf{b}_2, \mathbf{a}_2\mathbf{c}\boldsymbol{\tau} \rangle & \text{if } x = \mathbf{a}_1, \end{cases}$$

and $P$ is equal to one of these groups (the inclusion is an equality) by (4). But both of these groups are normal in $T^*$ — note that $\mathbf{b}_1(\mathbf{a}_2\mathbf{c}\boldsymbol{\tau})\mathbf{b}_1^{-1} \equiv (\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2)(\mathbf{a}_2\mathbf{c}\boldsymbol{\tau}) \pmod{T_0}$ — which contradicts (4). So this case is impossible.

Thus $P \cap \langle T_0, \mathbf{a}_1, \mathbf{b}_1\mathbf{b}_2 \rangle = T_0$. Since $[P : T_0] = 4$ by (4) again,

$$P = \langle T_0, \mathbf{c}x, \boldsymbol{\tau}y \rangle = \langle \mathbf{z}_1, \mathbf{a}_1\mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_3, \mathbf{c}x, \boldsymbol{\tau}y \rangle$$

for some $x, y \in \langle \mathbf{a}_1, \mathbf{b}_1\mathbf{b}_2 \rangle$. Since $\mathbf{b}_1\mathbf{c}\mathbf{b}_1^{-1} = \mathbf{a}_1^{-1}\mathbf{c}$, it suffices to consider the case where $x \in \langle \mathbf{b}_1\mathbf{b}_2 \rangle$. Then one of the following happens:

- $y \in \langle \mathbf{b}_1\mathbf{b}_2 \rangle$, $[\mathbf{b}_1\mathbf{b}_2, P] = \langle \mathbf{a}_1\mathbf{a}_2, \mathbf{z}_3 \rangle$, so $\mathrm{rk}([\mathbf{b}_1\mathbf{b}_2, P/\mathrm{Fr}(P)]) \le 1$ contradicting (2);
- $y = \mathbf{a}_1$, $(\boldsymbol{\tau}\mathbf{a}_1)^2 = \mathbf{a}_1\mathbf{a}_2 \in \mathrm{Fr}(P)$, contradicting (3); or
- $y = \mathbf{a}_1\mathbf{b}_1\mathbf{b}_2$, $(\boldsymbol{\tau}\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2)^2 = \mathbf{a}_1\mathbf{a}_2\mathbf{z}_2 \in \mathrm{Fr}(P)$, contradicting (3).

This finishes the proof. □

It remains to handle the subgroups of $T^*$ of index two in their normalizer.

**Proposition 7.3.** *The only critical subgroups in $T^*$ are $R_1$, $R_2$, $R_3$, $R_4$, $H_1$, and $H_2$.*

*Proof.* Fix a critical subgroup $P \le T^*$ of index two in its normalizer. By Lemma 3.4,

$$g \in N(P){\smallsetminus}P, \quad \Theta \text{ char } P \quad \Longrightarrow \quad [g,P] \nleq \Theta{\cdot}\mathrm{Fr}(P) \quad \text{or} \quad [g,\Theta] \nleq \mathrm{Fr}(P). \qquad (5)$$

In Step 1, we show that $\langle \mathbf{z}_1, \mathbf{z}_3, \mathbf{a}_3\rangle \le P$, and that $\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\rangle \le P$ if $P \le R_1$. In Step 2, we show $\mathbf{a}_1\mathbf{a}_2 \in P$. We then handle the cases where $P$ is not normal in $T^*$ in Step 3, and those where $P \lhd T^*$ (hence $[T^*{:}P] = 2$) in Step 4.

**Step 1:** Since $P$ is centric, $\mathbf{z}_3 \in Z(T^*) \le P$. Since $[\mathbf{z}_1, P] \le [\mathbf{z}_1, T^*] = \langle \mathbf{z}_3\rangle \le P$, and similarly for $\mathbf{a}_3$, $\langle \mathbf{z}_1, \mathbf{a}_3\rangle \le N(P)$. So if $\mathbf{z}_1 \notin P$, then $\mathbf{a}_3$ or $\mathbf{z}_1\mathbf{a}_3$ must be in $P$, since otherwise $|N(P)/P| \ge 4$. Then $\mathbf{z}_3 = \mathbf{a}_3^2 = (\mathbf{z}_1\mathbf{a}_3)^2 \in \mathrm{Fr}(P)$, so $[\mathbf{z}_1, P] \le \mathrm{Fr}(P)$, and this contradicts (5). This proves that $\mathbf{z}_1 \in P$.

Now assume $\mathbf{a}_3 \notin P$. By Lemma 3.6, applied with $z = \mathbf{z}_3$, $g = \mathbf{a}_3$, and $y = \mathbf{z}_1$, there is $h \in T^*$ such that $[h, \mathbf{a}_3] = \mathbf{z}_3 \notin \mathrm{Fr}(P)$, $h^2 = 1$, and $P = C_{T^*}(h)$. Also, $\mathbf{z}_1 \in Z(P)$ (since $\mathbf{z}_1\mathbf{a}_3 \notin P$), and $h$ is not $T^*$-conjugate to $\mathbf{z}_1 h$. Thus $h \in P \le R_1$, since $\mathbf{z}_1 \in Z(P)$. We return to the notation used at the beginning of the section, and write $h = [\![X_1, X_2, X_3]\!]$ for some $X_i \in \langle Y, B\rangle \cong Q_{16}$. Recall $A = Y^2$ and $\langle A, B\rangle \cong Q_8$.

The condition $[h, \mathbf{a}_3] = \mathbf{z}_3$ implies $[X_3, A] = -I$, and hence $X_3 \in \langle Y\rangle{\cdot}B$. Thus $X_3^2 = -I$, and hence $X_1^2 = X_2^2 = -I$ since $h^2 = 1$. Since $h$ is not $T^*$-conjugate to $\mathbf{z}_1 h$, $[X_1, A] \ne -I$ and $[X_1, B] \ne -I$ imply $X_1 = \pm I$, and thus $X_1^2 \ne -I$. Hence this situation is impossible, and we conclude that $\mathbf{a}_3 \in P$.

Now assume $P \le R_1 = \langle \mathbf{a}_i, \mathbf{b}_i, \mathbf{c} \,|\, i = 1,2,3\rangle$; we claim that $\mathbf{a}_1, \mathbf{a}_2 \in P$. This is clear if $P = R_1$, so we assume $P \lneqq R_1$. Then $N_{R_1}(P)/P \ne 1$, so $N(P) \le R_1$ since we are assuming $|N(P)/P| = 2$. Thus $P$ is also critical in $R_1$. In this situation, the same argument we just used to show $\mathbf{a}_3 \in P$ also applies to prove that $\mathbf{a}_1, \mathbf{a}_2 \in P$.

**Step 2:** We next prove that $\mathbf{a}_1\mathbf{a}_2 \in P$. Assume otherwise; then $\mathbf{a}_1\mathbf{a}_2 \in N_{T^*}(P){\smallsetminus}P$. Since $[\mathbf{a}_1\mathbf{a}_2, P] \le [\mathbf{a}_1\mathbf{a}_2, T^*] = \langle \mathbf{z}_1, \mathbf{z}_2\rangle$, and $\mathbf{z}_3 = \mathbf{a}_3^2 \in \mathrm{Fr}(P)$, $\mathbf{z}_1 \notin \mathrm{Fr}(P)$ by (5). By Step 1, $P \nleq R_1$; let $g \in R_1$ be such that $g\boldsymbol{\tau} \in P$.

By Lemma 3.5, there is $\alpha \in \mathrm{Aut}(P)$ of odd order, and $x \in [\mathbf{a}_1\mathbf{a}_2, P]$, such that $x \notin \mathrm{Fr}(P)$ and $[\mathbf{a}_1\mathbf{a}_2, \alpha(x)] \notin \mathrm{Fr}(P)$. Thus $x \in \{\mathbf{z}_1, \mathbf{z}_2\}$, and $[\mathbf{a}_1\mathbf{a}_2, \alpha(x)] \in \{\mathbf{z}_1, \mathbf{z}_2\}$. Set $y = \alpha(x) = [\![X_1, X_2, X_3]\!]\boldsymbol{\tau}^k$, where $X_i \in \langle Y, B\rangle \cong Q_{16}$ and $k = 0, 1$. Then $y^2 = \alpha(x^2) = 1$. The condition $[\mathbf{a}_1\mathbf{a}_2, y] \notin \langle \mathbf{z}_3\rangle$ means that $[A, X_1] \ne [A, X_2]$ (recall $\mathbf{a}_1\mathbf{a}_2 = [\![A, A, I]\!]$ and $A = Y^2$), and thus $X_1 \in \langle Y\rangle$ or $X_2 \in \langle Y\rangle$ but not both. If $k = 1$, then $y^2 = [\![X_1X_2, X_2X_1, X_3^2]\!] = 1$, which is impossible since $X_1X_2 \notin \langle Y\rangle$. Thus $k = 0$, and $y^2 = [\![X_1^2, X_2^2, X_3^2]\!] = 1$. Since $X_1$ or $X_2$ has order $\ge 4$, this implies $X_i^2 = -I$ for each $i = 1,2,3$. Also, $X_j \in \langle Y\rangle$ for $j = 1$ or $2$, so $X_j \in \langle A\rangle$, and thus $X_i \in \langle A, B\rangle \cong Q_8$ for each $i = 1,2,3$. We have now shown that $y = y_1y_2y_3$, where $y_i \in \langle \mathbf{a}_i, \mathbf{b}_i\rangle{\smallsetminus}\langle \mathbf{z}_i\rangle$, and where $y_1 \in \langle \mathbf{a}_1\rangle$ or $y_2 \in \langle \mathbf{a}_2\rangle$ but not both.

Thus $[y, g\boldsymbol{\tau}] \equiv [y, \boldsymbol{\tau}] \equiv \mathbf{b}_1\mathbf{b}_2 \pmod{\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\rangle}$. Since $[y, P] = \alpha([x, P]) = \langle \alpha(\mathbf{z}_3)\rangle$ has order 2 (recall $x \in \{\mathbf{z}_1, \mathbf{z}_2\}$), this implies $\mathbf{z}_3 \notin [y, P]$, so $[y, \mathbf{a}_3] = 1$, and $y_3 \in \langle \mathbf{a}_3\rangle$. But then $[y, g\boldsymbol{\tau}] \equiv \mathbf{b}_1\mathbf{b}_2 \pmod{\langle \mathbf{a}_1, \mathbf{a}_2\rangle}$, hence it has order four, which again is impossible since $[y, P]$ has order two. We conclude that $\mathbf{a}_1\mathbf{a}_2 \in P$.

**Step 3:** Assume $P \ntrianglelefteq T^*$. Set $T_1 = \langle \mathbf{z}_1, \mathbf{z}_3, \mathbf{a}_1\mathbf{a}_2, \mathbf{a}_3\rangle \le P$, and consider the extension

$$1 \longrightarrow \underset{=\langle \mathbf{a}_1, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\rangle \cong C_2^4}{R_0/T_1} \longrightarrow T^*/T_1 \longrightarrow \underset{=\langle \mathbf{c}, \tau\rangle \cong C_2^2}{T^*/R_0} \longrightarrow 1.$$

We want to apply Lemma 1.9, with $S = T^*/T_1$ and $S_0 = R_0/T_1 \cong C_2^4$, where we recall $R_0 = \langle \mathbf{a}_i, \mathbf{b}_i \,|\, i = 1,2,3\rangle$. Set $P_0 = P \cap R_0$. Since $[\langle \mathbf{a}_1, \mathbf{b}_1\mathbf{b}_2, \mathbf{b}_3\rangle, T^*] \le T_1 \le P$,

$$N_{T^*}(P) \ge T_2 \overset{\text{def}}{=} \langle T_1, \mathbf{a}_1, \mathbf{b}_1\mathbf{b}_2, \mathbf{b}_3\rangle \quad \text{and hence} \quad [T_2 : P \cap T_2] \le 2. \qquad (6)$$

Recall, in the notation of Lemma 1.9, that $m$ is the number of classes $xR_0 \in T^*/R_0$ such that $xR_0 \neq R_0$ and $[x, R_0] \leq P_0$. Since $[\boldsymbol{\tau}, R_0/T_1] = \langle \mathbf{b}_1\mathbf{b}_2T_1\rangle$, $[\mathbf{c}, R_0/T_1] = \langle \mathbf{a}_1T_1\rangle$, and $[\mathbf{c}\boldsymbol{\tau}, R_0/T_1] = \langle \mathbf{a}_1\mathbf{b}_1\mathbf{b}_2T_1\rangle$,

$$m = \bigl|\{\mathbf{a}_1, \mathbf{b}_1\mathbf{b}_2, \mathbf{a}_1\mathbf{b}_1\mathbf{b}_2\} \cap P\bigr|. \tag{7}$$

At least one of the elements $\mathbf{a}_1$, $\mathbf{b}_1\mathbf{b}_2$, or $\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2$ is in $P_0$ by (6), so $m \geq 1$.

By Lemma 1.9, we are left with the following cases, where $g \in N_{R_0}(P){\smallsetminus}P$:

**(b) $\mathbf{rk}(R_0/P_0) = 1$, $|P/P_0| = 2$, $m = 1$,** and $P_0 \ntrianglelefteq T^*$. By (7), $P_0$ contains exactly one of the elements $\mathbf{a}_1$, $\mathbf{b}_1\mathbf{b}_2$, or $\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2$. Fix $g \in \{\mathbf{a}_1, \mathbf{b}_1\mathbf{b}_2\}$ such that $g \notin P$. Let $h \in \{\mathbf{c}, \boldsymbol{\tau}, \mathbf{c}\boldsymbol{\tau}\}$ be such that $h \in PR_0$ ($|PR_0/R_0| = |P/P_0| = 2$).

Since $|R_0/P_0| = 2$, at least one of the elements $\mathbf{a}_1$, $\mathbf{b}_1$, or $\mathbf{a}_1\mathbf{b}_1$ is in $P_0$, and so $\mathbf{z}_1 = \mathbf{a}_1^2 = \mathbf{b}_1^2 = (\mathbf{a}_1\mathbf{b}_1)^2 \in \mathrm{Fr}(P)$. Since $\mathbf{z}_3 = \mathbf{a}_3^2 \in \mathrm{Fr}(P)$, $\mathrm{Fr}(P) \geq \langle \mathbf{z}_1, \mathbf{z}_2\rangle$.

By Lemma 3.5, there are elements $r, s \in P$ such that $s = \alpha(r)$ for some $\alpha \in \mathrm{Aut}(P)$, $r \in [g, P]$, $r \notin \mathrm{Fr}(P)$, and $[g, s] \notin \mathrm{Fr}(P)$. Since $[g, P] \leq [g, T^*] = \langle \mathbf{z}_1, \mathbf{a}_1\mathbf{a}_2\rangle$ (recall $g \in \{\mathbf{a}_1, \mathbf{b}_1\mathbf{b}_2\}$), this means that $r, [g, s] \in \{\mathbf{a}_1^i\mathbf{a}_2^j \,|\, i, j = \pm 1\}$. In particular, $[r, P] \leq \langle \mathbf{z}_1, \mathbf{z}_2\rangle$ has exponent two, so $[s, P] = \alpha([r, P])$ also has exponent two.

Now, $s \in P{\smallsetminus}R_0$ since $[g, R_0] \leq [R_0, R_0] \leq \langle \mathbf{z}_1, \mathbf{z}_2\rangle$, and thus $s = hs_0$ for some $s_0 \in R_0$. Since $|R_0/P_0| = 2$ and $g \in R_0{\smallsetminus}P_0$, $\mathbf{b}_1x \in P_0$ for some $x \in \langle g\rangle$. By the previous paragraph, $[hs_0, \mathbf{b}_1x] \in [s, P]$ has order at most 2. Set $K = \langle \mathbf{z}_1, \mathbf{a}_1\mathbf{a}_2\rangle \lhd T^*$. Then $[s_0, \mathbf{b}_1x] \in [R_0, R_0] \leq K$, $[h, x] \in [T^*, \langle \mathbf{a}_1, \mathbf{b}_1\mathbf{b}_2\rangle] \leq K$, and

$$[\boldsymbol{\tau}, \mathbf{b}_1] = \mathbf{b}_2\mathbf{b}_1^{-1}, \qquad [\mathbf{c}, \mathbf{b}_1] = (\mathbf{a}_1\mathbf{b}_1)\mathbf{b}_1^{-1} = \mathbf{a}_1, \qquad \text{and} \qquad [\mathbf{c}\boldsymbol{\tau}, \mathbf{b}_1] = (\mathbf{a}_2\mathbf{b}_2)\mathbf{b}_1^{-1}.$$

Thus $[s, \mathbf{b}_1x]$ is in one of the cosets $\mathbf{b}_1\mathbf{b}_2K$, $\mathbf{a}_1K$, or $\mathbf{b}_1\mathbf{a}_2\mathbf{b}_2K$. All of the elements in these cosets have order four, in contradiction with what was already shown. So this case is impossible.

**(c) $\mathbf{rk}(R_0/P_0) = 1$, $|P/P_0| = 2$, $m = 3$,** and $P_0 \lhd T^*$. Since $[\mathbf{c}, \boldsymbol{\tau}] = 1$, this would imply $[T^*, T^*] = [R_0, T^*] \leq P$, and hence $P \lhd T^*$.

**(e) $\mathbf{rk}(R_0/P_0) = 2$, $|P/P_0| = 4$, and $m = 1$.** By (6), $[T_2{:}P_0 \cap T_2] \leq 2$, where $T_2 = \langle T_1, \mathbf{a}_1, \mathbf{b}_1\mathbf{b}_2, \mathbf{b}_3\rangle$ has index two in $R_0$. Since $[R_0{:}P_0] = 4$, this implies that $P_0 \leq T_2$ with index two. Also, by (7), exactly one of the elements $\mathbf{a}_1$, $\mathbf{b}_1\mathbf{b}_2$, or $\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2$ is in $P_0$. This leaves the following possibilities for $P$:

- ($\mathbf{a}_1 \in P$) $P = \langle T_1, \mathbf{a}_1, \mathbf{b}_3x, \mathbf{c}y, \boldsymbol{\tau}z\rangle$ for some $x \in \langle \mathbf{b}_1\mathbf{b}_2\rangle$ and $y, z \in \langle \mathbf{b}_1, \mathbf{b}_2\rangle$. We take $g = \mathbf{b}_1\mathbf{b}_2$. In all of these cases, $\mathbf{z}_1 = \mathbf{a}_1^2$ and $\mathbf{a}_1\mathbf{a}_2 \equiv [\mathbf{a}_1, \boldsymbol{\tau}z] \pmod{\langle \mathbf{z}_1, \mathbf{z}_2\rangle}$ are both in $\mathrm{Fr}(P)$, and so $[g, P] \leq \langle \mathbf{z}_1, \mathbf{a}_1\mathbf{a}_2\rangle \leq \mathrm{Fr}(P)$.
- ($\mathbf{b}_1\mathbf{b}_2 \in P$) $P = \langle T_1, \mathbf{b}_1\mathbf{b}_2, \mathbf{b}_3x, \mathbf{c}y, \boldsymbol{\tau}z\rangle$ for some $x \in \langle \mathbf{a}_1\rangle$ and $y, z \in \langle \mathbf{a}_1, \mathbf{b}_1\rangle$. We take $g = \mathbf{a}_1$. Then $(\mathbf{c}y)^2 \in P_0$ implies $y \in \langle \mathbf{a}_1\rangle$, and $[\mathbf{c}y, \boldsymbol{\tau}z] \in P_0$ implies $z \in \langle \mathbf{a}_1\rangle$. We can also arrange that $y = 1$ by replacing $P$ by $\mathbf{b}_1P\mathbf{b}_1^{-1}$ if necessary. Then $\mathrm{Fr}(P)$ always contains $[\mathbf{c}, \mathbf{b}_1\mathbf{b}_2] = \mathbf{a}_1\mathbf{a}_2$. Since $(\mathbf{b}_3\mathbf{a}_1)^2 = \mathbf{z}_2$ and $[\boldsymbol{\tau}\mathbf{a}_1, \mathbf{b}_1\mathbf{b}_2] = \mathbf{z}_2$, $[g, P] \leq \langle \mathbf{z}_1, \mathbf{a}_1\mathbf{a}_2\rangle \leq \mathrm{Fr}(P)$ if either $x = \mathbf{a}_1$ or $z = \mathbf{a}_1$. If $x = z = 1$, then $P = H_1$.
- ($\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2 \in P$) $P = \langle T_1, \mathbf{a}_1\mathbf{b}_1\mathbf{b}_2, \mathbf{b}_3x, \mathbf{c}y, \boldsymbol{\tau}z\rangle$ for some $x \in \langle \mathbf{a}_1\rangle$ and $y, z \in \langle \mathbf{a}_1, \mathbf{b}_1\rangle$. We take $g = \mathbf{a}_1$. Then $(\mathbf{c}y)^2 \in P_0$ implies $y \in \langle \mathbf{a}_1\rangle$, and $[\mathbf{c}y, \boldsymbol{\tau}z] \in P_0$ implies $z \in \langle \mathbf{a}_1\rangle$. In all cases, $Z(P/\langle \mathbf{z}_1, \mathbf{z}_2\rangle) \leq T_1/\langle \mathbf{z}_1, \mathbf{z}_2\rangle$. Hence $Z(P) \leq C_{T_1}(\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2, \mathbf{b}_3x) = \langle \mathbf{z}_1, \mathbf{z}_2\rangle$, and $Z_2(P) \leq T_1$. Thus $[g, P] \leq \langle \mathbf{z}_1, \mathbf{a}_1\mathbf{a}_2\rangle \leq Z_2(T^*) \leq Z_2(P)$ and $[g, Z_2(P)] = 1$.

Thus in all cases except when $P$ is conjugate to $H_1$, $P$ fails to be critical by (5).

This finishes Step 3: the only critical subgroups of $T^*$ which are not normal are $H_1$ and $H_2$.

**Step 4:** It remains to handle the case where $P \lhd T^*$; i.e., where $P$ has index 2 in $T^*$. Thus $P$ contains

$$T^{*\prime} = [T^*, T^*] = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1\mathbf{b}_2 \rangle$$

(recall $[\mathbf{c}, \mathbf{b}_i] = \mathbf{a}_i$ by (1)). Also, $\mathrm{Fr}(P) \geq L_3(T^*) = [[T^*, T^*], T^*] = \langle \mathbf{z}_1, \mathbf{a}_1\mathbf{a}_2 \rangle$: $\mathbf{z}_1 = \mathbf{a}_1^2 \in \mathrm{Fr}(P)$, and $\mathbf{a}_1\mathbf{a}_2 \equiv [\mathbf{a}_1, \boldsymbol{\tau}] \equiv [\mathbf{a}_1, \boldsymbol{\tau}\mathbf{c}] \equiv [\mathbf{b}_1\mathbf{b}_2, \mathbf{c}] \pmod{\langle \mathbf{z}_1, \mathbf{z}_3 \rangle}$ (and one of the elements $\mathbf{c}$, $\boldsymbol{\tau}$, or $\boldsymbol{\tau}\mathbf{c}$ must be in $P$). For any $g \in N_{T^*}(P) \smallsetminus P$, $[g, P] \leq T^{*\prime}$ and $[g, T^{*\prime}] \leq L_3(T^*) \leq \mathrm{Fr}(P)$. Hence $T^{*\prime}$ is not characteristic in $P$ by (5).

Consider the group $S$ of Lemma 6.4. Set $x_1 = \mathbf{b}_3$, $x_2 = \mathbf{b}_3\mathbf{c}$, $x_3 = \mathbf{b}_1$, and $x_4 = \boldsymbol{\tau}$. These have the property that $x_i^2 \in L_3(T^*)$, the three commutators $[x_i, x_{i+1}]$ form a basis for $T^{*\prime}/L_3(T^*) \cong C_2^3$, and $[x_i, x_j] = 1$ when $|i - j| \geq 2$. Hence there is an epimorphism $\varphi\colon T^* \longrightarrow S$, defined by $\varphi(x_i) = g_i$, with $\mathrm{Ker}(\varphi) = L_3(T^*)$. By Lemma 6.4, either $\mathrm{Fr}(P) = T^{*\prime}$, in which case $T^{*\prime}$ is characteristic and $P$ is not critical; or $P$ is one of the groups

$$U_i = \langle T^{*\prime}, x_j \,|\, j \neq i \rangle \ (1 \leq i \leq 4), \quad U_{13} = \langle T^{*\prime}, x_1x_3, x_2, x_4 \rangle, \quad U_{24} = \langle T^{*\prime}, x_1, x_3, x_2x_4 \rangle.$$

Of these six cases, $U_2 = R_2$, $U_3 = R_4$, $U_4 = R_1$, and $U_{24} = R_3$. So it remains to show that $U_1$ and $U_{13}$ are not critical.

If $P = U_{13} = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1\mathbf{b}_2, \mathbf{b}_1\mathbf{b}_3, \mathbf{b}_3\mathbf{c}, \boldsymbol{\tau} \rangle$, then $\mathrm{Fr}(P) = \langle \mathbf{z}_1, \mathbf{a}_1\mathbf{a}_2, \mathbf{a}_1\mathbf{a}_3, \mathbf{b}_1\mathbf{b}_2 \rangle$. Set $H = \langle \mathbf{a}_1\mathbf{a}_2, \mathbf{a}_1\mathbf{a}_3 \rangle \cong C_4^2$. Thus is the unique abelian subgroup of index two in $\mathrm{Fr}(P)$, since any abelian subgroup not in $H$ is contained in $C_{\mathrm{Fr}(P)}(g) = \langle \mathbf{z}_1, \mathbf{z}_2, g \rangle$ for some $g \in \mathrm{Fr}(P) \smallsetminus H$. Hence $H$ is characteristic in $P$, and so is the subgroup $L_3(T^*) = \langle z_1, \mathbf{a}_1\mathbf{a}_2 \rangle$, since it is the subgroup of elements in $H$ which are inverted under conjugation by $\mathbf{b}_1\mathbf{b}_2$. Also, $Z(P/L_3(T^*)) = T^{*\prime}/L_3(T^*)$ by Lemma 6.4 again, so $T^{*\prime}$ is also characteristic, and $P$ is not critical.

If $P = U_1 = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\mathbf{c}, \boldsymbol{\tau} \rangle$, then $\mathrm{Fr}(P) = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1\mathbf{b}_2 \rangle$ again contains a unique abelian subgroup $H = \langle \mathbf{a}_1, \mathbf{a}_2 \rangle \cong C_4^2$ of index two. So $H$ and $\Omega_1(H) = \langle \mathbf{z}_1, \mathbf{z}_2 \rangle$ are characteristic in $P$. Also, $C_P(H) = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_3\mathbf{c} \rangle$ is characteristic, and since $(\mathbf{b}_3\mathbf{c})^2 = \mathbf{a}_1\mathbf{a}_2\mathbf{z}_3$, $\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \rangle / \langle \mathbf{z}_1, \mathbf{z}_2 \rangle$ is the 2-torsion subgroup of $C_P(H)/\langle \mathbf{z}_1, \mathbf{z}_2 \rangle$. So $\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \rangle$ is characteristic, $T^{*\prime} = \langle \mathrm{Fr}(P), \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \rangle$ is characteristic, and again $P$ is not critical. □

## 7.2 Automorphisms of critical subgroups

We first define automorphisms $\beta_1^*, \beta_2^*, \beta_3^*, \beta_4^* \in \mathrm{Aut}(T^*)$ via the following table:

| $g$ | $\mathbf{a}_1$ | $\mathbf{a}_2$ | $\mathbf{a}_3$ | $\mathbf{b}_1$ | $\mathbf{b}_2$ | $\mathbf{b}_3$ | $\mathbf{c}$ | $\boldsymbol{\tau}$ |
|---|---|---|---|---|---|---|---|---|
| $\beta_1^*(g)$ | $\mathbf{a}_1$ | $\mathbf{a}_2$ | $\mathbf{a}_3$ | $\mathbf{b}_1$ | $\mathbf{b}_2$ | $\mathbf{b}_3$ | $\mathbf{z}_3\mathbf{c}$ | $\boldsymbol{\tau}$ |
| $\beta_2^*(g)$ | $\mathbf{a}_1$ | $\mathbf{a}_2$ | $\mathbf{a}_3$ | $\mathbf{b}_1$ | $\mathbf{b}_2$ | $\mathbf{a}_3\mathbf{b}_3$ | $\mathbf{c}$ | $\boldsymbol{\tau}$ |
| $\beta_3^*(g)$ | $\mathbf{a}_1$ | $\mathbf{a}_2$ | $\mathbf{a}_3$ | $\mathbf{z}_3\mathbf{b}_1$ | $\mathbf{z}_3\mathbf{b}_2$ | $\mathbf{b}_3$ | $\mathbf{c}$ | $\boldsymbol{\tau}$ |
| $\beta_4^*(g)$ | $\mathbf{a}_1$ | $\mathbf{a}_2$ | $\mathbf{a}_3$ | $\mathbf{z}_3\mathbf{b}_1$ | $\mathbf{z}_1\mathbf{b}_2$ | $\mathbf{z}_2\mathbf{b}_3$ | $\mathbf{c}$ | $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\boldsymbol{\tau}$ |

Here, $\beta_1^*$ and $\beta_3^*$ are automorphisms since they have the form $\beta_i^*(g) = g{\cdot}\varphi_i(g)$ for some $\varphi_i \in \mathrm{Hom}(T^*, Z(T^*))$; and $\beta_2^*|_{R_2}$ and $\beta_4^*|_{R_1}$ are automorphisms for similar reasons. One easily checks that $\beta_2^*([\mathbf{c}, g]) = [\mathbf{c}, \beta_2^*(g)]$ for all $g \in R_2$, and hence that $\beta_2^* \in \mathrm{Aut}(T^*)$. As for $\beta_4^*$, since $(\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\boldsymbol{\tau})^2 = \boldsymbol{\tau}^2 = 1$, the only tricky point to check is that for $i = 1, 2, 3$

(and taking indices modulo 3, so that $\boldsymbol{\tau}\mathbf{b}_i\boldsymbol{\tau}^{-1} = \mathbf{b}_{-i}$):

$$\begin{aligned}[][\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\boldsymbol{\tau}, \beta_4^*(\mathbf{b}_i)] &= [\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\boldsymbol{\tau}, \mathbf{z}_{i-1}\mathbf{b}_i] = c_{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3}(\mathbf{z}_{-i+1}\mathbf{b}_{-i})\cdot(\mathbf{z}_{i-1}\mathbf{b}_i)^{-1} \\ &= (\mathbf{z}_{-i+1}\mathbf{z}_{-i}\mathbf{b}_{-i})(\mathbf{z}_{i-1}\mathbf{b}_i)^{-1} = \mathbf{z}_{-i-1}\mathbf{z}_{i-1}\mathbf{b}_{-i}\mathbf{b}_i^{-1} = \beta_4^*(\mathbf{b}_{-i}\mathbf{b}_i^{-1}) = \beta_4^*([\boldsymbol{\tau}, \mathbf{b}_i]).\end{aligned}$$

We will show in Lemma 7.4 that $\mathrm{Out}(T^*) \cong C_2^4$ with the elements $[\beta_1^*], \dots, [\beta_4^*]$ as generators.

Next let $\gamma \in \mathrm{Aut}(R_1)$ be the automorphism of order 3 where $\gamma([\![R,S,T]\!]) = [\![T,R,S]\!]$ for $R,S,T \in Q_{16}$. Thus $\gamma(\mathbf{a}_i) = \mathbf{a}_{i+1}$ and $\gamma(\mathbf{b}_i) = \mathbf{b}_{i+1}$, with indices taken modulo 3, and $\gamma(\mathbf{c}) = \mathbf{c}$. For all $i = 1,2,3,4$, set

$$\beta_i = \beta_i^*|_{R_1} \qquad \text{and} \qquad \beta_i' = \gamma\beta_i\gamma^{-1}.$$

Thus $\beta_4' = \beta_4$. We will see in the next lemma that as subgroups of $\mathrm{Out}(R_1)$, $\langle[\beta_i],[\beta_i']\rangle \cong C_2^2$ for $i = 1,2,3$, $\langle[\beta_4]\rangle = \langle[\beta_4']\rangle \cong C_2$, and each of these is normalized by $\gamma$.

For use in the following lemma, we define $\mathcal{Q}_i = \langle \mathbf{a}_i, \mathbf{b}_i\rangle \le T^*$ for $i = 1,2,3$. Thus $\mathcal{Q}_1 \cong \mathcal{Q}_2 \cong \mathcal{Q}_3 \cong Q_8$, $R_0 = \mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3$, the $\mathcal{Q}_i$ commute pairwise with each other, and the inclusions $\mathcal{Q}_i \le R_0$ define an isomorphism $R_0 \cong (Q_8)^3/C_2$. Also, we set

$$\mathrm{Aut}^0(R_0) = \bigl\{\alpha \in \mathrm{Aut}(R_0) \,\big|\, \alpha|_{Z(R_0)} = \mathrm{Id}\bigr\} \quad \text{and} \quad \mathrm{Out}^0(R_0) = \mathrm{Aut}^0(R_0)/\mathrm{Inn}(R_0).$$

**Lemma 7.4.** (a) *If $P \le T^*$ and $P \cong R_0$, then $P = R_0$.*

(b) *Each $\alpha \in \mathrm{Aut}^0(R_0)$ sends each of the subgroups $\mathcal{Q}_iZ(R_0)$ ($i = 1,2,3$) to itself.*

(c) *$\mathrm{Out}^0(R_0) \cong (\Sigma_4)^3$, and $\mathrm{Out}(R_0) = \mathrm{Out}^0(R_0) \rtimes \langle[\gamma], c_{\boldsymbol{\tau}}\rangle \cong \Sigma_4 \wr \Sigma_3$. The identification of $\Sigma_4$ with the group of automorphisms of $\mathcal{Q}_iZ(R_0) \cong Q_8 \times C_2$ which are the identity on its center is induced by the action of this automorphism group on the set of the four subgroups of $\mathcal{Q}_iZ(R_0)$ isomorphic to $Q_8$.*

(d) *$\mathrm{Out}(R_1) = \langle[\beta_1],[\beta_2],[\beta_3],[\beta_4],[\beta_1'],[\beta_2'],[\beta_3']\rangle \rtimes \langle[\gamma], c_{\boldsymbol{\tau}}\rangle \cong C_2^7 \rtimes \Sigma_3$. The conjugation action in $\mathrm{Out}(R_1)$ of $\langle[\gamma], c_{\boldsymbol{\tau}}\rangle \cong \Sigma_3$ on $\langle[\beta_i],[\beta_i']\rangle \cong C_2^7$ is the following: $[c_{\boldsymbol{\tau}},[\beta_i]] = 1$ for all $i$; $[[\gamma],[\beta_4]] = 1$; and for $i = 1,2,3$, $\gamma\beta_i\gamma^{-1} = \beta_i'$ and $\gamma\beta_i'\gamma^{-1} \equiv c_{\boldsymbol{\tau}}\beta_i'c_{\boldsymbol{\tau}}^{-1} \equiv \beta_i\beta_i'$ (mod $\mathrm{Inn}(R_1)$).*

(e) *For $k = 2,3$, restriction to $R_0$ induces an isomorphism*

$$\mathrm{Out}(R_k) \xrightarrow{\ \cong\ } C_{\mathrm{Out}^0(R_0)}(\mathrm{Out}_{R_k}(R_0)) \cong \begin{cases} \Sigma_4 \times \Sigma_4 & \text{if } k = 2 \\ \Sigma_4 \times C_2^2 & \text{if } k = 3. \end{cases}$$

(f) *$\mathrm{Out}(T^*) = \langle[\beta_1^*],[\beta_2^*],[\beta_3^*],[\beta_4^*]\rangle \cong C_2^4$. Every automorphism of $R_1$ which commutes with $c_{\boldsymbol{\tau}}$ in $\mathrm{Out}(R_1)$ extends to an automorphism of $T^*$, and every automorphism in $\mathrm{Aut}^0(R_0)$ which commutes with $c_{\boldsymbol{\tau}}$ and $c_{\mathbf{c}}$ in $\mathrm{Out}(R_0)$ extends to an automorphism of $T^*$.*

*Proof.* **(a)** Let $P \le T^*$ be any subgroup isomorphic to $R_0$. Set $\mathbf{A} = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{c}\rangle \cong C_4^3$. For each $H \lhd R_0$, either $\langle \mathbf{z}_1, \mathbf{z}_2\rangle \le H$ and $R_0/H$ is elementary abelian; or $\mathbf{z}_i \notin H$ for some $i$, $\mathcal{Q}_i \cap H = 1$, and so $R_0/H$ contains a subgroup $\cong Q_8$. Hence for each $H \lhd P$, either $H \ge Z(P)$ and $P/H$ is elementary abelian, or $P/H$ contains a subgroup $\cong Q_8$. When $H = P \cap \mathbf{A}$, then $P/H \cong P\mathbf{A}/\mathbf{A}$ is contained in $T^*/\mathbf{A} \cong C_2 \times D_8$ (Lemma 7.1(a)), which contains no subgroup isomorphic to $Q_8$. We conclude that $Z(P) \le H \le \mathbf{A}$, and $P\mathbf{A}/\mathbf{A}$ is elementary abelian. Also, $P \not\ge \mathbf{A}$, since $R_0$ contains no subgroup $C_4^3$. Thus $P\mathbf{A}/\mathbf{A} \cong C_2^k$, $k \le 3$ since $T^*/\mathbf{A} \cong C_2 \times D_8$, $|P \cap \mathbf{A}| = 2^{8-k} \le 2^5$; and hence $k = 3$ and $P \cap \mathbf{A} \cong C_4^2 \times C_2$.

Thus either $P\mathbf{A}/\mathbf{A} = \langle \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \rangle$ or $P\mathbf{A}/\mathbf{A} = \langle \mathbf{b}_1\mathbf{b}_2, \mathbf{b}_3, \boldsymbol{\tau} \rangle$. In either case, $\mathbf{b}_3 g \in P$ for some $g \in \mathbf{A}$. Hence $Z(P) \le C_{\mathbf{A}}(\mathbf{b}_3 g) = C_{\mathbf{A}}(\mathbf{b}_3) = \langle \mathbf{a}_1, \mathbf{a}_2 \rangle$, and so $Z(P) = \langle \mathbf{z}_1, \mathbf{z}_2 \rangle = Z(R_0)$ since it is 2-torsion. Thus $P \le C_{T^*}(\langle \mathbf{z}_1, \mathbf{z}_2 \rangle) = R_1$. Also,

$$R_1/Z(R_0) = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{c}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \rangle / Z(R_0) = (R_0/Z(R_0)) \cdot \langle \mathbf{c} \rangle \cong C_2^6 \rtimes C_2,$$

and $\mathbf{c}$ acts on $R_0/Z(R_0)$ centralizing only $\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \rangle$. Hence $R_0/Z(R_0)$ is the unique elementary abelian subgroup of rank six in $R_1/Z(R_0)$, so $P = R_0$, and this proves (a).

**(b)** Fix $\alpha \in \mathrm{Aut}^0(R_0)$. Since $(\alpha(\mathbf{a}_1))^2 = \alpha(\mathbf{a}_1^2) = \mathbf{z}_1$, either $\alpha(\mathbf{a}_1) \in \mathcal{Q}_1 Z(R_0)$, or $\alpha(\mathbf{a}_1) = x_2 x_3$ for some $x_i \in \mathcal{Q}_i$ ($i = 2, 3$) of order 4. In this last case, $[x_2 x_3, \mathcal{Q}_i] = [x_i, \mathcal{Q}_i] = \langle \mathbf{z}_i \rangle$ for $i = 2, 3$, so $[\alpha(\mathbf{a}_1), R_0] = Z(R_0)$. This is impossible, since $[\mathbf{a}_1, R_0] = \langle \mathbf{z}_1 \rangle$ has order two, and we conclude that $\alpha(\mathbf{a}_1) \in \mathcal{Q}_1 Z(R_0)$. Similar arguments show that $\alpha(\mathbf{a}_i), \alpha(\mathbf{b}_i) \in \mathcal{Q}_i Z(R_0)$ for each $i = 1, 2, 3$, and thus $\alpha(\mathcal{Q}_i) \le \mathcal{Q}_i Z(R_0)$.

**(c)** By (b), each $\alpha \in \mathrm{Aut}^0(R_0)$ leaves invariant each of the subgroups $\mathcal{Q}_i Z(R_0)$ for $i = 1, 2, 3$. The image of $\mathrm{Out}^0(R_0)$ under the projection to $\mathrm{Aut}(R_0/Z(R_0))$ is thus the group of automorphisms of $C_2^2 \times C_2^2 \times C_2^2$ which send each factor to itself, and hence is isomorphic to $(\Sigma_3)^3$. The group of automorphisms of $R_0$ which induce the identity on $R_0/Z(R_0)$ (and hence also on $Z(R_0)$) is isomorphic to $\mathrm{Hom}(R_0/Z(R_0), Z(R_0)) \cong C_2^{12}$, and this group contains $\mathrm{Inn}(R_0) \cong R_0/Z(R_0) \cong C_2^6$. We thus have an extension

$$1 \longrightarrow C_2^6 \longrightarrow \mathrm{Out}^0(R_0) \longrightarrow \Sigma_3 \times \Sigma_3 \times \Sigma_3 \longrightarrow 1.$$

In particular, $|\mathrm{Out}^0(R_0)| = 2^6 \cdot 6^3 = 24^3$.

We now make this more explicit. For each $i = 1, 2, 3$, define

$$\mathcal{Q}_{i1} = \mathcal{Q}_i = \langle \mathbf{a}_i, \mathbf{b}_i \rangle, \qquad \mathcal{Q}_{i2} = \langle \mathbf{a}_i\mathbf{z}_j, \mathbf{b}_i \rangle, \qquad \mathcal{Q}_{i3} = \langle \mathbf{a}_i, \mathbf{b}_i\mathbf{z}_j \rangle, \qquad \mathcal{Q}_{i4} = \langle \mathbf{a}_i\mathbf{z}_j, \mathbf{b}_i\mathbf{z}_j \rangle$$

for any $j \ne i$; these are the four subgroups of $\mathcal{Q}_i Z(R_0)$ isomorphic to $Q_8$. Let

$$\omega\colon \mathrm{Out}^0(R_0) \xrightarrow{\ \cong\ } \Sigma_4 \times \Sigma_4 \times \Sigma_4$$

be the isomorphism which sends $[\alpha]$, for $\alpha \in \mathrm{Aut}^0(R_0)$, to the triple of permutations of the $\mathcal{Q}_{ik}$ induced by $\alpha$. Thus, for example,

$$\begin{aligned} &\omega(\beta_2|_{R_0}) = \big(I, I, (2\,4)\big) && \omega(\beta_2'|_{R_0}) = \big((2\,4), I, I\big) \\ &\omega(\beta_3|_{R_0}) = \big((1\,3)(2\,4), (1\,3)(2\,4), I\big) && \omega(\beta_3'|_{R_0}) = \big(I, (1\,3)(2\,4), (1\,3)(2\,4)\big) \\ &\omega(\beta_4|_{R_0}) = \big((1\,3)(2\,4), (1\,3)(2\,4), (1\,3)(2\,4)\big) && \omega(c_{\mathbf{c}}) = \big((2\,4), (2\,4), (2\,4)\big)\ . \end{aligned}$$

If $\alpha$ is such that $[\alpha] \in \mathrm{Ker}(\omega)$, then $\alpha$ sends each $\mathcal{Q}_i$ to itself via the identity modulo $Z(\mathcal{Q}_i) = \langle \mathbf{z}_i \rangle$. Thus $\alpha|_{\mathcal{Q}_i} \in \mathrm{Inn}(\mathcal{Q}_i)$ for each $i$, and $\alpha \in \mathrm{Inn}(R_0)$. We conclude that $\omega$ is injective, and hence an isomorphism since the source and target both have order $24^3$.

Since $\langle \gamma|_{Z(R_0)}, c_{\boldsymbol{\tau}}|_{Z(R_0)} \rangle = \mathrm{Aut}(Z(R_0))$, $\mathrm{Aut}(R_0) = \mathrm{Aut}^0(R_0) \rtimes \langle \gamma, c_{\boldsymbol{\tau}} \rangle$, and similarly for $\mathrm{Out}(R_0)$. Hence $\omega$ extends to an isomorphism $\mathrm{Out}(R_0) \xrightarrow{\cong} \Sigma_4 \wr \Sigma_3$; for example, by regarding $\Sigma_4 \wr \Sigma_3$ as a group of permutations of the twelve subgroups $\mathcal{Q}_{ik}$.

**(d)** By Lemma 1.2 (and by (a)), there is an exact sequence

$$1 \longrightarrow H^1(R_1/R_0; Z(R_0)) \xrightarrow{\ \eta\ } \mathrm{Out}(R_1) \xrightarrow{\ \mathrm{Res}_{R_0}\ } C_{\mathrm{Out}(R_0)}(\langle c_{\mathbf{c}} \rangle)/\langle c_{\mathbf{c}} \rangle. \qquad (8)$$

Since $[\mathbf{c}, Z(R_0)] = 1$, $H^1(R_1/R_0; Z(R_0)) = \mathrm{Hom}(\langle \mathbf{c} \rangle, Z(R_0)) \cong (\mathbb{Z}/2)^2$. Hence $\mathrm{Im}(\eta) = \langle [\beta_1], [\beta_1'] \rangle$: since $[\beta_1] = \eta(\mathbf{c} \mapsto \mathbf{z}_3)$ (recall $\beta_1(\mathbf{c}) = \mathbf{z}_3\mathbf{c}$ and $\beta_1|_{R_0} = \mathrm{Id}$), and similarly

$[\beta_1'] = \eta(\mathbf{c} \mapsto \mathbf{z}_1)$. From the above table of values of $\omega(-)$, we see that

$$C_{\mathrm{Out}^0(R_0)}(c_{\mathbf{c}}) = \omega^{-1}\bigl(\{(\sigma_1,\sigma_2,\sigma_3)\,|\,\sigma_i \in \langle(1\,3),(2\,4)\rangle\}\bigr) \\ = \langle \beta_2|_{R_0}, \beta_2'|_{R_0}, \beta_3|_{R_0}, \beta_3'|_{R_0}, \beta_4|_{R_0}, c_{\mathbf{c}}\rangle \cong C_2^6.$$

Hence $C_{\mathrm{Out}(R_0)}(c_{\mathbf{c}})/\langle c_{\mathbf{c}}\rangle \cong C_2^5 \rtimes \Sigma_3$ is generated by the classes of the restrictions of the $\beta_i$ ($i = 2,3,4$), $\beta_i'$ ($i = 2,3$), $\gamma$, and $c_{\boldsymbol{\tau}}$ (recall $\langle\gamma, c_{\boldsymbol{\tau}}\rangle \cong \Sigma_3$). So by (8), $\mathrm{Out}(R_1)$ is generated by these elements together with $[\beta_1]$ and $[\beta_1']$.

In particular, this shows that the subgroup $\mathbf{A} = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{c}\rangle$ is characteristic in $R_1$ (in fact, it is the only subgroup of $T^*$ isomorphic to $C_4^3$). So by Lemma 1.2 again, there is an exact sequence

$$1 \longrightarrow H^1(R_1/\mathbf{A};\mathbf{A}) \longrightarrow \mathrm{Out}(R_1) \xrightarrow{\ \mathrm{Res}_{\mathbf{A}}\ } N_{\mathrm{Aut}(\mathbf{A})}(\mathrm{Aut}_{R_1}(\mathbf{A}))/\mathrm{Aut}_{R_1}(\mathbf{A})\ ,$$

where $\mathrm{Res}_{\mathbf{A}}$ is induced by restriction to $\mathbf{A}$. Hence $\mathrm{Ker}(\mathrm{Res}_{\mathbf{A}}) = \langle \beta_2,\beta_3,\beta_4,\beta_2',\beta_3'\rangle$ is abelian and normal in $\mathrm{Out}(R_1)$. Since $\langle\beta_1,\beta_1'\rangle$ is also normal and abelian, this proves that the subgroup of $\mathrm{Out}(R_1)$ generated by all seven automorphisms $\beta_i$ and $\beta_i'$ is abelian and normal. Also, this subgroup has exponent two: $(\beta_2)^2 = c_{\mathbf{a}_3}$, $(\beta_2')^2 = c_{\mathbf{a}_1}$, and the others have order 2 in $\mathrm{Aut}(R_1)$.

Thus $\mathrm{Out}(R_1) \cong C_2^7 \rtimes \Sigma_3$. The description of the action of $\langle\gamma, c_{\boldsymbol{\tau}}\rangle \cong \Sigma_3$ on the normal subgroup $\langle \beta_i, \beta_i' \,|\, 1 \le i \le 4\rangle \cong C_2^7$ follows from the splitting of this group as a product of two normal subgroups, together with the fact that the factor $\langle \beta_2,\beta_3,\beta_4,\beta_2',\beta_3'\rangle$ is sent injectively to $(\Sigma_4)^3$ via $\omega$.

**(e)** Fix $k = 2,3$. Set $x_2 = \boldsymbol{\tau}$ and $x_3 = \mathbf{c}\boldsymbol{\tau}$, so $R_k = \langle R_0, x_k\rangle$. Then $R_0$ is characteristic in $R_k$ by (a), and $Z(R_0) = \langle \mathbf{z}_1, \mathbf{z}_2\rangle$ is free as an module over $\mathrm{Out}_{R_k}(R_0) = \langle c_{x_k}\rangle$. By Corollary 1.3, restriction to $R_0$ induces an isomorphism

$$\mathrm{Out}(R_k) \xrightarrow[\cong]{\ \mathrm{Res}_{R_0}\ } C_{\mathrm{Out}(R_0)}(\langle c_{x_k}\rangle)/\langle c_{x_k}\rangle.$$

When $k = 2$ and $x_k = \boldsymbol{\tau}$, then $c_{\boldsymbol{\tau}} \in \mathrm{Out}(R_0)$ is the element which exchanges two of the factors $\Sigma_4$. Hence

$$\mathrm{Out}(R_2) \cong C_{\mathrm{Out}(R_0)}(c_{\boldsymbol{\tau}})/\langle c_{\boldsymbol{\tau}}\rangle \cong C_{\mathrm{Out}^0(R_0)}(c_{\boldsymbol{\tau}}) \cong \bigl\{(\alpha,\alpha,\beta)\,\big|\,\alpha,\beta \in \Sigma_4\bigr\} \cong \Sigma_4 \times \Sigma_4\ .$$

When $k = 3$ and $x_k = \mathbf{c}\boldsymbol{\tau}$, then the image of $c_{\mathbf{c}\boldsymbol{\tau}} \in \mathrm{Out}(R_0)$ in $\Sigma_4 \wr \Sigma_3$ is the product of the triple $\omega(\mathbf{c}) = \bigl((2\,4),(2\,4),(2\,4)\bigr)$ with the transposition of the first two factors $\Sigma_4$. Thus

$$\mathrm{Out}(R_3) \cong C_{\mathrm{Out}(R_0)}(c_{\mathbf{c}\boldsymbol{\tau}})/\langle c_{\mathbf{c}\boldsymbol{\tau}}\rangle \cong C_{\mathrm{Out}^0(R_0)}(c_{\mathbf{c}\boldsymbol{\tau}}) \\ \cong \bigl\{(\alpha,(2\,4)\alpha(2\,4),\beta)\,\big|\,\alpha\in\Sigma_4,\ \beta \in \langle(1\,3),(2\,4)\rangle\bigr\} \cong \Sigma_4 \times C_2^2\ .$$

**(f)** Now, $T^*/R_1 = \langle\boldsymbol{\tau}\rangle \cong C_2$, and $c_{\boldsymbol{\tau}}$ acts on $Z(R_1) = \langle \mathbf{z}_1,\mathbf{z}_2\rangle$ by exchanging $\mathbf{z}_1$ and $\mathbf{z}_2$. So by Corollary 1.3, restriction to $R_1$ induces an isomorphism

$$\mathrm{Out}(T^*) \cong N_{\mathrm{Out}(R_1)}(\mathrm{Out}_{T^*}(R_1))/\mathrm{Out}_{T^*}(R_1) = C_{\mathrm{Out}(R_1)}(c_{\boldsymbol{\tau}})/\langle c_{\boldsymbol{\tau}}\rangle\ .$$

By (d), this centralizer is generated by the $\beta_i$ and $c_{\boldsymbol{\tau}}$, and so $\mathrm{Out}(T^*) \cong C_2^4$ is generated by the $\beta_i^*$. This also proves that every automorphism of $R_1$ which commutes with $c_{\boldsymbol{\tau}}$ in $\mathrm{Out}(R_1)$ extends to $T^*$.

Finally, if $\beta \in \mathrm{Aut}^0(R_0)$ is such that $[\beta]$ commutes with $c_{\boldsymbol{\tau}}$ and $c_{\mathbf{c}}$, then by the computations of $\omega(-)$ in the proof of (c), $\omega([\beta]) = (\sigma_1,\sigma_2,\sigma_3)$ where $\sigma_1 = \sigma_2$ and $\sigma_i \in \langle(1\,3),(2\,4)\rangle$. Hence $[\beta]$ is in the subgroup generated by the classes of $\beta_2|_{R_0}$, $\beta_3|_{R_3}$, $\beta_4|_{R_0}$, and $c_{\mathbf{c}}$, and so $\beta$ extends to an automorphism of $T^*$. □

For $i = 1, 2, 3$, let $\eta_i \in \mathrm{Aut}^0(R_0)$ denote the automorphism of order 3: $\eta_i(\mathbf{a}_i) = \mathbf{b}_i$, $\eta_i(\mathbf{b}_i) = \mathbf{a}_i\mathbf{b}_i$, and $\eta_i$ fixes $\mathbf{a}_j$ and $\mathbf{b}_j$ for $j \neq i$. Also, set $\gamma_0 = \gamma|_{R_0}$: the automorphism which permutes the subgroups $\langle \mathbf{a}_i, \mathbf{b}_i \rangle$ cyclically. Thus $\langle [\eta_1], [\eta_2], [\eta_3] \rangle \cong C_3^3$ is a Sylow 3-subgroup of $\mathrm{Out}^0(R_0)$, and $\langle [\eta_1], [\eta_2], [\eta_3], [\gamma_0] \rangle \cong C_3 \wr C_3$ is a Sylow 3-subgroup of $\mathrm{Out}(R_0)$.

Define $\eta_{12}^{(2)}, \eta_3^{(2)} \in \mathrm{Aut}(R_2)$ by setting

$$\eta_{12}^{(2)}|_{R_0} = \eta_1\eta_2, \qquad \eta_3^{(2)}|_{R_0} = \eta_3, \qquad \text{and} \qquad \eta_{12}^{(2)}(\boldsymbol{\tau}) = \eta_3^{(2)}(\boldsymbol{\tau}) = \boldsymbol{\tau}.$$

These are well defined automorphisms since $\eta_1\eta_2$ and $\eta_3$ both commute with $c_{\boldsymbol{\tau}}$ in $\mathrm{Aut}(R_0)$.

Let $\eta^{(3)} \in \mathrm{Aut}(R_3)$ be any automorphism such that $\eta^{(3)}|_{R_0} = \eta_1\eta_2^{-1}$. The existence of such an automorphism, and its uniqueness modulo $\mathrm{Inn}(R_3)$, follows from Lemma 7.4(e) once we check that $c_{\mathbf{c}\boldsymbol{\tau}}$ commutes with $\eta_1\eta_2^{-1}$ in $\mathrm{Out}(R_0)$. Since each of the automorphisms $\eta_1\eta_2^{-1}$ and $c_{\mathbf{c}\boldsymbol{\tau}}\eta_1\eta_2^{-1}c_{\mathbf{c}\boldsymbol{\tau}}^{-1}$ sends each subgroup $\mathcal{Q}_i$ to itself, it suffices to show that they induce the same maps on each group $\mathcal{Q}_i/\langle \mathbf{z}_i \rangle$, and this is easily checked. Alternatively, under the explicit isomorphism $\omega\colon \mathrm{Out}^0(R_0) \xrightarrow{\cong} (\Sigma_4)^3$ defined in the proof of Lemma 7.4, they are both sent to $\bigl((4\,3\,2), (2\,3\,4), I\bigr)$. In fact, by a direct (and long) computation, one can show that $\eta^{(3)}$ can be chosen such that $\eta^{(3)}(\mathbf{c}\boldsymbol{\tau}) = \mathbf{c}\mathbf{a}_1^{-1}\mathbf{b}_2\boldsymbol{\tau}$, but that will not be needed here.

**Proposition 7.5.** *Let $\mathcal{F}$ be a saturated fusion system over $T^*$ for which $R_1$ is $\mathcal{F}$-essential. Then $|\mathrm{Out}_{\mathcal{F}}(R_0)| = 4{\cdot}3^n$ for some $1 \leq n \leq 4$, and we say $\mathcal{F}$ has "Type $n$". Also, $\mathcal{F}$ is isomorphic to a fusion system $\mathcal{F}'$ over $T^*$ for which the automorphism groups $\mathrm{Out}_{\mathcal{F}'}(R_i)$ ($i = 0, 1, 2, 3$) are as described in the following table:*

| | $\mathrm{Out}_{\mathcal{F}'}(R_0)$ | $\mathrm{Out}_{\mathcal{F}'}(R_1)$ | $\mathrm{Out}_{\mathcal{F}'}(R_2)$ | $\mathrm{Out}_{\mathcal{F}'}(R_3)$ |
|---|---|---|---|---|
| **Type 1** | $\langle c_{\mathbf{c}}, [\gamma_0], c_{\boldsymbol{\tau}} \rangle$ | $\langle [\gamma], c_{\boldsymbol{\tau}} \rangle$ | $\langle c_{\mathbf{c}} \rangle$ | $\langle c_{\mathbf{c}} \rangle$ |
| **Type 2** | $\langle [\eta_1\eta_2\eta_3], c_{\mathbf{c}}, [\gamma_0], c_{\boldsymbol{\tau}} \rangle$ | $\langle [\gamma], c_{\boldsymbol{\tau}} \rangle$ | $\langle [\eta_{12}^{(2)}\eta_3^{(2)}], c_{\mathbf{c}} \rangle$ | $\langle c_{\mathbf{c}} \rangle$ |
| **Type 3** | $\langle [\eta_1\eta_2^{-1}], [\eta_2\eta_3^{-1}], c_{\mathbf{c}}, [\gamma_0], c_{\boldsymbol{\tau}} \rangle$ | $\langle [\gamma], c_{\boldsymbol{\tau}} \rangle$ | $\langle [\eta_{12}^{(2)}\eta_3^{(2)}], c_{\mathbf{c}} \rangle$ | $\langle [\eta^{(3)}], c_{\mathbf{c}} \rangle$ |
| **Type 4** | $\langle [\eta_1], [\eta_2], [\eta_3], c_{\mathbf{c}}, [\gamma_0], c_{\boldsymbol{\tau}} \rangle$ | $\langle [\gamma], c_{\boldsymbol{\tau}} \rangle$ | $\langle [\eta_{12}^{(2)}], [\eta_3^{(2)}], c_{\mathbf{c}} \rangle$ | $\langle [\eta^{(3)}], c_{\mathbf{c}} \rangle$ |

*If $\mathcal{F}$ has type 1 or 2, then $V \stackrel{\text{def}}{=} \langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3, \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3 \rangle$ is $\mathrm{Out}_{\mathcal{F}'}(R_0)$-invariant.*

*Proof.* Since $\mathrm{Out}_{T^*}(R_1) = \langle c_{\boldsymbol{\tau}} \rangle \in \mathrm{Syl}_2(\mathrm{Out}_{\mathcal{F}}(R_1))$ intersects trivially with $O_2(\mathrm{Out}(R_1))$ by Lemma 7.4(d), $\mathrm{Out}_{\mathcal{F}}(R_1)$ is sent injectively under projection to the quotient group $\mathrm{Out}(R_1)/O_2(\mathrm{Out}(R_1)) \cong \Sigma_3$; and since $R_1$ is $\mathcal{F}$-essential, it is sent isomorphically. Thus

$$\mathrm{Out}(R_1) = O_2(\mathrm{Out}(R_1)){\cdot}\mathrm{Out}_{\mathcal{F}}(R_1) = O_2(\mathrm{Out}(R_1)){\cdot}\langle [\gamma], c_{\boldsymbol{\tau}} \rangle.$$

So by Proposition 1.8, and since $O_2(\mathrm{Out}(R_1))$ is abelian by Proposition 7.4(d), there is $\beta_1 \in O_2(\mathrm{Aut}(R_1))$ which commutes in $\mathrm{Out}(R_1)$ with $c_{\boldsymbol{\tau}}$ and such that $\mathrm{Out}_{\mathcal{F}}(R_1) = \langle [\beta_1\gamma\beta_1^{-1}], c_{\boldsymbol{\tau}} \rangle$. Any such $\beta_1$ extends to an automorphism $\beta$ of $T^*$ by Lemma 7.4(f). So upon replacing $\mathcal{F}$ by $\beta^{-1}\mathcal{F}\beta$, we can assume $\mathrm{Out}_{\mathcal{F}}(R_1) = \langle [\gamma], c_{\boldsymbol{\tau}} \rangle$. In particular, $\gamma_0 \in \mathrm{Aut}_{\mathcal{F}}(R_0)$.

Consider the following subgroups of $\mathrm{Out}(R_0)$:

$$Q = O_2(\mathrm{Out}(R_0)), \qquad H_0 = \langle c_{\mathbf{c}}, c_{\boldsymbol{\tau}}, [\gamma_0] \rangle, \qquad \widehat{H} = \langle [\eta_1], [\eta_2], [\eta_3], H_0 \rangle;$$

$$H^* = \mathrm{Out}_{\mathcal{F}}(R_0) \geq H_0, \qquad \text{and} \qquad H = H^*Q \cap \widehat{H}.$$

By Lemma 7.4(c), $\mathrm{Out}(R_0)/Q \cong \Sigma_3 \wr \Sigma_3$, and hence $\langle Q, [\eta_1], [\eta_2], [\eta_3], [\gamma_0]\rangle/Q$ is its only Sylow 3-subgroup which contains the class of $\gamma_0$. Since $[\gamma_0] \in \mathrm{Out}_{\mathcal{F}}(R_0) = H^*$, $H^*$ is generated by $H_0$ together with its Sylow 3-subgroup, and hence is contained in $\widehat{H}Q$. Thus $H^*Q \le \widehat{H}Q$, and so $H^*Q = HQ$. We are now in the situation of Proposition 1.8: there is $\varphi_0 \in \mathrm{Aut}(R_0)$ such that $[\varphi_0] \in C_Q(H_0)$ and $\varphi_0 H^* \varphi_0^{-1} = H$. By Lemma 7.4(f), $\varphi_0$ extends to some $\varphi \in \mathrm{Aut}(T^*)$. So upon replacing $\mathcal{F}$ by $\varphi\mathcal{F}\varphi^{-1}$, we can arrange that

$$\langle c_{\mathbf{c}}, c_{\boldsymbol{\tau}}, [\gamma_0]\rangle \le \mathrm{Out}_{\mathcal{F}}(R_0) = H \le \widehat{H} = \langle c_{\mathbf{c}}, c_{\boldsymbol{\tau}}, [\eta_1], [\eta_2], [\eta_3], [\gamma_0]\rangle.$$

The only proper $\gamma_0$-invariant subgroups of $\langle [\eta_1], [\eta_2], [\eta_3]\rangle \cong C_3^3$ are $\langle [\eta_1\eta_2^{-1}], [\eta_2\eta_3^{-1}]\rangle$ and $\langle [\eta_1\eta_2\eta_3]\rangle$. Hence $\mathrm{Out}_{\mathcal{F}}(R_0)$ must now be one of the four groups listed in the above table. Also, for $i = 2, 3$, $\mathrm{Out}_{\mathcal{F}}(R_i)$ is determined by $\mathrm{Out}_{\mathcal{F}}(R_0)$ as described in that table: for each $\beta \in \mathrm{Aut}_{\mathcal{F}}(R_0)$ such that $[\beta]$ is $\mathrm{Out}_{R_i}(R_0)$-invariant, $\beta$ extends to an element $\beta^* \in \mathrm{Aut}(R_i)$ (unique modulo $\mathrm{Inn}(R_i)$) by Proposition 7.4(e), and $\beta^* \in \mathrm{Aut}_{\mathcal{F}}(R_i)$ by the extension axiom. The last statement (about the subgroup $V$) follows since $V$ is normal in $T^*$, and is left invariant by $\gamma_0$ and by $\eta_1\eta_2\eta_3$. □

We next look at automorphisms of $\mathbf{Q}$, and of the subgroups $R_4$ and $H_i$ which contain $\mathbf{Q}$. Set $\overline{\mathbf{Q}} = \mathbf{Q}/Z(\mathbf{Q})$ for short. Let $\mathfrak{q}\colon \overline{\mathbf{Q}} \longrightarrow \mathbb{F}_2$ be the quadratic form where for any $\bar{x} = xZ(\mathbf{Q}) \in \overline{\mathbf{Q}}$, $\mathfrak{q}(\bar{x}) = 0$ if $x^2 = 1$ and $\mathfrak{q}(\bar{x}) = 1$ if $x^2 = \mathbf{z}_3$. Since $\mathrm{Inn}(\mathbf{Q})$ is the group of all automorphisms of $\mathbf{Q}$ which are the identity modulo $Z(\mathbf{Q})$, $\mathrm{Out}(\mathbf{Q}) \cong GO(\overline{\mathbf{Q}}, \mathfrak{q})$.

We want to choose an explicit isomorphism $\mathrm{Out}(\mathbf{Q}) \cong GO_6^+(2) \cong \Sigma_8$. Let $P_e(\underline{8})$ be the group of subsets of even order in $\underline{8} = \{1, 2, \ldots, 8\}$, regarded as an $\mathbb{F}_2$-vector space with addition given by symmetric difference $X + Y = ((X \smallsetminus Y) \cup (Y \smallsetminus X))$. Let $\mathfrak{q}$ be the quadratic form on $P_e(\underline{8})/\langle \underline{8}\rangle$ defined by $\mathfrak{q}(X) = \frac{1}{2}|X|$, associated to the bilinear form $\mathfrak{b}(X, Y) = |X \cap Y|$. The induced action of $\Sigma_8$ on $P_e(\underline{8})/\langle \underline{8}\rangle$ preserves the form, and thus defines a homomorphism $\Sigma_8 \longrightarrow SO(P_e(\underline{8})/\langle \underline{8}\rangle, \mathfrak{q})$ which is injective by the simplicity of $A_8$ and hence an isomorphism by counting.

Define $\kappa\colon \overline{\mathbf{Q}} \longrightarrow P_e(\underline{8})/\underline{8}$ by setting

$$\begin{array}{lll} \kappa(\mathbf{a}_1\mathbf{a}_2) = \{34\}, & \kappa(\mathbf{a}_3) = \{56\}, & \kappa(\mathbf{z}_1) = \{1234\} = \{5678\}, \\ \kappa(\mathbf{b}_1\mathbf{b}_2) = \{24\}, & \kappa(\mathbf{b}_3) = \{57\} & \kappa(\boldsymbol{\tau}) = \{1567\} = \{2348\}\,. \end{array}$$

This is motivated by the homomorphism $\rho\colon T^* \longrightarrow \Omega_7(3)$ defined at the beginning of the section: $\rho(\mathbf{Q})$ is the group of diagonal matrices, and $\kappa$ sends the class of $g \in \mathbf{Q}$ to the set of positions where $\rho(g)$ has $(-1)$ on the diagonal. So assuming $\rho$ lifts to a homomorphism $T^* \longrightarrow \mathrm{Spin}_7(3)$, $\kappa$ preserves the quadratic forms by standard commutator and squaring relations in the spinor groups (cf. [LO, Lemma A.4]). However, it is much easier to check this directly, by comparing values of the quadratic forms and associated bilinear forms on the basis used above to define $\kappa$.

Thus $\kappa$ induces an isomorphism

$$\chi_{\mathbf{Q}}\colon \mathrm{Out}(\mathbf{Q}) \cong GO(\overline{\mathbf{Q}}, \mathfrak{q}) \xrightarrow[\cong]{\kappa_*} GO(P_e(\underline{8})/\underline{8}; \mathfrak{q}) \xleftarrow{\cong} \Sigma_8.$$

To simplify notation, we also regard $\chi_{\mathbf{Q}}$ as a homomorphism on $\mathrm{Aut}(\mathbf{Q})$.

The images under $\chi_{\mathbf{Q}}$ of automorphisms in $\mathrm{Out}_{T^*}(\mathbf{Q})$, and also of the restrictions of $\eta_{12}^{(2)}, \eta_3^{(2)} \in \mathrm{Aut}(R_2)$, are given in the following table:

| $\alpha$ | $c_{\mathbf{a}_1}$ | $c_{\mathbf{c}}$ | $c_{\mathbf{b}_1}$ | $\eta_{12}^{(2)}\vert_{\mathbf{Q}}$ | $\eta_3^{(2)}\vert_{\mathbf{Q}}$ |
|---|---|---|---|---|---|
| $\chi_{\mathbf{Q}}(\alpha)$ | $(1\,2)(3\,4)$ | $(3\,4)(5\,6)$ | $(1\,3)(2\,4)$ | $(2\,3\,4)$ | $(5\,7\,6)$ |

(9)

For example, $c_{\mathbf{a}_1}$ sends $\mathbf{b}_1\mathbf{b}_2$ to $\mathbf{z}_1\mathbf{b}_1\mathbf{b}_2$, sends $\boldsymbol{\tau}$ to $\mathbf{a}_1\mathbf{a}_2^{-1}\boldsymbol{\tau} = \mathbf{a}_1\mathbf{a}_2\mathbf{z}_2\boldsymbol{\tau}$, and sends all of the other generators listed above to themselves. Hence $\chi_{\mathbf{Q}}(c_{\mathbf{a}_1}) \in \Sigma_8$ sends $\{24\}$ to $\{13\}$, sends $\{1567\}$ to $\kappa(\mathbf{z}_1\mathbf{a}_1\mathbf{a}_2\boldsymbol{\tau}) = \{2567\}$ (note $\mathbf{z}_2 \equiv \mathbf{z}_1 \pmod{Z(\mathbf{Q})}$), and sends each of $\{34\}$, $\{56\}$, $\{57\}$, and $\{1234\}$ to itself. So $\chi_{\mathbf{Q}}(c_{\mathbf{a}_1}) = (1\,2)(3\,4)$.

We now apply this to describe the automorphisms of $R_4$. In order to "see" better the symmetry of this subgroup, we give it the following, alternative description.

Define

$$\mathbb{S}' = \bigl\langle \mathbf{z}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3 \bigm| \mathbf{z}^2 = 1,\ \mathbf{r}_i^4 = \mathbf{z},\ \mathbf{s}_i^2 = \mathbf{z},\ \mathbf{s}_i\mathbf{r}_i\mathbf{s}_i^{-1} = \mathbf{r}_i^{-1}; \\ [\mathbf{r}_i, \mathbf{r}_j] = 1,\ [\mathbf{s}_i, \mathbf{r}_j] = 1,\ [\mathbf{s}_i, \mathbf{s}_j] = \mathbf{z} \ \text{ for all } i \neq j \bigr\rangle$$

Thus $\mathbb{S}'$ is generated by the three subgroups $\langle \mathbf{r}_i, \mathbf{s}_i \rangle \cong Q_{16}$ $(i = 1, 2, 3)$, which intersect in $Z(\mathbb{S}') = \langle \mathbf{z} \rangle$, and whose cyclic subgroups of order 8 commute with each other. This "twisted" product corresponds to the lifting to $\mathrm{Spin}_7(9)$ of three copies of $GO_2^-(3) \cong D_8$ in $SO_7(3) \leq \Omega_7(9)$.

Define an embedding $\psi\colon R_4 \longrightarrow \mathbb{S}'$ by setting

$$\psi(\mathbf{a}_1) = \mathbf{r}_1^{-1}\mathbf{r}_2 \qquad \psi(\mathbf{a}_2) = \mathbf{r}_1\mathbf{r}_2 \qquad \psi(\mathbf{a}_3) = \mathbf{r}_3^2 \qquad \psi(\mathbf{c}) = \mathbf{r}_2\mathbf{r}_3$$
$$\psi(\mathbf{b}_1\mathbf{b}_2) = \mathbf{r}_1^2\mathbf{r}_2^2\mathbf{s}_1\mathbf{s}_2 \qquad \psi(\mathbf{b}_3) = \mathbf{s}_3 \qquad \psi(\boldsymbol{\tau}) = \mathbf{r}_3^2\mathbf{s}_1\,.$$

Thus

$$\psi(R_4) = \langle \mathbf{r}_1^2, \mathbf{r}_1\mathbf{r}_2, \mathbf{r}_1\mathbf{r}_3, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3 \rangle \qquad \text{and} \qquad \psi(\mathbf{Q}) = \langle \mathbf{r}_1^2, \mathbf{r}_2^2, \mathbf{r}_3^2, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3 \rangle.$$

Also, $\psi([\![Y^i, Y^j, Y^k]\!]) = \mathbf{r}_1^{(j-i)/2}\mathbf{r}_2^{(j+i)/2}\mathbf{r}_3^k$ whenever $i \equiv j \equiv k \pmod 2$; and

$$\mathbf{s}_1 = \psi(\mathbf{a}_3^{-1}\boldsymbol{\tau}), \qquad \mathbf{s}_2 = \psi(\mathbf{z}_1\mathbf{a}_3\mathbf{b}_1\mathbf{b}_2\boldsymbol{\tau}) = \psi(\mathbf{b}_1(\mathbf{a}_3^{-1}\boldsymbol{\tau})\mathbf{b}_1^{-1}), \qquad \text{and} \qquad \mathbf{s}_3 = \psi(\mathbf{b}_3).$$

To simplify notation, we identify elements of $R_4$ with their images under $\psi$. Thus

$$\kappa(\mathbf{r}_1^2) = \{12\}, \qquad \kappa(\mathbf{r}_2^2) = \{34\}, \qquad \kappa(\mathbf{r}_3^2) = \{56\},$$
$$\kappa(\mathbf{s}_1) = \{17\}, \qquad \kappa(\mathbf{s}_2) = \{37\}, \qquad \kappa(\mathbf{s}_3) = \{57\}.$$

Define $\varepsilon_1, \varepsilon_2, \varepsilon_3, \theta_1, \theta_2, \xi \in \mathrm{Aut}(R_4)$ to be the restrictions of the automorphisms of $\mathbb{S}'$ defined in the following table, where, $\sigma, \tau \in \Sigma_3$ are the permutations $\sigma = (1\,2\,3)$ and $\tau = (1\,2)$; and $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$:

| $\alpha$ | $\varepsilon_j (j = 1, 2, 3)$ | $\theta_1$ | $\theta_2$ | $\xi$ | $c_{\mathbf{b}_1\mathbf{a}_1\mathbf{a}_2}$ |
|---|---|---|---|---|---|
| $\alpha(\mathbf{r}_i)$ | $\mathbf{z}^{\delta_{ij}}\mathbf{r}_i$ | $\mathbf{r}_i$ | $\mathbf{r}_i$ | $\mathbf{r}_{\sigma(i)}$ | $\mathbf{r}_{\tau(i)}$ |
| $\alpha(\mathbf{s}_i)$ | $\mathbf{s}_i$ | $\mathbf{r}_i^2\mathbf{s}_i$ | $\mathbf{r}_1^2\mathbf{r}_2^2\mathbf{r}_3^2\mathbf{s}_i$ | $\mathbf{s}_{\sigma(i)}$ | $\mathbf{s}_{\tau(i)}$ |
| $\chi_{\mathbf{Q}}(\alpha\vert_{\mathbf{Q}})$ | Id | $(1\,2)(3\,4)(5\,6)$ | $(7\,8)$ | $(1\,3\,5)(2\,4\,6)$ | $(1\,3)(2\,4)$ |

(10)

**Lemma 7.6.** $\mathrm{Out}(R_4) = \bigl(\langle [\varepsilon_1], [\varepsilon_2] \rangle \rtimes \langle [\xi], c_{\mathbf{b}_1} \rangle\bigr) \times \langle [\theta_1], [\theta_2] \rangle \cong \Sigma_4 \times C_2^2$, *where* $\varepsilon_1\varepsilon_2\varepsilon_3 = \mathrm{Id}_{R_4}$. *If $\mathcal{F}$ is a saturated fusion system over $T^*$ and $R_4$ is $\mathcal{F}$-essential, then* $\mathrm{Out}_{\mathcal{F}}(R_4)$ *is one of the groups* $\langle [\xi], c_{\mathbf{b}_1} \rangle$ *or* $\langle [\varepsilon_3\xi\varepsilon_3^{-1}], c_{\mathbf{b}_1} \rangle$, *both isomorphic to* $\Sigma_3$. *In either case, the image under* $\chi_{\mathbf{Q}}$ *of the restriction to* $\mathbf{Q}$ *of* $\mathrm{Aut}_{\mathcal{F}}(R_4)$ *is generated by* $\chi_{\mathbf{Q}}(\mathrm{Out}_{T^*}(\mathbf{Q}))$ *and* $(1\,3\,5)(2\,4\,6)$.

*Proof.* By Lemma 7.1(b), each automorphism of $R_4$ leaves $\mathbf{Q}$ invariant. Hence by Lemma 1.2, there is an exact sequence

$$1 \longrightarrow H^1(R_4/\mathbf{Q}; Z(\mathbf{Q})) \xrightarrow{\ \eta\ } \mathrm{Out}(R_4) \xrightarrow{\ \mathrm{Res}_{\mathbf{Q}}\ } N_{\mathrm{Out}(\mathbf{Q})}(\mathrm{Out}_{R_4}(\mathbf{Q}))/\mathrm{Out}_{R_4}(\mathbf{Q}).$$

Also, since $R_4/\mathbf{Q} = \langle \mathbf{r}_1\mathbf{r}_2, \mathbf{r}_2\mathbf{r}_3\rangle \cong C_2^2$ and $Z(\mathbf{Q}) = \langle \mathbf{z}\rangle \cong C_2$,

$$H^1(R_4/\mathbf{Q}; Z(\mathbf{Q})) \cong \mathrm{Hom}(R_4/\mathbf{Q}, Z(\mathbf{Q})) \cong C_2^2,$$

and $\eta$ sends a homomorphism $\varphi$ to the class of the automorphism $(g \mapsto g{\cdot}\varphi(g\mathbf{Q}))$. Thus $\mathrm{Ker}(\mathrm{Res}_{\mathbf{Q}}) = \langle [\varepsilon_1], [\varepsilon_2]\rangle$.

By (9), $\chi_{\mathbf{Q}}(\mathrm{Out}_{R_4}(\mathbf{Q})) = \langle (1\,2)(3\,4), (3\,4)(5\,6)\rangle$. The normalizer of this subgroup is the group of all permutations which leave $\{7, 8\}$ invariant, and permute the three subsets $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$. Hence

$$N_{\mathrm{Out}(\mathbf{Q})}(\mathrm{Out}_{R_4}(\mathbf{Q}))/\mathrm{Out}_{R_4}(\mathbf{Q}) \cong \Sigma_3 \times C_2^2,$$

where the direct factor $C_2^2$ is represented by the permutations $(1\,2)(3\,4)(5\,6) = \chi_{\mathbf{Q}}(\theta_1)$ and $(7\,8) = \chi_{\mathbf{Q}}(\theta_2)$ (see (10)). Also, $(1\,3\,5)(2\,4\,6) = \chi_{\mathbf{Q}}(\xi)$ and $(1\,3)(2\,4) = \chi_{\mathbf{Q}}(c_{\mathbf{b}_1})$ represent generators of the first factor. This proves that $\mathrm{Res}_{\mathbf{Q}}$ in the above exact sequence is onto, and also shows that $\mathrm{Out}(R_4)$ is generated by the classes of $\varepsilon_1$, $\varepsilon_2$, $\theta_1$, $\theta_2$, $\xi$, and $c_{\mathbf{b}_1}$.

The exact structure of this extension follows from (10) (and the relation $\varepsilon_1\varepsilon_2\varepsilon_3 = \mathrm{Id}$). Also, $\langle\theta_1, \theta_2\rangle$ is the subgroup of elements which restrict to the identity on $\mathbf{A} = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{c}\rangle = \langle \mathbf{r}_1^2, \mathbf{r}_1\mathbf{r}_2, \mathbf{r}_1\mathbf{r}_3\rangle$, and hence is normal in $\mathrm{Out}(R_4)$. Thus the subgroup $\langle [\varepsilon_1], [\varepsilon_2], [\theta_1], [\theta_2]\rangle$ in $\mathrm{Out}(R_4)$ is elementary abelian, and $\langle [\xi], c_{\mathbf{b}_1}\rangle \cong \Sigma_3$ permutes the three involutions $\varepsilon_i$ and acts trivially on $\langle [\theta_1], [\theta_2]\rangle$.

If $R_4$ is $\mathcal{F}$-essential for some saturated fusion system $\mathcal{F}$ over $T^*$, then $\mathrm{Out}_{\mathcal{F}}(R_4) = \langle \alpha, c_{\mathbf{b}_1}\rangle \cong \Sigma_3$ for some $\alpha \in \langle [\varepsilon_1], [\varepsilon_2], [\xi], c_{\mathbf{b}_1}\rangle \cong \Sigma_4$ of order three which is normalized by $c_{\mathbf{b}_1}$. Any transposition in $\Sigma_4$ normalizes exactly two subgroups of order 3, and in this case, these are easily seen to be the subgroups $\langle [\xi]\rangle$ and $\langle [\varepsilon_3\xi\varepsilon_3^{-1}]\rangle$. □

It remains to examine the outer automorphism groups of the $H_i$.

**Lemma 7.7.** (a) *For $i = 1$, 2, or 3, let $\alpha, \alpha' \in \mathrm{Aut}(H_i)$ be two automorphisms of order three such that $\chi_{\mathbf{Q}}(\alpha|_{\mathbf{Q}}) = \chi_{\mathbf{Q}}(\alpha'|_{\mathbf{Q}})$ in $\mathrm{Out}(\mathbf{Q})$. Then $[\alpha] = [\alpha']$ in $\mathrm{Out}(H_i)$.*

(b) *If $\mathcal{F}$ is a saturated fusion system over $T^*$ and $H_1$ and $H_2$ are $\mathcal{F}$-essential, then $\mathrm{Out}_{\mathcal{F}}(H_i) = \langle [\alpha_i], c_{\mathbf{c}}\rangle \cong \Sigma_3$ $(i = 1, 2)$ for some $\alpha_i \in \mathrm{Aut}(H_i)$ of order 3 such that $\chi_{\mathbf{Q}}(\alpha_1|_{\mathbf{Q}}) = (1\,2\,x)$ and $\chi_{\mathbf{Q}}(\alpha_2|_{\mathbf{Q}}) = (3\,4\,x)$ for the same $x = 7$ or 8.*

*Proof.* Set $x_1 = \mathbf{c}$, $x_2 = \mathbf{a}_1\mathbf{c}$, and $x_3 = \mathbf{a}_1$; thus $H_i = \langle \mathbf{Q}, x_i\rangle$ for $i = 1, 2, 3$. For each such $i$, $\mathbf{Q}$ is characteristic in $H_i$ by Lemma 7.1(b), so there is a well defined homomorphism

$$\mathrm{Res}_i \colon \mathrm{Out}(H_i) \longrightarrow C_{\mathrm{Out}(\mathbf{Q})}(c_{x_i})/\langle c_{x_i}\rangle$$

induced by restriction, and $\mathrm{Ker}(\mathrm{Res}_i) \cong H^1(H_i/\mathbf{Q}; Z(\mathbf{Q})) \cong H^1(C_2; \mathbb{Z}/2)$ has order 2 (Lemma 1.2). In particular, for any $[\alpha] \in \mathrm{Out}(\mathbf{Q})$ of order three which centralizes $c_{x_i}$, its class in the quotient lifts to at most one element of order three in $\mathrm{Out}(H_i)$, and this proves (a).

By (9), $\chi_{\mathbf{Q}}(c_{x_i}) = (3\,4)(5\,6)$ $(i = 1)$, $(1\,2)(5\,6)$ $(i = 2)$, or $(1\,2)(3\,4)$ $(i = 3)$. Thus $C_{\mathrm{Out}(\mathbf{Q})}(c_{x_i})/\langle c_{x_i}\rangle \cong C_2^2 \times \Sigma_4$ in all three cases, and $|\mathrm{Out}(H_i)| = 2^k$ or $3{\cdot}2^k$ for some $k$. By the Sylow axiom, for each $i$, $|\mathrm{Out}_{\mathcal{F}}(H_i)| = 2$ or 6. So if $H_1$ and $H_2$ are $\mathcal{F}$-essential, then for $i = 1, 2$, $\mathrm{Out}_{\mathcal{F}}(H_i) = \langle [\alpha_i], c_{x_3}\rangle \cong \Sigma_3$ for some $\alpha_i \in \mathrm{Aut}(H_i)$ of order 3 such that $\big[[\alpha_i|_{\mathbf{Q}}], c_{x_i}\big] = 1$ in $\mathrm{Out}(\mathbf{Q})$. Thus $\chi_{\mathbf{Q}}(\alpha_1|_{\mathbf{Q}})$ commutes with $\chi_{\mathbf{Q}}(c_{x_1}) = (3\,4)(5\,6)$ in $\Sigma_8$, and is also normalized by $\chi_{\mathbf{Q}}(c_{x_3}) = (1\,2)(3\,4)$. This is possible only if $\chi_{\mathbf{Q}}(\alpha_1|_{\mathbf{Q}}) = (1\,2\,x)$ for $x = 7$ or 8. Since $H_2 = \mathbf{b}_1 H_1 \mathbf{b}_1^{-1}$, and $\chi_{\mathbf{Q}}(c_{\mathbf{b}_1}) = (1\,3)(2\,4)$ by (9) again, we can choose $\alpha_2 = c_{\mathbf{b}_1}\alpha_1 c_{\mathbf{b}_1}^{-1}$, in which case $\chi_{\mathbf{Q}}(\alpha_2|_{\mathbf{Q}}) = (3\,4\,x)$. This proves (b). □

In fact, the homomorphisms $\mathrm{Res}_i$ in the above proof are surjective, and hence $|\mathrm{Out}(H_i)| = 3{\cdot}2^6$. This can be shown for $i = 3$ by checking that $C_{\mathrm{Out}(\mathbf{Q})}(c_{\mathbf{a}_1})/\langle c_{\mathbf{a}_1}\rangle$ is generated by restrictions of automorphisms of $R_2$ (those which leave $H_3$ invariant). It then follows for $i = 1, 2$ since $\xi \in \mathrm{Aut}(R_4)$ permutes transitively the $H_i$. However, this will not be needed here.

## 7.3 Fusion systems over $T^*$

We are now ready to list the saturated fusion systems over $T^*$.

**Theorem 7.8.** *Every nonconstrained centerfree fusion system over $T^*$ is isomorphic to the fusion system of* Sol(3)*,* $Co_3$*, or* $\mathrm{Aut}(PSp_6(3))$*.*

*Proof.* Let $\mathcal{F}$ be a nonconstrained fusion system over $T^*$ such that $\mathbf{z}_3$ is not central in $\mathcal{F}$. By Lemma 7.4(f), $\mathrm{Out}(T^*)$ is a 2-group, and hence $\mathrm{Out}_{\mathcal{F}}(T^*) = 1$. So by Proposition 7.3, all fusion in $\mathcal{F}$ is generated by fusion in $R_1$, $R_2$, $R_3$, $R_4$, $H_1$, and $H_2$. Since each of these except $R_1$ has center $\langle \mathbf{z}_3\rangle$, $R_1$ must be $\mathcal{F}$-essential, since otherwise $\mathbf{z}_3$ would be central in $\mathcal{F}$.

By Proposition 7.5, there is a fusion system $\mathcal{F}'$ over $T^*$ isomorphic to $\mathcal{F}$, such that that the groups $\mathrm{Out}_{\mathcal{F}'}(R_i)$ for $i = 0, 1, 2, 3$ are as in one of the four cases listed there. Assume for simplicity $\mathcal{F}' = \mathcal{F}$. Then $\mathrm{Out}_{\mathcal{F}}(R_1) = \langle [\gamma], c_{\boldsymbol{\tau}}\rangle$, $\mathrm{Out}_{\mathcal{F}}(R_2) \le \langle [\eta_{12}^{(2)}], [\eta_3^{(2)}], c_{\mathbf{c}}\rangle$, and $\mathrm{Out}_{\mathcal{F}}(R_0)$ determines $\mathrm{Out}_{\mathcal{F}}(R_2)$ and $\mathrm{Out}_{\mathcal{F}}(R_3)$.

If all $\mathcal{F}$-essential subgroups contain $R_0$, then $R_0$ must be normal in $\mathcal{F}$ (Lemma 7.4(a)), which would contradict the assumption that $\mathcal{F}$ is nonconstrained. Hence either $R_4$, or $H_1$ and $H_2$, are also $\mathcal{F}$-essential. (Recall that $H_1$ and $H_2$ are conjugate in $T^*$.) If $R_4$ is essential, then $H_1$, $H_2$, and $H_3$ are all $\mathcal{F}$-conjugate by Lemma 7.6, since $\xi$ and $\varepsilon_3\xi\varepsilon_3^{-1}$ both permute them transitively. Since $H_3 \lhd T^*$, this implies neither $H_1$ nor $H_2$ is fully normalized in $\mathcal{F}$, and hence neither is $\mathcal{F}$-essential.

The cases where $R_4$ is $\mathcal{F}$-essential will be handled in Step 1, and the cases where $H_1$ and $H_2$ are $\mathcal{F}$-essential in Step 2. Afterwards, the distinct (possible) fusion systems found in those two steps will be identified in Step 3.

**Step 1:** Assume $R_4$ is $\mathcal{F}$-essential. The automorphisms $\xi, \varepsilon_3 \in \mathrm{Aut}(R_4)$ both leave invariant the subgroup

$$V = \langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3, \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\rangle = \langle \mathbf{z}, \mathbf{r}_1^2\mathbf{r}_2^2, \mathbf{r}_1^2\mathbf{r}_3^3, \mathbf{s}_1\mathbf{s}_2\mathbf{s}_3\rangle \cong C_2^4,$$

and hence $\mathrm{Out}_{\mathcal{F}}(R_4)$ leaves leaves $V$ invariant by Lemma 7.6. Since we are assuming $\mathcal{F}$ is not constrained, $V$ is not normal in $\mathcal{F}$, and this implies $\mathrm{Out}_{\mathcal{F}}(R_0)$ does not leave $V$ invariant. So $\mathcal{F}$ must be of Type 3 or 4 by Proposition 7.5.

By Lemma 7.6 again, $\mathrm{Out}_{\mathcal{F}}(R_4)$ is one of the two groups $\langle [\xi], c_{\mathbf{b}_1}\rangle$ or $\langle [\varepsilon_3\xi\varepsilon_3^{-1}], c_{\mathbf{b}_1}\rangle$. So we are reduced to at most four different possibilities for $\mathcal{F}$. We claim that $\mathrm{Out}_{\mathcal{F}}(R_4) = \langle [\varepsilon_3\xi\varepsilon_3^{-1}], c_{\mathbf{b}_1}\rangle$ is the only possibility, given our choice of $\mathrm{Aut}_{\mathcal{F}}(R_1)$. This is closely related to [LO2, Lemma 1.2] (and to the error in [LO] which made a correction necessary), but we give a more direct argument here. Consider the subgroup $\mathbf{A} = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{c}\rangle \cong C_4^3$ of Lemma 7.1(a). For each $\alpha \in \mathrm{Aut}(\mathbf{A})$, let $M(\alpha) \in GL_3(\mathbb{Z}/4)$ be the matrix for $\alpha$ with respect to the ordered basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{c}\}$. Note that $\mathrm{Aut}_{\mathcal{F}}(\mathbf{A})$ is generated by restrictions of automorphisms in $\mathrm{Aut}_{\mathcal{F}}(R_4)$ and in $\mathrm{Aut}_{\mathcal{F}}(R_1) = \langle \mathrm{Inn}(R_1), \gamma, c_{\boldsymbol{\tau}}\rangle$. Then

$$M(\xi) = \begin{pmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix},\ M(\varepsilon_3\xi\varepsilon_3^{-1}) = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix},\ M(\gamma) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix},\ M(\xi\gamma) = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(where we drop "$|_{\mathbf{A}}$" to simplify the notation); and $M((\xi\gamma)^3) = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{smallmatrix}\right) \equiv \mathrm{Id} \pmod 2$. Since this has order two, $(\xi\gamma)^3$ must be conjugate in $\mathrm{Aut}(\mathbf{A})$ to some element of the Sylow 2-subgroup $\mathrm{Aut}_{T^*}(\mathbf{A}) = \langle c_{\mathbf{b}_1}, c_{\mathbf{b}_2}, c_{\mathbf{b}_3}, c_{\boldsymbol{\tau}}\rangle$. But this is impossible, since the only element of $\mathrm{Aut}_{T^*}(\mathbf{A})$ which is the identity modulo 2 is $c_{\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3}$, and $M(c_{\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3}) = -\mathrm{Id}$. So $\xi$ and $\gamma$ cannot both be in $\mathrm{Aut}_{\mathcal{F}}(\mathbf{A})$, and thus $\mathrm{Out}_{\mathcal{F}}(R_4) = \langle [\varepsilon_3\xi\varepsilon_3^{-1}], c_{\mathbf{b}_1}\rangle$. (In fact, $(\varepsilon_3\xi\varepsilon_3^{-1}\gamma|_{\mathbf{A}})^3 = 1$.)

Thus $\mathcal{F}$ must be isomorphic to one of two fusion systems, which we denote $\mathcal{F}_1$ (of Type 3) and $\mathcal{F}_2$ (of Type 4). The restriction of $\mathrm{Aut}_{\mathcal{F}}(R_4)$ to $\mathbf{Q}$ is generated by $\xi|_{\mathbf{Q}}$ (since $\varepsilon_3|_{\mathbf{Q}} = \mathrm{Id}$) and $\mathrm{Aut}_{T^*}(\mathbf{Q})$. Hence if we let $X_0 \le \mathrm{Out}_{\mathcal{F}}(\mathbf{Q})$ be the subgroup generated by $\mathrm{Out}_{T^*}(\mathbf{Q})$ and classes of restrictions of $\mathcal{F}$-automorphisms of $R_4$, then by (9) and (10),

$$\chi_{\mathbf{Q}}(X_0) = \Big\langle \underset{\chi_{\mathbf{Q}}(c_{\mathbf{a}_1})}{(1\,2)(3\,4)}, \underset{\chi_{\mathbf{Q}}(c_{\mathbf{c}})}{(3\,4)(5\,6)} \Big\rangle \rtimes \Big\langle \underset{\chi_{\mathbf{Q}}(\xi|_{\mathbf{Q}})}{(1\,3\,5)(2\,4\,6)}, \underset{\chi_{\mathbf{Q}}(c_{\mathbf{b}_1})}{(1\,3)(2\,4)} \Big\rangle.$$

Since $\chi_{\mathbf{Q}}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q}))$ is generated by $\chi_{\mathbf{Q}}(X_0)$ and restrictions of elements in $\mathrm{Out}_{\mathcal{F}}(R_2)$, Proposition 7.5 and (9) imply

$$\chi_{\mathbf{Q}}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q})) = \begin{cases} \langle \chi_{\mathbf{Q}}(X_0), (2\,3\,4)(5\,7\,6)\rangle & \text{if } \mathcal{F} = \mathcal{F}_1 \\ \langle \chi_{\mathbf{Q}}(X_0), (2\,3\,4), (5\,7\,6)\rangle & \text{if } \mathcal{F} = \mathcal{F}_2. \end{cases}$$

Since $\chi_{\mathbf{Q}}(X_0)$ contains all even permutations which fix 7 and 8 and permute the three subsets $\{1,2\}$, $\{3,4\}$, $\{5,6\}$, $\langle \chi_{\mathbf{Q}}(X_0), (2\,3\,4)\rangle$ contains all even permutations which fix 7 and 8, and so $\chi_{\mathbf{Q}}(\mathrm{Out}_{\mathcal{F}_2}(\mathbf{Q})) \cong A_7$ is the group of all even permutations which fix 8.

An isomorphism $\chi_{\mathbf{Q}}(\mathrm{Out}_{\mathcal{F}_1}(\mathbf{Q})) \cong GL_3(2)$ is defined via the bijection $\underline{\mathbf{8}} \xrightarrow{\;\cong\;} \mathbb{F}_2^3$ which sends $n \in \underline{\mathbf{8}}$ to the three digits in the binary expansion of $8-n$. Thus 1 is sent to $(1,1,1)$, 2 to $(1,1,0)$, 8 to $(0,0,0)$, etc.

**Step 2:** Now assume $H_1$ and $H_2$ are $\mathcal{F}$-essential (and $R_4$ is not). By Lemma 7.7(b), $\mathrm{Out}_{\mathcal{F}}(H_i) \cong \Sigma_3$ for $i = 1,2$, and there are elements $\alpha_i \in \mathrm{Aut}_{\mathcal{F}}(H_i)$ of order three such that $\chi_{\mathbf{Q}}(\alpha_1|_{\mathbf{Q}}) = (1\,2\,x)$ and $\chi_{\mathbf{Q}}(\alpha_2|_{\mathbf{Q}}) = (3\,4\,x)$ for some fixed $x \in \{7,8\}$. Thus $\chi_{\mathbf{Q}}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q}))$ contains $\langle (1\,2\,x), (3\,4\,x)\rangle$, which is the group of all even permutations of the set $\{1,2,3,4,x\}$.

Now, since $\chi_{\mathbf{Q}}(\eta_{12}^{(2)}|_{\mathbf{Q}}) = (2\,3\,4)$ (where $\eta_{12}^{(2)} \in \mathrm{Aut}(R_2)$), this implies $\eta_{12}^{(2)}|_{\mathbf{Q}} \in \mathrm{Aut}_{\mathcal{F}}(\mathbf{Q})$, and hence (by the extension axiom) that $\eta_{12}^{(2)}$ (or some other automorphism with the same restriction) is in $\mathrm{Aut}_{\mathcal{F}}(R_2)$. So $\mathcal{F}$ has Type 4 by Proposition 7.5, and $\chi_{\mathbf{Q}}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q}))$ also contains $\chi_{\mathbf{Q}}(\eta_3^{(2)}|_{\mathbf{Q}}) = (5\,7\,6)$.

If $x = 7$, then $\chi_{\mathbf{Q}}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q}))$ contains all even permutations which fix the element 8. In particular, it contains $\chi_{\mathbf{Q}}(\xi|_{\mathbf{Q}})$, where $\xi \in \mathrm{Aut}(R_4)$ has order three (see Step 1). By the extension axiom, $\xi|_{\mathbf{Q}} \in \mathrm{Aut}_{\mathcal{F}}(\mathbf{Q})$ extends to an automorphism in $\mathrm{Aut}_{\mathcal{F}}(R_4)$ of order 3, so $R_4$ is $\mathcal{F}$-essential, contradicting our original assumption.

Thus $x = 8$, and

$$\chi_{\mathbf{Q}}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q})) = \Big\langle \underset{\chi_{\mathbf{Q}}(c_{\mathbf{a}_1})}{(1\,2)(3\,4)}, \underset{\chi_{\mathbf{Q}}(c_{\mathbf{b}_1})}{(1\,3)(2\,4)}, \underset{\chi_{\mathbf{Q}}(c_{\mathbf{c}})}{(3\,4)(5\,6)}, \underset{\chi_{\mathbf{Q}}(\alpha_1)}{(1\,2\,8)}, \underset{\chi_{\mathbf{Q}}(\alpha_2)}{(3\,4\,8)}, \underset{\chi_{\mathbf{Q}}(\eta_{12}^{(2)})}{(2\,3\,4)}, \underset{\chi_{\mathbf{Q}}(\eta_3^{(2)})}{(5\,7\,6)} \Big\rangle$$

is the group of all even permutations which leave the sets $\{1,2,3,4,8\}$ and $\{5,6,7\}$ invariant. By Lemma 7.7(a), $[\alpha_i] \in \mathrm{Out}(H_i)$ is determined by $\chi_{\mathbf{Q}}(\alpha_i|_{\mathbf{Q}}) \in \mathrm{Out}(\mathbf{Q})$ for $i = 1,2$. Hence there is only one fusion system satisfying these conditions, and we denote it by $\mathcal{F}_3$.

**Step 3:** By the construction in [LO], $T^*$ is a Sylow 2-subgroup of $\mathrm{Spin}_7(3)$, and also of the exotic fusion system Sol(3) constructed there. The sporadic simple group $Co_3$ contains $2{\cdot}Sp_6(2)$ with odd index (cf. [Fi, Theorem 2]). The orthogonal group $\Omega_7(3)$ contains $Sp_6(2)$ with odd index (as a subgroup of index two in the Weyl group of $E_7$), and hence $\mathrm{Spin}_7(3)$ contains $2{\cdot}Sp_6(2)$ with odd index. Thus $Co_3$ also has Sylow 2-subgroups isomorphic to $T^*$.

Since $Sp_6(9)$ contains the wreath product $Sp_2(9) \wr \Sigma_3$ with odd index, and since $Sp_2(9) \cong SL_2(9)$ has Sylow 2-subgroups isomorphic to $Q_{16}$, the group $\mathbb{S}$ defined at the beginning of the section is isomorphic to a Sylow 2-subgroup of $PSp_6(9)$, and its subgroup $R_2 = \langle R_0, \boldsymbol{\tau} \rangle$ is isomorphic to a Sylow 2-subgroup of $PSp_6(3)$. The group $\mathrm{Aut}(PSp_6(3))$ is the extension of $PSp_6(3)$ by its diagonal automorphisms, hence the normalizer of $PSp_6(3)$ in $PSp_6(9)$, and contains $PSp_6(3)$ with index two. Its Sylow 2-subgroup is thus isomorphic to a subgroup of index four in $\mathbb{S}$ of the form $\langle P, \boldsymbol{\tau} \rangle$ for some $R_0 \le P \le \mathbb{S}_0$ which is invariant under the action of $\Sigma_3$ permuting the central factors; and this can only be $P = \langle R_0, \mathbf{c} \rangle = R_1$. The Sylow 2-subgroups of $\mathrm{Aut}(PSp_6(3))$ are thus isomorphic to $\langle R_1, \boldsymbol{\tau} \rangle = T^* \le \mathbb{S}$.

We showed in Steps 1 and 2 that every nonconstrained centerfree saturated fusion system over $T^*$ is isomorphic to one of the fusion systems $\mathcal{F}_1$, $\mathcal{F}_2$, or $\mathcal{F}_3$. Hence the fusion systems of $Co_3$, Sol(3), and $\mathrm{Aut}(PSp_6(3))$ must be among these, and it remains to identify them.

As shown in Steps 1 and 2, the $\mathcal{F}_i$ are distinguished by $\mathrm{Out}_{\mathcal{F}_i}(\mathbf{Q})$:

$$\mathrm{Out}_{\mathcal{F}_1}(\mathbf{Q}) \cong GL_3(2), \quad \mathrm{Out}_{\mathcal{F}_2}(\mathbf{Q}) \cong A_7, \quad \text{and} \quad \mathrm{Out}_{\mathcal{F}_3}(\mathbf{Q}) \cong (A_5 \times C_3) \rtimes C_2.$$

By Lemma 7.1(b), $\mathbf{Q}$ is the only subgroup of $T^*$ of order $2^7$ with quotient group $C_2^6$. Hence to determine $\mathrm{Out}_G(\mathbf{Q})$ for any finite group $G$ with Sylow 2-subgroups isomorphic to $T^*$, it suffices to find any subgroup $C_2^6$ in $C_G(z)/\langle z \rangle$ for some involution $z$ in $G$, and determine its automorphism group.

The centralizer in $Co_3$ of a Sylow central involution is isomorphic to $2{\cdot}Sp_6(2)$ (cf. [Fi, Lemma 4.4]), and $Sp_6(2)$ contains a maximal subgroup $C_2^6 \rtimes GL_3(2)$ (the stabilizer subgroup of an isotropic plane). Hence $\mathcal{F}_S(Co_3) \cong \mathcal{F}_1$ for $S \in \mathrm{Syl}_2(Co_3)$.

The centralizer in Sol(3) of any involution is the fusion system of $\mathrm{Spin}_7(3)$ [LO, Theorem 2.1], and $\Omega_7(3)$ contains a maximal subgroup $C_2^6 \rtimes A_7$ (the elements which leave an orthonormal basis invariant up to sign and permutation). Hence $\mathcal{F}_{T^*}(\mathrm{Sol}(3)) \cong \mathcal{F}_2$.

Finally, the group $\mathrm{Aut}(PSp_6(3))$ has involution centralizer $(SL_2(3) \times_{C_2} Sp_4(3)) \rtimes C_2$ (the elements which leave invariant an orthogonal decomposition $\mathbb{F}_3^2 \times \mathbb{F}_3^4$ of the vector space). Since $PSL_2(3) \times PSp_4(3)$ contains a subgroup $(C_2^2 \rtimes C_3) \times (C_2^4 \rtimes A_5)$, this shows that $\mathcal{F}_S(\mathrm{Aut}(PSp_6(3))) \simeq \mathcal{F}_3$ for $S \in \mathrm{Syl}_2(\mathrm{Aut}(PSp_6(3)))$. □

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