FUSION SYSTEMS AND AMALGAMS

KASPER K. S. ANDERSEN, BOB OLIVER, AND JOANA VENTURA

Abstract. We study reduced fusion systems from the point of view of their essential subgroups, using the classification by Goldschmidt and Fan of amalgams of prime index to analyze certain pairs of such subgroups. Our results are applied here to study reduced fusion systems over 2-groups of order at most 64, and also reduced fusion systems over 2-groups having abelian subgroups of index two. More applications are given in later papers.

A *saturated fusion system* over a finite p-group S is a category whose objects are the subgroups of S, whose morphisms are monomorphisms between subgroups, and which satisfy certain axioms first formulated by Puig [Pg2] and motivated by conjugacy relations among p-subgroups of a given finite group. A saturated fusion system is *reduced* if it has no proper normal subsystem of p-power index, no proper normal subsystem of index prime to p, and no nontrivial normal p-subgroup. (All three of these concepts are defined by analogy with finite groups.) Reduced fusion systems need not be simple, in that they can have proper nontrivial normal subsystems. They were introduced by us in [AOV] as forming a class of fusion systems which is small enough to be manageable, but still large enough to detect any fusion systems (reduced or not) which are "exotic" (not defined via conjugacy relations in any finite group).

When G is a finite group and $S \in \mathrm{Syl}_p(G)$, the version of Alperin's fusion theorem shown by Goldschmidt [Gd1] says that all G-conjugacy relations among subgroups of S are generated by $\mathrm{Aut}_G(S)$ (automorphisms induced by conjugation in G), together with $\mathrm{Aut}_G(P)$ for certain "essential" proper subgroups of S, and restrictions of such automorphisms. There is a version of this result for abstract fusion systems (see Theorem 1.2), which says that a fusion system $\mathcal{F}$ is generated by $\mathcal{F}$-automorphisms of $\mathcal{F}$-essential subgroups (Definition 1.1). Our goal in this and our other papers is to study, and to classify in certain cases, reduced fusion systems from the point of view of their essential subgroups and generating automorphisms.

This point of view was introduced in [OV], where two of us described how fusion systems over a given 2-group S could be classified by first listing the subgroups of S which potentially could be essential, using Bender's theorem on groups with strongly embedded subgroups. When we try to extend those methods to larger classes of groups, it is useful to search for pairs of essential subgroups via theorems of Goldschmidt and Fan classifying certain types of amalgams.

2010 *Mathematics Subject Classification.* Primary 20D20. Secondary 20E06, 20E45.

Key words and phrases. Finite groups, fusion, amalgams, fusion systems.

K. K. S. Andersen was partially supported by the Danish National Research Foundation (DNRF) through the Centre for Symmetry and Deformation.

B. Oliver is partially supported by UMR 7539 of the CNRS, and by project ANR BLAN08-2_338236, HGRT.

J. Ventura was partially supported by FCT through program POCI 2010/FEDER and project PTDC/MAT/098317/2008.

The situation we want to study is the following. Assume $\mathcal{F}$ is a saturated fusion system over a finite 2-group S, and $P_1, P_2 < S$ are $\mathcal{F}$-essential subgroups of index two in their normalizer. In addition, we assume either that P_1 and P_2 have index two in S, or that $N_S(P_1) = N_S(P_2) < S$ and P_1, P_2 are S-conjugate to each other. Set $P = P_1P_2 = N_S(P_i)$. Then there are finite groups $G_i > P$ such that $P_i \trianglelefteq G_i$, $G_i/P_i \cong D_{2p_i}$ for some odd prime p_i, and $\mathrm{Aut}_{G_i}(P_i) \le \mathrm{Aut}_{\mathcal{F}}(P_i)$. By applying the classifications of amalgams by Goldschmidt and Fan to the triple $(G_1 > P < G_2)$, we get information about S and the P_i. In this paper, we only deal with certain cases (see Theorems 4.5 and 4.6), but these are the cases which occur most often in "small" examples.

Applications of these results are given in Section 5. The reduced fusion systems over 2-groups of order at most 32, and the groups of order 64 which support reduced fusion systems, are all listed in Theorems 5.3 and 5.4, respectively. These are preceded by Propositions 5.1 and 5.2, which list various conditions on a reduced fusion system over a 2-group S which imply that S is dihedral of order at least 8, semidihedral of order at least 16, or a wreath product $C_{2^n} \wr C_2$ for $n \ge 2$. Furthermore, in these cases, $\mathcal{F}$ is isomorphic to the fusion system of $PSL_2(q)$ for some $q \equiv \pm 1 \pmod 8$, or of $PSL_3(q)$ for some odd q. For example, by Proposition 5.2(a,b,c,e), these conclusions hold whenever $\mathcal{F}$ is a reduced fusion system over a 2-group S, where either

- S contains an abelian subgroup of index two; or
- $[S, S]$ is cyclic; or
- there is a subgroup $Q < S$ such that $|N_S(Q)/Q| = 2$, $\mathrm{Out}_S(Q) \not\le O_2(\mathrm{Out}(Q))$, and either Q is abelian or $|Q| \le 16$.

These results are applied in a later paper by the same authors, where we combine them with a computer search to list reduced fusion systems over 2-groups of order at most 2^9. They have also been applied by the second author when classifying reduced fusion systems over 2-groups of sectional rank at most four.

Notation: For any group G, we let $G^{\mathrm{ab}} = G/[G, G]$ denote its abelianization. Also, C_n denotes a (multiplicative) cyclic group of order n, and D_{2^m}, SD_{2^m}, and Q_{2^m} denote dihedral, semidihedral, and quaternion groups of order 2^m. As usual, when P is a finite p-group for some prime p, then $\Omega_1(P) = \langle g \in P \,|\, g^p = 1\rangle$, and $\mathrm{Fr}(P) = \langle a^p, [a, b] \,|\, a, b \in P\rangle$ (the Frattini subgroup). For any finite group G, $O_p(G)$ is the largest normal p-subgroup of G, and $O^p(G)$ is the smallest normal subgroup of p-power index.

When G acts on a group X, we let $C_X(G)$ be the subgroup of elements of X fixed by G. When $A \subseteq G$ and $B \subseteq X$ are subsets, we set $[A, B] = \langle g(x)x^{-1} \,|\, g \in A,\ x \in B\rangle$. When g, h are elements of any group G, we write their commutator $[g, h] = ghg^{-1}h^{-1}$.

1. Saturated fusion systems

When G is a finite group and $S \in \mathrm{Syl}_p(G)$, the *fusion system* of G over S is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S, and where $\mathrm{Mor}_{\mathcal{F}_S(G)}(P, Q) = \mathrm{Hom}_G(P, Q)$ is the set of monomorphisms from P to Q induced by conjugation in G. An abstract fusion system $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, whose morphisms are monomorphisms of groups including all those induced by conjugation in S, and where for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q) = \mathrm{Mor}_{\mathcal{F}}(P, Q)$, φ restricts to an $\mathcal{F}$-isomorphism from P to $\varphi(P) \le Q$. A fusion system is *saturated* if it satisfies certain additional conditions. Rather than listing those conditions here, we refer to [AKO, Definition I.2.2] or our earlier paper [AOV].

In particular, for any finite G with $S \in \mathrm{Syl}_p(G)$, $\mathcal{F}_S(G)$ is a saturated fusion system (cf. [AKO, Theorem I.2.3]). An abstract fusion system $\mathcal{F}$ over S is called *realizable* if $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with $S \in \mathrm{Syl}_p(G)$, and is called *exotic* otherwise.

If G is a finite group and p is a prime, then a proper subgroup $H < G$ is *strongly p-embedded* in G if $p \big| |H|$, and for each $g \in G \smallsetminus H$, $p \nmid |H \cap gHg^{-1}|$. We refer to [AKO, Proposition A.7] for a very brief survey of some of the properties of strongly p-embedded subgroups, and to [A1, § 46] or [Sz2, § 6.4] for more details.

Definition 1.1. *Fix a prime p, a finite p-group S, and a saturated fusion system $\mathcal{F}$ over S. Let $P \leq S$ be any subgroup. Set $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$.*

- *$P^{\mathcal{F}}$ denotes the set of subgroups of S which are $\mathcal{F}$-*conjugate* to P; i.e., isomorphic to P in the category $\mathcal{F}$. For each $g \in S$, $g^{\mathcal{F}}$ denotes the $\mathcal{F}$-conjugacy class of g.*
- *P is* fully normalized in $\mathcal{F}$ *if $|N_S(P)| \geq |N_S(Q)|$ for each $Q \in P^{\mathcal{F}}$.*
- *P is* $\mathcal{F}$-centric *if $C_S(Q) = Z(Q)$ for all $Q \in P^{\mathcal{F}}$.*
- *P is* $\mathcal{F}$-essential *if $P < S$, P is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup.*
- *P is* normal in $\mathcal{F}$ *($P \trianglelefteq \mathcal{F}$) if $P \trianglelefteq S$ and every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ in $\mathcal{F}$ extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ,PR)$ such that $\overline{\varphi}(P) = P$.*
- *$O_p(\mathcal{F})$ denotes the largest subgroup of S which is normal in $\mathcal{F}$.*
- *$N_{\mathcal{F}}(P) \subseteq \mathcal{F}$ denotes the largest fusion subsystem over $N_S(P)$ (i.e., the largest subcategory of $\mathcal{F}$ which is a fusion system over $N_S(P)$) which contains P as normal subgroup.*
- *For each $\varphi \in \mathrm{Aut}(S)$, $\varphi\mathcal{F}\varphi^{-1}$ is the fusion system over S defined by*

$$\mathrm{Hom}_{\varphi\mathcal{F}\varphi^{-1}}(P,Q) = \bigl\{ (\varphi|_{\varphi^{-1}(Q)}) \circ \psi \circ (\varphi|_{\varphi^{-1}(P)})^{-1} \bigm| \psi \in \mathrm{Hom}_{\mathcal{F}}(\varphi^{-1}(P), \varphi^{-1}(Q)) \bigr\}$$

for all $P, Q \leq S$.

It follows immediately from the definition of a normal subgroup in $\mathcal{F}$ that the maximal subgroup $O_p(\mathcal{F}) \trianglelefteq \mathcal{F}$ is uniquely defined. The notation is, of course, chosen by analogy with that for finite groups.

We now look at essential subgroups of a fusion system.

Theorem 1.2. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then each morphism in $\mathcal{F}$ is a composite of restrictions of morphisms in $\mathrm{Aut}_{\mathcal{F}}(S)$, and of morphisms in $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for $\mathcal{F}$-essential subgroups $P \leq S$.*

Proof. See, e.g., [O1, Proposition 1.10(a,b)]. In fact, a proper subgroup $P < S$ fully normalized in $\mathcal{F}$ is $\mathcal{F}$-essential exactly when $\mathrm{Aut}_{\mathcal{F}}(P)$ is *not* generated by restrictions of morphisms between strictly larger subgroups of S. (See [OV, Proposition 2.5] or [AKO, Proposition I.3.3(b)] for more details.) □

The next proposition follows easily from Theorem 1.2, together with the definition of a normal p-subgroup.

Proposition 1.3. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and fix $Q \leq S$. Assume, whenever $P = S$ or P is $\mathcal{F}$-essential, that $Q \leq P$ and each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ sends Q to itself. Then $Q \trianglelefteq \mathcal{F}$.*

Proof. See, e.g., [AKO, Proposition I.4.5] for details. □

The next two lemmas are our main tools for detecting essential subgroups, or rather, for proving that certain subgroups are not essential.

Lemma 1.4 ([OV, Lemma 3.4])**.** *Fix a prime p, a finite p-group S, a subgroup $P \leq S$, and a characteristic subgroup $\Theta \leq P$. Assume there is $g \in N_S(P)\smallsetminus P$ such that*

(a) $[g, P] \leq \Theta{\cdot}\mathrm{Fr}(P)$, *and*

(b) $[g, \Theta] \leq \mathrm{Fr}(P)$.

Then $c_g \in O_p(\mathrm{Aut}(P))$. Hence P is not $\mathcal{F}$-essential for any saturated fusion system $\mathcal{F}$ over S.

In fact, [OV, Lemma 3.4] is stated in terms of "(semi)critical subgroups" of a finite p-group S rather than essential subgroups. We refer to [OV, Definition 3.1] for the definition of critical subgroups, and just note here that by [OV, Proposition 3.2], each $\mathcal{F}$-essential subgroup (for any saturated fusion system $\mathcal{F}$ over S) is critical in S. This remark also applies to the next lemma, which is a special case of [OV, Proposition 3.3(c)].

Lemma 1.5. *Let S be a finite 2-group. Assume that $P \leq S$ is $\mathcal{F}$-essential for some saturated fusion system $\mathcal{F}$ over S, and also that $|N_S(P)/P| \geq 4$. Then $\mathrm{rk}(P/\mathrm{Fr}(P)) \geq 4$, and $\mathrm{rk}([s, P/\mathrm{Fr}(P)]) \geq 2$ for all $s \in N_S(P)\smallsetminus P$.*

We next recall the definitions of the focal and hyperfocal subgroups of a saturated fusion system, defined by analogy with the finite group case.

Definition 1.6. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. The* focal subgroup *of $\mathcal{F}$ is the subgroup*

$$\begin{aligned}\mathfrak{foc}(\mathcal{F}) &\stackrel{\text{def}}{=} \langle g^{-1}h \,|\, g, h \in S \text{ and } h \in g^{\mathcal{F}}\rangle \\ &= \langle g^{-1}\alpha(g) \,|\, g \in P \leq S,\ P = S \text{ or } P \text{ is } \mathcal{F}\text{-essential},\ \alpha \in \mathrm{Aut}_{\mathcal{F}}(P)\rangle.\end{aligned}$$

The hyperfocal subgroup *of $\mathcal{F}$ is the subgroup*

$$\mathfrak{hyp}(\mathcal{F}) = \langle g^{-1}\alpha(g) \,|\, g \in P \leq S, \alpha \in O^p(\mathrm{Aut}_{\mathcal{F}}(P))\rangle.$$

The two definitions of $\mathfrak{foc}(\mathcal{F})$ are equivalent by Theorem 1.2. In the definition of $\mathfrak{hyp}(\mathcal{F})$, we could equivalently restrict to automorphisms of order prime to p. When $\mathcal{F} = \mathcal{F}_S(G)$ for a finite group G and $S \in \mathrm{Syl}_p(G)$, then $\mathfrak{foc}(\mathcal{F}) = S \cap [G, G]$ by the focal subgroup theorem (cf. [G, Theorem 7.3.4]), and $\mathfrak{hyp}(\mathcal{F}) = S \cap O^p(G)$ by the hyperfocal theorem of Puig [Pg1, § 1.1].

Next recall the following definitions from [5a2].

Definition 1.7. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathcal{F}_0 \subseteq \mathcal{F}$ be a saturated fusion subsystem over a subgroup $S_0 \leq S$.*

(a) *$\mathcal{F}_0$ has* p-power index *in $\mathcal{F}$ if $\mathfrak{hyp}(\mathcal{F}) \leq S_0 \leq S$, and $\mathrm{Aut}_{\mathcal{F}_0}(P) \geq O^p(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P \leq S_0$.*

(b) $\mathcal{F}_0$ *has* index prime to p *in* $\mathcal{F}$ *if* $S_0 = S$, *and* $\mathrm{Aut}_{\mathcal{F}_0}(P) \geq O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ *for all* $P \leq S$.

By [5a2, Theorems 4.3 & 5.4], each saturated fusion system $\mathcal{F}$ over a finite p-group S contains a unique minimal saturated fusion subsystem $O^p(\mathcal{F})$ of p-power index (over $\mathfrak{hyp}(\mathcal{F})$), and a unique minimal saturated fusion subsystem $O^{p'}(\mathcal{F})$ of index prime to p (over S). Furthermore:

Proposition 1.8. *For any saturated fusion system* $\mathcal{F}$ *over a finite* p*-group* S,

$$\mathcal{F} = O^p(\mathcal{F}) \iff \mathfrak{hyp}(\mathcal{F}) = S \iff \mathfrak{foc}(\mathcal{F}) = S .$$

Proof. See, e.g., [AKO, Corollary I.7.5]. The second equivalence follows upon checking that the image of $\mathfrak{foc}(\mathcal{F})$ in $S/\mathfrak{hyp}(\mathcal{F})$ is precisely its commutator subgroup (cf. [AKO, Lemma I.7.2]). □

2. Reduced fusion systems

This paper is centered around the special class of *reduced* fusion systems, which are defined as follows.

Definition 2.1. *A* reduced fusion system *is a saturated fusion system* $\mathcal{F}$ *such that*

- $\mathcal{F}$ *has no nontrivial normal* p*-subgroups,*
- $\mathcal{F}$ *has no proper subsystem of* p*-power index, and*
- $\mathcal{F}$ *has no proper subsystem of index prime to* p.

Equivalently, $\mathcal{F}$ *is reduced if* $O_p(\mathcal{F}) = 1$, $O^p(\mathcal{F}) = \mathcal{F}$, *and* $O^{p'}(\mathcal{F}) = \mathcal{F}$.

Definition 2.1 was originally formulated in [AOV], and was motivated by Theorems A and B in that paper. Very roughly, those theorems describe a way to "detect" exotic fusion systems while looking only at reduced fusion systems.

In this section, we give some conditions on a fusion system which are necessary for it to be reduced (equivalently, conditions which imply that it is not reduced). We begin with two very general results.

Lemma 2.2. *If* $\mathcal{F}$ *is a reduced fusion system over a nontrivial finite* 2*-group* S, *and* $\mathcal{E}$ *is the set of* $\mathcal{F}$*-essential subgroups of* S, *then* $|\mathcal{E}| \geq 2$ *and* $[S{:}\langle\mathcal{E}\rangle] \neq 2$.

Proof. By Proposition 1.3, $S \trianglelefteq \mathcal{F}$ if $\mathcal{E} = \varnothing$, while $P \trianglelefteq \mathcal{F}$ if $\mathcal{E} = \{P\}$ for some P. So $\mathcal{F}$ is not reduced ($O_p(\mathcal{F}) \neq 1$) if $|\mathcal{E}| \leq 1$.

If $[S{:}\langle\mathcal{E}\rangle] = 2$, then $[\mathrm{Aut}_{\mathcal{F}}(S), S] \leq \langle\mathcal{E}\rangle$ since $\mathrm{Aut}_{\mathcal{F}}(S)$ acts trivially on $S/\langle\mathcal{E}\rangle \cong C_2$. Since $\mathfrak{foc}(\mathcal{F})$ is generated by $[\mathrm{Aut}_{\mathcal{F}}(S), S]$ and the $[\mathrm{Aut}_{\mathcal{F}}(P), P]$ for $P \in \mathcal{E}$, $\mathfrak{foc}(\mathcal{F}) \leq \langle\mathcal{E}\rangle < S$. So $\mathcal{F}$ is not reduced by Proposition 1.8. □

The next proposition is a simple application of a transfer homomorphism for fusion systems.

Proposition 2.3. *Let* $\mathcal{F}$ *be a saturated fusion system over a finite* 2*-group* S.

(a) *Assume there is* $g \in \Omega_1(Z(S)){\smallsetminus}[S,S]$ *such that each* $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ *sends the coset* $g[S,S]$ *to itself. Then* $g \notin \mathfrak{foc}(\mathcal{F})$.

(b) *More generally, let $U \trianglelefteq S$ be such that each element of $\mathrm{Aut}_{\mathcal{F}}(S)$ sends U to itself, and $U \leq [P,P]$ for each $P < S$ of index two. Assume there is $g \in S{\smallsetminus}[S,S]$ such that $[g,S] \leq U$, $g^2 \in U$, and each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ sends the coset $g[S,S]$ to itself. Then $g \notin \mathfrak{foc}(\mathcal{F})$.*

In either case, $\mathcal{F}$ is not reduced.

Proof. We refer to [AKO, § I.8] for some of the properties of the transfer homomorphism $\mathrm{trf}_{\mathcal{F}}\colon S/\mathfrak{foc}(\mathcal{F}) \longrightarrow S^{\mathrm{ab}}$ when $\mathcal{F}$ is a saturated fusion system over S. Let $[g] \in S^{\mathrm{ab}}$ be the class of g. Since (a) is a special case of (b) (the case $U = 1$), and was shown in [AKO, Corollary I.8.5], we assume g satisfies the conditions of (b).

For $P < S$, let $\mathrm{trf}_P^S\colon S^{\mathrm{ab}} \longrightarrow P^{\mathrm{ab}}$ be the usual transfer homomorphism (cf. [AKO, Lemma I.8.1(b)]). If $[S{:}P] = 2$, then $\mathrm{trf}_P^S([g]) = [g^2]$ if $g \notin P$, and $\mathrm{trf}_P^S([g]) = [gxgx^{-1}]$ if $g \in P$ and $x \in S{\smallsetminus}P$. This follows from the construction in [AKO] upon taking coset representatives $\{1,x\}$. Since $g^2 \in U$, $gxgx^{-1} = g^2[g^{-1},x] \in U$, and $U \leq [P,P]$, $\mathrm{trf}_P^S([g]) = 1$. Since this holds for each $P < S$ of index two, $\mathrm{trf}_P^S([g]) = 1$ for each $P < S$ since transfers compose (cf. [AKO, Lemma I.8.1(d)]).

By assumption, for each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$, $\alpha([g]) = [g]$. So by [AKO, Proposition I.8.4(a)], $\mathrm{trf}_{\mathcal{F}}([g]) = [g]^k \neq 1$ where $k = |\mathrm{Out}_{\mathcal{F}}(S)|$ is odd. Thus $g \notin \mathfrak{foc}(\mathcal{F})$ since $\mathrm{trf}_{\mathcal{F}}$ is well defined, so $\mathfrak{foc}(\mathcal{F}) < S$, $O^2(\mathcal{F}) \neq \mathcal{F}$ by Proposition 1.8, and $\mathcal{F}$ is not reduced. □

The next lemma is an application of Lemma A.8, together with the transfer homomorphism for fusion systems (cf. [AKO, § I.8]). Recall that a finite group G is *metacyclic* if it has a normal cyclic subgroup $H \trianglelefteq G$ such that G/H is also cyclic.

Lemma 2.4. *Let S be a finite 2-group, and let $\mathcal{F}$ be a saturated fusion system over S. Let $\mathcal{E}$ be the set of $\mathcal{F}$-essential subgroups of S.*

(a) *Assume $P \in \mathcal{E}$ is such that $[N_S(P),P]$ is cyclic. Then there are decompositions $P = P_0P_1$ and $\mathrm{Out}_{\mathcal{F}}(P) = \Gamma_0 \times \Gamma_1$, where for $i = 0,1$, $[P,P] \leq P_i \trianglelefteq P$, Γ_i sends P_i to itself and acts trivially on $P_{1-i}/[P,P]$, Γ_0 has odd order, and $\Gamma_1 \cong \Sigma_3$. Either*
   (i) *P is abelian, $P_1 \cong C_{2^n} \times C_{2^n}$ for some $n \geq 1$, $C_{P_1}(N_S(P)) \cong C_{2^n}$ and $[N_S(P),P] \cong C_{2^n}$ are both direct factors of P_1, and $P_0 \cap P_1 = 1$; or*
   (ii) *$P_1 \cong Q_8$, $[P_0,P_1] = 1$, and $P_0 \cap P_1 = [P,P] = Z(P_1)$.*

(b) *If the image of $\mathrm{Aut}_{\mathcal{F}}(S)$ in $\mathrm{Aut}(S/Z(S))$ is a 2-group, then $\mathrm{Aut}_{\mathcal{F}}(S) = \Delta \times \mathrm{Inn}(S)$ for some (unique) subgroup $\Delta \leq \mathrm{Aut}_{\mathcal{F}}(S)$ of odd order.*

(c) *Assume $\mathcal{F}$ is reduced and the image of $\mathrm{Aut}_{\mathcal{F}}(S)$ in $\mathrm{Aut}(S/Z(S))$ is a 2-group. Assume also, for each $P \in \mathcal{E}$, that $[N_S(P),P]$ is cyclic and the factor P_0 of point (a) is contained in $Z(S)$. Then $\mathrm{Out}_{\mathcal{F}}(S) = 1$, and $\Omega_1(Z(S)) \leq [S,S]$.*

Proof. **(a)** By Lemma 1.5, $|\mathrm{Out}_S(P)| = |N_S(P)/P| = 2$. If P is abelian, then by Lemma A.8, P and $\mathrm{Aut}_{\mathcal{F}}(P)$ have decompositions as described in (i). More precisely, Lemma A.8 says that $P_1 \cong C_{2^n} \times C_{2^n}$ and $[N_S(P),P] = [N_S(P),P_1] \cong C_{2^n}$ for some $n \geq 1$, and hence that $[N_S(P),P]$ is a direct factor of P_1. Also, for $x \in N_S(P){\smallsetminus}P$, $C_{P_1}(N_S(P)) = C_{P_1}(x)$ is the kernel of the map $P_1 \longrightarrow P_1$ which sends g to $[x,g]$, so $P_1/C_{P_1}(x) \cong [x,P_1] \cong C_{2^n}$, and $C_{P_1}(x) \cong C_{2^n}$ is also a direct factor of P_1.

Assume P is nonabelian. By Lemma A.2 (applied with $P_0 = [P,P]$), the kernel of the action of $\mathrm{Aut}_{\mathcal{F}}(P)$ on P^{ab} is a 2-group. Also, $O_2(\mathrm{Out}_{\mathcal{F}}(P)) = 1$ (i.e., $\mathrm{Out}_S(P) \ntrianglelefteq$

$\mathrm{Out}_{\mathcal{F}}(P)$), since $\mathrm{Out}_{\mathcal{F}}(P)$ has a strongly 2-embedded subgroup. Thus $\mathrm{Out}_{\mathcal{F}}(P)$ acts faithfully on P^{ab}.

Set $P' = [P,P] \neq 1$. By Lemma A.8, applied to the $\mathrm{Out}_{\mathcal{F}}(P)$-action on $P^{\mathrm{ab}} = P/P'$, there are decompositions $P = P_0P_1$ and $\mathrm{Out}_{\mathcal{F}}(P) = \Gamma_0 \times \Gamma_1$ such that $P_i \trianglelefteq P$ is $\mathrm{Aut}_{\mathcal{F}}(P)$-invariant, $P_0 \cap P_1 = P'$, $P_1/P' \cong C_{2^n} \times C_{2^n}$ for some $n \geq 1$, Γ_0 has odd order and acts trivially on P_1/P', and $\Gamma_1 \cong \Sigma_3$ acts trivially on P_0/P'. Also (by the same lemma), $[N_S(P), P^{\mathrm{ab}}] = [N_S(P), P_1/P'] \cong C_{2^n}$, so

$$P_1/[N_S(P),P] \cong (P_1/P')/[N_S(P),P_1/P'] \cong C_{2^n}.$$

Since $[N_S(P),P]$ is cyclic by assumption, P_1 is metacyclic.

Any $[\alpha] \in \Gamma_1 \cong \Sigma_3$ of order 3 lifts to some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$, and upon replacing α by α^k for appropriate k, we can assume α has order 3. If P_1 is abelian, then $P_1 \cong C_{2^n} \times C_{2^m}$ where $m > n$ (since $P' \neq 1$), which is impossible by Corollary A.3(a). So by Lemma A.7, $P_1 \cong Q_8$. Hence $P_1/P' \cong C_2^2$, and $P' = [P_1,P_1] = Z(P_1) \cong C_2$.

For $x \in P_0$ and $y \in P_1$, $[x,y] = \alpha([x,y]) = [x,\alpha(y)]$ since $|P'| = 2$ and α acts trivially on P_0/P'. Hence $[x, y^{-1}\alpha(y)] = 1$, and $[P_0,P_1] = 1$ since $[\alpha,P_1] = P_1$.

(b) Assume that the image of $\mathrm{Aut}_{\mathcal{F}}(S)$ in $\mathrm{Aut}(S/Z(S))$ is a 2-group. Then for each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ of odd order, α induces the identity on $S/Z(S)$. So for each $g \in S$, there is $x \in Z(S)$ such that $\alpha(g) = xg$, and hence $\alpha c_g \alpha^{-1} = c_{\alpha(g)} = c_g$.

Thus each element of odd order in $\mathrm{Aut}_{\mathcal{F}}(S)$ commutes with $\mathrm{Inn}(S)$, so $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Inn}(S)C_{\mathrm{Aut}_{\mathcal{F}}(S)}(\mathrm{Inn}(S))$. Since $\mathrm{Inn}(S)$ is 2-centric in $\mathrm{Aut}_{\mathcal{F}}(S)$, $C_{\mathrm{Aut}_{\mathcal{F}}(S)}(\mathrm{Inn}(S)) = Z(\mathrm{Inn}(S)) \times \Delta$ where Δ has odd order (cf. [BLO1, Lemma A.4]). Thus $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Inn}(S) \times \Delta$.

(c) Let Δ be as in (b), and set $Q = [\Delta, S] \leq Z(S)$. We first show that $Q \trianglelefteq \mathcal{F}$.

Fix $P \in \mathcal{E}$, and let $P = P_0P_1$ and $\mathrm{Out}_{\mathcal{F}}(P) = \Gamma_0 \times \Gamma_1$ be the decompositions of (a). For each $\delta \in \Delta$, $\delta(P) = P$ since $[\Delta,S] = Q \leq Z(S) \leq P$, so $\delta|_P \in \mathrm{Aut}_{\mathcal{F}}(P)$. Also, $[\delta|_P] \in N_{\mathrm{Out}_{\mathcal{F}}(P)}(\mathrm{Out}_S(P)) = \Gamma_0 \times \mathrm{Out}_S(P)$ since $\delta(N_S(P)) = N_S(P)$, so $[\delta|_P] \in \Gamma_0$ since it has odd order. Thus $[\delta, P] \leq P_0$. Hence

$$Q = [\Delta,S] = [\Delta,[\Delta,S]] \leq [\Delta,P] \leq P_0,$$

where the second equality holds by [G, Theorem 5.3.6].

Fix $\beta \in \mathrm{Aut}_{\mathcal{F}}(P)$ of odd order, and let $[\beta]$ be its class in $\mathrm{Out}_{\mathcal{F}}(P)$. If $[\beta] \in \Gamma_0$, then since $P_0 \leq Z(S)$ by assumption, $\beta|_{P_0}$ extends to an element of odd order in $\mathrm{Aut}_{\mathcal{F}}(S)$ by the extension axiom (i.e., since P_0 is fully centralized), and thus extends to an element of Δ. So $\beta(Q) = Q$ in this case. If $[\beta] \in \Gamma_1$, then β induces the identity on $P_0/[P,P]$ and on $[P,P]$ (since $|[P,P]| \leq 2$), and hence $\beta|_{P_0} = \mathrm{Id}_{P_0}$ (and $\beta|_Q = \mathrm{Id}_Q$) by Lemma A.2. Since $\mathrm{Aut}_S(P)$ acts trivially on $Z(S)$, this proves that all elements of $\mathrm{Aut}_{\mathcal{F}}(P)$ send Q to itself.

Since this holds for each $P \in \mathcal{E}$, $Q \trianglelefteq \mathcal{F}$ by Proposition 1.3. Hence $Q = [\Delta,S] = 1$ since $\mathcal{F}$ is reduced, so $\mathrm{Out}_{\mathcal{F}}(S) \cong \Delta = 1$, and $\Omega_1(Z(S)) \leq [S,S]$ by Proposition 2.3(a) (and since $\mathcal{F}$ is reduced). □

The next proposition will be greatly generalized in Section 5, as a consequence of the results in Section 4 using amalgams.

Proposition 2.5. *Let S be any finite nonabelian 2-group such that $[S,S]$ is cyclic and S has an abelian subgroup of index two. Then either $S \cong D_{2^n}$ ($n \geq 3$), SD_{2^n} ($n \geq 4$), or $C_{2^n} \wr C_2$ ($n \geq 2$); or there is no reduced fusion system over S.*

Proof. Let $A < S$ be abelian of index two in S. By Lemma A.6(a), all elements in $(S/Z(S))\smallsetminus(A/Z(S))$ have order two, and $A/Z(S) \cong [S,S]$. Since $[S,S]$ is cyclic, $S/Z(S)$ is dihedral (or $\cong C_2^2$).

Assume $\mathcal{F}$ is a reduced fusion system over S, and let $\mathcal{E}$ be the set of $\mathcal{F}$-essential subgroups of S. We first show that the hypotheses of Lemma 2.4(c) hold. For each $P \in \mathcal{E}$, $[N_S(P), P]$ is cyclic since $[S,S]$ is cyclic, and hence Lemma 2.4(a) applies to P. Let $P_0, P_1 \trianglelefteq P$ be as in that lemma; thus $P = P_0P_1$ and $P_0 \cap P_1 = [P,P]$.

If P is abelian, it must be maximal abelian since it is centric. So either $P = A$, in which case $Z(S) = C_P(S) = C_P(N_S(P))$ and $P/Z(S) = A/Z(S)$ is cyclic; or $PA = S$, in which case $P \cap A = Z(S)$, so $|P/Z(S)| = 2$, $Z(S) \le C_P(N_S(P)) < P$, and hence $Z(S) = C_P(N_S(P))$. In either case, $Z(S) = C_P(N_S(P)) \ge P_0$ and $P/Z(S)$ is cyclic.

If P is nonabelian, then by Lemma 2.4(a.ii), $P_1 \cong Q_8$, $[P,P] = Z(P_1)$, and $[P_0, P_1] = 1$. In particular, $Z(S) \le C_P(P_1) = P_0$. If $Z(S) < P_0$, then $P/Z(S)$ contains a subgroup isomorphic to Q_8 (if $Z(P_1) \nleq Z(S)$) or C_2^3, both of which are impossible since $S/Z(S)$ is dihedral. Hence $Z(S) = P_0$.

Recall that $S/Z(S)$ is dihedral of order ≥ 4. If $|S/Z(S)| \ge 8$, then $\mathrm{Aut}(S/Z(S))$ is a 2-group by Corollary A.3(b). If $|S/Z(S)| = 4$, then subgroups A_1, A_2, A_3 of index two in S which contain $Z(S)$ are all abelian, they are the only proper subgroups centric in S, and hence the only subgroups which could be in $\mathcal{E}$. If $A_i \in \mathcal{E}$ (recall $\mathcal{E} \ne \varnothing$ by Lemma 2.2), then by Lemma 2.4(a.i), it contains a direct factor $A_{i1} \cong C_{2^m} \times C_{2^m}$ (some $m \ge 1$), which in turn contains $C_{A_{i1}}(S) \cong C_{2^m}$ and $[S,S] = [S, A_i] \cong C_{2^m}$ as direct factors. Thus $Z(S) = C_{A_i}(S) \cong A_{i0} \times C_{2^m}$ and $[S,S]$ are both direct factors of A_i, and so $[S,S] \nleq \mathrm{Fr}(Z(S))$. By Lemma A.6(d), there is no automorphism of S which permutes the A_i transitively, and thus the image of $\mathrm{Aut}_{\mathcal{F}}(S)$ in $\mathrm{Aut}(S/Z(S)) \cong \mathrm{Aut}(C_2^2)$ is a 2-group.

The hypotheses of Lemma 2.4(c) thus hold, and so $\Omega_1(Z(S)) \le [S,S]$. Since $[S,S]$ is cyclic, this implies that $|\Omega_1(Z(S))| = 2$, and hence that $Z(S)$ is cyclic.

If $|Z(S)| = 2$, then $Z(S) = \Omega_1(Z(S)) \le [S,S]$. So $S^{\mathrm{ab}} \cong (S/Z(S))^{\mathrm{ab}} \cong C_2^2$, which implies S is dihedral, semidihedral, or quaternion (cf. [G, Theorem 5.4.5]). If $S \cong Q_{2^n}$, then by Lemma 2.4(a), for each $P \in \mathcal{E}$, $P \cong Q_8$ and $Z(P) = Z(S)$. Hence $Z(S) \trianglelefteq \mathcal{F}$ by Proposition 1.3, which contradicts the assumption that $\mathcal{F}$ is reduced.

Now assume $|Z(S)| = 2^m$ for $m \ge 2$; we will show that $S \cong C_{2^m} \wr C_2$. If $P \le S$ is any nonabelian subgroup, then since $[S,S]$ is cyclic, $\Omega_1([S,S]) = \Omega_1([P,P])$ is characteristic in P. So if all $\mathcal{F}$-essential subgroups are nonabelian, then $\Omega_1([S,S])$ is characteristic in each of them, and hence is normal in $\mathcal{F}$ by Proposition 1.3. Since this contradicts the assumption that $\mathcal{F}$ is reduced, there is an abelian subgroup $P \in \mathcal{E}$, and we already saw that this implies $P/Z(S)$ is cyclic. Since $Z(S)$ is also cyclic, P has rank two, and $P \cong (C_{2^n})^2$ by Lemma 2.4(a.i) (i.e., $P_0 = 1$). Then $n = m \ge 2$, since $Z(S)$ and $P/Z(S)$ are cyclic of order 2^m and 2^{2n-m}, respectively. Also, $P/Z(S) \cong C_{2^m}$ is normal in the dihedral group $S/Z(S)$, and hence $P \trianglelefteq S$. So $[S{:}P] = 2$ by Lemma 1.5 (and since $\mathrm{rk}(P) = 2$). Choose any $t \in S \smallsetminus P$, fix $a \in P$ such that $aZ(S)$ generates $P/Z(S)$, and set $b = tat^{-1}$. Then $a^{2^{m-1}} \notin Z(S)$ implies $1 \ne [t, a^{2^{m-1}}] = (ba^{-1})^{2^{m-1}}$, so $|ba^{-1}| = 2^m$, and $P = \langle a, b\rangle$. Also, $tbt^{-1} = a$ since $t^2 \in P$, so $Z(S) = \langle ab\rangle$. Let i be such that $t^2 = (ab)^i$; then $(a^{-i}t)^2 = 1$, and this finishes the proof that $S \cong C_{2^m} \wr C_2$. □

In Section 5, as applications of our main theorems, we will generalize Proposition 2.5 by giving different (weaker) conditions on a 2-group each of which implies the conclusion of Proposition 2.5. For example, by Proposition 5.2(a,b), the same conclusion holds if S has an abelian subgroup of index 2 *or* $[S,S]$ is cyclic.

3. Two examples

For use in Section 4, we determine here the essential subgroups of the simple groups $PSU_3(3)$ and M_{12}. At the same time, since reduced fusion systems over wreath products $C_{2^n}\wr C_2$ play an important role in Section 5 (and in Proposition 2.5), we determine all reduced fusion systems over such groups.

We begin with the wreath products. Let $v_2(-)$ denote the 2-adic valuation: $v_2(n)=k$ if $2^k|n$ and $2^{k+1}\nmid n$.

Proposition 3.1. *Assume $S=\langle a,b,t\rangle\cong C_{2^m}\wr C_2$ for some $m\ge 2$, where $A\overset{\text{def}}{=}\langle a,b\rangle\cong C_{2^m}\times C_{2^m}$, $t^2=1$, and $tat^{-1}=b$. Set*

$$Q=\langle ab,a^{2^{m-1}},t\rangle\cong C_{2^m}\times_{C_2}D_8\cong C_{2^m}\times_{C_2}Q_8\ .$$

(a) *If $\mathcal{F}$ is a saturated fusion system over S, then the only subgroups of S which could be $\mathcal{F}$-essential are A and the subgroups S-conjugate to Q. If $O^2(\mathcal{F})=\mathcal{F}$, then all of these subgroups are $\mathcal{F}$-essential.*

(b) *Up to isomorphism, there is a unique saturated fusion system $\mathcal{F}$ over S such that $O^2(\mathcal{F})=\mathcal{F}$. Also, $\mathcal{F}$ is reduced, and is isomorphic to the fusion system of $PSL_3(q)$ for any prime power q such that $v_2(q-1)=m$, and to the fusion system of $PSU_3(q)$ for any prime power q such that $v_2(q+1)=m$.*

Proof. Let $\mathcal{F}$ be a saturated fusion system over S, and let $\mathcal{E}$ be the set of $\mathcal{F}$-essential subgroups of S. If $P\in\mathcal{E}$ and $|N_S(P)/P|\ge 4$, then $\operatorname{rk}(P/\operatorname{Fr}(P))\ge 4$ by Lemma 1.5. Since $P\cap A$ is abelian of rank ≤ 2 and has index ≤ 2 in P, this is impossible. So $|N_S(P)/P|=2$.

Assume $P\ne A$. Since P is centric in S, $Z(S)=\langle ab\rangle\le P$. Also, $P\not\le A$ since P is centric, and $P\ngeq A$ since $P\notin\{A,S\}$. Thus $P\cap A=\langle ab,a^{2i}\rangle$ for some i, and $P=\langle ab,a^{2i},a^jt\rangle$ for some j. If $a^{2i}=1$, then P is abelian with a cyclic subgroup $Z(S)$ of index two and order ≥ 4, and $\operatorname{Aut}(P)$ is a 2-group by Corollary A.3(a). If $|a^{2i}|\ge 4$, then $Z(P)=\langle ab\rangle=Z(S)$ is cyclic and $P/Z(P)$ is dihedral of order $2\cdot|a^{2i}|\ge 8$ (all elements in $(S/Z(S))\smallsetminus(A/Z(S))$ have order two by Lemma A.6(a)), so $\operatorname{Aut}(P)$ is a 2-group by Lemma A.2 and Corollary A.3(b). Thus P can be essential only if $P=\langle ab,a^{2i},a^jt\rangle$ where $|a^{2i}|=2$. If j is odd, then $(a^jt)^2=(ab)^j$ generates $Z(S)$, $P^{\rm ab}\cong C_2\times C_{2^m}$, and $\operatorname{Aut}(P)$ is a 2-group by Corollary A.3(a,c). This leaves only the possiblity $P=\langle ab,a^{2^{m-1}},a^jt\rangle=a^{j/2}Qa^{-j/2}$ for even j. Thus $\mathcal{E}\subseteq\{A\}\cup\mathcal{Q}$, where $\mathcal{Q}$ denotes the S-conjugacy class of Q.

Now assume $O^2(\mathcal{F})=\mathcal{F}$. Then $\mathfrak{foc}(\mathcal{F})=S$ by Proposition 1.8. By Corollary A.3(a,c) (and since $S^{\rm ab}\cong C_{2^m}\times C_2$), $\operatorname{Aut}(S)$ is a 2-group, and so $\operatorname{Out}_{\mathcal{F}}(S)=1$. Hence $\mathfrak{foc}(\mathcal{F})\le\big\langle[S,S],\mathcal{E}\big\rangle$. Since the images of A and of Q are both properly contained in $S^{\rm ab}$, $\mathcal{E}=\{A\}\cup\mathcal{Q}$. This proves (a).

Each of the three abelian subgroups of index two in Q is isomorphic to $C_{2^m}\times C_2$ and contains exactly two elements of order 4 not in $Z(S)$. Hence Q contains exactly six such elements, and they generate a subgroup Q_0 which is the unique subgroup of Q isomorphic to Q_8. Since $Q=Q_0Z(S)$ and $Q_0\cap Z(S)=\langle(ab)^{2^{m-1}}\rangle$ (and $Z(S)=Z(Q)$), $\operatorname{Out}(Q)=\operatorname{Out}(Z(S))\times\operatorname{Out}(Q_0)$, where $\operatorname{Out}(Q_0)\cong\Sigma_3$, and $\operatorname{Out}(Z(S))$ is a 2-group since $Z(S)$ is cyclic. Hence $\mathrm{Out}_{\mathcal{F}}(Q) \cong \Sigma_3$, and $\mathrm{Aut}_{\mathcal{F}}(Q)$ acts via the identity on $Z(S)$ and the full automorphism group of Q_0. Thus $\mathrm{Aut}_{\mathcal{F}}(Q)$ is uniquely determined.

By Lemma A.2, $\mathrm{Aut}(A)/O_2(\mathrm{Aut}(A)) \cong \mathrm{Aut}(A/\mathrm{Fr}(A)) \cong \Sigma_3$. Hence $\mathrm{Aut}_{\mathcal{F}}(A) \cong \Sigma_3$. Set $c = (ab)^{-1} \in Z(S)$, and let $\{a', b', c\}$ be its $\mathrm{Aut}_{\mathcal{F}}(A)$-orbit. Thus a', b', c represent the three involutions in $A/\mathrm{Fr}(A)$, and any two of them generate A. Also, $a'b'c = 1$ since $\mathrm{Aut}_{\mathcal{F}}(A)$ fixes $a'b'c$, and hence $c = (a'b')^{-1}$.

Since $c \in Z(S) = C_A(t)$, c_t exchanges a' and b' and fixes c. We can thus assume the a and b were chosen so that $a = a'$ and $b = b'$. So up to an automorphism of S (i.e., a relabelling of its generators), $\mathrm{Aut}_{\mathcal{F}}(A)$ is uniquely determined. Thus $\mathcal{F}$ is uniquely determined up to isomorphism by (a) and Theorem 1.2.

Now, $O^{2'}(\mathcal{F}) = \mathcal{F}$ since $\mathrm{Out}_{\mathcal{F}}(S) = 1$ ($\mathcal{F}$ is generated by $O^{2'}(\mathcal{F})$ and $\mathrm{Aut}_{\mathcal{F}}(S)$ by Theorem 1.2). If $P \trianglelefteq \mathcal{F}$, then P is contained in all $\mathcal{F}$-essential subgroups (cf. [AKO, Proposition I.4.5]), and hence is contained in their intersection $\langle ab, a^{2^{m-1}} \rangle$. Then $P \leq \langle ab \rangle$ since it is $\mathrm{Aut}_{\mathcal{F}}(Q)$-invariant, and so $P = 1$ since it is $\mathrm{Aut}_{\mathcal{F}}(A)$-invariant. Thus $O_2(\mathcal{F}) = 1$, and $\mathcal{F}$ is reduced.

If q is a prime power with $v_2(q-1) = m$, then $GL_2(q) \leq SL_3(q)$ and hence $PSL_3(q)$ have Sylow 2-subgroups isomorphic to S; while if $v_2(q+1) = m$, then $GU_2(q) \leq SU_3(q)$ and $PSU_3(q)$ have Sylow 2-subgroups isomorphic to S (cf. [CF, pp. 142–143]). Set $G = PSL_3(q)$ or $PSU_3(q)$, as appropriate, and identify $S \in \mathrm{Syl}_2(G)$. Since G is simple, $\mathfrak{foc}(\mathcal{F}_S(G)) = S \cap [G, G] = S$ by the focal subgroup theorem (cf. [G, Theorem 7.3.4]), and hence $\mathcal{F}_S(G) \cong \mathcal{F}$. □

We now look at 2-groups of type M_{12}.

Proposition 3.2. *Consider the group $S = \underset{a,b}{C_4^2} \rtimes \underset{r,t}{C_2^2}$, where $rar^{-1} = a^{-1}$, $rbr^{-1} = b^{-1}$, $tat^{-1} = b$, and $tbt^{-1} = a$. Set $A = \langle a, b \rangle \cong C_4^2$, $R = \langle A, r \rangle \cong C_4^2 \rtimes C_2$, and*

$$Q = \langle a^2, ab, r, t \rangle = \langle ab^{-1}, a^2t \rangle \times_{\langle a^2b^2 \rangle} \langle ab, a^2rt \rangle \cong Q_8 \times_{C_2} Q_8 \ .$$

For any saturated fusion system $\mathcal{F}$ over S, the set of $\mathcal{F}$-essential subgroups is contained in $\{Q, R\}$, with equality if $\mathcal{F}$ is reduced.

Proof. Let $\mathcal{E}$ be the set of $\mathcal{F}$-essential subgroups of S. If $\mathcal{F}$ is reduced, then $|\mathcal{E}| \geq 2$ by Lemma 2.2. So it suffices to prove that $\mathcal{E} \subseteq \{R, Q\}$ for each saturated $\mathcal{F}$.

Fix $P \in \mathcal{E}$. Then $Z(S) = \langle a^2b^2 \rangle \leq P$. Assume first that $\mathrm{Fr}(S) = \langle a^2, ab \rangle \nleq P$, and fix $g \in \mathrm{Fr}(S) \smallsetminus P$. Then $[g, P] \leq [g, S] = \langle a^2b^2 \rangle$, so $|N_S(P)/P| = 2$ by Lemma 1.5, and $a^2b^2 \notin \mathrm{Fr}(P)$ by Lemma 1.4 (applied with $\Theta = 1$). In particular, since $(ab)^2 = (ab^{-1})^2 = a^2b^2$, neither ab nor ab^{-1} is in P. Hence $a^2 \in P$, since $|N_S(P)/P| = 2$. Also, $P \leq C_S(a^2) = \langle a, b, r \rangle$ since $a^2b^2 \notin \mathrm{Fr}(P)$, $N_S(P) \geq \langle a, b, r \rangle$ since $\langle a, b, r \rangle / \langle a^2, b^2 \rangle$ is abelian and $P \geq \langle a^2, b^2 \rangle$, and hence $[\langle a, b, r \rangle{:}P] \leq 2$. So up to S-conjugacy, $P = \langle a, b^2, r \rangle$ or $\langle a, b^2, br \rangle$ (recall $\langle a^2, ab \rangle \nleq P$). In either case, $P \cong C_2 \times D_8$, so $\mathrm{Aut}(P)$ is a 2-group by Lemma A.2 and since P contains a unique subgroup isomorphic to $C_2 \times C_4$, and hence $P \notin \mathcal{E}$.

Thus $P \geq \mathrm{Fr}(S)$, so $P \trianglelefteq S$. If $[S{:}P] \geq 4$, then $|P| \leq 16$, $\mathrm{rk}(P/\mathrm{Fr}(P)) \geq 4$ by Lemma 1.5, and hence $P \cong C_2^4$. This is impossible since $P \geq \mathrm{Fr}(S) \cong C_4 \times C_2$, and so $[S{:}P] = 2$. If $P = \langle a, b, t \rangle$ or $\langle a, b, rt \rangle$, then $P/[P, P] \cong C_4 \times C_2$, $\mathrm{Aut}(P)$ is a 2-group by Corollary A.3(a,c), and hence $P \notin \mathcal{E}$. So $R = \langle a, b, r \rangle$ is the only (possible) subgroup in $\mathcal{E}$ which contains A.

Now assume $P = P_{ij} = \langle ab, a^2, a^i r, a^j t\rangle$ for $i, j = 0, 1$: these are the remaining four subgroups of index 2 in S. Let $Z_2(P) \trianglelefteq P$ be the subgroup such that $Z_2(P)/Z(P) = Z(P/Z(P))$. Then $Z(P) = \langle a^2b^2\rangle$ and $Z_2(P) \geq \langle a^2, ab\rangle$. If $(i, j) \neq (0, 0)$, then the relations

$$[r, at] = a^2, \qquad [ar, t] = ab^{-1}, \qquad [ar, at] = a^{-1}b^{-1}$$

show that $Z_2(P) = \langle a^2, ab\rangle$. So $[a, P] \leq Z_2(P)$, $[a, Z_2(P)] = 1$, and $P \notin \mathcal{E}$ by Lemma 1.4. Thus $Q = P_{00}$ is the only possible subgroup in $\mathcal{E}$ which does not contain A. □

With a little more work, one can show that the only reduced fusion systems over S (as above) are those of M_{12} and $G_2(3)$. But we leave that for a later paper.

4. Detecting essential subgroups via amalgams

We are now ready to describe how theorems of Goldschmidt and Fan [Gd2, Fn] on amalgams can be used to get information about essential subgroups of index two in their normalizer for saturated fusion systems over 2-groups. Throughout the section, in the statements of lemmas and in the proofs of Theorems 4.5 and 4.6, we will refer repeatedly to the following set of hypotheses.

(∗) Assume $P_1, P_2 \leq P \leq G_1, G_2$ are finite groups such that the following hold:

- P is a 2-group, $[P{:}P_1] = [P{:}P_2] = 2$, and $P = P_1P_2$.
- For $i = 1, 2$, $P_i \trianglelefteq G_i$, $G_i/P_i \cong D_{2p_i}$ for some odd prime p_i, and $C_{G_i}(P_i) \leq P_i$.

Set $P_{12} = P_1 \cap P_2$, and let $T \leq P_{12}$ be the largest subgroup which is normal in both G_1 and G_2.

Clearly, hypotheses (∗) imply that $[G_i{:}P] = p_i$ and $P_i = O_2(G_i)$ for $i = 1, 2$. In particular, in the terminology of Goldschmidt [Gd2], the triple $\big(G_1 > P < G_2\big)$ is an amalgam of index (p_1, p_2).

The following lemma helps to explain the motivation for these hypotheses.

Lemma 4.1. *Fix a finite 2-group S and a saturated fusion system $\mathcal{F}$ over S.*

(a) *Assume $P_1, P_2 \leq S$ are distinct $\mathcal{F}$-essential subgroups of index two in S. Then there are groups $G_1 > S < G_2$, and odd primes p_1 and p_2, such that $\mathrm{Out}_{G_i}(P_i) \leq \mathrm{Out}_{\mathcal{F}}(P_i)$, and such that hypotheses (∗) hold with $P = S$.*

(b) *Let $P_1 \leq S$ be an $\mathcal{F}$-essential subgroup which is not normal and has index two in its normalizer. Set $P = N_S(P_1) < S$, choose $x \in N_S(P) \smallsetminus P$ such that $x^2 \in P$, and set $P_2 = xP_1x^{-1}$. Then there are groups $G_1 > P < G_2$ and an odd prime p, such that $\mathrm{Out}_{\mathcal{F}}(P_i) \geq \mathrm{Out}_{G_i}(P_i)$ and hypotheses (∗) hold with $p_1 = p_2 = p$. Also, $x \in N_S(T)$, where T is as defined in (∗), and there is an isomorphism $\beta\colon G_1 \xrightarrow{\cong} G_2$ such that $\beta|_P = c_x|_P$.*

Proof. Assume the hypotheses of (a) or (b), where $P = S$ in (a). Since each P_i is $\mathcal{F}$-essential (and $|N_S(P_i)/P_i| = 2$), $\mathrm{Out}_S(P_i) \in \mathrm{Syl}_2(\mathrm{Out}_{\mathcal{F}}(P_i))$ and is not the only Sylow 2-subgroup. Fix $1 \neq g \in \mathrm{Out}_S(P_i)$, and let $h \in \mathrm{Out}_{\mathcal{F}}(P_i)$ be any other involution. Then $\langle g, h\rangle$ is dihedral since it is generated by two involutions, and it has order $2n$ for some odd integer $n > 1$ since $4 \nmid |\mathrm{Out}_{\mathcal{F}}(P_i)|$. Upon choosing an appropriate subgroup

$\Gamma_i \le \langle g, h \rangle$, we can arrange that $\mathrm{Out}_S(P_i) \le \Gamma_i \le \mathrm{Out}_{\mathcal{F}}(P_i)$ and $\Gamma_i \cong D_{2p_i}$ for some odd prime p_i.

Fix $i = 1, 2$, and set $\mathcal{F}_i = N_{\mathcal{F}}(P_i)$ (see Definition 1.1). This is a saturated fusion system over $N_S(P_i) = P$ (cf. [AKO, Theorem I.5.5]), and is constrained in the sense of [AKO, Definition I.4.8] since $P_i \trianglelefteq \mathcal{F}_i$ and $C_P(P_i) \le P_i$. By the model theorem in the form of [AKO, Theorem III.5.10(a)], there is a finite group $G_i^* \ge P$ such that $P_i \trianglelefteq G_i^*$, $C_{G_i^*}(P_i) \le P_i$, $P \in \mathrm{Syl}_2(G_i^*)$, and $\mathcal{F}_P(G_i^*) = \mathcal{F}_i$. In particular, $\mathrm{Out}_{G_i^*}(P_i) = \mathrm{Out}_{\mathcal{F}}(P_i)$. Let $G_i \le G_i^*$ be the unique subgroup such that $P \le G_i$ and $\mathrm{Out}_{G_i}(P_i) = \Gamma_i$. Then these groups satisfy hypotheses $(*)$.

In the situation of (b), we can assume that Γ_2 is chosen so that $\Gamma_2 = [c_x]\Gamma_1[c_x]^{-1} \le \mathrm{Out}_{\mathcal{F}}(P_2)$. Choose G_1 as in (a), and then choose G_2 together with an isomorphism $\beta \in \mathrm{Iso}(G_1, G_2)$ such that $\beta|_P = c_x|_P$. Since T is the unique largest subgroup of P_{12} which is normal in G_1 and G_2, $x \in N_S(T)$ since c_x exchanges P_1 and P_2 (recall $x^2 \in P = N_S(P_1)$). $\square$

As usual, we say T is *centric* in a group $X \ge T$ if $C_X(T) \le T$. In general, our results using hypotheses $(*)$ split into separate cases, depending on whether or not T is centric in P.

We say that a finite group G is *strictly p-constrained* for a prime p if $O_p(G)$ is centric in G. The question of whether one or both of the groups G_i/T (under hypotheses $(*)$) is strictly 2-constrained plays an important role in Fan's classification of amalgams of type (p_1, p_2) [Fn].

Lemma 4.2. *Assume hypotheses $(*)$. Then the following hold.*

(a) *If T is centric in P, then T is also centric in G_1 and G_2.*

(b) *If $O_{p_1}(G_1/T) \ne 1$ or $O_{p_2}(G_2/T) \ne 1$, then T is centric in P.*

(c) *For $i = 1, 2$, G_i/T is strictly 2-constrained if and only if $O_{p_i}(G_i/T) = 1$.*

(d) *Assume T is centric in P, and let $S \ge P$ be any finite 2-group such that $N_S(P_i) = P$ for $i = 1, 2$. Then T is centric in S.*

(e) *Assume T is centric in P, and let $S \ge P$ be as in (d). Set $\widehat{G}_i = \mathrm{Out}_{G_i}(T)$, $\widehat{S} = \mathrm{Out}_S(T)$, and $\widehat{G} = \langle \widehat{G}_1, \widehat{G}_2, \widehat{S} \rangle \le \mathrm{Out}(T)$, and assume $\widehat{S} \in \mathrm{Syl}_2(\widehat{G})$. Then $O_2(\widehat{G}) = 1$, and $\widehat{G}$ acts faithfully on $T/\mathrm{Fr}(T)$.*

Proof. Let $S \ge P$ be any finite 2-group as in (d); i.e., such that $N_S(P_i) = P$ for $i = 1, 2$. Set

$$P^0 = C_P(T), \qquad G_i^0 = C_{G_i}(T) \quad (i = 1, 2), \qquad \text{and} \qquad S^0 = C_S(T)\,.$$

Note that $P^0 \trianglelefteq P$, $G_i^0 \trianglelefteq G_i$, and $P^0 \in \mathrm{Syl}_2(G_i^0)$ since T is normal in P and the G_i.

If $p_i \nmid |G_i^0|$, then G_i^0 is a 2-group, and $G_i^0 \le O_2(G_i) = P_i$ since $G_i^0 \trianglelefteq G_i$. If $P^0 = G_1^0 = G_2^0$, then $P^0 \le P_{12}$ and is normal in G_1 and G_2, so $P^0 \le T$ by the maximality of T. To summarize:

$$p_i \nmid |G_i^0| \implies P^0 = G_i^0 \le P_i \qquad \text{and} \qquad P^0 = G_1^0 = G_2^0 \implies P^0 \le T\,. \tag{1}$$

We next claim that when $i = 1$ or 2,

$$O_{p_i}(G_i/T) \ne 1 \quad \text{or} \quad P^0 \le T \quad \implies \quad p_i \nmid |G_i^0|\,. \tag{2}$$

To see this, assume $p_i \big| |G_i^0|$, and fix $g_i \in G_i^0$ of order p_i. Then $[g_i, T] = 1$, while $[g_i, P_i] \ne 1$ by $(*)$ ($C_{G_i}(P_i) \le P_i$). So by Lemma A.2, g_i acts nontrivially on P_i/T;

i.e., $[g_i, P_i] \not\leq T$. Thus $\langle T, g_i\rangle \ntrianglelefteq G_i$, so $O_{p_i}(G_i/T) = 1$. Also, $G_i^0 T \trianglelefteq G_i$ contains the normal closure of $\langle T, g_i\rangle$ in G_i, so $G_i^0 T > \langle T, g_i\rangle$, $P^0 T > T$, and thus $P^0 \not\leq T$.

(a) If T is centric in P (i.e., $P^0 \leq T$), then it is centric in the G_i by (2) and (1).

(b) Let $H_i \trianglelefteq G_i$ be the subgroup such that $H_i/T = O_{p_i}(G_i/T)$. If $H_1/T \neq 1$ and $H_2/T \neq 1$, then $P^0 \leq T$ by (2) and (1) again. So assume $H_1/T \neq 1$ and $H_2/T = 1$. By (2) and (1), $P^0 = G_1^0 \leq P_1$. Also, $P_{12} \trianglelefteq G_1 = H_1 P$ since

$$[P_{12}, H_1] \leq [P_1, H_1] \leq P_1 \cap H_1 = T \leq P_{12}$$

(recall P_1/T is a 2-group and $|H_1/T| = p_1$). Since $G_1^0 \cap P = P^0 = G_2^0 \cap P$ and $G_1^0 \leq P_1$,

$$G_2^0 \cap P_2 = G_1^0 \cap P_2 = G_1^0 \cap P_{12} \trianglelefteq G_1 .$$

Also, $G_2^0 \cap P_2 \trianglelefteq G_2$, so $G_2^0 \cap P_2 \leq T$ by the maximality of T, and the induced map $G_2^0 T/T \longrightarrow G_2/P_2 \cong D_{2p_2}$ is injective. Then $O_{p_2}(G_2^0 T/T) = 1$ since $G_2^0 T/T \trianglelefteq G_2/T$ and $O_{p_2}(G_2/T) = 1$, and hence $G_2^0 T/T = 1$ since it is isomorphic to a normal subgroup of D_{2p_2}. Thus $P^0 \leq G_2^0 \leq T$, so T is centric in P.

(c) By assumption, for $i = 1, 2$, G_i/T is solvable of order $2^n p_i$ for some n. Hence the Fitting subgroup $F(G_i/T) = O_2(G_i/T) O_{p_i}(G_i/T)$ is always centric in G_i/T (cf. [G, Theorem 6.1.3]). So G_i/T is strictly 2-constrained if and only if $O_{p_i}(G_i/T) = 1$.

(d) Assume T is centric in P. Then $S^0 \trianglelefteq N_S(T)$ and $S^0 \cap P = P^0 \leq T$. Since $N_S(P_i) = P$ by assumption, $N_{S^0 P_i}(P_i) = P \cap S^0 P_i = P_i$, so $S^0 P_i = P_i$ by Lemma A.1. Thus $C_S(T) = S^0 \leq P$, so $C_S(T) = C_P(T)$, and T is centric in S since it is centric in P.

(e) Assume T is centric in P, and hence also centric in G_1, G_2, and S by (a) and (d). Assume $\widehat{S} \in \mathrm{Syl}_2(\widehat{G})$, and set $Q = O_2(\widehat{G}) \leq \widehat{S}$ for short. For $i = 1, 2$, $Q \cap \widehat{G}_i \leq O_2(\widehat{G}_i) = \widehat{P}_i$, and hence

$$N_{Q\widehat{P}_i}(\widehat{P}_i) = N_{\widehat{S}}(\widehat{P}_i) \cap Q\widehat{P}_i = \widehat{P} \cap Q\widehat{P}_i = (\widehat{P} \cap Q)\widehat{P}_i = \widehat{P}_i .$$

By Lemma A.1, this implies $Q\widehat{P}_i = \widehat{P}_i$, and hence $Q \leq \widehat{P}_i$.

Thus $Q \leq \widehat{P}_{12}$. Hence $Q = \widehat{R} = \mathrm{Out}_R(T)$ for some unique $R \leq P_{12}$ such that $R \geq T$, and $R \trianglelefteq G_i$ $(i = 1, 2)$ since $Q = O_2(\widehat{G}) \trianglelefteq \widehat{G}$. Thus $R = T$ by definition of T, and $Q = 1$.

Since $\widehat{G} \leq \mathrm{Out}(T)$, the kernel of the induced $\widehat{G}$-action on $T/\mathrm{Fr}(T)$ is a 2-group by Lemma A.2, and is trivial since $O_2(\widehat{G}) = 1$. So the action is faithful. □

The next lemma will be needed to handle the cases involving amalgams whose maximal normal subgroup is not centric. As usual, when $\mathcal{F}$ is a fusion system over a finite p-group S, a subgroup $P \leq S$ is *strongly closed* in $\mathcal{F}$ if no element of P is $\mathcal{F}$-conjugate to an element of $S \smallsetminus P$. For example, if $\mathcal{F} = \mathcal{F}_S(G)$ for a finite group G, and $H \trianglelefteq G$, then $S \cap H$ is strongly closed in $\mathcal{F}$.

Lemma 4.3. *Assume hypotheses* $(*)$*, and also that* T *is not centric in* P*. Set* $\overline{P} = P/T$ *and* $\overline{G}_i = G_i/T$ *for short, and let* $\mathcal{F}^*$ *be the smallest fusion system over* $\overline{P}$ *which contains* $\mathcal{F}_{\overline{P}}(\overline{G}_1)$ *and* $\mathcal{F}_{\overline{P}}(\overline{G}_2)$*. Assume*

(a) $\mathcal{F}^* = \mathcal{F}_{\overline{P}}(\Gamma)$ *for some finite perfect group* Γ *for which* $\overline{P} \in \mathrm{Syl}_2(\Gamma)$*;*

(b) *no nontrivial proper subgroup of* $\overline{P}$ *is strongly closed in* $\mathcal{F}^*$*;*

(c) $\overline{G}_i/O^2(\overline{G}_i)$ *is abelian for* $i=1,2$*; and*

(d) $(\overline{P}\cap[\overline{G}_1,\overline{G}_1])\cap(\overline{P}\cap[\overline{G}_2,\overline{G}_2])\le[\overline{P},\overline{P}]$.

Then $C_P(T){\cdot}T=P$*, and* $C_{G_i}(T){\cdot}T=G_i$ *for* $i=1,2$*. If, in addition, we define*

$$U_i=P\cap O^2(G_i)\ ,\quad U=U_1U_2\trianglelefteq P\ ,\quad \text{and}\quad Z=U\cap T\ ,$$

then $[U,T]=1$*,* $P=UT$*, and* $U\in\mathrm{Syl}_2(\widetilde{\Gamma})$ *for some finite perfect group* $\widetilde{\Gamma}$ *such that* $Z\le Z(\widetilde{\Gamma})$ *and* $\widetilde{\Gamma}/Z\cong\Gamma$*.*

Proof. In general, for $X\le G_i$ or $g\in G_i$ $(i=1,2)$, we let $\overline{X}=XT/T$ or $\bar{g}=gT$ denote the image of X or g, respectively, in $\overline{G}_i$. Write $P^0=C_P(T)$ and $G_i^0=C_{G_i}(T)$ for short. Then $G_i^0\trianglelefteq G_i$ since $T\trianglelefteq G_i$, and hence $\overline{G_i^0}\trianglelefteq\overline{G}_i$. So $\overline{P^0}=\overline{P}\cap\overline{G_i^0}$ is strongly closed in $\mathcal{F}_{\overline{P}}(\overline{G}_1)$ and $\mathcal{F}_{\overline{P}}(\overline{G}_2)$, and hence is strongly closed in $\mathcal{F}^*=\langle\mathcal{F}_{\overline{P}}(\overline{G}_1),\mathcal{F}_{\overline{P}}(\overline{G}_2)\rangle$. Since T is not centric in P by assumption, $P^0=C_P(T)\not\le T$, so $\overline{P^0}\ne1$, and hence $\overline{P^0}=\overline{P}$ by (b). This in turn implies that $\overline{G_i^0}\ge\overline{P}$ for $i=1,2$, and hence $\overline{G_i^0}=\overline{G}_i$ since $\overline{G}_i$ is the normal closure of $\overline{P}$ (recall that $\overline{G}_i/\overline{P}_i\cong D_{2p_i}$). We have now shown that

$$C_P(T){\cdot}T=P^0T=P\qquad\text{and}\qquad C_{G_i}(T){\cdot}T=G_i^0T=G_i\quad(i=1,2)\,.$$

In particular, $O^2(G_i)=O^2(G_i^0)$, and hence $U_i=P\cap O^2(G_i^0)$. Set

$$U_i^\bullet=P\cap[G_i^0,G_i^0]\ ,\quad U^\bullet=U_1^\bullet U_2^\bullet\trianglelefteq P^0\ ,\quad\text{and}\quad Z^\bullet=U^\bullet\cap T\ .$$

Then $U_i\le U_i^\bullet$ (see Lemma A.4), and hence $U\le U^\bullet$ and $Z\le Z^\bullet$.

Now, $\overline{U_i}=\overline{P}\cap O^2(\overline{G}_i)=\overline{P}\cap[\overline{G}_i,\overline{G}_i]=\overline{U_i^\bullet}$, where the first and third equalities hold since $P\ge T$, and the second holds by (c) and Lemma A.4. Hence

$$\overline{U}=\overline{U_1}\,\overline{U_2}=\ \big\langle\overline{P}\cap[\overline{G}_1,\overline{G}_1],\overline{P}\cap[\overline{G}_2,\overline{G}_2]\big\rangle=\mathfrak{foc}(\mathcal{F}^*)=\mathfrak{foc}(\mathcal{F}_{\overline{P}}(\Gamma))=\overline{P}\,,$$

where the third and fifth equalities hold by the focal subgroup theorem (cf. [G, Theorem 7.3.4]) applied to $\overline{G}_1$, $\overline{G}_2$, and $\overline{\Gamma}$ (and since Γ is perfect by (a)). Thus $UT=P$. So after taking intersections with $P^0=C_P(T)$, we get $UZ(T)=P^0$. Since $[U,Z(T)]\le[U^\bullet,T]=1$ by definition,

$$[U,U]=[P^0,P^0]\ .$$

Assume $u=u_1u_2\in T\cap U^\bullet=Z(T)\cap U^\bullet$, where $u_i\in U_i^\bullet$. Then $\bar{u}=1$ and $\bar{u}_i\in\overline{P}\cap[\overline{G}_i,\overline{G}_i]$, so $\bar{u}_1=\bar{u}_2^{-1}\in[\overline{P},\overline{P}]$ by (d). Thus $u_i\in P^0\cap[P^0T,P^0T]T=[P^0,P^0]Z(T)$ (recall that $[P^0,T]=1$). Write $u_i=g_it_i$ where $g_i\in[P^0,P^0]\le U_i^\bullet$ and $t_i\in Z(T)\cap U_i^\bullet$. Since $Z(T)\cap U_i^\bullet=Z(T)\cap[G_i^0,G_i^0]=Z(T)\cap[P^0,P^0]$, the last equality by Proposition A.5, we see that g_1g_2, t_1, and t_2 all lie in $Z(T)\cap[P^0,P^0]$. This proves that

$$Z^\bullet=T\cap U^\bullet=Z(T)\cap[P^0,P^0]=Z(T)\cap[U,U]\le T\cap U=Z\,.$$

Hence $Z=Z^\bullet$, since we already saw that $Z\le Z^\bullet$.

Let $\tau\in H^2(\overline{P};Z(T))$ and $\tau_i\in H^2(\overline{G}_i;Z(T))$ be the classes of the central extensions P^0 and G_i^0, respectively. Thus $\tau=\tau_i|_{\overline{P}}$, so τ is stable with respect to fusion in $\overline{G}_i$, and hence is stable with respect to the fusion system $\mathcal{F}^*$. Since $\mathcal{F}^*=\mathcal{F}_{\overline{P}}(\Gamma)$ by (a), τ is the restriction of a unique element $\tau_\Gamma\in H^2(\Gamma;Z(T))$, where Γ acts trivially on $Z(T)$ (cf. [CE, Theorem XII.10.1]). Let $\widehat{\Gamma}$ be the corresponding central extension of $Z(T)$ by Γ. Thus $P^0\in\mathrm{Syl}_2(\widehat{\Gamma})$.

Set $\widetilde{\Gamma} = [\widehat{\Gamma}, \widehat{\Gamma}]$. Since $\widehat{\Gamma}/Z(T) \cong \Gamma$ is perfect, $\widehat{\Gamma} = \widetilde{\Gamma}{\cdot}Z(T)$ where $[\widetilde{\Gamma}, Z(T)] = 1$, so $\widetilde{\Gamma} = [\widehat{\Gamma}, \widehat{\Gamma}] = [\widetilde{\Gamma}, \widetilde{\Gamma}]$. Thus $\widetilde{\Gamma}$ is perfect. Since $P^0 \in \mathrm{Syl}_2(\widehat{\Gamma})$, we have

$$Z = Z^\bullet = Z(T) \cap [P^0, P^0] = Z(T) \cap [\widehat{\Gamma}, \widehat{\Gamma}] = Z(T) \cap \widetilde{\Gamma} ,$$

the third equality by Proposition A.5. Recall that $P^0 = UZ(T)$. Hence $P^0/U \cong Z(T)/Z \cong \Gamma/\widetilde{\Gamma}$, and so $U \in \mathrm{Syl}_2(\widetilde{\Gamma})$. □

We recall the terminology which will be used in the statements of our main theorems. By an *amalgam* is meant here a triple of groups $\mathcal{G} = (G_1 > H < G_2)$. A (proper) *completion* of $\mathcal{G}$ is a group G, together with injections $\rho_i\colon G_i \longrightarrow G$, such that $\rho_1|_H = \rho_2|_H$ and $G = \langle \rho_1(G_1), \rho_2(G_2)\rangle$. The universal completion is the amalgamated free product $G_1 \ast_H G_2$. When H has prime index in G_1 and in G_2, then the amalgam $\mathcal{G}$ is *primitive* if no nontrivial subgroup of H is normal in G_1 and in G_2. An *isomorphism of amalgams* from $(G_1 > H < G_2)$ to $(G_1^* > H^* < G_2^*)$ is a triple of isomorphisms $\alpha\colon H \xrightarrow{\cong} H^*$ and $\beta_i\colon G_i \xrightarrow{\cong} G_i^*$ such that $\beta_1|_H = \alpha = \beta_2|_H$.

Lemma 4.4. *For each $n = 1, \ldots, 6$, let $\overline{G}_1 = \overline{G}_1^{(n)}$, $\overline{G}_2 = \overline{G}_2^{(n)}$, and $\Gamma = \Gamma^{(n)}$ be the groups listed in case (n) of Table 4.1. Then there is a primitive amalgam $(\overline{G}_1 > \overline{S} < \overline{G}_2)$ with completion Γ, where $\overline{S} = \overline{S}^{(n)} \in \mathrm{Syl}_2(\overline{G}_i)$ and $[\overline{G}_i{:}\overline{S}] = 3$ for $i = 1, 2$, and this amalgam is uniquely determined up to isomorphism. Set $\mathcal{F}^{(n)} = \big\langle \mathcal{F}_{\overline{S}^{(n)}}(\overline{G}_1^{(n)}), \mathcal{F}_{\overline{S}^{(n)}}(\overline{G}_2^{(n)})\big\rangle$.*

(a) *For each $n = 1, 3, 5$, $\Gamma^{(n)}$ can be identified with a subgroup of index two in $\Gamma^{(n+1)}$ in such a way that the amalgam $\big(\overline{G}_1^{(n)} > \overline{S}^{(n)} < \overline{G}_2^{(n)}\big)$ is contained in $\big(\overline{G}_1^{(n+1)} > \overline{S}^{(n+1)} < \overline{G}_2^{(n+1)}\big)$ with index two, and the normal closure of $\overline{S}^{(n)}$ in $\overline{G}_i^{(n+1)}$ equals $\overline{G}_i^{(n)}$ for $i = 1, 2$.*

(b) *For each $1 \le n \le 6$, if $Q \le \overline{S}^{(n)}$ is strongly closed in $\mathcal{F}^{(n)}$, then either $Q = 1$ or $Q = \overline{S}^{(n)}$, or n is even and $Q = \overline{S}^{(n-1)}$.*

(c) *For $n = 1, 3, 5$, $\mathcal{F}^{(n)} = \mathcal{F}_{\overline{S}^{(n)}}(\Gamma^{(n)})$. (This also holds when $n = 2, 4, 6$, but we will not need that.)*

Proof. These are the amalgams denoted G_i and G_i^1 for $i = 3, 4, 5$ in [Gd2, Table 1]. In all but the first case, Goldschmidt's choice of completion is the same as the one listed here in Table 4.1. (Note that $\mathrm{Aut}(U_3(3)) \cong G_2(2)$.) In case (1), Goldschmidt lists $L_3(2)$ as a completion, but A_6 is easily seen to be a completion for the same amalgam. The uniqueness of the amalgams (for given $\overline{G}_i$ and Γ) follows from the classification in [Gd2, Theorem A].

Point (a) follows from Goldschmidt's construction of the amalgams [Gd2, 3.5, 3.7, 3.8], and also by a direct inspection of the groups in question.

Point (b) can be checked case-by-case. When n is even, it follows from point (a), point (b) for $n{-}1$, and the observation that no central subgroup of order two in $\overline{S}^{(n)}$ is strongly closed. So assume n is odd. Since $Z(\overline{S}^{(n)})$ is cyclic in all three cases ($n = 1, 3, 5$), each nontrivial normal subgroup of $\overline{S} = \overline{S}^{(n)}$ contains $\Omega_1(Z(\overline{S}))$.

Consider, for example, the case $n = 5$, where $\overline{S} = \langle a, b, r, t\rangle$ in the notation of Proposition 3.2. Set $A = \langle a, b\rangle \cong C_4^2$, $A_0 = \Omega_1(A) = \langle a^2, b^2\rangle$, $\overline{P}_1 = O_2(\overline{G}_1) =$

---

[1]The groups $\overline{G}_1$ in cases (4) and (5) are not isomorphic. See [Gd2, Table 1] for more details.

	$(\overline{G}_1, \overline{G}_2) \cong$	Γ	$U \cong$	$\lvert Z\rvert$	$[S{:}UT]$
(1)	(Σ_4, Σ_4)	A_6	D_8	1	1
			Q_{16}	2	1
(2)	$(C_2 \times \Sigma_4, C_2 \times \Sigma_4)$	Σ_6	D_8	1	2
			Q_{16}	2	2
(3)	$\bigl((Q_8 \times_{C_2} C_4).\Sigma_3, C_4^2 \rtimes \Sigma_3\bigr)$	$U_3(3)$	$C_4 \wr C_2$	1	1
(4)	$\bigl((Q_8 \times_{C_2} Q_8) \rtimes \Sigma_3, C_4^2 \rtimes D_{12}\bigr)$[1]	$\mathrm{Aut}(U_3(3))$	$C_4 \wr C_2$	1	2
(5)	$\bigl((Q_8 \times_{C_2} Q_8) \rtimes \Sigma_3, C_4^2 \rtimes D_{12}\bigr)$[1]	M_{12}	$\mathrm{Syl}_2(M_{12})$	1	1
			$\mathrm{Syl}_2(2M_{12})$	2	1
(6)	$\bigl((Q_8 \times_{C_2} Q_8).D_{12}, C_4^2 \rtimes ((C_2^2 \times C_3) \rtimes C_2)\bigr)$	$\mathrm{Aut}(M_{12})$	$\mathrm{Syl}_2(M_{12})$	1	2
			$\mathrm{Syl}_2(2M_{12})$	2	2

TABLE 4.1

$\langle a^2, ab, r, t\rangle \cong Q_8 \times_{C_2} Q_8$, and $\overline{P}_2 = O_2(\overline{G}_2) = \langle A, r\rangle \cong C_4^2 \rtimes C_2$. If $1 \neq P_* \le \overline{S}$ is strongly closed, then $P_* \ge Z(\overline{S}) = \langle a^2b^2\rangle$ and $P_* \cap \overline{P}_2 \trianglelefteq \overline{G}_2$ imply $P_* \ge A_0$ (since $A_0 = \langle a^2, b^2\rangle$ is the normal closure in $\overline{G}_2$ of $\langle a^2b^2\rangle$). If $P_*/A_0 \le Z(\overline{S}/A_0) = \langle A_0, ab, r\rangle/A_0$, then $P_* \le \overline{P}_1 \cap \overline{P}_2$ and $P_* \trianglelefteq \overline{G}_i$ for $i = 1, 2$, which contradicts the assumption that the amalgam is primitive. Thus $P_*/A_0 \nleq Z(\overline{S}/A_0)$ and $P_* \trianglelefteq \overline{S}$. Every normal subgroup of $\overline{S}/A_0 \cong D_8 \times C_2$ is either contained in its center or contains its commutator subgroup, so $P_* \ge [\overline{S}, \overline{S}] = \langle A_0, ab\rangle$. But then $P_* \ge A$ since $P_* \cap \overline{P}_2 \trianglelefteq \overline{G}_2$, and so $P_* = \overline{S}$ since $P_* \cap \overline{P}_1 \trianglelefteq \overline{G}_1$ (and by [A2, Lemma 5.3(2)]).

Point (c) is clear for $n = 1$. When $n = 3$ or 5, then $\mathcal{F}^{(n)} \subseteq \mathcal{F}_{\overline{S}}(\Gamma)$ since Γ is a proper completion of $(\overline{G}_1 > \overline{S} < \overline{G}_2)$, and $\overline{P}_i \overset{\text{def}}{=} O_2(\overline{G}_i)$ for $i = 1, 2$ are the only (possible) $\mathcal{F}_{\overline{S}}(\Gamma)$-essential subgroups of Γ by Propositions 3.1 or 3.2, respectively. So to show that $\mathcal{F}^{(n)} = \mathcal{F}_{\overline{S}}(\Gamma)$, it suffices to show that $\mathrm{Out}_{\overline{G}_i}(\overline{P}_i) \cong \mathrm{Out}_\Gamma(\overline{P}_i)$ for $i = 1, 2$. When $n = 5$ and $i = 1$, this is shown in [A2, Lemma 5.3(2)], and in all of the other cases, it follows since $\mathrm{Aut}(\overline{P}_i)/O_2(\mathrm{Aut}(\overline{P}_i)) \cong \Sigma_3$ by Lemma A.2. □

Theorem 4.5. *Let $\mathcal{F}$ be a saturated fusion system over a finite 2-group S, with distinct $\mathcal{F}$-essential subgroups P_1 and P_2 of index two in S. Then there are finite groups $G_1 > S < G_2$ such that for $i = 1, 2$, $P_i \trianglelefteq G_i$, $G_i/P_i \cong \mathrm{Out}_{G_i}(P_i) \cong D_{2p_i}$ for some odd prime p_i, and $\mathrm{Aut}_{G_i}(P_i) \le \mathrm{Aut}_{\mathcal{F}}(P_i)$. Let $T \le P_1 \cap P_2$ be the largest subgroup normal in both G_1 and G_2. Then these groups satisfy hypotheses $(*)$ with $P = S$.*

If T is not centric in S, then $p_1 = p_2 = 3$, and $(G_1/T > S/T < G_2/T)$ is one of the amalgams listed in Table 4.1. Set

$$U_i = S \cap O^2(G_i) \quad (i = 1, 2), \qquad U = U_1U_2, \qquad \text{and} \qquad Z = U \cap T.$$

Then $U \trianglelefteq S$, $[U, T] = 1$; and $\overline{G}_i = G_i/T$, U, Z, and $[S{:}UT]$ are as listed in Table 4.1.

Proof. Set $P = S$. By Lemma 4.1(a), there are groups $G_1 > P < G_2$ as described such that the $P_i < P < G_i$ satisfy hypotheses $(*)$. Let T be as in $(*)$, and set $\overline{G}_i = G_i/T$ and $\overline{S} = S/T$.

Assume T is not centric in S. By Lemma 4.2(b,c), $O_{p_i}(\overline{G}_i) = 1$, and $\overline{G}_i$ is strictly 2-constrained, for $i = 1, 2$. So by a theorem of Fan [Fn, Theorem 1], either $(\overline{G}_1 > \overline{S} < \overline{G}_2)$ is one of the amalgams listed by Goldschmidt in [Gd2, Table 1], or it is the ${}^2F_4(2)'$- or ${}^2F_4(2)$-amalgam (points (2) and (3) in [Fn, Theorem 1]). It cannot be either of the last two, since that would require that $G_i/O_2(G_i) \cong Sz(2) \cong C_5 \rtimes C_4$ for $i = 1$ or 2 (see, e.g., [Wi, Theorem 1], or the discussion in [Car, § 8.5] of the Levi decomposition of parabolic subgroups of groups of Lie type). Hence it is one of the six amalgams listed in Table 4.1, since the others listed by Goldschmidt involve groups which are not strictly 2-constrained.

Assume we are in case (n) in Table 4.1. When n is odd, we apply Lemma 4.3. Conditions (a) and (b) in the lemma were shown in Lemma 4.4(c,b), respectively, and (c) and (d) are easily checked case-by-case. So by that lemma, $[U, T] = 1$, $UT = S$, and $U \in \mathrm{Syl}_2(\widetilde{\Gamma})$ for some finite perfect group $\widetilde{\Gamma}$ such that $Z \le Z(\widetilde{\Gamma})$ and $\widetilde{\Gamma}/Z \cong \Gamma$. Here, $\Gamma \cong A_6$, $U_3(3)$, or M_{12}, when $n = 1$, 3, or 5. These groups have Schur multiplier C_6, 1, and C_2, respectively (see [A1, (33.15)], [Gr, Theorem 2], and [Mz]). Hence $|Z| \le 2$, with equality possible only when $n = 1$ or 5.

When n is even, then by Lemma 4.4(a), there are subgroups $\widetilde{G}_i < G_i$ and $\widetilde{S} < S$ of index two, all containing T, such that $\widetilde{S} = S \cap \widetilde{G}_i$, and such that $\big(\widetilde{G}_1/T > \widetilde{S}/T < \widetilde{G}_2/T\big)$ is an amalgam of type $(n-1)$. Hence $U_i = S \cap O^2(G_i) = \widetilde{S} \cap O^2(\widetilde{G}_i)$, and so $U = U_1U_2$ plays the same role for the new amalgam as for the original one. Also, $(*)$ holds for $\big(\widetilde{G}_1 > \widetilde{S} < \widetilde{G}_2\big)$, and T is the largest subgroup normal in $\widetilde{G}_1$ and $\widetilde{G}_2$ since the quotient amalgam is primitive. If T were centric in $\widetilde{S}$, then it would be centric in the $\widetilde{G}_i$ by Lemma 4.2(a), and since T is not centric in S, $T{\cdot}C_S(T) = T{\cdot}C_{G_i}(T) > T$ would be a strictly larger subgroup normal in the G_i. Since this contradicts the choice of T, we conclude that T is not centric in $\widetilde{S}$, and hence that the result follows by the argument in the last paragraph applied to $(\widetilde{G}_1 > \widetilde{S} < \widetilde{G}_2)$. □

Theorem 4.5 will be used when looking for pairs of essential subgroups of index two in S. We next turn to the problem of identifying essential subgroups which have index two in their normalizer but are not normal. The idea is to apply the classification of amalgams by Goldschmidt and Fan to a pair of essential subgroups which are conjugate in S, and have index two in their common normalizer.

Theorem 4.6. *Let $\mathcal{F}$ be a saturated fusion system over a finite 2-group S. Let $P_1 \le S$ be an $\mathcal{F}$-essential subgroup which is not normal in S, and such that $|N_S(P_1)/P_1| = 2$. Set $P = N_S(P_1) < S$. Choose $x \in N_S(P){\smallsetminus}P$ such that $x^2 \in P$, and set $P_2 = xP_1x^{-1}$. Then there are finite groups $G_1 > P < G_2$, an odd prime p, and an isomorphism $\beta\colon G_1 \xrightarrow{\cong} G_2$ such that $\beta|_P = c_x \in \mathrm{Aut}_S(P)$, and for $i = 1, 2$, $\mathrm{Out}_{G_i}(P_i) \le \mathrm{Out}_{\mathcal{F}}(P_i)$, $P_i \trianglelefteq G_i$, and $\mathrm{Out}_{G_i}(P_i) \cong G_i/P_i \cong D_{2p}$. Let $T \le P_1 \cap P_2$ be the largest subgroup which is normal in G_1 and in G_2.*

(a) *If T is not centric in S, then $p = 3$. Set $U_i = P \cap O^2(G_i) \trianglelefteq P$, $U = U_1U_2$, and $Z = T \cap U$. Set $W = \mathrm{Fr}(U)$, $S_* = N_S(W)$, and let Δ be the normal closure of U in S_*. Then the following hold.*

(i) $[T,U]=1$; *and either* $U_i\cong C_2^2$, $U\cong D_8$, *and* $Z=1$; *or* $U_i\cong Q_8$, $U\cong Q_{16}$, *and* $Z=Z(U)$.

(ii) *There is a subgroup* $T^\bullet\trianglelefteq P$ *such that* $T\le T^\bullet<P_1\cap P_2$, $[T^\bullet{:}T]\le 2$;

(ii.1) $P_i=T^\bullet U_i$ *and* $T^\bullet\cap U=T\cap U=Z$; *and*

(ii.2) $[S_*{:}T^\bullet\Delta]=2$, $C_{S_*}(U_1)=T^\bullet U_1$ *if* $U\cong D_8$, $C_{S_*}(U_1)=T^\bullet$ *if* $U\cong Q_{16}$.

Also, G_i/T, $|Z|$, U_i, U, Δ, $[T^\bullet{:}T]$, *and* $[T^\bullet,U]$ *are as described in Table 4.2.*

(b) *If* T *is centric in* P *or (equivalently) centric in* S, *then* $[S,S]$ *is nonabelian,* $O_2(\mathrm{Out}_{\mathcal{F}}(T))=1$, *and* $\mathrm{Out}_{\mathcal{F}}(T)$ *acts faithfully on* $T/\mathrm{Fr}(T)$.

(c) *In the situation of (a), if* $[S,S]$ *is abelian, or if* $T^\bullet/Z$ *is abelian, or more generally if* $T^\bullet$ *contains no quaternion subgroup of order* 16 *and* $T^\bullet/Z$ *contains no dihedral subgroup of order* 8, *then* $S_*=S$.

$G_i/T\cong$	$\lvert Z\rvert$	$U_i\cong$	$U\cong$	$\Delta\cong$	$\lvert T^\bullet/T\rvert$	$[T^\bullet,U]$
Σ_4	1	C_2^2	D_8	$D_{2^n},\ n\ge 3$	1	1
Σ_4	2	Q_8	Q_{16}	$Q_{2^n},\ n\ge 4$	1	1
$\Sigma_4\times C_2$	1	C_2^2	D_8	$D_{2^n},\ n\ge 3$	2	1
$\Sigma_4\times C_2$	2	Q_8	Q_{16}	$Q_{2^n},\ n\ge 4$	2	Z

TABLE 4.2

Proof. By Lemma 4.1(b), there are finite groups $G_1>P<G_2$ such that $P_i\trianglelefteq G_i$, $G_i/P_i\cong D_{2p}$ for some odd prime p, $\mathrm{Out}_S(P_i)\le\mathrm{Out}_{G_i}(P_i)\le\mathrm{Out}_{\mathcal{F}}(P_i)$, and $\beta|_P=c_x|_P$ for some $\beta\in\mathrm{Iso}(G_1,G_2)$. Thus $(*)$ holds with $p_1=p_2=p$. Also, $x\in N_S(T)$, where $T\le P_1\cap P_2$ is the largest subgroup normal in G_1 and G_2.

(a) If T is not centric in S, then it is not centric in P by Lemma 4.2(d). Hence the G_i/T are strictly 2-constrained by Lemma 4.2(b,c). By [Fn, Theorem 1], they are among the groups listed in [Gd2, Table 1], and hence are isomorphic to Σ_4 or $C_2\times\Sigma_4$ (and $p=3$).

Case 1: $G_i/T\cong\Sigma_4$. We apply Lemma 4.3 with $\Gamma=A_6$ (see Table 4.1). Conditions (a) and (b) follow from Lemma 4.4(c,b), and conditions (c) and (d) are easily checked. Recall that $U_i=P\cap O^2(G_i)$, $U=U_1U_2$, and $Z=T\cap U$. By the lemma, $[T,U]=1$ and $P=UT$; and $U\in\mathrm{Syl}_2(\widetilde{\Gamma})$, where $\widetilde{\Gamma}$ is a finite perfect group such that $Z\le Z(\widetilde{\Gamma})$ and $\widetilde{\Gamma}/Z\cong A_6$. Thus $\widetilde{\Gamma}$ is isomorphic to $A_6\cong PSL_2(9)$ or its 2-fold central extension $SL_2(9)$ (cf. [A1, 33.15]), and $U\cong D_8$ or Q_{16}. The image of U_i in $P/T\cong U/Z$ is a Sylow 2-subgroup of $O^2(G_i/T)\cong A_4$, so $P_i/T=O_2(G_i/T)=U_iT/T\cong C_2^2$, and thus $P_i=U_iT$. Since each nontrivial subgroup of Q_{16} contains its center,

$$\text{either } U\cong D_8,\ U_i\cong C_2^2,\text{ and } Z=1;\text{ or } U\cong Q_{16},\ U_i\cong Q_8,\text{ and } |Z|=2.$$

Set $T^\bullet=T$. We have now proven (i) and (ii.1) in this case, and proven the information on the first two lines in Table 4.2, except for that about Δ.

Case 2: $G_i/T\cong C_2\times\Sigma_4$. By Lemma 4.4(a), there are subgroups $\widetilde{G}_i<G_i$, $\widetilde{P}_i<P_i$, and $\widetilde{P}<P$ of index two, all containing T and satisfying $(*)$, such that

$\widetilde{G}_1 > \widetilde{P} < \widetilde{G}_2$, $\widetilde{P}_i = O_2(\widetilde{G}_i)$, and $\bigl(\widetilde{G}_1/T > \widetilde{P}/T < \widetilde{G}_2/T\bigr)$ is an amalgam of the type handled in Case 1. Also, $U_i = O^2(G_i) \cap P = O^2(\widetilde{G}_i) \cap \widetilde{P}_i$ (i.e., U_i and U play the same role for the smaller amalgam as for the original one), and T is not centric in $\widetilde{P}$ by a similar argument to the one used at the end of the proof of Theorem 4.5. So the conclusions in Case 1 hold after replacing P and P_i by $\widetilde{P}$ and $\widetilde{P}_i$. In particular, (i) holds.

Let $T^\bullet/T \le P_1/T \cong C_2^3$ be the subgroup of order two which is fixed by the conjugation action of $G_1/P_1 \cong \Sigma_3$. Then $P_1/T = (T^\bullet/T){\cdot}(TU_1/T)$ and $P_2/T = (T^\bullet/T){\cdot}(TU_2/T)$ (but note that $T^\bullet/T$ is *not* fixed by the action of $\mathrm{Aut}_{G_2}(P_2)$ on P_2/T). So $P_1 = T^\bullet U_1$, $P_2 = T^\bullet U_2$, and $T^\bullet \cap U = T \cap U = Z$. This finishes the proof of (ii.1).

Now, $[T^\bullet, U] \le T^\bullet \cap U = Z$ since $T^\bullet$ and U are both normal in P. Hence $[T^\bullet, U] = 1$ if $|Z| = 1$. Now assume $|Z| = 2$, $U \cong Q_{16}$, and $W = \mathrm{Fr}(U) \cong C_4$, and fix $g \in T^\bullet \smallsetminus T$. Since Z is central in U, $[g, -]$ is a homomorphism from U to Z, and so $[g, W] = 1$. Also, $gT \in Z(G_1/T)$ by definition of $T^\bullet$. Choose $a \in G_1$ such that aT has order three in G_1/T and $W \ne aWa^{-1}$. Then $aga^{-1} \in gT$, $aWa^{-1} \le P \cap O^2(G_1) = U_1$, and $[g, aWa^{-1}] = [aga^{-1}, aWa^{-1}] = 1$ since $[T, U] = 1$ by (i) and $[g, W] = 1$. Since $U_1 = \langle W, aWa^{-1}\rangle$, we conclude that $[T^\bullet, U_1] = [g, U_1] = 1$.

By a similar argument, there is $g' \in P_2 \smallsetminus T$ such that $g'T \in P_2/T \cong C_2^3$ is the involution fixed by the action of $G_2/P_2 \cong \Sigma_3$, and such that $[g', U_2] = 1$. Then gT and $g'T$ are both in the center of $P/T \cong C_2 \times D_8$, not in $[P/T, P/T] = WT/T$, and $gT \ne g'T$ since otherwise $\langle T, g\rangle$ would be normal in G_1 and in G_2 (contradicting the maximality of T). So $g' \in gwT$ for some $w \in W \smallsetminus Z$. Thus $[gw, U_2] = 1$, and since $[w, U_2] = Z$ (w has order 4 in $U_2 \cong Q_8$), $[T^\bullet, U] = [g, U_2] = Z$.

This finishes the proof of the information in Table 4.2, except for that about Δ.

Both cases. If $U \cong D_8$, then by Lemma B.4, $\Delta \trianglelefteq S_*$ is dihedral, $T^\bullet \cap \Delta = 1$, $[S_*{:}T^\bullet\Delta] = 2$, and all noncentral involutions in Δ are S_*-conjugate. Fix $y \in U_1 \smallsetminus Z(U)$. For $g \in S_*$, $c_g(y)$ is Δ-conjugate to y if $g \in T^\bullet\Delta$ (recall $[T^\bullet, U_1] = 1$), so $c_g(y)$ lies in the other Δ-conjugacy class of noncentral involutions if $g \notin T^\bullet\Delta$. Thus $C_{S_*}(U_1) = C_{T^\bullet\Delta}(U_1) = T^\bullet U_1$.

If $U \cong Q_{16}$, then by Lemma B.4 applied to $U/Z \le S_*/Z$, $\Delta/Z \trianglelefteq S_*/Z$ is dihedral, $T^\bullet \cap \Delta = Z$, $[S_*{:}T^\bullet\Delta] = 2$, and all noncentral involutions in Δ/Z are S_*/Z-conjugate. Since all involutions in Δ/Z are S_*/Z-conjugate to elements of U_1/Z where $U_1 \cong Q_8$, there are no involutions in $\Delta \smallsetminus Z$, and Δ is quaternion. Since $[T^\bullet, U_1] = 1$, and since the non-normal subgroups of order four in Δ are S_*-conjugate, a similar argument to that used in the last paragraph (applied with $y \in U_1 \smallsetminus \mathrm{Fr}(U)$) shows that $C_{S_*}(U_1) = C_{T^\bullet\Delta}(U_1) = T^\bullet$.

This proves (ii.2), and the description of Δ in Table 4.2.

(c) Assume $S > S_*$, so $N_S(S_*) > S_*$ by Lemma A.1. Fix $y \in N_S(S_*) \smallsetminus S_*$ such that $y^2 \in S_*$. Set $\Delta^* = y\Delta y^{-1}$, $Z^* = yZy^{-1}$, $W^* = yWy^{-1}$, and $N = \Delta \cap \Delta^*$. The subgroups Δ and Δ^* are both normal in S_*, so $N \trianglelefteq S_*$ is normal in each of them, and $[\Delta, \Delta^*] \le N$. If $Z = 1$ (if Δ is dihedral), then $W^* \ne W$, $W^* \le Z(S_*)$ (since $W \le Z(S_*)$) and hence $W^* \nleq \Delta$, and $N = 1$ since each nontrivial normal subgroup of Δ^* contains W^*. Thus $[\Delta^*, \Delta] = 1$, $\Delta^* \le C_{S_*}(\Delta) \le T^\bullet W$, and thus $T^\bullet \cong T^\bullet W/W$ contains a dihedral subgroup of order 8. Also, since the commutator subgroup of $\langle \Delta, y\rangle$ contains all elements of the form $g^{-1}(ygy^{-1})$ for $g \in \Delta$, it surjects onto Δ under the projection from $\Delta{\cdot}\Delta^*$, and thus is nonabelian. So $[S, S]$ is nonabelian in this case.

If $|Z|=2$ (if Δ is quaternion) and $Z^*\neq Z$, then a similar argument shows that $\Delta^*\le T^\bullet$ and hence $T^\bullet$ contains a quaternion subgroup of order 16. If $Z^*=Z$, we apply the argument in the last paragraph to the conjugation action of y on S_*/Z, where $y(W/Z)y^{-1}\neq(W/Z)$ by assumption. Thus $[\Delta^*,\Delta]\le Z$, so $\Delta^*\le C_{S_*}(U_1){\cdot}W=T^\bullet W$ since $\operatorname{Ker}\bigl[\operatorname{Aut}(\Delta)\to\operatorname{Aut}(\Delta/Z)\bigr]\cong C_2^2$ is generated by $\operatorname{Aut}_W(\Delta)$ and an automorphism which is the identity on U_1. So in this case, $T^\bullet W/W\cong T^\bullet/Z$ contains a dihedral subgroup of order 8. In both cases, $[S,S]$ is nonabelian by the argument used in the last paragraph.

(b) If T is centric in P, then it is centric in S by Lemma 4.2(d), while the converse is immediate. Assume both of these hold.

If T is not fully normalized, then there is some $\varphi\in\operatorname{Hom}_{\mathcal F}(N_S(T),S)$ such that $\varphi(T)$ is fully normalized (cf. [AKO, Lemma I.2.6(c)]). Upon replacing T by $\varphi(T)$, P_i by $\varphi(P_i)$, P by $\varphi(P)$, etc., we can assume T is fully normalized (and $(*)$ still holds and T is still centric in P).

Set $\widehat S=\operatorname{Out}_S(T)$, $\widehat G_i=\operatorname{Out}_{G_i}(T)$, and similarly for subgroups of S, and set $\widehat G=\langle\widehat G_1,\widehat G_2,\widehat S\rangle\le\operatorname{Out}_{\mathcal F}(T)$. Then $\widehat S\in\operatorname{Syl}_2(\widehat G)$ by the Sylow axiom. Set $V=T/\operatorname{Fr}(T)$, written additively, and regarded as an $\mathbb F_2[\operatorname{Out}(T)]$-module. By Lemma 4.2(e), $O_2(\widehat G)=1$ and $\widehat G$ acts faithfully on V. Hence $O_2(\operatorname{Out}_{\mathcal F}(T))=1$, and by Lemma A.2, $\operatorname{Out}_{\mathcal F}(T)$ acts faithfully on V.

It remains to show that $[S,S]$ is nonabelian. Set $V_0=[\widehat S,V]$. Since every element of $\widehat S$ lifts to an element of S, and every element of V_0 lifts to an element of $[S,S]$, it will suffice to show that $[\widehat S,\widehat S]$ acts nontrivially on V_0. Assume otherwise: assume $[\widehat S,\widehat S]$ acts trivially on V_0. If $g\in[\widehat S,\widehat S]$, then for each $v\in V$, g acts trivially on $[g,v]=gv-v\in V_0$, so $(g-1)^2v=(g^2-1)v=0$, and $g^2v=v$. Thus $g^2=1$ since $\widehat S$ acts faithfully on V. It follows that $[\widehat S,\widehat S]$ is elementary abelian.

If $\widehat G_1\cong\widehat G_2$ are strictly 2-constrained, then by the argument used in the second paragraph in the proof of Theorem 4.5, $\bigl(\widehat G_1>\widehat P<\widehat G_2\bigr)$ must be one of the amalgams listed in Table 4.1. Hence $\widehat P\cong D_8$ or $D_8\times C_2$ (the only cases in the table with $\overline G_1\cong\overline G_2$), and $\widehat P_i=O_2(\widehat G_i)\cong C_2^2$ or C_2^3, respectively. Note that $\widehat P_{12}=Z(\widehat P)$ in either case. Choose $x_1\in\widehat P_1\smallsetminus\widehat P_{12}$. Then x_1 is $\widehat S$-conjugate to some $x_2\in\widehat P_2\smallsetminus\widehat P_{12}$, and $x_1x_2^{-1}\in[\widehat S,\widehat S]$ has order 4. So this case is impossible.

Thus $\widehat G_1\cong\widehat G_2$ are not strictly 2-constrained. By Lemma 4.2(c), $O_p(\widehat G_i)\neq1$ for $i=1,2$. Thus $[\widehat P_i,O_p(\widehat G_i)]=1$ since both are normal in $\widehat G_i$ (and have relatively prime order), so $\widehat P_{12}\trianglelefteq\widehat G_i$ for $i=1,2$. Hence $P_{12}\trianglelefteq G_i$, and $P_{12}=T$ by definition of T. In other words, $\widehat P_{12}=1$, $\widehat P_i\cong C_2$, and so $\widehat P\cong C_2^2$.

Let $a_i\in\widehat P_i\cong C_2$ be the generator, and let $g\in O_p(\widehat G_1)$ be an element of order p. Thus $\widehat G_1=\langle a_1,a_2,g\rangle\cong D_{4p}$, $\langle g,a_2\rangle\cong D_{2p}$, and $[g,a_1]=1$. Choose $t\in\widehat S$ such that $ta_it^{-1}=a_{3-i}$. By assumption, and since $a_1a_2\in[\widehat S,\widehat S]$, $[a_1a_2,[\widehat S,V]]=0$.

Set $W=[g,V]$; $W\neq0$ since $\widehat G_1$ acts faithfully on V. We claim that

$$C_W(a_2)=[a_2,W]\qquad\text{and}\qquad[a_1,W]=0\,. \tag{1}$$

If $w\in C_W(a_2)$, then $a_2(g^iw)=g^{-i}a_2w=g^{-i}w$ for each i, and $\sum_{i=0}^{p-1}g^iw=0$ since $w\in W=[g,V]$. Thus $w=(1+a_2)\bigl(\sum_{i=1}^{(p-1)/2}g^iw\bigr)\in[a_2,W]$, proving the first statement, and in fact proving that $C_{W_0}(a_2)=[a_2,W_0]$ for each $\widehat G_1$-invariant subgroup $W_0\le W$.

So if we set $W_0 = [a_1, W]$, then $C_{W_0}(a_2) = [a_2, W_0] = [a_1a_2, W_0] \le \bigl[a_1a_2, [\widehat{S}, V]\bigr]$, where the last group is zero by assumption. Hence $W_0 = 0$, finishing the proof of (1).

Since $W = [g, V]$ is a $\widehat{G}_1$-invariant direct summand of V, $W \cap [a_1, V] = [a_1, W] = 0$ by (1). So if $v \in W \cap t(W)$, then $a_1(v) = v$ since $v \in W$, $v \in C_{t(W)}(a_1) = [a_1, t(W)]$ by (1), and so $v \in W \cap [a_1, V] = 0$. Thus $W \cap t(W) = 0$. So if we choose any $w \in W$ such that $a_2(w) \ne w$, then $t(w) - w \in [\widehat{S}, V]$, and hence

$$a_1a_2(t(w) - w) = t(w) - w \quad\Longrightarrow\quad w - a_1a_2(w) = t(w) - ta_1a_2(w) \in W \cap t(W) = 0\,.$$

Thus $a_1(w) \ne a_1a_2(w) = w$, which contradicts (1). □

5. Some applications

We finish the paper with some applications of Theorems 4.5 and 4.6. Following the terminology of [G, § 16.7], we say that a 2-group S is *wreathed* if $S \cong C_{2^m} \wr C_2$ for some $m \ge 2$. These groups arise as Sylow 2-subgroups of $GL_2(q)$ and $(P)SL_3(q)$ when $q \equiv 1$ (mod 4).

Proposition 5.1. *Fix a finite 2-group S containing a normal subgroup $\Delta \trianglelefteq S$ which is dihedral or quaternion of order ≥ 8. Assume*

(a) *for some dihedral or quaternion subgroup $\Delta_0 \le \Delta$ of order 8, $C_S(\Delta_0)$ is abelian; and*

(b) *two of the three subgroups of index two in Δ are S-conjugate.*

Then either S is dihedral, semidihedral, or wreathed, or there is no reduced fusion system over S.

Proof. We first fix some notation for elements and subgroups of S. Let $A \trianglelefteq \Delta$ be the unique cyclic subgroup of index two in Δ which is normal in S (A is characteristic in Δ unless $\Delta \cong Q_8$). Set $Z = Z(\Delta)$, and let $W \le A$ be the subgroup of order 4. Fix a generator $a \in A$, and fix $b \in \Delta_0 \smallsetminus A$. Thus $\Delta = \langle a, b\rangle$ and $\Delta_0 = \langle W, b\rangle$. Set $A_0 = \langle a^2\rangle$, and set $T = C_S(\Delta_0)$. To summarize these definitions for later reference,

$$\underset{=Z(\Delta)}{Z} \le \underset{=\langle a^2\rangle}{A_0} < \underset{=\langle a\rangle}{A} < \underset{=\langle a,b\rangle}{\Delta} \trianglelefteq S\,, \quad \underset{\cong C_4}{W} \le A\,, \quad \Delta_0 = \langle W, b\rangle, \quad T = C_S(\Delta_0).$$

By (a) and (b), we are in the situation of Lemma B.3. In particular,

$$[S{:}T\Delta] = 2 \qquad\text{and}\qquad \forall\, g \in S,\ gbg^{-1} = a^jb \text{ where } \begin{cases} j \text{ is even} & \text{if } g \in T\Delta \\ j \text{ is odd} & \text{if } g \notin T\Delta \end{cases} \tag{1}$$

by Lemma B.3(b). Note also that

$$\forall\, g \in S,\ gag^{-1} = a^i \quad\text{where}\quad \begin{cases} i \in 4\mathbb{Z}+1 & \text{if } g \in C_S(W) \\ i \in 4\mathbb{Z}-1 & \text{if } g \notin C_S(W)\,. \end{cases} \tag{2}$$

We claim that the following hold.

$$TA_0 \trianglelefteq S, \quad S/TA_0 \cong D_8, \quad\text{and}\quad Z(S/TA_0) = TA/TA_0\,. \tag{3}$$

$$[S, S] \text{ is abelian, and } [S, S] = \langle A, [y, T]\rangle \text{ for each } y \in S\smallsetminus T\Delta\,. \tag{4}$$

$$[S, S] = A \quad\Longrightarrow\quad \forall\, g \in S\smallsetminus C_S(W),\ C_{TA}(g)\cdot A = TA \text{ and } |C_{TA}(g)| = |T|. \tag{5}$$

Point (3) and the first statement in (4) are shown in Lemma B.3(c,d). The last statement in (4) holds since $[S,S] \geq A$ by Lemma B.3(a), $S = \langle \Delta, T, y\rangle$, and hence $[S,S] = \langle [\Delta,S], [y,T]\rangle = \langle A, [y,T]\rangle$.

To see (5), fix $g \in S\smallsetminus C_S(W)$ and $t \in T$. By (2), $gag^{-1} = a^{4k-1}$ for some $k \in \mathbb{Z}$. Also, $t^{-1}gt = a^j g$ for some j since $[S,S] = A$, and j is even ($a^j \in A_0$) since $TA_0 \trianglelefteq S$ by (3). Choose i such that $i(1-2k) \equiv (j/2) \pmod{|A|}$. Then $a^i g a^{-i} = a^j g = t^{-1}gt$, so $ta^i \in C_{TA}(g)$. This proves the first statement in (5), and the second then follows since $C_{TA}(g) \cap A = C_A(g) = Z = T \cap A$.

Set $\overline{S} = S/TA_0$ ($TA_0 \trianglelefteq S$ by (3)). We write $\bar{g} = gTA_0 \in \overline{S}$ for $g \in S$, and $\overline{Q} = QTA_0/TA_0$ for $Q \leq S$: the images of g and Q in $\overline{S}$.

Fix $x \in S\smallsetminus T\Delta$. Upon replacing x by bx if necessary, we can assume $x \in C_S(W)\smallsetminus T\Delta$. Also, $xbx^{-1} = a^j b$ for some odd j by (1), and upon replacing x by an appropriate element of xA, we can assume $xbx^{-1} = ab$. By (2), $xax^{-1} = a^i$ for some $i \in 1 + 4\mathbb{Z}$. Then $x^2bx^{-2} = a^{i+1}b$ where $i+1 \in 2+4\mathbb{Z}$; $x^2 \in TA$ since $S/TA \cong \overline{S}/Z(\overline{S}) \cong C_2^2$ by (3); and hence $x^2 \in TA_0a$. To summarize,

$$S = \langle TA, b, x\rangle, \quad C_S(W) = \langle TA, x\rangle, \quad xbx^{-1} = ab, \quad \bar{x}^2 = \bar{a} \in \overline{S}. \tag{6}$$

Assume $\mathcal{F}$ is a reduced fusion system over S, and let $\mathcal{E}$ be the set of $\mathcal{F}$-essential subgroups of S. Define

$$b_0 = x, \quad b_1 = b, \quad b_2 = bx, \quad H_i = \langle TA, b_i\rangle, \quad \mathcal{E}_i = \{P \in \mathcal{E} \mid P \leq H_i\}$$

for $i = 0,1,2$. Thus $H_0 = C_S(W)$ and $H_1 = T\Delta$. We will prove the following statements.

$$\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2, \quad \text{and} \quad P \in \mathcal{E}_i \implies \text{the normal closure of } P \text{ is } H_i. \tag{7}$$

$$\mathcal{E}_0 \neq \varnothing \implies S \text{ is wreathed.} \tag{8}$$

$$\mathcal{E}_2 \neq \varnothing \implies [S,S] = A,\ \exists\, b^\bullet \in TAb_2 \text{ s.t. } \langle A, b^\bullet\rangle \text{ dihedral or quaternion.} \tag{9}$$

$$\mathcal{E}_0 = \varnothing \implies \mathcal{E}_1 \neq \varnothing,\ \mathcal{E}_2 \neq \varnothing, \text{ and } S \text{ is dihedral or semidihedral.} \tag{10}$$

The proposition then follows from (8) and (10).

Points (7), (8), and (9) will be shown in Steps 1 and 2: essential subgroups of index two in S will be handled in Step 1, and the others in Step 2. It is in Step 2 that Theorem 4.6 plays a crucial role. Point (10) will be shown in Step 3.

Step 1: Fix $P \in \mathcal{E}$ such that $[S{:}P] = 2$. Thus $P \geq [S,S] \geq A$.

We consider seven cases. In the first four, we show P cannot be $\mathcal{F}$-essential using Lemma 1.4 applied with Θ one of the following characteristic subgroups $\Theta_i \leq P$:

$$\Theta_1 = Z(P)\ , \quad \Theta_2 = \big\langle g \in P \,\big|\, [P{:}C_P(g)] \leq 2 \big\rangle\ , \quad \text{or} \quad \Theta_3 = \big\langle \beta(A) \,\big|\, \beta \in \mathrm{Aut}(P) \big\rangle\ .$$

By Lemma 1.4, $P \notin \mathcal{E}$ if for some $i = 1,2,3$ and some $g \in S\smallsetminus P$, $[g,P] \leq \Theta_i\mathrm{Fr}(P)$ and $[g,\Theta_i] \leq \mathrm{Fr}(P)$.

Since $P \geq A$, either $P \ngeq T$, or P is one of the H_i. In the first two cases below, $P \ngeq T$, and we set $T_0 = P \cap T$, choose $t^\bullet \in T\smallsetminus T_0 \subseteq S\smallsetminus P$, and let $t_1, t_2 \in \{1, t^\bullet\}$ be such that $P = \langle T_0A, bt_1, xt_2\rangle$. Since $[\overline{xt_2}, \overline{bt_1}] = [\bar{x}, \bar{b}] = \bar{a}$, $[xt_2, bt_1] \in [S,S] \cap TA_0a$.

- $\boldsymbol{P \ngeq T}$**,** $\boldsymbol{[t^\bullet, P] \leq AZ(S)}$**, and** $\boldsymbol{\Theta = \Theta_1 = Z(P)}$**.** Then $\overline{P} = \overline{S}$, so $\overline{Z(P)} \leq Z(\overline{S}) = \overline{TA}$ by (3), and $Z(P) \leq TA$. Thus $[t^\bullet, Z(P)] \leq [t^\bullet, TA] \leq \langle a^4\rangle \leq \mathrm{Fr}(P)$. Also, $[t^\bullet, P] \leq [t^\bullet, S] \leq TA_0$ since $t^\bullet \in TA_0 \trianglelefteq S$ by (3). So by assumption (and since $Z(S) \le C_S(\Delta_0) = T$), $[t^\bullet, P] \le TA_0 \cap AZ(S) = Z(S)A_0 \le \Theta_1\mathrm{Fr}(P)$. Thus $P \notin \mathcal{E}$.

- $\boldsymbol{P \not\ge T,\ [t^\bullet, P] \not\le AZ(S),}$ **and** $\boldsymbol{\Theta = \Theta_2}$**.** Then $[t^\bullet, P] \le TA_0 \cap [S,S]$ since $t^\bullet \in TA_0 \trianglelefteq S$. Since $[t^\bullet, T\Delta] = [t^\bullet, A] \le A_0$ (recall $\langle T, b\rangle$ is abelian) and $[t^\bullet, P] \not\le AZ(S)$, $[t^\bullet, x] = [t^\bullet, xt_2] = sa^{2i}$ for some $s \in T_0 \smallsetminus AZ(S)$. Then $[s, A] = 1$ since $sa^{2i} \in [S,S]$, $A \le [S,S]$, and $[S,S]$ is abelian by (4). Hence $[s, T\Delta] = 1$, and $[s, g] \ne 1$ for $g \in S \smallsetminus T\Delta$ since $s \notin Z(S)$.

  Now, $s \in \Theta$ since $C_P(s) = T\Delta \cap P$. Also, $[t^\bullet, T\Delta] \le A_0 \trianglelefteq S$, so $[t^\bullet, P] \le \langle A_0, [t^\bullet, x]\rangle = \langle A_0, s\rangle \le \Theta\mathrm{Fr}(P)$. For all $g \in P \smallsetminus T\Delta$, $C_P(g) \le \langle T_0 A, g\rangle$ since $C_{\overline{S}}(\bar g) = \langle \bar a, \bar g\rangle$, and this inclusion is strict since $[s, g] \ne 1$. Thus $g \notin \Theta$. So $\Theta \le T\Delta$, and hence $[t^\bullet, \Theta] \le [T, T\Delta] \le \langle a^4\rangle \le \mathrm{Fr}(P)$. Thus $P \notin \mathcal{E}$.

- $\boldsymbol{P \in \{H_0, H_2\}}$ **and** $\boldsymbol{\Theta = \Theta_3 \le TA}$**.** Then $b \in S \smallsetminus P$, $[b, P] \le A \le \Theta$ since $bA \in Z(S/A)$, and $[b, \Theta] \le [b, TA] \le A_0 \le \mathrm{Fr}(P)$. Thus $P \notin \mathcal{E}$.

- $\boldsymbol{P = H_1,\ [S,S] = A,}$ **and** $\boldsymbol{\Theta = \Theta_3 \le TA}$**.** Then $x \in S \smallsetminus P$ and $[x, P] \le [S,S] = A \le \Theta$. Also, since $\overline{TA} = Z(\overline{S})$ and hence $[S, TA] \le TA_0$, $[x, \Theta] \le [S, TA] \le A \cap TA_0 = A_0 \le \mathrm{Fr}(P)$. Thus $P \notin \mathcal{E}$.

- $\boldsymbol{P = H_1,\ [S,S] > A,}$ **and** $\boldsymbol{\Theta_3 \le TA}$**.** We will see later that we don't need to consider this case separately.

- $\boldsymbol{P = H_0,}$ **and** $\boldsymbol{\Theta_3 \not\le TA}$**.** Since $\Theta_3 \not\le TA$, there is $\beta \in \mathrm{Aut}(P)$ such that $\beta(A) \not\le TA$. Set $u = \beta(a) \in P \smallsetminus TA = TAx$ and $U = \langle u\rangle = \beta(A) \trianglelefteq P$. Then $\bar u \in \{\bar x, \overline{ax}\}$, so $\bar u^2 = \bar a$ by (6), and $u^2 = ta^i$ where $t \in T$ and i is odd. Set $C = A \cap U$ and $2^m = [A{:}C] = [U{:}C]$, so $[A, U] \le C$, $C = \langle a^{2^m}\rangle = \langle u^{2^m}\rangle$, and $AU/C \cong (C_{2^m})^2$. Then $t^{2^{m-1}} = (u^2a^{-i})^{2^{m-1}} \equiv u^{2^m}a^{-i\cdot 2^{m-1}} \equiv a^{2^{m-1}} \pmod{C}$ since i is odd, so $t^{2^{m-1}} = a^{k\cdot 2^{m-1}}$ for some odd k, $a^{k\cdot 2^{m-1}} \in A \cap T = Z$ has order ≤ 2, and hence $|a| = 2^m$. Thus $C = 1$, and $AU \cong A \times A \cong (C_{2^m})^2$.

  Now, $[T, u^2] \le [T, TA] \le A$, and $[T, u^2] \le U$ since $U \trianglelefteq P$. Hence $[T, u^2] = 1$, and $[T, A] = [T, a^i] = 1$ since $u^2 = ta^i$ for i odd. Thus $[A, P] = [A, TAU] = 1$, $[U, P] = \beta([A, P]) = 1$, and so $P = TAU$ is abelian. Also, $T \le Z(S)$ since $S = \langle P, b\rangle$ and $[T, b] = 1$. Hence $[S,S] = A$ by (4).

  Thus $[S,S] = A$ is cyclic and $H_0 = P$ is abelian of index two in S, so we are in the situation of Proposition 2.5. Since $S \ge AU \cong A \times A$, S is not dihedral or semidihedral, and hence is wreathed.

- $\boldsymbol{P \in \{H_1, H_2\}}$ **and** $\boldsymbol{\Theta_3 \not\le TA}$**.** Let $i = 1, 2$ be such that $P = H_i = \langle TA, b_i\rangle$. Since $\Theta_3 \not\le TA$, there is $\beta \in \mathrm{Aut}(P)$ such that $\beta(a) \notin TA$. Set $b^\bullet = \beta(a)$; thus $b^\bullet \in TAb_i \subseteq S \smallsetminus C_S(W)$, so $b^\bullet a b^{\bullet-1} = a^j$ for some $j \in -1 + 4\mathbb{Z}$ by (2). Then
$$\beta(A) \ge \beta([a, P]) = [b^\bullet, P] \ge [b^\bullet, A] = \langle a^2\rangle = A_0$$
  since $A \ge [A, P]$, $[\beta(A){:}A_0] = 2$ since $|\beta(A)| = |A|$, and so $b^{\bullet 2} = a^{2\ell}$ for some odd ℓ. Thus $a^2 = b^\bullet a^2 b^{\bullet-1} = a^{2j}$, so $j \equiv 1 \pmod{\frac12|a|}$. Since $j \equiv 3 \pmod 4$, this proves that $|a| = |b^\bullet| = 4$, $[b^\bullet, a] = a^2$, and thus $\langle A, b^\bullet\rangle = \langle a, b^\bullet\rangle \cong Q_8$. Also, $A = W$, so $[A, T] = 1$, and TA is abelian.

  If $P = H_2 = \langle TA, bx\rangle$, then $[b^\bullet, T] \le (\langle b^\bullet\rangle \cap TA) = Z$ since $\langle b^\bullet\rangle = \beta(A) \trianglelefteq P$. Hence $[S,S] = \big\langle A, [b^\bullet, T]\big\rangle = A$ by (4).

To summarize, if $P \in \mathcal{E}$ has index 2 in S, then either $P = H_0$ and S is wreathed, or $P = H_2$ and $[S,S] = A$, or $P = H_1$. Also, if $P \in \{H_1, H_2\}$ and $[S,S] = A$, then $P = T\Delta^\bullet$ for some $Q_8 \cong \Delta^\bullet \trianglelefteq P$, and $P = C_{TA}(\Delta^\bullet)\Delta^\bullet$ by (5). So this proves (7), (8), and (9) for essential subgroups of index two in S.

Step 2: Now assume $P \in \mathcal{E}$ where $[S{:}P] > 2$. If $|N_S(P)/P| > 2$, then by Lemma 1.5, $\mathrm{rk}([g, P/\mathrm{Fr}(P)]) \geq 2$ for all $g \in N_S(P){\smallsetminus}P$. For each $i \in \mathbb{Z}$, $[a^i, S] = \langle a^{2i}\rangle$, so $a^i \in N_S(P)$ if $a^{2i} \in P$, which implies $a^i \in P$ since $[a^i, S]$ is cyclic. Thus $A \leq P$ by induction on $|A \cap P|$, so $a^2 \in \mathrm{Fr}(P)$. For $t \in T$, $[t, S] \leq \langle a^4, [t, x]\rangle$, and thus $P \geq TA$. Hence $P = TA$ since $[S{:}P] \geq 4$, so $[b, P] = \langle a^2\rangle \leq \mathrm{Fr}(P)$ (and $b \in N_S(P)$), which contradicts Lemma 1.5. We conclude that $|N_S(P)/P| = 2$.

In particular, P is not normal in S. By Theorem 4.6(b,c), and since $[S, S]$ is abelian by (4), P is of the type described in Theorem 4.6(a) with $S_* = S$. Thus $P = C_S(U^\bullet)U^\bullet$, where $U^\bullet \cong C_2^2$ or Q_8, and the normal closure $\Delta^\bullet$ of $U^\bullet$ in S is dihedral of order ≥ 8 or quaternion of order ≥ 16. Let $A^\bullet \trianglelefteq \Delta^\bullet$ be the unique cyclic subgroup of index two, and fix $b^\bullet \in U^\bullet{\smallsetminus}A^\bullet$. Thus $\Delta^\bullet = \langle A^\bullet, b^\bullet\rangle$.

Since $A^\bullet \leq \Delta^\bullet$ are both normal in S, the coset $A^\bullet b^\bullet = \Delta^\bullet{\smallsetminus}A^\bullet$ is a union of S-conjugacy classes, and its elements are all conjugate since otherwise $[b^\bullet, S] = [b^\bullet, A^\bullet]$ has index two in $A^\bullet$ and the normal closure of $U^\bullet$ is strictly contained in $\Delta^\bullet$. Hence

$$A^\bullet = [b^\bullet, S] \leq [S, S] \leq TA$$

(recall that $S/TA \cong C_2^2$ is abelian by (3)). We claim that $\Delta^\bullet \not\leq TA$, and hence $b^\bullet \notin TA$. Assume otherwise: then $Z(\Delta^\bullet) \leq [\Delta^\bullet, \Delta^\bullet] \leq [TA, TA] \leq A$, so $Z(\Delta^\bullet) = Z$, and the dihedral group $\Delta^\bullet/Z(\Delta^\bullet)$ is generated by elements of order two in TA/Z. Since each element of order two in $TA/Z \cong (A/Z) \rtimes (T/Z)$ lies in TW/Z, $\Delta^\bullet$ is contained in the abelian group TW, which is impossible.

Thus $b^\bullet \in H_i{\smallsetminus}TA = TAb_i$ for some unique $i = 0, 1, 2$. Since $b^\bullet \notin TA$, $\overline{b^\bullet}$ is not central in $\overline{S} \cong D_8$ (see (3)), so $C_{\overline{S}}(\overline{b^\bullet}) = \langle \overline{b^\bullet}, \bar{a}\rangle = \overline{H}_i$. Hence $C_S(b^\bullet) \leq H_i = \langle TA, b^\bullet\rangle$, so $P = C_S(U^\bullet){\cdot}U^\bullet \leq C_S(b^\bullet)A^\bullet \leq H_i$, and $P \in \mathcal{E}_i$. Since the normal closure of P has index two in $S = S_*$ by Theorem 4.6(a) again, it must be equal to H_i. This proves (7).

Assume $i = 0$ (so $b^\bullet \in TAx$), or $i = 2$ (so $b^\bullet \in TAbx$). Since $[b, TA] = A_0 = \langle a^2\rangle$ and $[x, b] = a$, we get $\langle [b^\bullet, b]\rangle = A$ in all cases. Thus $A^\bullet = [b^\bullet, S] \geq A$, $\overline{A^\bullet} = [\overline{S}, \overline{S}] = \overline{A}$, and hence $A^\bullet = A$. (If $A^\bullet > A$, then since both are cyclic, they could not have the same nontrivial image in any quotient group.) So $\Delta^\bullet = \langle A, b^\bullet\rangle$. Also, since $S = \langle T\Delta, b^\bullet\rangle$ in these cases, (4) implies that

$$[S, S] = \langle A, [b^\bullet, T]\rangle = A\ .$$

Together with Step 1, this proves (9).

The subgroup $W \leq A^\bullet = A$ of order 4 is central in H_0 and not central in $\Delta^\bullet$. Thus $P \notin \mathcal{E}_0$. Together with Step 1, this finishes the proof of (8): $\mathcal{E}_0 = \varnothing$ or $\mathcal{E}_0 = \{H_0\}$, and in the latter case, S is wreathed.

Step 3: We have now proven (7), (8), and (9), and it remains to prove (10). So assume $\mathcal{E}_0 = \varnothing$. Then $\mathcal{E}_1 \neq \varnothing$ and $\mathcal{E}_2 \neq \varnothing$ by (7) and Lemma 2.2, and $[S, S] = A$ by (9). It remains to prove that S is dihedral or semidihedral.

Fix $P_2 = C_S(U^\bullet)U^\bullet \in \mathcal{E}_2$ with normal closure H_2, as described in Step 2. Choose $b^\bullet \in U^\bullet{\smallsetminus}A$, so that $\langle A, b^\bullet\rangle$ is the normal closure of $U^\bullet$ and is dihedral or quaternion. Thus $b^\bullet \in TAbx$, and $\widehat{\Delta} \overset{\text{def}}{=} \langle A, b, b^\bullet\rangle \trianglelefteq S$ is dihedral, semidihedral, or quaternion by Lemma B.3(e). Also, $T\widehat{\Delta} = \langle TA, b, b^\bullet\rangle = S$. Let $\widehat{A} < \widehat{\Delta}$ be the cyclic subgroup of index two (so $\widehat{\Delta} = \langle \widehat{A}, b\rangle$, $[\widehat{A}{:}A] = 2$, and $\widehat{A} \trianglelefteq S$).

If S contains an abelian subgroup of index two, then since $[S, S] = A$ is cyclic, S is dihedral, semidihedral, or wreathed by Proposition 2.5. Since $[S, S] = A$ is strictly contained in the larger cyclic subgroup $\widehat{A}$, S cannot be wreathed.

Assume S does not contain an abelian subgroup of index two. Thus $T\widehat{A}$ is nonabelian, so $[T,\widehat{A}]\neq 1$ (T and $\widehat{A}$ are both abelian), and the homomorphism $c\colon T \longrightarrow \mathrm{Aut}(\widehat{A})$ induced by conjugation is nontrivial. Also, $\mathrm{Im}(c)$ is cyclic (each $t\in T$ acts via $(a\mapsto a^i)$ for $i\in 1+4\mathbb{Z}$ by (2)), and $C_T(\widehat{A})=\mathrm{Ker}(c)=Z(S)$ since $S=\langle T\widehat{A},b\rangle$, $[T,T]=[T,b]=1$, and $T\overset{\text{def}}{=}C_S(\Delta_0)\geq Z(S)$. Also, $C_T(A)=\{t\in T\mid |c(t)|\leq 2\}$, since for $i\in 4\mathbb{Z}+1$, $i\equiv 1 \pmod{|A|}$ if and only if $i^2\equiv 1 \pmod{|\widehat{A}|}$. So $[C_T(A){:}C_T(\widehat{A})]=2$. Thus

$$C_S(A)/Z(S)=C_T(A)\widehat{A}/C_T(\widehat{A})\cong (C_T(A)/C_T(\widehat{A}))\times(\widehat{A}/Z)\cong C_2\times(\widehat{A}/Z)$$

since $[C_T(A),\widehat{A}]=Z$, so $\mathrm{Aut}(C_S(A)/Z(S))$ is a 2-group by Corollary A.3(a). Hence each element of odd order in $\mathrm{Aut}_{\mathcal{F}}(S)$ induces the identity on $C_S(A)/Z(S)$, and so the action of the odd order group $\mathrm{Out}_{\mathcal{F}}(S)$ on $S^{\mathrm{ab}}=S/A$ induces the identity on $C_S(A)/Z(S)A$.

We want to apply Proposition 2.3(b), with $U=Z=[S,S]\cap Z(S)$. For each $P<S$ of index two, $1\neq[P,P]\leq[S,S]=A$, so $[P,P]\geq Z$ since A is cyclic. Choose $g_0\in C_T(A){\smallsetminus}C_T(\widehat{A})$, and let g_1 be such that $g_1A=\prod_{\alpha\in\mathrm{Out}_{\mathcal{F}}(S)}\alpha(g_0A)$. Since $\mathrm{Out}_{\mathcal{F}}(S)$ acts trivially on $C_S(A)/Z(S)A$, $g_1\equiv g_0^k \pmod{Z(S)A}$, where $k=|\mathrm{Out}_{\mathcal{F}}(S)|$ is odd, and we can assume $g_1\in T$ and $g_1\equiv g_0^k \pmod{Z(S)}$. Thus $|c(g_1)|=|c(g_0^k)|=2$ since k is odd and $[g_0,\widehat{A}]=Z$, so $g_1\notin Z$. Fix $g\in\langle g_1\rangle$ such that $g\notin Z$ but $g^2\in Z$. Then $|c(g)|\leq 2$, so $[g,S]=[g,\widehat{A}]\leq Z$. By construction, every element of $\mathrm{Aut}_{\mathcal{F}}(S)$ sends gA to itself.

Thus $\mathcal{F}$ is not reduced by Proposition 2.3, applied with $U=Z$ and g as above. □

We now prove some other versions of Proposition 5.1, by listing different hypotheses which give the same conclusion.

Proposition 5.2. *Let S be a finite nonabelian 2-group which satisfies at least one of the following conditions.*

(a) *There is an abelian subgroup $A\trianglelefteq S$ of index two.*

(b) *$[S,S]$ is cyclic.*

(c) *There is an abelian subgroup $Q<S$ such that $|N_S(Q)/Q|=2$ and $\mathrm{Out}_S(Q)\not\leq O_2(\mathrm{Out}(Q))$.*

(d) *There is a subgroup $Q=Z(Q)U$ in S, where $U\cong Q_8$, $|N_S(Q)/Q|=2$, and $\mathrm{Out}_S(Q)$ exchanges two of the three abelian subgroups of index two in Q.*

(e) *There is a subgroup $Q<S$ such that $|Q|\leq 16$, $|N_S(Q)/Q|=2$, and $\mathrm{Out}_S(Q)\not\leq O_2(\mathrm{Out}(Q))$.*

Then either S is dihedral, semidihedral, or wreathed, or there is no reduced fusion system over S.

Proof. Assume there is a reduced fusion system $\mathcal{F}$ over S, and let $\mathcal{E}$ be the set of all $\mathcal{F}$-essential subgroups. Let $\mathcal{DSW}$ be the class of all dihedral, semidihedral, and wreathed 2-groups; we must show $S\in\mathcal{DSW}$.

(a) Let $A\leq S$ be an abelian subgroup of index two. If $R\in\mathcal{E}$ and $R\neq A$, then $R\not\leq A$ since R is centric in S, and so $AR=S$. So there is $g\in A\cap N_S(R){\smallsetminus}R$, and $[g,R]$ is cyclic, generated by $[g,h]$ for any $h\in R{\smallsetminus}A$. Hence by Lemma 1.5, $|N_S(R)/R|=2$.

For each $x \in S \smallsetminus A$, $C_S(x) = \langle C_A(x), x\rangle$ is abelian. Hence

$$R \le S,\ R \nleq A \implies C_S(R) \text{ is abelian.} \tag{1}$$

By Theorem 4.6(b,c), and since $[S,S] \le A$ is abelian, each $R \in \mathcal{E}$ either has index two in S, or has the form described in Theorem 4.6(a) with $S_* = S$. In the latter case, there is $\Delta \trianglelefteq S$ which is dihedral of order ≥ 8 or quaternion of order ≥ 16, and the noncyclic subgroups of index two in Δ are S-conjugate since Δ is the normal closure of a subgroup $U_1 \cong C_2^2$ or Q_8. Also, for $\Delta_0 \le \Delta$ dihedral or quaternion of order 8, $C_S(\Delta_0)$ is abelian by (1). The hypotheses of Proposition 5.1 thus hold, and so $S \in \mathcal{DSW}$.

We are left with the case where each $R \in \mathcal{E}$ has index two in S. If all $\mathcal{F}$-essential subgroups are abelian, then S has at least two abelian subgroups of index two (Lemma 2.2), so $|[S,S]| = 2$ by Lemma A.6(c), and $S \in \mathcal{DSW}$ by Proposition 2.5.

Assume $R \in \mathcal{E}$ is nonabelian. If $A \cap R$ is the only abelian subgroup of index two in R, then it is characteristic in R. For $g \in A \smallsetminus R$, $[g,R] \le R \cap A$, $[g, R \cap A] = 1$, and this contradicts Lemma 1.4 (applied with $\Theta = R \cap A$).

Thus by Lemma A.6(b,c), R has three abelian subgroups of index two, $[R{:}Z(R)] = 4$, and $|[R,R]| = 2$. Set $Z = [R,R] \le Z(R)$, and fix a generator $z \in Z$. Fix $g \in A \smallsetminus R$ and $h \in R \smallsetminus A$, and set $y = [h,g]$. Thus $[g,R] = \langle y\rangle$, and

$$hgh^{-1} = gy \implies g = h^2gh^{-2} = gy(hyh^{-1}) \implies [h,y] = hyh^{-1}y^{-1} = y^{-2}.$$

If $y^2 = 1$, then $[h,y] = 1$, so $[g,R] = \langle y\rangle \le Z(R)$, $[g, Z(R)] = 1$ since $Z(R) \le A$, and this again contradicts Lemma 1.4 (applied with $\Theta = Z(R)$). Thus $y^2 = [h,y]^{-1} \ne 1$, so $y^2 = z \in [R,R]$, and $[S,R] = \langle y, [R,R]\rangle = \langle y\rangle$ is cyclic.

By Lemma 2.4(a.ii) (and since R is nonabelian), there are $R_1 \trianglelefteq R$ and $\mathrm{Out}_S(R) \le \Gamma_1 \trianglelefteq \mathrm{Out}_{\mathcal{F}}(R)$ such that $R_1 \cong Q_8$, $\mathrm{Aut}_{\mathcal{F}}(R)$ sends R_1 to itself, and $\Gamma_1 \cong \Sigma_3$ acts faithfully on $R_1/\mathrm{Fr}(R_1)$. Thus $R_1 \trianglelefteq S$, $\mathrm{Aut}_S(R)$ exchanges two of the three subgroups of index two in R_1, and $C_S(R_1)$ is abelian by (1). Hence $S \in \mathcal{DSW}$ by Proposition 5.1.

(b) Assume $[S,S]$ is cyclic, and let $Z \le [S,S]$ be the subgroup of order 2. If some $P \in \mathcal{E}$ has index ≥ 4 in its normalizer, then $\mathrm{rk}([s, P/\mathrm{Fr}(P)]) \ge 2$ for each $s \in N_S(P) \smallsetminus P$ by Lemma 1.5, which is impossible since $[S,S]$ is cyclic. If $P \in \mathcal{E}$ is nonabelian, then $1 \ne [P,P] \le [S,S]$, and $Z = \Omega_1([P,P])$ is characteristic in P. So if each $P \in \mathcal{E}$ is nonabelian, then Z is characteristic in S and in each $P \in \mathcal{E}$, hence $Z \trianglelefteq \mathcal{F}$ by Proposition 1.3, and $\mathcal{F}$ is not reduced.

Thus some $R \in \mathcal{E}$ is abelian and has index 2 in its normalizer. If $[S{:}R] = 2$, then the result follows from (a). Otherwise, by Theorem 4.6, there is $\Delta \trianglelefteq S$ which is dihedral of order ≥ 8 ($R \ngeq Q_8$ since it is abelian), and the noncyclic subgroups of index two in Δ are S-conjugate since Δ is the normal closure of a subgroup $U_1 \cong C_2^2$ in R. Also, for $\Delta_0 \le \Delta$ dihedral of order 8 containing U_1, $C_S(\Delta_0) \le C_S(U_1) \le R$ is abelian. The hypotheses of Proposition 5.1 thus hold, and so $S \in \mathcal{DSW}$.

(c) If S contains an abelian subgroup of index two, then $S \in \mathcal{DSW}$ by (a). If there is an abelian subgroup $Q < S$ which satisfies the assumptions in (c) and is not normal, then by Lemma B.5(a), there are subgroups $\Delta_0 \le \Delta \trianglelefteq S$ such that Δ is dihedral, $\Delta_0 \cong D_8$, $C_S(\Delta_0) \le Q$ is abelian, and the noncentral involutions in Δ are all S-conjugate. Then $S \in \mathcal{DSW}$ by Proposition 5.1.

(d) Fix $Q = Z(Q)U \le S$, where $U \cong Q_8$, $|N_S(Q)/Q| = 2$, and $\mathrm{Out}_S(Q)$ exchanges two abelian subgroups A_1 and A_2 of index two in Q. Set $Z = Z(U) = [Q,Q] \cong C_2$, let $z \in Z$ be the generator, choose $u_1 \in A_1 \cap U$ of order 4, and set $u_2 = xu_1x^{-1} \in A_2$ for some $x \in N_S(Q) \smallsetminus Q$. Then $u_2^2 = u_1^2 = z$ since $[x, z] = 1$, $u_2 \notin Z(Q)$, and so $[u_1, u_2] = z$ and $\langle u_1, u_2 \rangle \cong Q_8$. So upon replacing U by $\langle u_1, u_2 \rangle$, we can assume $U \trianglelefteq N_S(Q)$.

Set $S_* = C_S(Z)$, and let Δ be the normal closure of U in S_*. Then Q/Z is abelian, and $[N_{S_*/Z}(Q/Z){:}Q/Z] = 2$ since $[N_S(Q){:}Q] = 2$ and $N_{S_*}(Q) = N_S(Q)$. Also, $U/Z \cong C_2^2$ is a direct factor in Q/Z, and $[N_S(Q)/Z, U/Z] \neq 1$ since $\mathrm{Out}_S(Q)$ exchanges two of the abelian subgroups of U.

If $\Delta > U$, then Δ/Z is dihedral by Lemma B.5(b), and all involutions in Δ/Z are S_*-conjugate to elements of U/Z. Hence there are no involutions in $\Delta \smallsetminus Z$, and Δ is quaternion.

Set $R = N_{C_S(U)}(Z(Q))$. Since conjugation by each $y \in N_S(Q) \smallsetminus Q$ exchanges A_1 and A_2, $C_S(U) \cap N_S(Q) \leq Q$. Also, R normalizes $Z(Q)U = Q$, so $R \leq C_S(U) \cap N_S(Q) \leq Q$. Hence $R \leq C_Q(U) = Z(Q)$, so $C_S(U) = Z(Q)$ by Lemma A.1.

If $S_* < S$, then choose $y \in N_S(S_*) \smallsetminus S_*$, and set $\beta = c_y \in \mathrm{Aut}(S_*)$. Thus $\beta(Z) \neq Z$, and $\Delta \cap \beta(\Delta) = 1$ since Z and $\beta(Z)$ contain the only elements of order two in Δ and $\beta(\Delta)$, respectively. So $[\Delta, \beta(\Delta)] \leq \Delta \cap \beta(\Delta) = 1$ since both subgroups are normal, which is impossible since $C_S(\Delta) \leq C_S(U) = Z(Q)$ is abelian. Thus $S_* = S$.

If $\Delta = U$, then by assumption, $\mathrm{Aut}_S(Q)$ exchanges two of the abelian subgroups of index two in $Q = Z(Q)U$, and hence exchanges two of the index two subgroups of $U = \Delta$. If $\Delta > U$, the two noncyclic subgroups of index two in Δ are S-conjugate since Δ is generated by the S-conjugates of $U < \Delta$. Since $C_S(U) = Z(Q)$ is abelian, $S \in \mathcal{DSW}$ by Proposition 5.1, applied with $\Delta_0 = U$.

(e) Assume $Q < S$ is such that $|Q| \leq 16$, $|N_S(Q)/Q| = 2$, and $\mathrm{Out}_S(Q) \nleq O_2(\mathrm{Out}(Q))$. In particular, $\mathrm{Out}(Q)$ is not a 2-group. By Corollary A.3(a,c), $Q^{\mathrm{ab}} \ncong C_4 \times C_2$, so either Q is abelian or Q^{ab} is elementary abelian. If Q is abelian, then $S \in \mathcal{DSW}$ by (c). If Q is nonabelian, then by Corollary A.3(b) and the list of groups of order 16 (cf. [Bu, § 74]), Q is isomorphic to Q_8, $C_2 \times Q_8$, or $C_4 \times_{C_2} Q_8$. (Note that $\mathrm{Aut}(C_2 \times D_8)$ is a 2-group by Lemma A.2, since it contains a unique subgroup isomorphic to $C_2 \times C_4$.) In each of these cases, by Lemma A.2 applied to the chain $1 < \mathrm{Fr}(Q) \leq Z(Q) < Q$, $O_2(\mathrm{Aut}(Q))$ contains all $\alpha \in \mathrm{Aut}(Q)$ which act trivially on $Q/Z(Q)$. Since $\mathrm{Aut}_S(Q) \nleq O_2(\mathrm{Aut}(Q))$, $\mathrm{Aut}_S(Q)$ acts nontrivially on $Q/Z(Q)$ and hence contains elements which exchange two of the three abelian subgroups of index two in Q. So $S \in \mathcal{DSW}$ by (d). □

Each reduced fusion system over a dihedral, semidihedral, or wreathed 2-group is isomorphic to the fusion system of $PSL_2(q)$ for $q \equiv \pm 1 \pmod 8$, or of $PSU_3(q)$ or $PSL_3(q)$ for $q \equiv 1 \pmod 4$. Fusion systems over dihedral and semidihedral 2-groups have been listed by several people; cf. [AOV, § 4.1] for the reduced case. For wreathed 2-groups, this was shown in Proposition 3.1. The fusion systems (at the prime 2) of $PSU_3(q)$ and $PSL_3(q)$ for $q \equiv 3 \pmod 4$ also have this form (see, e.g., [BMO, Theorem A(d)]).

Proposition 5.2(e) has as an easy consequence:

Theorem 5.3. *Let $\mathcal{F}$ be a reduced fusion system over a nontrivial finite 2-group S of order at most 32. Then S is dihedral, semidihedral, or wreathed, and $\mathcal{F}$ is isomorphic to the fusion system of $PSL_2(q)$ for $q \equiv \pm 1$ (mod 8), or that of $PSL_3(q)$ for q odd.*

Proof. By Lemma 2.2, there exists an $\mathcal{F}$-essential subgroup $Q < S$. Then $|Q| \leq 16$, and $|N_S(Q)/Q| = 2$, since otherwise $|S| \geq |N_S(Q)| \geq 64$ by Lemma 1.5. Since $\mathrm{Out}_{\mathcal{F}}(Q)$ contains a strongly 2-embedded subgroup, $O_2(\mathrm{Out}_{\mathcal{F}}(Q)) = 1$ (cf. [AKO, Proposition

A.7(c)]). Hence $\mathrm{Out}_S(Q) \not\leq O_2(\mathrm{Out}(Q))$, so S is dihedral, semidihedral or wreathed by Theorem 5.2(e). The description of $\mathcal{F}$ follows from the above remarks. □

We finish with a slightly less easy consequence of the results in this section and in Section 4: a list of those groups of order 64 which support reduced fusion systems. For all n and q, $UT_n(q)$ denotes the group of strictly upper triangular matrices over $\mathbb{F}_q$ (i.e., those with 1's on the diagonal). A fusion system over a p-group S is *indecomposable* if it is not isomorphic to a product of fusion systems over nontrivial subgroups of S.

Theorem 5.4. *Let $\mathcal{F}$ be a reduced, indecomposable fusion system over S, where $|S| = 64$. Then S is isomorphic to one of the groups D_{64}, SD_{64}, $UT_4(2)$, $UT_3(4)$, or to a Sylow 2-subgroup of M_{12}.*

Proof. Fix $\mathcal{F}$ and S, and let $\mathcal{E}$ be the set of $\mathcal{F}$-essential subgroups of S. Assume S is neither dihedral nor semidihedral; it cannot be wreathed since $|S|$ is a power of 4.

Case 1: Assume there is $P \in \mathcal{E}$ such that $P \ntrianglelefteq S$ and $|N_S(P)/P| = 2$. Then $|P| \leq 16$, $\mathrm{Out}_S(P) \not\leq O_2(\mathrm{Out}(P))$ since P is essential, and this is impossible by Proposition 5.2(e).

Case 2: Assume that we are not in the situation of Case 1, and that there is $P \in \mathcal{E}$ such that $|N_S(P)/P| \geq 4$. By Lemma 1.5, $\mathrm{rk}(P/\mathrm{Fr}(P)) \geq 4$. Thus $P \cong C_2^4$ and $P \trianglelefteq S$. Using Bender's classification of groups with strongly 2-embedded subgroups [Bd, Satz 1], we see that either $\mathrm{Aut}_{\mathcal{F}}(P) \geq A_5$, or $\mathrm{Aut}_{\mathcal{F}}(P) \cong C_5 \rtimes C_4$, $C_{15} \rtimes C_4$, or $C_3^2 \rtimes C_4$. Also, there are exactly two conjugacy classes of subgroups isomorphic to A_5 in $GL_4(2) \cong A_8$ (cf. [Ta, Corollary 6.7]), corresponding to the A_5-orbits A_5/A_4 and A_5/D_{10}.

Case 2A: Assume $A_5 \leq \mathrm{Aut}_{\mathcal{F}}(P)$, acting via the reduced permutation action (i.e., the permutation action on $\mathbb{F}_2^5$ modulo its fixed subspace). Then $\mathrm{Aut}_S(P) \cong C_2^2$ permutes freely a basis for $P \cong C_2^4$, so the extension of P by S/P splits, and $S \cong UT_4(2)$.

Case 2B: Assume $A_5 \leq \mathrm{Aut}_{\mathcal{F}}(P)$, acting via the canonical action of $A_5 \cong SL_2(4)$ on $\mathbb{F}_4^2$. With respect to a suitable basis, each $g \in S \smallsetminus P$ acts via multiplication by a triangular matrix $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ for some $x \in \mathbb{F}_4 \smallsetminus \{0\}$, so $C_P(g) = [g, P] = Z(S) \cong C_2^2$. Thus $g^2 \in C_P(g) = Z(S)$ for each $g \in S$, so $S/Z(S)$ has exponent two and hence is abelian. It follows that $[S, S] = Z(S)$, and in particular, that each $R \in \mathcal{E}$ is normal in S.

Assume $R \in \mathcal{E}$ with $[S{:}R] = 2$. If $P \leq R$, then $[g, P] = [S, S]$ for any $g \in R \smallsetminus P$. If $P \not\leq R$, then $S = PR$, $\mathrm{rk}(P \cap R) = 3 > \mathrm{rk}(Z(S))$, and for $g \in (P \cap R) \smallsetminus Z(S)$, $[R, R] \geq [R, g] = [PR, g] = [\mathrm{Aut}_S(P), g] = [S, S]$. Thus $[R, R] = [S, S]$ in both cases. Hence $[S, R] = [R, R] \leq \mathrm{Fr}(R)$, which contradicts Lemma 1.4 (applied with $\Theta = 1$).

By Lemma 2.2, there is $R \in \mathcal{E}$ with $R \neq P$, and we just showed that $[S{:}R] > 2$. Hence $|N_S(R)/R| \geq 4$, so $R \cong C_2^4$ by Lemma 1.5. Also, $Z(RP) = Z(S) \cong C_2^2$ since $C_P(g) = Z(S)$ for each $g \in S \smallsetminus P$, so $|R \cap P| \leq |Z(RP)| = 4$, and $RP = S$. In particular, the extension of P by $S/P \cong C_2^2$ is split, and so $S \cong UT_3(4)$.

Case 2C: Assume $\mathrm{Aut}_S(P) \cong S/P \cong C_4$, and thus $\mathrm{Aut}_{\mathcal{F}}(P) \cong C_5 \rtimes C_4$, $C_{15} \rtimes C_4$, or $C_3^2 \rtimes C_4$. In either case, there is a basis of $P \cong C_2^4$ permuted freely by $S/P \cong C_4$, and so $C_P(S) \leq [S, P] \cong C_2^3$. If $x \in S$ is such that $S = \langle P, x \rangle$, then $x^4 \in C_P(S) \leq [S, P]$, and so $[S, S] = [S, P]$ and $S/[S, S] \cong C_2 \times C_4$.

Let $Q = \langle P, x^2 \rangle < S$ be the (unique) subgroup of index two which contains P. Thus $Q/[S, S] \cong C_2^2$, and in particular, each elementary abelian subgroup of S is contained in Q.

For each $R \in \mathcal{E}$, we have shown that either $[S{:}R] = 2$ or $R \cong C_2^4$. If $R \cong C_2^4$, then $R \le Q$, and $RP = Q$ if $R \ne P$. (In fact, there is no such subgroup.) If $[S{:}R] = 2$ and $R \ne Q$, then $R \cap P \cong C_2^3$, so $R \cap P = [S, P]$ since it is normal in S, $RP = S$, and $[R, R] = [S, R\cap P]$ has index two in $R\cap P$. So $R/[R, R] \cong C_2 \times C_4$, $\mathrm{Aut}(R)$ is a 2-group by Corollary A.3(a,c), and $R \notin \mathcal{E}$. Thus $RP = Q$ for each $R \in \mathcal{E}\smallsetminus\{P\}$. So $\mathcal{E} = \{P\}$ or $\langle\mathcal{E}\rangle = Q$, either of which contradicts Lemma 2.2. Hence this case is impossible.

Case 3: We are left with the case where each $\mathcal{F}$-essential subgroup has index two in S. Since $|\mathcal{E}| \ge 2$ by Lemma 2.2, we can choose distinct $P_1, P_2 \in \mathcal{E}$. By Theorem 4.5, we can choose $G_1 > S < G_2$ and $T \le P_{12} = P_1 \cap P_2$ which satisfy conditions $(*)$ at the beginning of Section 4, with $P = S$ and $\mathrm{Aut}_{G_i}(P_i) \le \mathrm{Aut}_{\mathcal{F}}(P_i)$. Thus $\mathrm{Aut}_{G_i}(T) \le \mathrm{Aut}_{\mathcal{F}}(T)$.

Case 3A: Assume T is not centric in S. Let U and $Z = T\cap U$ be as in Theorem 4.5; $[T, U] = 1$ in all cases. There are six cases to consider, listed in Table 4.1.

(1) $|T| = 8$, and $S \cong T\times D_8$ or $T \times_{C_2} Q_{16}$. The first is impossible by [O1, Theorem B] (and since $\mathcal{F}$ is indecomposable), and the second by Proposition 5.2(b) ($[S, S] = [U, U]$ is cyclic).

(2) $|T| = 4$, $[S{:}UT] = 2$, $U \trianglelefteq S$, $(U, Z) \cong (D_8, 1)$ or (Q_{16}, C_2), and $S/T \cong C_2 \times D_8$. Then $S = \langle UT, x\rangle$ for some x such that $xT \in Z(S/T)$. If $U \cong D_8$, then $S = U \times \langle T, x\rangle$, which is impossible by [O1, Theorem B] again. Hence $U \cong Q_{16}$, $[x, U] \le T\cap U = Z$ since $[x, S] \le T$ and $U \trianglelefteq S$, and $[x, T] \le Z$ since $[T{:}Z] = 2$ and $Z \trianglelefteq S$. Thus $[S, S] = [U, U]$ is cyclic, and this is impossible by Proposition 5.2(b).

(3) $U \cong C_4 \wr C_2$, $Z = 1$, and $S = UT$. Then $S \cong U \times C_2$ has an abelian subgroup of index two, which is impossible by Proposition 5.2(a).

(4) $U \cong C_4 \wr C_2$, $Z = 1$, and $[S{:}UT] = 2$. Thus $T = 1$, and S is of type $\mathrm{Aut}(U_3(3))$. By the descriptions of this amalgam in [Gd2, Table 1 & (3.7)], S is isomorphic to a Sylow 2-subgroup of M_{12}.

(5) $S = U$ is a Sylow subgroup of M_{12}.

(6) $|U| \ge 2^7$, so this is impossible.

In the remaining cases, we assume T is centric in S. Set $\widehat{S} = \mathrm{Out}_S(T)$, $\widehat{G}_i = \mathrm{Out}_{G_i}(T)$, $\widehat{G} = \langle\widehat{G}_1, \widehat{G}_2\rangle \le \mathrm{Out}_{\mathcal{F}}(T)$, and $\widehat{P}_i = \mathrm{Out}_{P_i}(T)$. Thus $\widehat{S} \in \mathrm{Syl}_2(\widehat{G})$ by the Sylow axiom since $T \trianglelefteq S$ (hence T is fully normalized). Let Γ be a model for $T \trianglelefteq N_{\mathcal{F}}(T)$ (cf. [AKO, Theorem III.5.10(a)]), and let $G \le \Gamma$ be such that $T \le G$ and $\mathrm{Out}_G(T) = \widehat{G}$. Thus G is a finite group such that

$$S \in \mathrm{Syl}_2(G), \quad T \trianglelefteq G, \quad C_G(T) \le T, \quad \text{and} \quad \widehat{G} = \mathrm{Out}_G(T).$$

Case 3B: Assume $[S{:}T] = 4$. Thus $\widehat{S} \cong S/T \cong C_2^2$ by $(*)$, $T = P_{12}$, $P_i/T \cong C_2$ is normal in $\widehat{G}_i \cong G_i/T$ with quotient group $G_i/P_i \cong D_{2p_i}$, and hence $\widehat{G}_i \cong C_2 \times D_{2p_i}$ contains an element of order $2p_i$. Also, $\widehat{G}$ acts faithfully on $T/\mathrm{Fr}(T)$ by Lemma 4.2(e), $GL_4(2) \cong A_8$ contains no element of order $2p_i$ for primes $p_i \ge 5$, and $GL_3(2)$ contains no element of order 6. It follows that $\mathrm{rk}(T/\mathrm{Fr}(T)) = 4$, so $T \cong C_2^4$, and $p_1 = p_2 = 3$.

Set $\widehat{H}_i = O^2(\widehat{G}_i) \cong C_3$. It is not hard to see, by examining normalizers of subgroups of order three in $GL_4(2) \cong A_8$, that there are exactly two conjugacy classes of subgroups $C_2 \times \Sigma_3$ in $GL_4(2)$. Hence for some decomposition $T = V_1 \times V_2$, where $V_i \cong C_2^2$, the action of $\widehat{G}_1 \cong C_2 \times \Sigma_3$ on T has the following form: either

(i) Σ_3 acts faithfully on each V_i, and C_2 switches the two factors; or

(ii) Σ_3 acts faithfully on V_1 and trivially on V_2, and C_2 acts faithfully on V_2 and trivially on V_1.

In case (i), $\widehat{S}$ acts by permuting freely a basis of T, and hence $S \cong UT_4(2)$ as seen earlier.

In case (ii), there is a basis of T permuted by $\widehat{S}$ in two orbits of length 2. Hence the three involutions in $\widehat{S}$ cannot be conjugate in $\widehat{G}$, so $\mathcal{F}_{\widehat{S}}(\widehat{G})$ is the fusion system of $\widehat{S} \cong C_2^2$. Thus $[\widehat{G}, \widehat{G}] \cap \widehat{S} = 1$ by the focal subgroup theorem (cf. [G, Theorem 7.3.4]), so there is $\widehat{H} \trianglelefteq \widehat{G}$ of odd order and index four. Also, $\widehat{P}_i \trianglelefteq \widehat{G}_i$ for $i = 1, 2$, where $\widehat{P}_i \cong C_2$ and $\widehat{G}_i \cong C_2 \times \Sigma_3$, so $\widehat{G}_1 \neq \widehat{G}_2$. Thus $\widehat{G} = \langle \widehat{G}_1, \widehat{G}_2 \rangle$ contains two distinct subgroups of order 3, they are contained in $\widehat{H}$, and hence $|\widehat{H}| > 3$. Also, $\widehat{H}$ cannot contain a normal subgroup $\widehat{K} \trianglelefteq \widehat{H}$ of order 7, since that would imply $\widehat{G} \leq N_{\mathrm{Aut}(T)}(\widehat{K}) \cong C_7 \rtimes C_3$. Since $|GL_4(2)| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$, $\widehat{H}$ contains C_3^2 or C_5, and in either case, $C_T(\widehat{H}) = 1$. Let $H < G$ be such that $H \geq T$ and $\mathrm{Aut}_H(T) = \widehat{H}$. By Lemma A.9, applied to the triple $T \trianglelefteq H \trianglelefteq G$, G (and hence S) splits as a semidirect product over T. Thus $S \cong D_8 \times D_8$, which is impossible by [O1, Theorem B] again ($\mathcal{F}$ is indecomposable).

Case 3C: Now assume $[S{:}T] \geq 8$. Since $\widehat{S}$ acts faithfully on $T/\mathrm{Fr}(T)$, $\mathrm{rk}(T/\mathrm{Fr}(T)) \geq 3$, so $T \cong C_2^3$, $|\widehat{S}| = |S/T| = 8$, and $\widehat{S} \cong D_8$ since this is a Sylow 2-subgroup of $GL_3(2)$. Neither $\widehat{P}_1$ nor $\widehat{P}_2$ can be cyclic of order four, since the normalizer in $\mathrm{Aut}(T) \cong GL_3(2)$ of such a subgroup has order 8. Hence $\widehat{P}_i \cong C_2^2$ for $i = 1, 2$, $\widehat{G}_i \cong \Sigma_4$ (the normalizer in $\mathrm{Aut}(T) \cong GL_3(2)$ of $\widehat{P}_i$), and $\widehat{G} = \langle \widehat{G}_1, \widehat{G}_2 \rangle = \mathrm{Aut}(T)$. Thus G is an extension of C_2^3 by $GL_3(2)$. By [GH, Lemma II.3.4], either the extension is split and $S \cong UT_4(2)$, or it is not split and S is a Sylow 2-subgroup of M_{12}. □

Appendix A. Background on groups

We list here some elementary results about finite groups which are needed throughout the paper, beginning with a well known property of p-groups.

Lemma A.1. *If $Q < P$ are finite p-groups for some prime p, then $Q < N_P(Q)$.*

Proof. See, e.g., [Sz1, Theorem 2.1.6]. □

We next look at automorphisms of finite p-groups.

Lemma A.2. *Fix a prime p, a finite p-group P, a subgroup $P_0 \leq \mathrm{Fr}(P)$, and a sequence of subgroups*

$$P_0 \trianglelefteq P_1 \trianglelefteq \cdots \trianglelefteq P_k = P.$$

Set

$$\mathcal{A} = \bigl\{ \alpha \in \mathrm{Aut}(P) \bigm| x^{-1}\alpha(x) \in P_{i-1},\ \text{all } x \in P_i,\ \text{all } i = 1, \ldots, k \bigr\} \leq \mathrm{Aut}(P):$$

the group of automorphisms which leave each P_i invariant, and which induce the identity on each quotient group P_i/P_{i-1}. Then $\mathcal{A}$ is a p-group. If the P_i are all characteristic in P, then $\mathcal{A} \trianglelefteq \mathrm{Aut}(P)$, and hence $\mathcal{A} \leq O_p(\mathrm{Aut}(P))$.

Proof. See, for example, [G, Theorems 5.1.4 & 5.3.2]. □

As an easy exercise, Lemma A.2 implies the following corollary, which contains a list of 2-groups whose automorphism groups are 2-groups. Recall that $G^{\mathrm{ab}} = G/[G,G]$ denotes the abelianization of a group G.

Corollary A.3. *For a finite 2-group P, $\mathrm{Aut}(P)$ is a 2-group if at least one of the following hold:*

(a) $P \cong C_{2^{k_1}} \times C_{2^{k_2}} \times \cdots \times C_{2^{k_r}}$, *where $k_1, \ldots, k_r$ are pairwise distinct.*

(b) *P is dihedral of order ≥ 8, or semidihedral or quaternion of order ≥ 16.*

(c) $\mathrm{Aut}(P^{\mathrm{ab}})$ *is a 2-group.*

The next two results involve the intersection of a Sylow subgroup with the commutator subgroup.

Lemma A.4. *For any finite group G with $S \in \mathrm{Syl}_p(G)$, $S \cap O^p(G) \leq S \cap [G,G]$, with equality if $G/O^p(G)$ is abelian.*

Proof. Set $G' = [G,G]$. Since G/G' is abelian, its largest p-group quotient $G/O^p(G)G'$ is isomorphic to its Sylow p-subgroup $SG'/G' \cong S/(S \cap G')$ (and the isomorphism is induced by the inclusion $S \leq G$). Hence $S \cap G' = S \cap O^p(G)G'$, and so $S \cap O^p(G) \leq S \cap G'$ with equality if $G' \leq O^p(G)$. □

The following proposition goes essentially back to Schur [Sch, IX–X].

Proposition A.5. *Fix a finite group G with $S \in \mathrm{Syl}_p(G)$. Let $Z \leq Z(G)$ be a p-subgroup. Then $Z \cap [G,G] = Z \cap [S,S]$.*

Proof. This follows as an application of the transfer in (co)homology. See, e.g., [Hu, Satz IV.2.2]. □

The next lemma describes nonabelian 2-groups with abelian subgroup of index two.

Lemma A.6. *Let S be a finite nonabelian 2-group containing an abelian subgroup $A \trianglelefteq S$ of index two. Then the following hold.*

(a) $[S,S] \cong A/Z(S)$, *and all elements of $(S/Z(S)) \smallsetminus (A/Z(S))$ have order two.*

(b) *If $|[S,S]| = 2$, then $S/Z(S) \cong C_2^2$, and S contains exactly three abelian subgroups of index two.*

(c) *If $|[S,S]| \geq 4$, then $|S/Z(S)| \geq 8$, and A is the unique abelian subgroup of index two in S.*

(d) *If S contains three abelian subgroups of index two which are permuted transitively by some automorphism of S, then either $Z(S)$ is not a direct factor of A, or $[S,S] \leq \mathrm{Fr}(Z(S))$.*

Proof. **(a)** For each $x \in S \smallsetminus A$, $x^2 \in C_A(x) = Z(S)$, and thus $xZ(S)$ has order two in $S/Z(S)$. Also, $[S,S] = [x,A]$ is the image of $\mathrm{Id}-c_x$ as a homomorphism from A to itself, and $Z(S)$ is its kernel. Hence $[S,S] \cong A/Z(S)$.

(b) Assume $|[S,S]| = 2$. Then by (a), $|S/Z(S)| = 4$ and $S/Z(S) \cong C_2^2$. Each abelian subgroup of index two contains $Z(S)$, and each subgroup of index two which contains $Z(S)$ is abelian. So there are exactly three abelian subgroups of index two in S.

(c) Now assume $|[S,S]| \geq 4$, so $|S/Z(S)| \geq 8$ by (a). If $B \neq A$ is another abelian subgroup of index two in S, then $AB = S$, so $Z(S) \geq A \cap B$, and $|S/Z(S)| \leq 4$, a contradiction. Thus A is the only abelian subgroup of index two.

(d) Assume S contains three abelian subgroups $A = A_1, A_2, A_3$ of index two which are permuted transitively by some automorphism of S. Thus $|[S,S]| = 2$ and $S/Z(S) \cong C_2^2$ by (b,c). Fix a generator $z \in [S,S]$. If $Z(S)$ is a direct factor of $A = A_1$, then it is a direct factor of each A_i, and there are elements $a_i \in A_i \smallsetminus Z(S)$ of order 2. Then $a_1a_2a_3 \in Z(S)$ since the a_i represent the three nonidentity elements in $S/Z(S) \cong C_2^2$, $[a_i, a_j] = z$ for distinct $i, j \in \{1,2,3\}$ since $S = \langle a_i, a_j, Z(S)\rangle$ is nonabelian, and so $(a_1a_2a_3)^2 = z^3 = z$. Thus $z \in \mathrm{Fr}(Z(S))$. □

Recall that a group G is *metacyclic* if it contains a normal cyclic subgroup $H \trianglelefteq G$ such that G/H is also cyclic.

Lemma A.7. *Let S be a finite metacyclic 2-group with an automorphism $\gamma \neq \mathrm{Id}_S$ of odd order. Then either $S \cong C_{2^n} \times C_{2^n}$ for some $n \geq 1$ and $C_S(\gamma) = 1$, or $S \cong Q_8$.*

Proof. If S is abelian, then $\mathrm{rk}(S) \leq 2$, and $S \cong C_{2^n} \times C_{2^n}$ by Corollary A.3(a) (and since $\mathrm{Aut}(S)$ is not a 2-group). Also, $C_S(\gamma) = 1$ since γ acts nontrivially on $\Omega_1(S) \cong C_2^2$ (see [G, Theorem 5.2.4]).

Now assume S is nonabelian. Fix $A = \langle a\rangle \trianglelefteq S$ such that S/A is cyclic, and let $b \in S$ be such that $S = \langle a, b\rangle$. Set $S' = [S,S] \leq A$. Then $\gamma|_{S'} = \mathrm{Id}$ since $S' \leq A$ is cyclic. Let $\bar\gamma \in \mathrm{Aut}(S/S')$ be the automorphism induced by γ. Then $\bar\gamma \neq \mathrm{Id}_{S/S'}$ by Lemma A.2, so $\bar\gamma$ fixes only the identity element by the first paragraph, and $C_S(\gamma) = S'$. Also, $Z(S)$ is abelian of rank ≤ 2 and $\gamma|_{Z(S)}$ has nonidentity fixed points (since $S' \cap Z(S) \neq 1$), so $\gamma|_{Z(S)} = \mathrm{Id}$ by the first paragraph, and $Z(S) \leq S'$. Since γ does not induce the identity on $S/\mathrm{Fr}(S) \cong C_2^2$ (and $a \notin \mathrm{Fr}(S)$), $S = \langle a, \gamma(a), \mathrm{Fr}(S)\rangle$, so $S = \langle a, \gamma(a)\rangle$ since $\mathrm{Fr}(S)$ is the intersection of all maximal subgroups. Thus $[S', S] = 1$, and $S' = Z(S)$. Also, $[a^i, b^j] = [a,b]^{ij}$ for all $i, j \geq 1$ since $[a,b] \in Z(S)$.

Set $S_0 = \mathrm{Fr}(S) = \langle a^2, b^2\rangle \geq S'$ (since $[a,b] \in \langle a^2\rangle$). If $S_0 > S'$, then $\gamma|_{S_0} \neq \mathrm{Id}$ and $C_{S_0}(\gamma) \neq 1$. Hence S_0 is nonabelian, and we just showed this implies $Z(S_0) = [S_0, S_0]$. But $[S_0, S_0] = \big\langle [a^2, b^2]\big\rangle = \big\langle [a,b]^4\big\rangle < S' \leq Z(S_0)$, so this is impossible. We conclude that $S_0 = S'$, and hence is generated by $[a,b]$. Also, $b^2 \in A$ implies $[a,b]^2 = [a, b^2] = 1$, so $|S'| = 2$, $|S| = 8$, and $S \cong Q_8$. □

The following result about actions on abelian 2-groups is very useful in certain situations.

Lemma A.8. *Fix a finite abelian 2-group A and a subgroup $G \leq \mathrm{Aut}(A)$ with Sylow subgroup $S \in \mathrm{Syl}_2(G)$ of order two. Assume $S \not\leq Z(G)$, and $[S, A] \cong C_{2^n}$ for some $n \geq 1$. Then there are unique decompositions $A = A_0 \times A_1$ and $G = G_0 \times G_1$, such that the G-action on A splits as a product of G_i-actions on A_i; and such that $|G_0|$ is odd, $G_1 \cong \Sigma_3$, and $A_1 \cong C_{2^n} \times C_{2^n}$.*

Proof. See, e.g., [O1, Proposition 2.3]. □

Lemma A.9. *Fix a prime p, and a finite group G with subgroups $A \trianglelefteq B \trianglelefteq G$, both normal in G, such that A is an abelian p-group, B/A has order prime to p, and $C_A(B) = 1$. Then G splits as a semidirect product $G = A \rtimes H$, where $H \cong G/A$.*

Proof. By the spectral sequence for the extension $1 \to B/A \to G/A \to G/B \to 1$, $H^i(G/A;A) = 0$ for each $i \geq 0$ since $H^0(B/A;A) = C_A(B) = 0$ (and since $(|B/A|, |A|) = 1$). In particular, $G \cong A \rtimes (G/A)$.

Alternatively, since $|B/A|$ is prime to $|A|$, by the Schur-Zassenhaus theorem [G, Theorem 6.2.1], there is $K \leq B$ such that $KA = B$ and $K \cap A = 1$, and each such subgroup is B-conjugate to K. Hence the B-conjugacy class of K is its G-conjugacy class, so $[N_G(K){:}N_B(K)] = [G{:}B]$. If $a \in A \cap N_G(K)$, then $[a, K] \leq A \cap K = 1$, so $a = 1$ since $C_A(K) = C_A(B) = 1$. Thus $N_G(K) \cap A = 1$, so $N_B(K) = K$, and $G = AN_G(K)$. □

Appendix B. Finite 2-groups with normal dihedral or quaternion subgroups

We prove here some elementary results about certain finite 2-groups which have normal dihedral or quaternion subgroups, and their automorphisms. We begin by stating the following general proposition about automorphisms of products of p-groups.

Proposition B.1 ([O1, Proposition 3.2(a)])**.** *Fix a pair of finite p-groups S_1 and S_2, set $S = S_1 \times S_2$, and let $\mathrm{pr}_i \in \mathrm{Hom}(S, S_i)$ be the projection. Let $\alpha \in \mathrm{Aut}(S)$ be such that $\alpha(\Omega_1(Z(S_1))) = \Omega_1(Z(S_1))$. Then for $i = 1, 2$, $\mathrm{pr}_i(\alpha(S_i)) = S_i$ and $\alpha(S_i Z(S_{3-i})) = S_i Z(S_{3-i})$.*

We next look at semidirect products with normal dihedral subgroup.

Lemma B.2. *Fix a finite 2-group S, and subgroups $\Delta \trianglelefteq S$ and $T \leq S$. Assume that $S = T\Delta$, $T \cap \Delta = 1$, Δ is dihedral of order ≥ 8, and $[T, \Delta_0] = 1$ for some dihedral subgroup $\Delta_0 \leq \Delta$ of order 8. Let $A, \Delta_1, \Delta_2 \leq \Delta$ be the three subgroups of index two where A is cyclic, and let $A_0 = [\Delta, \Delta]$ be the subgroup of index two in A. Set $Z = Z(\Delta)$ for short. Fix $b \in \Delta_0 \smallsetminus A$.*

(a) *Assume $\varphi \in \mathrm{Hom}(\Delta, S)$ is such that $\varphi(Z) = Z$ and $\varphi(A) \trianglelefteq S$. Then $\varphi(A_0) = A_0$, $\varphi(A) \leq TA$, and $\varphi(b) \in TAb$.*

(b) *Assume $\alpha \in \mathrm{Aut}(S)$ is such that $\alpha(Z) = Z$. Then either α sends each of the subgroups $T\Delta_1$ and $T\Delta_2$ to itself or it exchanges them.*

Proof. Fix a generator $a \in A$; thus $A_0 = \langle a^2 \rangle$. Since $\Delta \trianglelefteq S$ and A is characteristic in Δ, $A \trianglelefteq S$. For all $t \in T$, $[t, \Delta_0] = 1$ by assumption, so $[t, b] = 1$, and $[t, a^k] = 1$ if $|a^k| \leq 4$. Hence $tat^{-1} = a^{4j+1}$ for some j, so $[t, a] \in \langle a^4 \rangle$. Thus $[T, \Delta] \leq \langle a^4 \rangle$, and so $\langle T, a^4 \rangle \trianglelefteq S$.

(a) Assume $\varphi \in \mathrm{Hom}(\Delta, S)$ is such that $\varphi(Z) = Z$ and $\varphi(A) \trianglelefteq S$. Since $A_0 = [\Delta, \Delta]$, $\varphi(A_0) \leq [S, S] \leq TA_0$. If $\varphi(A_0) \leq \langle T, a^4 \rangle$, then there are distinct elements $a^i \neq a^j$ in A_0 such that $\varphi(a^i), \varphi(a^j) \in Ta^k$ for some k. Then $1 \neq a^{i-j}$ and $\varphi(a^{i-j}) \in T$, which is impossible since $Z \leq \langle a^{i-j} \rangle$ and $\varphi(Z) = Z \not\leq T$.

Thus $\varphi(a^2) \notin \langle T, a^4 \rangle$, and the image of $\varphi(a)$ in $S/\langle T, a^4 \rangle \cong D_8$ has order 4. Hence $\varphi(a) = ta^i$ for some $t \in T$ and some odd i. Also, the image of $\varphi(b)$ in $S/\langle T, a^4 \rangle$ must invert that of $\varphi(a)$, and so $\varphi(b) \in TAb$. Since $\varphi(A) \trianglelefteq S$, $b(ta^i)b^{-1} = ta^{-i}$, and $(ta^{-i})^{-1} ta^i = a^{2i}$, we have $a^{2i} \in \varphi(A)$. Thus $A_0 = \langle a^{2i} \rangle \leq \varphi(A)$, so $\varphi(A_0) = A_0$. This finishes the proof of (a).

(b) Now assume $\alpha \in \mathrm{Aut}(S)$ is such that $\alpha(Z) = Z$. By (a), applied with $\varphi = \alpha|_\Delta$, $\alpha(A) \le TA$ and $\alpha(A_0) = A_0$. Hence $\alpha(\langle a^4 \rangle) = \langle a^4 \rangle$, and α induces an automorphism $\bar\alpha$ of $S/\langle a^4\rangle \cong T \times D_8$ which sends $Z(\Delta/\langle a^4\rangle) = A_0/\langle a^4\rangle$ to itself. By Proposition B.1, $\bar\alpha(TA_0/\langle a^4\rangle) = TA_0/\langle a^4\rangle$, and thus $\alpha(TA_0) = TA_0$. Since $\alpha(TA) = TA$, it now follows that $\bar\alpha$ either sends the two subgroups $T\Delta_i/TA_0$ of $S/TA_0 \cong C_2^2$ to themselves $(i = 1, 2)$ or switches them. □

The next lemma involves a similar situation.

Lemma B.3. *Fix a finite 2-group S with a normal dihedral or quaternion subgroup $\Delta \trianglelefteq S$ of order ≥ 8. Assume two of the three subgroups of index two in Δ are S-conjugate. Let $\Delta_0 \le \Delta$ be a dihedral or quaternion subgroup of order 8, and set $T = C_S(\Delta_0)$. Let $A \trianglelefteq \Delta$ be the cyclic subgroup of index two: the one which is normal in S if $\Delta \cong Q_8$. Fix a generator $a \in A$, and choose $b \in \Delta_0 \smallsetminus A$. Let $Z \le A_0 < A$ be the subgroups of order two and index two, respectively. Then the following hold.*

(a) *$A \le [S, S]$.*

(b) *$[S{:}T\Delta] = 2$. For each $g \in S$, $gbg^{-1} = a^i b$ where i is even if $g \in T\Delta$ and i is odd if $g \in S \smallsetminus T\Delta$.*

(c) *$TA_0 \trianglelefteq S$, $S/TA_0 \cong D_8$, and $Z(S/TA_0) = TA/TA_0$.*

(d) *If T is abelian, then $[S, S]$ is abelian.*

(e) *Assume $b_1, b_2 \in S$ are in distinct cosets of TA, $[b_1, A] = A_0 = [b_2, A]$, $b_i^2 \in A$, and $\langle A, b_i\rangle \trianglelefteq S$. Then $\langle A, b_1, b_2\rangle$ is dihedral, semidihedral, or quaternion.*

(f) *Assume that Δ is dihedral and Z is a direct factor of T. Let $\alpha \in \mathrm{Aut}(S)$ be such that $\alpha^2 \in \mathrm{Inn}(S)$, $\alpha(Z) = Z$, and $\alpha(T\Delta) \ne T\Delta$. Set $\widehat\Delta = \alpha(\Delta){\cdot}\Delta$. Then $\widehat\Delta$ is dihedral, $\widehat\Delta \trianglelefteq S$, $[\widehat\Delta{:}\Delta] = 2$, $T \cap \widehat\Delta = Z$, and $T\widehat\Delta = S$.*

Proof. By assumption, $\Delta \trianglelefteq S$. Hence $A \trianglelefteq S$ since A is characteristic in Δ, except when $\Delta \cong Q_8$ in which case A was chosen to be the unique subgroup of index two in Δ normal in S.

(a) By assumption, the subgroups $\langle A_0, b\rangle$ and $\langle A_0, ba\rangle$ are S-conjugate. Hence there is $x \in S$ such that $c_x(bA_0) = baA_0$, so $xbx^{-1} = ba^i$ for some odd i, and $A = \langle a^i\rangle \le [S, S]$.

(b) By definition (and since $A = \langle a\rangle \trianglelefteq S$),

$$T = C_S(\Delta_0) = \bigl\{g \in S \bigm| gbg^{-1} = b,\ gag^{-1} = a^{4j+1} \text{ some } j\bigr\}. \tag{1}$$

Hence

$$T\Delta = \bigl\{g \in S \,|\, c_g \in \mathrm{Aut}_T(\Delta)\mathrm{Inn}(\Delta)\bigr\} = \bigl\{g \in S \bigm| gbg^{-1} = a^i b \text{ some } i \equiv 0 \pmod 2\bigr\}.$$

Since $\Delta \trianglelefteq S$, this proves that $gbg^{-1} = a^i b$ with i odd whenever $g \in S \smallsetminus T\Delta$. Also, $T\Delta < S$ since $\langle A_0, b\rangle$ and $\langle A_0, ab\rangle$ are S-conjugate, and hence $[S{:}T\Delta] = 2$.

(c) If $\Delta = \Delta_0$ has order 8, then $TA_0 = T = C_S(\Delta)$ is normal in S since $\Delta \trianglelefteq S$. So assume $|\Delta| > 8$. Since $A \trianglelefteq S$, $1 \ne \langle a^4\rangle \trianglelefteq S$. Also, $[T, A] \le \langle a^4\rangle$ by (1), and hence $T\Delta/\langle a^4\rangle \cong (T/Z)\times D_8$. For $x \in S$, $c_x(T\Delta) = T\Delta$ by (b), so $c_x(TA_0/\langle a^4\rangle) = TA_0/\langle a^4\rangle$ by Proposition B.1 applied with $\alpha = c_x \in \mathrm{Aut}(T\Delta/\langle a^4\rangle)$. Thus $TA_0 \trianglelefteq S$.

Throughout the rest of the proof of the lemma, we set $\overline{S} = S/TA_0$, and let $\overline{P} \le \overline{S}$ or $\bar g \in \overline{S}$ be the image of $P \le S$ or $g \in S$. Thus $\overline{T\Delta} = \langle \bar a, \bar b\rangle \cong C_2^2$.

For $x \in S \smallsetminus T\Delta$, $c_x(\bar{a}) = \bar{a}$ and $c_x(\bar{b}) = \overline{ab}$ by (b). Thus $\overline{S} \cong D_8$, with center $\langle \bar{a} \rangle = \overline{TA}$.

(d) Assume T is abelian. Since A is cyclic, $\mathrm{Aut}(A)$ is abelian, and hence $[S,S]$ is in the kernel of the map $S \longrightarrow \mathrm{Aut}(A)$ induced by conjugation. Thus $[S,S] \le C_S(A)$. Also, $[S,S] \le TA$ since TA is normal of index 4 in S. Since T is abelian, $C_{TA}(A)$ is also abelian, and so is $[S,S]$.

(e) Fix b_1, b_2 as above, and set $\Delta_i = \langle A, b_i \rangle \trianglelefteq S$ and $\widehat{\Delta} = \Delta_1\Delta_2$. Then $\widehat{\Delta}/A \cong C_2^2$ since the Δ_i are normal and contain A with index two, so $(b_1b_2)^2 \in A$. Also, $(\overline{b_1}\overline{b_2})^2 = \bar{a}$ in $\overline{S}$ since $\overline{b_1}$ and $\overline{b_2}$ are in distinct nonidentity cosets of $\overline{A} = Z(\overline{S})$ and have order 2 in $\overline{S}$, so $(b_1b_2)^2 = a^j$ for some odd j. Thus $\widehat{A} \overset{\text{def}}{=} \langle b_1b_2 \rangle$ is cyclic of index two in $\widehat{\Delta}$, conjugation by b_1 inverts the subgroup A of index two in $\widehat{A}$, and hence $\widehat{\Delta}$ is dihedral, semidihedral, or quaternion (cf. [G, Theorem 5.4.4]).

(f) Now assume that Δ is dihedral, and that $T = T_0Z$ where $T_0 \cap Z = 1$. Let $\alpha \in \mathrm{Aut}(S)$ be such that $\alpha^2 \in \mathrm{Inn}(S)$, $\alpha(Z) = Z$, and $\alpha(T\Delta) \neq T\Delta$. Set $\Delta^* = \alpha(\Delta)$, and $\widehat{\Delta} = \Delta\Delta^*$. Since Δ and Δ^* are both normal in S and are exchanged by α (since $\alpha^2 \in \mathrm{Inn}(S)$), $\widehat{\Delta}$ is normal in S.

Set $a_* = \alpha(a)$ and $b_* = \alpha(b)$. By (c), $TA_0 \trianglelefteq S$, and $\overline{S} = S/TA_0$ is dihedral of order 8 with center TA/TA_0.

Since $A \le [S,S]$ by (a), $\alpha(A) \le [S,S] \le TA$. If $b_* \in T\Delta$, then $\alpha(\Delta) \le T\Delta$, and by Lemma B.2(a) (applied with $\varphi = \alpha|_\Delta$ and T replaced by T_0), $b_* \in T_0Ab$. So $\overline{b_*} \in \{\bar{b}, \overline{ab}\}$. Since $[T,b] = 1$ (recall $b \in \Delta_0$), $[\overline{\alpha(T)}, \overline{b_*}] = 1$, so $\overline{\alpha(T)} \le \langle \bar{a}, \bar{b} \rangle = \overline{T\Delta}$. Thus $\alpha(T) \le T\Delta$, so $\alpha(T\Delta) = T\Delta$, and this contradicts our original assumption about α.

Thus $b_* \notin T\Delta$. Hence by (b), $b_*bb_*^{-1} = a^ib$ for some odd i. Set $\widehat{a} = b_*b$; then $\widehat{a}^2 = a^i$, and thus $\langle b_*, b \rangle = \langle \widehat{a}, b \rangle$ is a dihedral group which contains Δ with index two. Since $\alpha^2 \in \mathrm{Inn}(S)$, $\alpha(b_*) = \alpha^2(b) = a^jb$ for some j, and so $\alpha(\widehat{a}) = a^jbb_* = a^j(b_*b)^{-1} = a^j\widehat{a}^{-1}$ is in $\langle \widehat{a} \rangle$. So $a_* \in \langle a \rangle$, $\langle b_*, b \rangle = \Delta\Delta^*$ is dihedral, and it contains Δ with index two. Also, $T\widehat{\Delta} = S$ since $[S{:}T\Delta] = 2$ and $\widehat{\Delta} \not\le T\Delta$, and $T \cap \widehat{\Delta} = T \cap \Delta = Z$ since $T\Delta \cap \widehat{\Delta} = \Delta$. □

This will now be applied to prove the following lemma.

Lemma B.4. *Fix a finite 2-group S, and subgroups $T, \Delta_0 \le S$ such that Δ_0 is dihedral of order 8, $T \cap \Delta_0 = 1$, and $[T, \Delta_0] = 1$. Let $U, V \le \Delta_0$ be the two noncyclic subgroups of order four, and set $Z = U \cap V = Z(\Delta_0)$. Assume $N_S(TU) = T\Delta_0$, $N_S(\Delta_0) > T\Delta_0$, and either $Z \le Z(S)$ or T contains no subgroup isomorphic to D_8. Let $\Delta \trianglelefteq S$ be the normal closure of Δ_0 in S. Then Δ is dihedral, $T \cap \Delta = 1$, $[S{:}T\Delta] = 2$, and all noncentral involutions in Δ are S-conjugate.*

Proof. **Case 1:** Assume first that $Z \le Z(S)$. Set $\Delta_{-1} = U$. We will construct subgroups $\Delta_0 < \Delta_1 < \cdots < \Delta_m < S$, all normalized by T, such that $[S{:}T\Delta_m] = 2$, and such that for each $0 \le i \le m$,

(i) Δ_i is dihedral of order 2^{3+i};

(ii) $T \cap \Delta_i = 1$ and $T\Delta_i = N_S(T\Delta_{i-1}) = N_S(\Delta_{i-1})$; and

(iii) $N_S(\Delta_i) > T\Delta_i$.

To simplify notation, we set $S_i = T\Delta_i$ whenever Δ_i has been defined.

When $i = 0$, the only condition which is not immediate from the hypotheses is that $T\Delta_0 = N_S(U)$. One inclusion is clear: $N_S(U) \geq T\Delta_0$ since $[U,T] = 1$ and $U \trianglelefteq \Delta_0$. If $g \in N_{N_S(U)}(T\Delta_0)$, then since g normalizes U and $T\Delta_0$ and $C_{T\Delta_0}(U) = TU$, $g \in N_S(TU) = T\Delta_0$. Thus $N_{N_S(U)}(T\Delta_0) = T\Delta_0$, and so $N_S(U) = T\Delta_0$ by Lemma A.1.

Assume, for some $i \geq 0$, that we have constructed Δ_i which satisfies (i)–(iii). If $[S{:}S_i] = 2$ (recall $S_i = T\Delta_i$), then $\Delta_i \trianglelefteq S$ since $N_S(\Delta_i) > S_i$. For $g \in S \smallsetminus S_i$, $g \notin N_S(\Delta_{i-1}) = S_i$ implies that g exchanges the two conjugacy classes of noncentral involutions in Δ_i, and hence the noncentral involutions in Δ_i are all S-conjugate. If $i > 0$, then the normal closure of Δ_0 in Δ_i is Δ_{i-1} which is not normal in S, and thus Δ_i is the normal closure of Δ_0 in S (this is trivial if $i = 0$). So the subgroup $\Delta = \Delta_i$ satisfies the conditions in the statement of the lemma.

Now assume $[S{:}S_i] \geq 4$. Set $S_{i+1} = N_S(\Delta_i) > S_i$. Let Δ_{i-1} and Δ^*_{i-1} be the two dihedral subgroups of index two in Δ_i. Since $S_i = N_S(\Delta_{i-1})$ and $\Delta_i \trianglelefteq S_{i+1}$, conjugation by any element of $S_{i+1} \smallsetminus S_i$ exchanges Δ_{i-1} with Δ^*_{i-1}. The product of any two elements of $S_{i+1} \smallsetminus S_i$ thus lies in S_i, so $[S_{i+1}{:}S_i] = 2$, and $S_i \trianglelefteq S_{i+1} < S$.

For each $x \in N_S(S_i)$, since $c_x(Z) = Z$ by assumption ($Z \leq Z(S)$), Lemma B.2(b) implies that c_x either leaves $T\Delta_{i-1}$ and $T\Delta^*_{i-1}$ invariant or exchanges them; and leaves them invariant only if $x \in N_S(T\Delta_{i-1}) = S_i$. Thus $[N_S(S_i){:}S_i] = 2$. Since $S_i \trianglelefteq S_{i+1}$ with index two, this implies $N_S(S_i) = S_{i+1}$.

Choose any $g \in N_S(S_{i+1}) \smallsetminus S_{i+1}$ such that $g^2 \in S_{i+1}$, and set $\alpha = c_g \in \mathrm{Aut}(S_{i+1})$. Then $\alpha(Z) = Z$ since $Z \leq Z(S)$, and $\alpha(S_i) \neq S_i$ since $g \notin S_{i+1} = N_S(S_i)$. Also, $C_S(\Delta_0) = TZ$ since $N_S(U) = T\Delta_0$ by (ii) when $i = 0$ (and since $C_S(\Delta_0) \leq C_S(U) \leq N_S(U)$). The hypotheses of Lemma B.3(f) thus hold (but where T in Lemma B.3 corresponds to TZ here). So if we define $\Delta_{i+1} = \Delta_i{\cdot}\alpha(\Delta_i)$, then $\Delta_{i+1} \trianglelefteq S_{i+1}$ is dihedral, $[\Delta_{i+1}{:}\Delta_i] = 2$, $T \cap \Delta_{i+1} = 1$, and $T\Delta_{i+1} = S_{i+1}$. Thus (i) and (ii) hold, (iii) holds since $g \in N_S(\Delta_{i+1})$, and this finishes the induction step in the proof.

Case 2: Now assume $Z \nleq Z(S)$, and set $S_* = C_S(Z) < S$. By Case 1, S_* has the form described in the lemma: it contains a normal dihedral subgroup $\Delta \trianglelefteq S_*$ (the normal closure of Δ_0 in S_*), $T \cap \Delta = 1$, and $[S_*{:}T\Delta] = 2$. We prove that T contains a subgroup isomorphic to D_8, contradicting the assumptions.

Choose any $g \in N_S(S_*) \smallsetminus S_*$ such that $g^2 \in S_*$, and set $\beta = c_g \in \mathrm{Aut}(S_*)$. Then $\beta(Z) \neq Z$. Assume first that $\beta(\Delta) \nleq T\Delta$, and choose $x \in \beta(\Delta) \smallsetminus T\Delta$. Let $A \trianglelefteq \Delta$ be the cyclic subgroup of index two ($A \trianglelefteq S_*$), and choose $b \in U \smallsetminus Z$. (Recall $C_2^2 \cong U \leq \Delta_0$.) If $c_x(b) \in \Delta \smallsetminus A$ is Δ-conjugate to b, then $c_{ax}(U) = U$ for some $a \in A$, which is impossible since we showed in the proof of Case 1 that $N_S(U) = T\Delta_0$. Thus b is not Δ-conjugate to xbx^{-1}, so $[x,b]$ generates A, and hence $A \leq \beta(\Delta)$ since $x \in \beta(\Delta) \trianglelefteq S_*$. Since A is cyclic of order ≥ 4, this is possible only if $\beta(A) = A$, which is impossible since $\beta(Z) \neq Z$.

Thus $\beta(\Delta) \leq T\Delta$. Let $\psi \in \mathrm{Hom}(\Delta, T)$ be the composite $\Delta \xrightarrow{\ \beta\ } T\Delta \twoheadrightarrow T\Delta/\Delta \cong T$. Since $\beta(Z) \neq Z$ and $\beta(Z) \leq Z(T\Delta)$, $\beta(Z) \nleq \Delta$, and hence $Z \nleq \mathrm{Ker}(\psi)$. Since any nontrivial normal subgroup of Δ contains Z, this implies that ψ is injective, and thus that T contains a subgroup isomorphic to D_8. □

We also need the following corollary to Lemma B.4.

Lemma B.5. *Fix a finite 2-group S, and an abelian subgroup $P \le S$ such that $|N_S(P)/P| = 2$ and $P \ntrianglelefteq S$. Assume either*

(a) $\mathrm{Aut}_S(P) \nleq O_2(\mathrm{Aut}(P))$, *or*

(b) *there is a direct factor $U \le P$ such that $U \cong C_2^2$ and $1 \ne [N_S(P), U] \le U$.*

Then there are subgroups $\Delta_0 \le \Delta \trianglelefteq S$ such that Δ is dihedral, $\Delta_0 \cong D_8$, $C_S(\Delta_0) \le P$, and the noncentral involutions in Δ are all S-conjugate. In case (b), Δ can be taken to be the normal closure of U in S.

Proof. Set $\widehat{P} = N_S(P) < S$, fix $x \in N_S(\widehat{P}) \smallsetminus \widehat{P}$ such that $x^2 \in \widehat{P}$, and set $Q = xPx^{-1}$. Set $Z = [\widehat{P}, \widehat{P}]$. Since $\widehat{P}$ is nonabelian, and P, Q are distinct abelian subgroups of index two, Lemma A.6(c,a) implies that $|Z| = 2$, and $Z(\widehat{P}) = P \cap Q$ has index 2 in P and in Q. Also, $Z \le Z(\widehat{P})$.

We first show that (a) implies (b). Consider the subgroups

$$\Theta_1 = \bigl\{\alpha \in \mathrm{Aut}(P) \bigm| [\alpha, P] \le \mathrm{Fr}(P)\bigr\} \quad \text{and} \quad \Theta_2 = \bigl\{\alpha \in \mathrm{Aut}(P) \bigm| \alpha|_{\Omega_1(P)} = \mathrm{Id}\bigr\}.$$

Both are normal in $\mathrm{Aut}(P)$, and they are 2-subgroups by Lemma A.2 and [G, Theorem 5.2.4], respectively. Thus $\Theta_1\Theta_2 \le O_2(\mathrm{Aut}(P))$. Since $\mathrm{Aut}_{\widehat{P}}(P) = \mathrm{Aut}_S(P) \nleq O_2(\mathrm{Aut}(P))$ by assumption, $\mathrm{Aut}_{\widehat{P}}(P) \nleq \Theta_1\Theta_2$. Hence for $y \in \widehat{P} \smallsetminus P$, $Z = [y, P] \nleq \mathrm{Fr}(P)$ and $Z(\widehat{P}) = C_P(y) \ngeq \Omega_1(P)$.

Thus there are $T < Z(\widehat{P})$ and $g \in P \smallsetminus Z(\widehat{P})$ such that $|g| = 2$ and $P = Z \times T \times \langle g \rangle$ ($T = \mathrm{Ker}(f)$ for any $f\colon Z(\widehat{P}) \to Z(\widehat{P})/\mathrm{Fr}(Z(\widehat{P})) \to C_2$ with $Z \nleq \mathrm{Ker}(f)$). Set $U = \langle Z, g \rangle$. Then $U \cong C_2^2$, $P = TU$, and $T \cap U = 1$. So (b) holds.

Now assume (b). Thus $P = TU$ where $U \cong C_2^2$, $T \cap U = 1$, and $Z = [\widehat{P}, \widehat{P}] \le U$. By the argument used in the last paragraph, we can assume that $T < Z(\widehat{P})$.

Set $V = xUx^{-1}$ and $\Delta_0 = UV$. Then $\Delta_0 \cong D_8$ since $[U, V] = Z$ and $U \cong V \cong C_2^2$, $T \cap \Delta_0 = T \cap Z = 1$, and $[T, \Delta_0] \le [T, \widehat{P}] = 1$ since $T \le Z(\widehat{P})$. Also, $N_S(TU) = N_S(P) = \widehat{P} = T\Delta_0$, and $N_S(\Delta_0) > T\Delta_0$ since $x \in N_S(\Delta_0) \smallsetminus T\Delta_0$. Since T is abelian, it contains no subgroup isomorphic to D_8. So by Lemma B.4, the normal closure Δ of Δ_0 in S is dihedral, and all noncentral involutions in Δ are S-conjugate.

Set $R = N_{C_S(\Delta_0)}(Z(\widehat{P}))$. Then R normalizes $Z(\widehat{P})U = P$, so $R \le N_S(P) = \widehat{P}$. Hence $R \le C_{\widehat{P}}(\Delta_0) = Z(\widehat{P})$, so by Lemma A.1, $C_S(\Delta_0) = Z(\widehat{P}) \le P$. □

References

[AOV] K. Andersen, B. Oliver, & J. Ventura, Reduced, tame, and exotic fusion systems, Proc. London Math. Soc. (to appear)

[A1] M. Aschbacher, Finite group theory, Cambridge Univ. Press (1986)

[A2] M. Aschbacher, A 2-local characterization of M_{12}, Illinois J. Math. 47 (2003), 31–47

[AKO] M. Aschbacher, R. Kessar, & B. Oliver, Fusion systems in algebra and topology, Cambridge Univ. Press (to appear)

[BLO1] C. Broto, R. Levi, & B. Oliver, Homotopy equivalences of p-completed classifying spaces of finite groups, *Invent. Math.*, **151** (2003), 611–664.

[Bd] H. Bender, Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festlässt, J. Algebra 17 (1971), 527–554

[5a2] C. Broto, N. Castellana, J. Grodal, R. Levi, & B. Oliver, Extensions of p-local finite groups, Trans. Amer. Math. Soc. 359 (2007), 3791–3858

[BMO] C. Broto, J. Møller, & B. Oliver, Equivalences between fusion systems of finite groups of Lie type, Jour. Amer. Math. Soc. (to appear)
[Bu] W. Burnside, The theory of groups of finite order, Cambridge Univ. Press (1897)
[CE] H. Cartan & S. Eilenberg, Homological algebra, Princeton Univ. Press (1956)
[Car] R. Carter, Simple groups of Lie type, Wiley (1972)
[CF] R. Carter & P. Fong, The Sylow 2-subgroups of the finite classical groups, J. Algebra 1 (1964), 139–151
[Fn] P. Fan, Amalgams of prime index, J. Algebra 98 (1986), 375–421
[Gd1] D. Goldschmidt, A conjugation family for finite groups, J. Algebra 16 (1970), 138–142
[Gd2] D. Goldschmidt, Automorphisms of trivalent graphs, Annals of Math. 111 (1980), 377–406
[G] D. Gorenstein, Finite groups, Harper & Row (1968)
[GH] D. Gorenstein & K. Harada, Finite groups whose 2-subgroups are generated by at most 4 elements, Memoirs Amer. Math. Soc. 147 (1974)
[Gr] R. Griess, Schur multipliers of finite simple groups of Lie type, Trans. Amer. Math. Soc. 183 (1973), 355–421
[Hu] B. Huppert, Endliche Gruppen I, Springer-Verlag (1967)
[Mz] P. Mazet, Sur les multiplicateurs de Schur des groupes de Mathieu, J. Algebra 77 (1982), 552–576
[O1] B. Oliver, Splitting fusion systems over 2-groups, proceedings of the 2009 Skye conference on algebraic topology, group theory and representation theory, Proc. Edinburgh Math. Soc. (to appear).
[OV] B. Oliver & J. Ventura, Saturated fusion systems over 2-groups, Trans. Amer. Math. Soc. 361 (2009), 6661–6728.
[Pg1] L. Puig, The hyperfocal subalgebra of a block, Invent. math. 141 (2000), 365–397
[Pg2] L. Puig, Frobenius categories, J. Algebra 303 (2006), 309–357
[Sch] J. Schur, Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, J. reine angew. Math. 127 (1904), 20–50
[Sz1] M. Suzuki, Group theory I, Springer-Verlag (1982)
[Sz2] M. Suzuki, Group theory II, Springer-Verlag (1986)
[Ta] D. Taylor, The geometry of the classical groups, Heldermann (1992)
[Wi] R. Wilson, The geometry and maximal subgroups of the simple groups of A. Rudvalis and J. Tits, Proc. London Math. Soc. 48 (1984), 533–563

Center for Mathematical Sciences, LTH, Box 118, 22100 Lund, Sweden

E-mail address: kksa@maths.lth.se

LAGA, Institut Galilée, Av. J-B Clément, 93430 Villetaneuse, France

E-mail address: bobol@math.univ-paris13.fr

Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049–001 Lisboa, Portugal

E-mail address: jventura@math.ist.utl.pt