REDUCED, TAME, AND EXOTIC FUSION SYSTEMS

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Abstract. We define here two new classes of saturated fusion systems, reduced fusion systems and tame fusion systems. These are motivated by our attempts to better understand and search for exotic fusion systems: fusion systems which are not the fusion systems of any finite group. Our main theorems say that every saturated fusion system reduces to a reduced fusion system which is tame only if the original one is realizable, and that every reduced fusion system which is not tame is the reduction of some exotic (nonrealizable) fusion system.

When G is a finite group and $S \in \mathrm{Syl}_p(G)$, the fusion category of G is the category $\mathcal{F}_S(G)$ whose objects consist of all subgroups of S, and where

$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q) \stackrel{\mathrm{def}}{=} \{c_g \in \mathrm{Hom}(P,Q) \,|\, g \in G,\ gPg^{-1} \le Q\} .$$

This provides a means of encoding the *p-local structure* of G: the conjugacy relations among the subgroups of the Sylow p-subgroup S. An abstract "saturated fusion system" over a finite p-group S is a category whose objects are the subgroups of S, whose morphisms are certain monomorphisms of groups between the subgroups, and which satisfies certain conditions formulated by Puig [Pg2] and stated here in Definition 1.1. In particular, for any finite G as above, $\mathcal{F}_S(G)$ is a saturated fusion system. A saturated fusion system is called *realizable* if it is isomorphic to the fusion system of some finite group G, and is called *exotic* otherwise.

It turns out to be very difficult in general to construct exotic fusion systems, especially over 2-groups. This says something about how close Puig's definition is to the properties of fusion systems of finite groups.

This paper is centered around the problem of identifying exotic fusion systems. A first step towards doing this was taken in [OV2], where two of the authors developed methods for listing saturated fusion systems over any given 2-group. However, it quickly became clear that in order to have any chance of making a systematic search through all 2-groups (or p-groups) of a given type, one must first find a way to limit the types of fusion systems under consideration, and do so without missing any possible exotic ones.

This leads to the concept of a *reduced fusion system.* A saturated fusion system is reduced if it contains no nontrivial normal p-subgroups, and also contains no proper normal subsystems of p-power index or of index prime to p. These last concepts will be defined precisely in Definitions 1.2 and 1.21; for now, we just remark that they are analogous to requiring a finite group to have no nontrivial normal p-subgroups and no proper normal subgroups of p-power index or of index prime to p. Thus it is very far from requiring that the fusion system be simple in any sense, but it is adequate for our purposes.

The second concept which plays a central role in our results is that of a *tame fusion system.* Roughly, a fusion system $\mathcal{F}$ is tame if it is realized by a finite group G for which all

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automorphisms of $\mathcal{F}$ are induced by automorphisms of G. The precise (algebraic) definition is given in Definition 2.5. In terms of classifying spaces, $\mathcal{F}$ is tame if it is realized by a finite group G such that the natural map from $\mathrm{Out}(G)$ to $\mathrm{Out}(BG^\wedge_p)$ is split surjective, where $\mathrm{Out}(BG^\wedge_p)$ is the group of homotopy classes of self homotopy equivalences of the space $BG^\wedge_p$.

For any saturated fusion system $\mathcal{F}$ over a finite p-group S, there is a canonical reduction $\mathfrak{red}(\mathcal{F})$ of $\mathcal{F}$ (Definition 2.1). The analogy for a finite group G with maximal normal p-subgroup Q would be to set $G_0 = C_G(Q)/Q$, and then let $\mathfrak{red}(G) \trianglelefteq G_0$ be the smallest normal subgroup such that $G_0/\mathfrak{red}(G)$ is p-solvable. Our first main theorem is the following.

Theorem A. *For any saturated fusion system $\mathcal{F}$ over a finite p-group S, if $\mathfrak{red}(\mathcal{F})$ is tame, then $\mathcal{F}$ is also tame, and in particular $\mathcal{F}$ is realizable.*

Thus Theorem A says that reduced fusion systems detect all possible exotic fusion systems. If one wants to find all exotic fusion systems over p-groups of order $\le p^k$ for some p and k, then one first searches for all reduced fusion systems over p-groups of order $\le p^k$ which are not tame, and then for all other fusion systems which reduce to them.

The proof of Theorem A uses the uniqueness of linking systems associated to the fusion system of a finite group, and through that depends on the classification of finite simple groups. In order to make it clear exactly which part of the result depends on the classification theorem and which part is independent, we introduce another (more technical) concept, that of "strongly tame" fusion systems (Definition 2.9). Using the classification, together with results in [O1] and [O2], we prove that all tame fusion systems are strongly tame (Theorem 2.10). (In fact, the definition of "strongly tame" is such that any tame fusion system which we're ever likely to be working with can be shown to be strongly tame without using the classification.) Independently of that, and without using the classification theorem, we prove in Theorem 2.20 that $\mathcal{F}$ is tame whenever $\mathfrak{red}(\mathcal{F})$ is strongly tame; and this together with Theorem 2.10 imply Theorem A.

Alternatively, one can also avoid using the classification theorem by restating Theorem A in terms of fusion systems together with associated linking systems.

Albert Ruiz has constructed examples [Rz] which show that the reduction of a tame fusion system need not be tame, and in fact, can be exotic. So there is no equivalence between the tameness of $\mathcal{F}$ and tameness of $\mathfrak{red}(\mathcal{F})$. The next theorem does, however, provide a weaker converse to Theorem A, by saying that for every non-tame reduced fusion system, there is some associated exotic fusion system in the background.

Theorem B. *Let $\mathcal{F}$ be a reduced fusion system which is not tame. Then there is an exotic fusion system whose reduction is isomorphic to $\mathcal{F}$.*

As remarked above, reduced fusion systems can be very far from being simple in any sense. For example, a product of reduced fusion systems is always reduced (Proposition 3.4). The next theorem handles reduced fusion systems which factor as products.

Theorem C. *Each reduced fusion system $\mathcal{F}$ over a finite p-group S has a unique maximal factorization $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_m$ as a product of indecomposable fusion systems $\mathcal{F}_i$ over subgroups $S_i \trianglelefteq S$. If $\mathcal{F}_i$ is tame for each i, then $\mathcal{F}$ is tame.*

Here, "unique" means that the indecomposable subsystems are unique as subcategories, not only up to isomorphism. By Theorem C, in order to find minimal reduced fusion systems which are not tame, it suffices to look at those which are indecomposable. In practice, it seems that any reduced indecomposable fusion system which is not simple (which has no proper normal fusion subsystems in the sense of Definition 1.18 or of [Asch, § 6]) has to be over a p-group of very large order. The smallest example of this type we know of is the fusion system of $A_6 \wr A_5$, over a group of order 2^{17}.

Using these results and those in [OV2] as starting point, we have started to undertake a systematic computer search for reduced fusion systems over small 2-groups. So far, while details have yet to be rechecked carefully, we seem to have shown that each reduced fusion system over a 2-group of order ≤ 512 is the fusion system of a finite simple group, and is tame. We hope to be able to extend this soon to 2-groups of larger order.

What we really would like to find is an example of a realizable fusion system which is not tame. It seems very likely that such a fusion system exists, but so far, our attempts to find one have been unsuccessful.

The theorems stated above will all be proven in Sections 2 and 3: Theorems A and B as Theorems 2.20 and 2.6, and Theorem C as Proposition 3.6 and Theorem 3.7. They are preceded by a first section containing mostly background definitions and results, and are followed by a fourth section with examples of how to prove certain fusion systems are tame.

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Contents

1. Fusion and linking systems

We first collect the basic results about fusion and linking systems and their automorphisms which will be needed in the rest of the paper. Most of this is taken directly from earlier papers, such as [BLO2], [BCGLO1], [BCGLO2] and [O3].

1.1. Background on fusion systems.

We first recall very briefly the definition of a saturated fusion system, in the form given in [BLO2]. In general, for any group G and any pair of subgroups $H, K \le G$, $\mathrm{Hom}_G(H,K)$ denotes the set of all homomorphisms from H to K induced by conjugation by some element of G. When G is finite and $S \in \mathrm{Syl}_p(G)$, $\mathcal{F}_S(G)$ (the *fusion category* of G) is the category whose objects are the subgroups of S, and where for each pair of objects $\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)$.

A *fusion system* over a finite p-group S is a category $\mathcal{F}$, where $\mathrm{Ob}(\mathcal{F})$ is the set of all subgroups of S, such that for all $P, Q \le S$,

$$\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q);$$

and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ is the composite of an isomorphism in $\mathcal{F}$ followed by an inclusion. Here, $\mathrm{Inj}(P,Q)$ denotes the set of injective homomorphisms from P to Q. If $\mathcal{F}$ is a fusion system over a finite p-subgroup S, then two subgroups $P, Q \le S$ are *$\mathcal{F}$-conjugate* if they are isomorphic as objects of the category $\mathcal{F}$.

Definition 1.1 ([Pg2], see [BLO2, Definition 1.2])**.** *Let $\mathcal{F}$ be a fusion system over a finite p-group S.*

- *A subgroup $P \le S$ is* fully centralized in $\mathcal{F}$ *if $|C_S(P)| \ge |C_S(P^*)|$ for each $P^* \le S$ which is $\mathcal{F}$-conjugate to P.*
- *A subgroup $P \le S$ is* fully normalized in $\mathcal{F}$ *if $|N_S(P)| \ge |N_S(P^*)|$ for each $P^* \le S$ which is $\mathcal{F}$-conjugate to P.*
- *$\mathcal{F}$ is a* saturated fusion system *if the following two conditions hold:*
  - (I) (Sylow axiom) *For each $P \le S$ which is fully normalized in $\mathcal{F}$, P is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.*
  - (II) (Extension axiom) *If $P \le S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ are such that $\varphi(P)$ is fully centralized in $\mathcal{F}$, and if we set*

    $$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\},$$

    then there is $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar\varphi|_P = \varphi$.

If G is a finite group and $S \in \mathrm{Syl}_p(G)$, then the category $\mathcal{F}_S(G)$ is a saturated fusion system (cf. [BLO2, Proposition 1.3]).

We now list some classes of subgroups of S which play an important role when working with fusion systems over S. Here and elsewhere, for any fusion system $\mathcal{F}$ over a finite p-group S, we write for each $P \le S$,

$$\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P) \le \mathrm{Out}(P)\ .$$

Definition 1.2. *Fix a prime p, a finite p-group S, and a fusion system $\mathcal{F}$ over S. Let $P \le S$ be any subgroup.*

- *P is $\mathcal{F}$-*centric *if $C_S(P^*) = Z(P^*)$ for each P^* which is $\mathcal{F}$-conjugate to P.*

- *P is* $\mathcal{F}$-radical *if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$; i.e., if $\mathrm{Out}_{\mathcal{F}}(P)$ contains no nontrivial normal p-subgroups.*
- *P is* central in $\mathcal{F}$ *if $P \trianglelefteq S$, and every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ in $\mathcal{F}$ extends to a morphism $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ,PR)$ such that $\bar\varphi|_P = \mathrm{Id}_P$.*
- *P is* normal in $\mathcal{F}$ *($P \trianglelefteq \mathcal{F}$) if $P \trianglelefteq S$, and every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ in $\mathcal{F}$ extends to a morphism $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ,PR)$ such that $\bar\varphi(P) = P$.*
- *P is* strongly closed in $\mathcal{F}$ *if no element of P is $\mathcal{F}$-conjugate to an element of $S \smallsetminus P$.*
- *$Z(\mathcal{F}) \le Z(S)$ and $O_p(\mathcal{F}) \le S$ denote the largest subgroups of S which are central in $\mathcal{F}$ and normal in $\mathcal{F}$, respectively.*

It follows directly from the definitions that if P_1 and P_2 are both central (normal) in $\mathcal{F}$, then so is P_1P_2. This is why there always is a largest central subgroup $Z(\mathcal{F})$, and a largest normal subgroup $O_p(\mathcal{F})$.

Several forms of Alperin's fusion theorem have been shown for saturated fusion systems, starting with Puig in [Pg2, § 5]. The following version suffices for what we need in most of this paper. A stronger version will be given in Theorem 4.1.

Theorem 1.3 ([BLO2, Theorem A.10])**.** *For any saturated fusion system $\mathcal{F}$ over a finite p-group S, each morphism in $\mathcal{F}$ is a composite of restrictions of automorphisms in $\mathrm{Aut}_{\mathcal{F}}(P)$, for subgroups P which are fully normalized in $\mathcal{F}$, $\mathcal{F}$-centric, and $\mathcal{F}$-radical.*

The following elementary result is useful for identifying subgroups which are centric and radical in a fusion system.

Lemma 1.4. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. If $P \le S$ is $\mathcal{F}$-centric and $\mathcal{F}$-radical, then there is no $g \in N_S(P) \smallsetminus P$ such that $c_g \in O_p(\mathrm{Aut}_{\mathcal{F}}(P))$. Conversely, if $P \le S$ is fully normalized in $\mathcal{F}$, and there is no $g \in N_S(P) \smallsetminus P$ such that $c_g \in O_p(\mathrm{Aut}_{\mathcal{F}}(P))$, then P is $\mathcal{F}$-centric and $\mathcal{F}$-radical.*

Proof. Assume P is $\mathcal{F}$-centric and $\mathcal{F}$-radical. Fix $g \in N_S(P)$ such that $c_g \in O_p(\mathrm{Aut}_{\mathcal{F}}(P))$. Then $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$ since P is $\mathcal{F}$-radical, so $c_g \in \mathrm{Inn}(P)$, and $g \in P{\cdot}C_S(P) = P$ since P is $\mathcal{F}$-centric. This proves the first statement.

Now assume P is fully normalized in $\mathcal{F}$. If P is not $\mathcal{F}$-centric, then $C_S(P) \not\le P$ (since P is fully centralized), and hence there is $g \in N_S(P) \smallsetminus P$ with $c_g = 1$. If P is not $\mathcal{F}$-radical, then $O_p(\mathrm{Out}_{\mathcal{F}}(P)) \ne 1$. This subgroup is contained in each Sylow p-subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$, and in particular is contained in $\mathrm{Out}_S(P)$. Thus each nontrivial element of $O_p(\mathrm{Out}_{\mathcal{F}}(P))$ is induced by conjugation by some element of $N_S(P) \smallsetminus P$. □

Proposition 1.5. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. For any normal subgroup $Q \trianglelefteq S$, Q is normal in $\mathcal{F}$ if and only if Q is strongly closed and contained in all subgroups which are centric and radical in $\mathcal{F}$.*

Proof. This is shown in [BCGLO1, Proposition 1.6]. Note, however, that wherever "$\mathcal{F}$-radical" appears in the statement and proof of that proposition, it should be replaced by "$\mathcal{F}$-centric and $\mathcal{F}$-radical". □

Lemma 1.4 shows the importance of being able to identify elements of the subgroup $O_p(\mathrm{Aut}_{\mathcal{F}}(P))$. The following, very well known property of automorphisms of p-groups is useful in many cases when doing this.

Lemma 1.6. *Fix a prime p, a finite p-group P, and a group $\mathcal{A} \le \mathrm{Aut}(P)$ of automorphisms of P. Assume $1 = P_0 \trianglelefteq P_1 \trianglelefteq \cdots \trianglelefteq P_m = P$ is a sequence of normal subgroups such that $\alpha(P_i) = P_i$ for all $\alpha \in \mathcal{A}$ and all i. For $1 \le i \le m$, let $\Psi_i\colon \mathcal{A} \longrightarrow \mathrm{Aut}(P_i/P_{i-1})$ be the homomorphism which sends $\alpha \in \mathcal{A}$ to the induced automorphism of P_i/P_{i-1}. Then for all $\alpha \in \mathcal{A}$, $\alpha \in O_p(\mathcal{A})$ if and only if $\Psi_i(\alpha) \in O_p(\Psi_i(\mathcal{A}))$ for all $i = 1, \ldots, m$.*

Proof. Set $\Psi = (\Psi_1, \ldots, \Psi_m)$, as a homomorphism from $\mathcal{A}$ to $\prod_{i=1}^m \mathrm{Aut}(P_i/P_{i-1})$. Then $\mathrm{Ker}(\Psi)$ is a p-group (cf. [G, Corollary 5.3.3]). If $\Psi_i(\alpha) \in O_p(\Psi_i(\mathcal{A}))$ for all $i = 1, \ldots, m$, then $\Psi(\alpha) \in O_p(\Psi(\mathcal{A}))$, and so $\alpha \in O_p(\mathcal{A})$. Conversely, if $\alpha \in O_p(\mathcal{A})$, then clearly $\Psi_i(\alpha) \in O_p(\Psi_i(\mathcal{A}))$ for all i. □

Another elementary lemma which is frequently useful when working with centric and radical subgroups is the following:

Lemma 1.7. *Let P and Q be p-subgroups of a finite group G such that $P \le N_G(Q)$ and $Q \not\le P$. Then $N_{QP}(P) \gneqq P$, and $(Q \cap N_G(P)) \not\le P$.*

Proof. Since P normalizes Q, QP is also a p-group, and $QP \gneqq P$ by assumption. Hence $N_{QP}(P) \gneqq P$ (cf. [Sz1, Theorem 2.1.6]). Since $N_{QP}(P) = P{\cdot}(Q \cap N_{QP}(P))$, we have $(Q \cap N_{QP}(P)) \not\le P$. □

We also need to work with certain quotient fusion systems. When $\mathcal{F}$ is a saturated fusion system over S and $Q \trianglelefteq S$ is strongly closed in $\mathcal{F}$, we define the quotient fusion system $\mathcal{F}/Q$ over S/Q by setting

$$\mathrm{Hom}_{\mathcal{F}/Q}(P/Q, R/Q) = \mathrm{Im}\big[\mathrm{Hom}_{\mathcal{F}}(P, R) \longrightarrow \mathrm{Hom}(P/Q, R/Q)\big]$$

for all $P, R \le S$ containing Q.

Proposition 1.8. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $Q \trianglelefteq S$ be a strongly closed subgroup. Then $\mathcal{F}/Q$ is a saturated fusion system. For each $P \le S$ containing Q, P is fully normalized in $\mathcal{F}$ if and only if P/Q is fully normalized in $\mathcal{F}/Q$. If Q is central in $\mathcal{F}$, then $P \trianglelefteq \mathcal{F}$ if and only if $P/Q \trianglelefteq \mathcal{F}/Q$.*

Proof. By [O1, Lemma 2.6], $\mathcal{F}/Q$ is a saturated fusion system, and P is fully normalized if and only if P/Q is. So it remains only to prove the last statement.

By Proposition 1.5, P is normal in $\mathcal{F}$ if and only if it is strongly closed in $\mathcal{F}$, and contained in each subgroup which is $\mathcal{F}$-centric and $\mathcal{F}$-radical. For $P \le S$ containing Q, clearly P is strongly closed in $\mathcal{F}$ if and only if P/Q is strongly closed in $\mathcal{F}/Q$.

We apply the criterion in Lemma 1.4 for detecting subgroups which are centric and radical. Let $\rho\colon \mathrm{Aut}_{\mathcal{F}}(P) \longrightarrow \mathrm{Aut}_{\mathcal{F}/Q}(P/Q)$ be the homomorphism induced by projection. Then ρ is surjective by definition of $\mathcal{F}/Q$. For $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$, we have $\alpha|_Q = \mathrm{Id}_Q$ since Q is central in $\mathcal{F}$, so by Lemma 1.6, $\alpha \in O_p(\mathrm{Aut}_{\mathcal{F}}(P))$ if and only if $\rho(\alpha) \in O_p(\mathrm{Aut}_{\mathcal{F}/Q}(P/Q))$.

If $Q \le R \le S$, and R^* is $\mathcal{F}$-conjugate to R and fully normalized in $\mathcal{F}$, then R^*/Q is $\mathcal{F}/Q$-conjugate to R/Q and fully normalized in $\mathcal{F}/Q$. So by Lemma 1.4, R and R^* are centric and radical in $\mathcal{F}$ if and only if R/Q and R^*/Q are centric and radical in $\mathcal{F}/Q$. Upon combining this with the above criterion for normality, we see that $P \trianglelefteq \mathcal{F}$ if and only if $P/Q \trianglelefteq \mathcal{F}/Q$. □

1.2. **Background on linking systems.**

We next define abstract linking systems associated to a fusion system $\mathcal{F}$. We use the definition given in [O3], which is more flexible in the choice of objects than the earlier definitions in [BLO2] and [BCGLO1]. This definition also differs slightly from the one given in [BCGLO1, Definition 3.3], in that we include a choice of inclusion morphisms as part of the data in the linking system. All of these definitions are, however, equivalent, aside from having greater freedom in the choice of objects.

For any finite group G and any $S \in \mathrm{Syl}_p(G)$, let $\mathcal{T}_S(G)$ denote the *transporter category* of G: the category whose objects are the subgroups of S, and where for all $P, Q \le S$,

$$\mathrm{Mor}_{\mathcal{T}_S(G)}(P,Q) = N_G(P,Q) \stackrel{\mathrm{def}}{=} \{ g \in G \,|\, gPg^{-1} \le Q \} \ .$$

If $\mathcal{H}$ is a set of subgroups of S, then $\mathcal{T}_{\mathcal{H}}(G) \subseteq \mathcal{T}_S(G)$ denotes the full subcategory with object set $\mathcal{H}$.

Definition 1.9 ([O3, Definition 3])**.** *Let $\mathcal{F}$ be a fusion system over a finite p-group S. A* linking system *associated to $\mathcal{F}$ is a finite category $\mathcal{L}$, together with a pair of functors*

$$\mathcal{T}_{\mathrm{Ob}(\mathcal{L})}(S) \xrightarrow{\ \delta\ } \mathcal{L} \xrightarrow{\ \pi\ } \mathcal{F} \,,$$

satisfying the following conditions:

(A) $\mathrm{Ob}(\mathcal{L})$ *is a set of subgroups of S closed under $\mathcal{F}$-conjugacy and overgroups, and includes all subgroups which are $\mathcal{F}$-centric and $\mathcal{F}$-radical. Each object in $\mathcal{L}$ is isomorphic (in $\mathcal{L}$) to one which is fully centralized in $\mathcal{F}$. Also, δ is the identity on objects, and π is the inclusion on objects. For each $P, Q \in \mathrm{Ob}(\mathcal{L})$ such that P is fully centralized in $\mathcal{F}$, $C_S(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ via δ_P and right composition, and $\pi_{P,Q}$ induces a bijection*

$$\mathrm{Mor}_{\mathcal{L}}(P,Q)/C_S(P) \xrightarrow{\ \cong\ } \mathrm{Hom}_{\mathcal{F}}(P,Q) \ .$$

(B) *For each $P, Q \in \mathrm{Ob}(\mathcal{L})$ and each $g \in N_S(P,Q)$, $\pi_{P,Q}$ sends $\delta_{P,Q}(g) \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ to $c_g \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$.*

(C) *For all $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ and all $g \in P$, the diagram*

$$\begin{array}{ccc} P & \xrightarrow{\ \psi\ } & Q \\ {\scriptstyle \delta_P(g)}\big\downarrow & & \big\downarrow{\scriptstyle \delta_Q(\pi(\psi)(g))} \\ P & \xrightarrow{\ \psi\ } & Q \end{array}$$

commutes in $\mathcal{L}$.

If $\mathcal{L}^*$ is another linking system associated to $\mathcal{F}$ with the same set of objects as $\mathcal{L}$, then an isomorphism *of linking systems is an isomorphism of categories $\mathcal{L} \xrightarrow{\cong} \mathcal{L}^*$ which commutes with the structural functors: those coming from $\mathcal{T}_{\mathrm{Ob}(\mathcal{L})}(S)$ and those going to $\mathcal{F}$.*

Note that we do not assume in this definition that $\mathcal{F}$ is saturated, since we want to at least be able to talk about linking systems associated to $\mathcal{F}$ without first proving $\mathcal{F}$ is saturated. This leads to some pretty exotic examples; for example, a linking system could be empty. In practice, however, we only work with linking systems associated to saturated fusion systems.

A *p-local finite group* is defined to be a triple $(S, \mathcal{F}, \mathcal{L})$, where $\mathcal{F}$ is a saturated fusion system over a finite p-group S, and where $\mathcal{L}$ is a *centric linking system* associated to $\mathcal{F}$ (i.e., one whose objects are the $\mathcal{F}$-centric subgroups of S).

For $P \le Q$ in $\mathrm{Ob}(\mathcal{L})$, we usually write $\iota_P^Q = \delta_{P,Q}(1)$, and regard this as the "inclusion" of P into Q. The definition in [BCGLO1, Definition 3.3] of a (quasicentric) linking system does not include these inclusions, but it is explained there how to choose inclusions in a way so that a functor $\delta\colon \mathcal{T}_{\mathrm{Ob}(\mathcal{L})}(S) \longrightarrow \mathcal{L}$ can be defined in a unique way with the above properties ([BCGLO1, Lemma 3.7]). Note also that because the above definition includes a choice of inclusions, condition $(\mathrm{D})_q$ in [BCGLO1, Definition 3.3] is not needed here.

We have defined linking systems to be as flexible as possible in the choice of objects, but one cannot avoid completely discussing quasicentric subgroups in this context. First recall the definition of the centralizer fusion system (cf. [BLO2, § 2] or [AKO, § I.5]): if $\mathcal{F}$ is a fusion system over a finite p-group S and $Q \le S$, then $C_{\mathcal{F}}(Q)$ is the fusion system over $C_S(Q)$ for which

$$\mathrm{Hom}_{C_{\mathcal{F}}(Q)}(P,R) = \bigl\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,R) \bigm| \varphi = \bar\varphi|_P,\ \bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ,RQ),\ \bar\varphi|_Q = \mathrm{Id}_Q\bigr\}.$$

This is a special case of the normalizer fusion systems which will be defined in Section 1.4. If $\mathcal{F}$ is saturated and Q is fully centralized in $\mathcal{F}$, then $C_{\mathcal{F}}(Q)$ is also saturated (cf. [AKO, Theorem I.5.5]).

Definition 1.10. (a) *For any finite group G, a p-subgroup $P \le G$ is G-*quasicentric *if $O^p(C_G(P))$ has order prime to p.*

(b) *For any saturated fusion system $\mathcal{F}$ over a finite p-group S, a subgroup $P \le S$ is $\mathcal{F}$-*quasicentric *if for each P^* which is fully centralized in $\mathcal{F}$ and $\mathcal{F}$-conjugate to P, $C_{\mathcal{F}}(P^*)$ is the fusion system of the p-group $C_S(P^*)$. Equivalently, for each such P^* and each $Q \le P^*{\cdot}C_S(P^*)$ containing P^*, $\{\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q) \mid \alpha|_{P^*} = \mathrm{Id}\}$ is a p-group.*

The equivalence of the two definitions in (b) is shown in [AKO, Lemma III.4.6(a)].

For any saturated fusion system $\mathcal{F}$, the set of $\mathcal{F}$-quasicentric subgroups is closed under $\mathcal{F}$-conjugacy and overgroups (see [AKO, Lemma III.4.6(d)]). So a quasicentric linking system as defined in [BCGLO1, § 3] is a linking system in the sense defined here.

Fix a finite group G and $S \in \mathrm{Syl}_p(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. Then a subgroup $P \le S$ is G-quasicentric if and only if it is $\mathcal{F}$-quasicentric (cf. [AKO, Lemma III.4.6(e)]). For any set $\mathcal{H}$ of G-quasicentric subgroups of S, define $\mathcal{L}_S^{\mathcal{H}}(G)$ to be the category with object set $\mathcal{H}$, and where for each $P, Q \in \mathcal{H}$,

$$\mathrm{Mor}_{\mathcal{L}_S^{\mathcal{H}}(G)}(P,Q) = N_G(P,Q)/O^p(C_G(P)).$$

Composition is well defined, since for each $g \in N_G(P,Q)$, $g^{-1}Qg \ge P$, so $g^{-1}C_G(Q)g \le C_G(P)$, and thus $g^{-1}O^p(C_G(Q))g \le O^p(C_G(P))$. When $\mathcal{H}$ is closed under $\mathcal{F}$-conjugacy and overgroups and contains all subgroups of S which are $\mathcal{F}$-centric and $\mathcal{F}$-radical, then $\mathcal{L}_S^{\mathcal{H}}(G)$ is a linking system associated to $\mathcal{F}$. When $\mathcal{H}$ is the set of $\mathcal{F}$-centric subgroups of S, we write $\mathcal{L}_S^c(G) = \mathcal{L}_S^{\mathcal{H}}(G)$.

Proposition 1.11. *The following hold for any linking system $\mathcal{L}$ associated to a saturated fusion system $\mathcal{F}$ over a finite p-group S.*

(a) *For each $P, Q \in \mathrm{Ob}(\mathcal{L})$, the subgroup $E(P) \overset{\text{def}}{=} \mathrm{Ker}[\mathrm{Aut}_{\mathcal{L}}(P) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(P)]$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ via right composition, and $\pi_{P,Q}$ induces a bijection*

$$\mathrm{Mor}_{\mathcal{L}}(P,Q)/E(P) \xrightarrow{\ \cong\ } \mathrm{Hom}_{\mathcal{F}}(P,Q)\ .$$

(a′) *A morphism $\psi \in \mathrm{Mor}(\mathcal{L})$ is an isomorphism if and only if $\pi(\psi) \in \mathrm{Mor}(\mathcal{F})$ is an isomorphism.*

(b) *For every morphism* $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$*, and every* $P_0, Q_0 \in \mathrm{Ob}(\mathcal{L})$ *such that* $P_0 \le P$*,* $Q_0 \le Q$*, and* $\pi(\psi)(P_0) \le Q_0$*, there is a unique morphism* $\psi|_{P_0,Q_0} \in \mathrm{Mor}_{\mathcal{L}}(P_0,Q_0)$ *(the "restriction" of* ψ*) such that* $\psi \circ \iota_{P_0}^{P} = \iota_{Q_0}^{Q} \circ \psi|_{P_0,Q_0}$.

(b′) *For each* $P, Q \in \mathrm{Ob}(\mathcal{L})$ *and each* $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$*, if we set* $Q_0 = \pi(\psi)(P)$*, then there is a unique* $\psi_0 \in \mathrm{Iso}_{\mathcal{L}}(P,Q_0)$ *such that* $\psi = \iota_{Q_0}^{Q} \circ \psi_0$.

(c) *The functor* δ *is injective on all morphism sets.*

(d) *If* $P \in \mathrm{Ob}(\mathcal{L})$ *is fully normalized in* $\mathcal{F}$*, then* $\delta_P(N_S(P)) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{L}}(P))$.

(e) *Let* $P, Q, \overline{P}, \overline{Q} \in \mathrm{Ob}(\mathcal{L})$ *and* $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ *be such that* $P \trianglelefteq \overline{P}$ *and* $Q \le \overline{Q}$*. Then there is a morphism* $\overline{\psi} \in \mathrm{Mor}_{\mathcal{L}}(\overline{P},\overline{Q})$ *such that* $\overline{\psi}|_{P,Q} = \psi$ *if and only if*

$$\forall\, g \in \overline{P} \quad \exists\, h \in \overline{Q} \quad \text{such that} \quad \iota_{Q}^{\overline{Q}} \circ \psi \circ \delta_P(g) = \delta_{Q,\overline{Q}}(h) \circ \psi. \qquad (1)$$

If such a morphism $\overline{\psi}$ *exists, then it is unique.*

(f) *All morphisms in* $\mathcal{L}$ *are monomorphisms and epimorphisms in the categorical sense.*

(g) *All objects in* $\mathcal{L}$ *are* $\mathcal{F}$*-quasicentric.*

Proof. Most of this is contained in [O3, Proposition 4]. Point (a′) follows from (a), which implies that if $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,P)$ and $\pi_P(\psi) = \mathrm{Id}_P$, then ψ is an automorphism. Point (b′) is a special case of (b), where $\psi_0 \overset{\text{def}}{=} \psi|_{P,Q_0}$ is an isomorphism by (a′). In (e), the implication that the existence of $\overline{\psi}$ implies (1) follows from axiom (C) in Definition 1.9 (where $h = \pi(\overline{\psi})(g)$). □

We will also have use for the following "linking system version" of Alperin's fusion theorem.

Theorem 1.12. *For any saturated fusion system* $\mathcal{F}$ *over a finite* p*-group* S *and any linking system* $\mathcal{L}$ *associated to* $\mathcal{F}$*, each morphism in* $\mathcal{L}$ *is a composite of restrictions of automorphisms in* $\mathrm{Aut}_{\mathcal{L}}(P)$*, where* P *is fully normalized in* $\mathcal{F}$*,* $\mathcal{F}$*-centric, and* $\mathcal{F}$*-radical.*

Proof. Using Theorem 1.3 together with Proposition 1.11(a), we are reduced to proving the theorem for automorphisms in $E(P) = \mathrm{Ker}[\mathrm{Aut}_{\mathcal{L}}(P) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(P)]$ for $P \in \mathrm{Ob}(\mathcal{L})$. If P is fully centralized, then $E(P) = \{\delta_P(g) \,|\, g \in C_S(P)\}$ by axiom (A), and each element $\delta_P(g)$ is the restriction of $\delta_S(g) \in \mathrm{Aut}_{\mathcal{L}}(S)$. If P is arbitrary, and Q is fully centralized in $\mathcal{F}$ and $\mathcal{F}$-conjugate to P, then there is some $\psi \in \mathrm{Iso}_{\mathcal{L}}(P,Q)$ which satisfies the conclusion of the theorem (choose any $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P,Q)$, write it as a composite of restrictions of automorphisms, and lift each of those automorphisms to $\mathcal{L}$). Then each element of $E(P)$ has the form $\psi^{-1}\delta_Q(g)\psi$ for some $g \in C_S(Q)$, and hence satisfies the conclusion of the theorem. □

1.3. **Automorphisms of fusion and linking systems.**

Recall that for any linking system $\mathcal{L}$ associated to a fusion system $\mathcal{F}$ over S, and any pair $P \le Q$ of objects in $\mathcal{L}$, the *inclusion* of P into Q is the morphism $\iota_P^Q = \delta_{P,Q}(1) \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$. By Proposition 1.11(b′), each morphism in $\mathcal{L}$ splits uniquely as the composite of an isomorphism followed by an inclusion.

As usual, an *equivalence* of small categories is a functor $\Phi\colon \mathcal{C} \longrightarrow \mathcal{D}$ which induces a bijection between the sets of isomorphism classes of objects and bijections between each pair of morphism sets. It is not hard to see that for each such equivalence, there is an "inverse" $\Psi\colon \mathcal{D} \longrightarrow \mathcal{C}$ such that both composites $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are naturally isomorphic to the identities. In particular, the quotient monoid $\mathrm{Out}(\mathcal{C})$ of all self equivalences of $\mathcal{C}$ modulo natural isomorphisms of functors is a group.

Definition 1.13 ([BLO2, § 8]). *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathcal{L}$ be a linking system associated to $\mathcal{F}$.*

(a) *An automorphism $\beta\in\mathrm{Aut}(S)$ is* fusion preserving *if for each $P,Q\le S$ and each $\varphi\in\mathrm{Hom}_{\mathcal{F}}(P,Q)$, $(\beta|_{Q,\beta(Q)})\varphi(\beta|_{P,\beta(P)})^{-1}$ lies in $\mathrm{Hom}_{\mathcal{F}}(\beta(P),\beta(Q))$. In particular, each such β normalizes $\mathrm{Aut}_{\mathcal{F}}(S)$. Let $\mathrm{Aut}(S,\mathcal{F})$ be the group of all fusion preserving automorphisms of S, and set $\mathrm{Out}(S,\mathcal{F})=\mathrm{Aut}(S,\mathcal{F})/\mathrm{Aut}_{\mathcal{F}}(S)$. Note that $\mathrm{Out}(S,\mathcal{F})$ is a* subquotient *of $\mathrm{Out}(S)$.*

(b) *An equivalence of categories $\alpha\colon\mathcal{L}\longrightarrow\mathcal{L}$ is* isotypical *if $\alpha(\delta_P(P))=\delta_{\alpha(P)}(\alpha(P))$ for each $P\in\mathrm{Ob}(\mathcal{L})$.*

(c) *Let $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ be the group of classes of isotypical self equivalences of $\mathcal{L}$ modulo natural isomorphisms of functors.*

(d) *Let $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ be the group of isotypical equivalences of $\mathcal{L}$ which send inclusions to inclusions.*

Since $\mathrm{Out}(\mathcal{L})$ is a group by the above remarks, and is finite since $\mathrm{Mor}(\mathcal{L})$ is finite, $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ is a submonoid of a finite group and hence itself a group. Another proof of this, as well as a proof that $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ is a group, will be given in Lemma 1.14.

One of the main results in [BLO2] (Theorem 8.1) says that for any p-local finite group $(S,\mathcal{F},\mathcal{L})$, $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})\cong\mathrm{Out}(|\mathcal{L}|^\wedge_p)$: the group of homotopy classes of self homotopy equivalences of $|\mathcal{L}|^\wedge_p$. This helps to explain the importance of $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$, among other groups of automorphisms of $(S,\mathcal{F},\mathcal{L})$ which we might have chosen.

The next lemma gives an alternative description of $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$, and also of $\mathrm{Out}(G)$ — descriptions which will be useful later. For each $\mathcal{L}$ associated to $\mathcal{F}$ over S, and each $\gamma\in\mathrm{Aut}_{\mathcal{L}}(S)$, let $c_\gamma\in\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ be the automorphism which sends $P\in\mathrm{Ob}(\mathcal{L})$ to $\gamma(P)=\pi(\gamma)(P)$, and sends $\psi\in\mathrm{Mor}_{\mathcal{L}}(P,Q)$ to $(\gamma|_{Q,\gamma(Q)})\circ\psi\circ(\gamma|_{P,\gamma(P)})^{-1}$. This is clearly isotypical, since for $g\in P\in\mathrm{Ob}(\mathcal{L})$, $c_\gamma(\delta_P(g))=\delta_{\gamma(P)}(\pi(\gamma)(g))$ by axiom (C). For $P\le Q$ in $\mathrm{Ob}(\mathcal{L})$, c_γ sends ι_P^Q to $\iota_{\gamma(P)}^{\gamma(Q)}$ by definition of restriction, and thus $c_\gamma\in\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$.

Lemma 1.14. (a) *For any saturated fusion system $\mathcal{F}$ over a finite p-group S, and any linking system $\mathcal{L}$ associated to $\mathcal{F}$, the sequence*

$$1\longrightarrow Z(\mathcal{F})\xrightarrow{\ \delta_S\ }\mathrm{Aut}_{\mathcal{L}}(S)\xrightarrow{\ \gamma\mapsto c_\gamma\ }\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})\longrightarrow\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})\longrightarrow 1$$

is exact. All elements of $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ are automorphisms of $\mathcal{L}$, and hence $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ and $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ are both groups.

(b) *For any finite group G and any $S\in\mathrm{Syl}_p(G)$, the sequence*

$$1\longrightarrow Z(G)\xrightarrow{\ \mathrm{incl}\ }N_G(S)\xrightarrow{\ g\mapsto c_g\ }\mathrm{Aut}(G,S)\longrightarrow\mathrm{Out}(G)\longrightarrow 1$$

is exact, where $\mathrm{Aut}(G,S)=\{\alpha\in\mathrm{Aut}(G)\,|\,\alpha(S)=S\}$.

Proof. **(a)** Each equivalence of $\mathcal{L}$ (isotypical or not) sends S to itself, since S is the only object which is the target of morphisms from all other objects.

If $\alpha\in\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$, then for each $P\in\mathrm{Ob}(\mathcal{L})$, $\alpha_{P,S}$ sends ι_P^S to $\iota_{\alpha(P)}^S$, and α_P sends $\delta_P(P)$ to $\delta_{\alpha(P)}(\alpha(P))$. Hence α_S sends $\delta_S(P)$ to $\delta_S(\alpha(P))$, and thus determines the action of α on $\mathrm{Ob}(\mathcal{L})$. In particular, α permutes the objects of $\mathcal{L}$ bijectively, and hence is an automorphism of $\mathcal{L}$. This proves that $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ is a group; and that $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ is also a group if the above sequence is exact.

We next show that each isotypical equivalence $\alpha\colon \mathcal{L} \longrightarrow \mathcal{L}$ is naturally isomorphic to an isotypical equivalence which sends inclusions to inclusions. For each $P \in \mathrm{Ob}(\mathcal{L})$, let $\alpha(\iota_P^S) = \iota_{\beta(P)}^S \circ \omega(P)$ be the unique decomposition of $\alpha(\iota_P^S)$ as a composite of an isomorphism $\omega(P) \in \mathrm{Iso}_{\mathcal{L}}(\alpha(P), \beta(P))$ followed by an inclusion (Proposition 1.11(b$'$)). In particular, $\omega(S) = \mathrm{Id}$. Let β be the automorphism of $\mathcal{L}$ which on objects sends P to $\beta(P)$, and which on morphisms sends $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ to $\omega(Q) \circ \alpha(\varphi) \circ \omega(P)^{-1}$ in $\mathrm{Mor}_{\mathcal{L}}(\beta(P), \beta(Q))$. Then β is isotypical by axiom (C) (and since α is isotypical); it sends inclusions to inclusions by construction (and since $\omega(S) = \mathrm{Id}$); and $\omega(-)$ defines a natural isomorphism from α to β.

This proves that the natural homomorphism from $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ to $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ is onto. If $\alpha \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ is in the kernel, then it is naturally isomorphic to the identity, via some $\omega(-)$ which consists of isomorphisms $\omega(P) \in \mathrm{Iso}_{\mathcal{L}}(P, \alpha(P))$ such that for each $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$, $\alpha(\psi) \circ \omega(P) = \omega(Q) \circ \psi$. Since α sends ι_P^S to $\iota_{\alpha(P)}^S$, $\omega(P) = \omega(S)|_{P,\alpha(P)}$, and thus α is conjugation by $\omega(S) \in \mathrm{Aut}_{\mathcal{L}}(S)$.

Conversely, if $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S)$, then c_γ is naturally isomorphic to $\mathrm{Id}_{\mathcal{L}}$, by the natural isomorphism which sends $P \in \mathrm{Ob}(\mathcal{L})$ to $\gamma|_{P,\pi(\gamma)(P)}$. This finishes the proof that the above sequence is exact at $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$.

It remains to show, for $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S)$, that $c_\gamma = \mathrm{Id}_{\mathcal{L}}$ if and only if $\gamma \in \delta_S(Z(\mathcal{F}))$. If $c_\gamma = \mathrm{Id}_{\mathcal{L}}$, then since $\gamma\delta_S(g)\gamma^{-1} = \delta_S(g)$ for all $g \in S$, $\pi(\gamma) = \mathrm{Id}_S$ by axiom (C) and the injectivity of δ_S. So by axiom (A), there is $a \in Z(S)$ such that $\gamma = \delta_S(a)$. For each $P, Q \in \mathrm{Ob}(\mathcal{L})$ and each $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$, $\delta_Q(a) \circ \psi = \psi \circ \delta_P(a)$ implies there is $\bar{\psi} \in \mathrm{Mor}_{\mathcal{L}}(\langle P,a\rangle, \langle Q,a\rangle)$ such that $\bar{\psi}|_{P,Q} = \psi$ (Proposition 1.11(e)), and $\pi(\bar{\psi})(a) = a$ by axiom (C) and the injectivity of δ. Together with Theorem 1.3, this proves that each morphism in $\mathcal{F}$ extends to one which sends a to itself, and hence that $a \in Z(\mathcal{F})$.

Conversely, if $a \in Z(\mathcal{F})$, then each $\psi \in \mathrm{Mor}(\mathcal{L})$ extends to some $\bar{\psi}$ such that $\pi(\bar{\psi})(a) = a$, $\bar{\psi}$ commutes with $\gamma = \delta_S(a)$ by axiom (C) again, and so $c_\gamma(\psi) = \psi$. Thus $c_\gamma = \mathrm{Id}_{\mathcal{L}}$.

(b) The natural homomorphism from $\mathrm{Aut}(G,S)$ to $\mathrm{Out}(G)$ is onto by the Frattini argument (the Sylow p-subgroups of G are permuted transitively by inner automorphisms). The kernel of that map clearly consists of conjugation by elements of $N_G(S)$. □

In particular, the group $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ defined here is the same as that defined in [O3], where it was defined explicitly as a group of automorphisms of $\mathcal{L}$ rather than of equivalences.

The next lemma describes how elements of $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ induce automorphisms of the associated fusion system. For $\beta \in \mathrm{Aut}(S,\mathcal{F})$, let $c_\beta \in \mathrm{Aut}(\mathcal{F})$ be the automorphism of the category $\mathcal{F}$ which sends $P \le S$ to $\beta(P)$, and sends $\varphi \in \mathrm{Mor}(\mathcal{F})$ to $\beta\varphi\beta^{-1}$.

Lemma 1.15 ([O3, Proposition 6]). *Let $\mathcal{L}$ be a linking system associated to a saturated fusion system $\mathcal{F}$ over a finite p-group S, with structure functors $\mathcal{T}_{\mathrm{Ob}(\mathcal{L})}(S) \xrightarrow{\ \delta\ } \mathcal{L} \xrightarrow{\ \pi\ } \mathcal{F}$. Fix $\alpha \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$. Let $\beta \in \mathrm{Aut}(S)$ be such that $\alpha(\delta_S(g)) = \delta_S(\beta(g))$ for all $g \in S$. Then $\beta \in \mathrm{Aut}(S,\mathcal{F})$, $\alpha(P) = \beta(P)$ for $P \in \mathrm{Ob}(\mathcal{L})$, and $\pi \circ \alpha = c_\beta \circ \pi$.*

Proof. See [O3, Proposition 6] (and note that $\mathcal{F}$ is, in fact, assumed to be saturated in the proof of that proposition). The relation $\alpha(P) = \beta(P)$ is not in the statement of the proposition, but it is shown in its proof. It is really part of the statement $\pi \circ \alpha = c_\beta \circ \pi$ (since π is the inclusion on objects). □

Lemma 1.15 motivates the following definition. For any saturated fusion system $\mathcal{F}$ over a finite p-group S, and any linking system $\mathcal{L}$ associated to $\mathcal{F}$, define

$$\widetilde{\mu}_{\mathcal{L}}\colon \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}) \longrightarrow \mathrm{Aut}(S,\mathcal{F})$$

by setting $\widetilde{\mu}_{\mathcal{L}}(\alpha) = \delta_S^{-1} \circ \alpha_S \circ \delta_S \in \mathrm{Aut}(S)$ for $\alpha \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$. By Lemma 1.15, $\mathrm{Im}(\widetilde{\mu}_{\mathcal{L}}) \le \mathrm{Aut}(S,\mathcal{F})$. For $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S)$, $\widetilde{\mu}_{\mathcal{L}}(c_\gamma) = \pi(\gamma) \in \mathrm{Aut}_{\mathcal{F}}(S)$ by axiom (C) in Definition 1.9. So by Lemma 1.14(a), $\widetilde{\mu}_{\mathcal{L}}$ induces a homomorphism

$$\mu_{\mathcal{L}}\colon \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \longrightarrow \mathrm{Out}(S,\mathcal{F})$$

by sending the class of α to that of $\widetilde{\mu}_{\mathcal{L}}(\alpha)$. When $\mathcal{L} = \mathcal{L}_S^{\mathcal{H}}(G)$ for some finite group G and some set of objects $\mathcal{H}$, we write $\widetilde{\mu}_G^{\mathcal{H}} = \widetilde{\mu}_{\mathcal{L}}$ and $\mu_G^{\mathcal{H}} = \mu_{\mathcal{L}}$ for short. When $\mathcal{L} = \mathcal{L}_S^c(G)$ is the centric linking system, we write $\widetilde{\mu}_G = \widetilde{\mu}_{\mathcal{L}}$ and $\mu_G = \mu_{\mathcal{L}}$.

Lemma 1.16. *For any linking system $\mathcal{L}$ associated to a saturated fusion system $\mathcal{F}$,* $\mathrm{Ker}(\mu_{\mathcal{L}})$ *is a finite p-group.*

Proof. Assume $\mathcal{F}$ is a fusion system over the finite p-group S. Since $\mathcal{L}$ is a finite category, $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ and $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ are finite groups. So it suffices to prove that each element of $\mathrm{Ker}(\mu_{\mathcal{L}})$ has p-power order.

Fix $\alpha \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ such that $[\alpha] \in \mathrm{Ker}(\mu_{\mathcal{L}})$. Thus $\widetilde{\mu}_{\mathcal{L}}(\alpha) \in \mathrm{Aut}_{\mathcal{F}}(S)$, and $\widetilde{\mu}_{\mathcal{L}}(\alpha) = \pi(\gamma)$ for some $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S)$. So upon replacing α by $c_\gamma^{-1} \circ \alpha$, we can assume $\alpha \in \mathrm{Ker}(\widetilde{\mu}_{\mathcal{L}})$.

Thus $\alpha_S|_{\delta_S(S)} = \mathrm{Id}$. Since α sends inclusions to inclusions, $\alpha_P|_{\delta_P(P)} = \mathrm{Id}$ for all $P \in \mathrm{Ob}(\mathcal{L})$. Assume also that P is $\mathcal{F}$-centric. For each $\psi \in \mathrm{Aut}_{\mathcal{L}}(P)$, ψ and $\alpha(\psi)$ have the same conjugation action on $\delta_P(P)$, so $\psi^{-1}\alpha(\psi) \in C_{\mathrm{Aut}_{\mathcal{L}}(P)}(\delta_P(P)) \le \delta_P(Z(P))$. Hence $\alpha(\psi) = \psi\delta_P(g)$ for some $g \in Z(P)$, and $\alpha^k(\psi) = \psi\delta_P(g^k)$ for all k since α is the identity on $\delta_P(P)$.

Choose $m \ge 0$ such that $g^{p^m} = 1$ for all $g \in S$. Then α^{p^m} is the identity on $\mathrm{Aut}_{\mathcal{L}}(P)$ for each $P \in \mathrm{Ob}(\mathcal{L})$ which is $\mathcal{F}$-centric. So by Theorem 1.12, $\alpha^{p^m} = \mathrm{Id}_{\mathcal{L}}$. □

The kernel of $\mu_{\mathcal{L}}$ will be studied much more closely in Proposition 4.2.

Since we will need to work with linking systems with different sets of objects associated to the same fusion system, it will be important to know they have the same automorphisms.

Lemma 1.17. *Fix a saturated fusion system $\mathcal{F}$ over a finite p-group S. Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be a pair of linking systems associated to $\mathcal{F}$. Set $\mathcal{H}_0 = \mathrm{Ob}(\mathcal{L}_0)$ and $\mathcal{H} = \mathrm{Ob}(\mathcal{L})$, and assume $\mathcal{H}_0 \subseteq \mathcal{H}$ are both* $\mathrm{Aut}(S,\mathcal{F})$*-invariant. Then restriction defines an isomorphism*

$$\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \xrightarrow[\cong]{R} \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_0).$$

Proof. Using axiom (A), one sees that $\mathcal{L}_0$ must be a full subcategory of $\mathcal{L}$. Set $\mathcal{P} = \mathcal{H} \smallsetminus \mathcal{H}_0$. We can assume, by induction on $|\mathcal{H}| - |\mathcal{H}_0|$, that all subgroups in $\mathcal{P}$ have the same order. Thus all morphisms in $\mathcal{L}$ between subgroups in $\mathcal{P}$ are isomorphisms.

Since $\mathcal{H}_0$ is $\mathrm{Aut}(S,\mathcal{F})$-invariant and $\mathcal{L}_0$ is a full subcategory, there is a well defined restriction homomorphism

$$\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}) \xrightarrow{\mathrm{Res}} \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_0).$$

By assumption, $\mathcal{H}_0$ contains all subgroups which are $\mathcal{F}$-centric and $\mathcal{F}$-radical. Hence Theorem 1.12 implies that all morphisms in $\mathcal{L}$ are composites of restrictions of morphisms in $\mathcal{L}_0$. Since each $\alpha \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ sends inclusions to inclusions, it also sends restrictions to restrictions, and hence $\alpha|_{\mathcal{L}_0} = \mathrm{Id}_{\mathcal{L}_0}$ only if $\alpha = \mathrm{Id}_{\mathcal{L}}$. Thus Res is injective. We next show it is surjective, and hence an isomorphism.

Let $\mathcal{P}_* \subseteq \mathcal{P}$ be a subset consisting of one fully normalized subgroup from each $\mathcal{F}$-conjugacy class in $\mathcal{P}$. For each $P \in \mathcal{P}_*$, $\delta_P(N_S(P)) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{L}}(P))$ by Proposition 1.11(d), so there is a unique $\widehat{P} \le N_S(P)$ such that $\delta_P(\widehat{P}) = O_p(\mathrm{Aut}_{\mathcal{L}}(P))$. Since $P \notin \mathcal{H}_0$, P is either not $\mathcal{F}$-centric or not $\mathcal{F}$-radical. In either case, $\widehat{P} \gneqq P$ by Lemma 1.4. By Proposition 1.11(e), each $\psi \in \mathrm{Aut}_{\mathcal{L}}(P)$ extends to a unique automorphism $\widehat{\psi} \in \mathrm{Aut}_{\mathcal{L}}(\widehat{P})$.

Let $\nu\colon \mathcal{P} \longrightarrow \mathcal{P}_*$ be the map which sends P to the unique subgroup $\nu(P) \in \mathcal{P}_*$ which is $\mathcal{F}$-conjugate to P. For each $P \in \mathcal{P}$, $\delta_{\nu(P)}(N_S(\nu(P))) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{L}}(\nu(P)))$ by Proposition 1.11(d) (and since $\nu(P)$ is fully normalized), and hence there is $\lambda_P \in \mathrm{Iso}_{\mathcal{L}}(P,\nu(P))$ such that

$$\lambda_P\delta_P(N_S(P))\lambda_P^{-1} \le \delta_{\nu(P)}(N_S(\nu(P))) \ .$$

By Proposition 1.11(e) again, λ_P extends to a unique $\widehat{\lambda}_P \in \mathrm{Mor}_{\mathcal{L}}(N_S(P), N_S(\nu(P)))$. When $P \in \mathcal{P}_*$ (so $\nu(P) = P$), we set $\lambda_P = \mathrm{Id}_P$, and hence $\widehat{\lambda}_P = \mathrm{Id}_{N_S(P)}$.

Fix any $\alpha_0 \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_0)$; we want to extend α_0 to $\mathcal{L}$. By Lemma 1.15, α_0 induces some $\beta \in \mathrm{Aut}(S,\mathcal{F})$, and $\alpha_0(P) = \beta(P)$ for all $P \in \mathcal{H}_0$. So define $\alpha(P) = \beta(P)$ for $P \in \mathcal{H}$; this is possible since $\mathcal{H}$ is $\mathrm{Aut}(S,\mathcal{F})$-invariant by assumption. By Lemma 1.15 again, for each $P,Q \in \mathcal{H}_0$ and each $\psi \in \mathrm{Mor}_{\mathcal{L}_0}(P,Q)$, $\pi(\alpha_0(\psi)) = c_\beta(\pi(\psi)) = \beta(\pi(\psi))\beta^{-1}$. In other words,

$$\psi \in \mathrm{Mor}_{\mathcal{L}_0}(P,Q),\ g \in P,\ \pi(\psi)(g) = h \in Q \Longrightarrow \pi(\alpha_0(\psi))(\beta(g)) = \beta(h) \ . \tag{2}$$

We next define α on isomorphisms between subgroups in $\mathcal{P}$. Fix $P_1, P_2 \in \mathcal{P}$ and $\psi \in \mathrm{Iso}_{\mathcal{L}}(P_1,P_2)$, and set $P_* = \nu(P_1) = \nu(P_2)$. There is a unique $\psi_* \in \mathrm{Aut}_{\mathcal{L}}(P_*)$ such that $\psi = \lambda_{P_2}^{-1} \circ \psi_* \circ \lambda_{P_1}$, and we set

$$\alpha(\psi) = \bigl(\alpha_0(\widehat{\lambda}_{P_2})|_{\alpha(P_2),\alpha(P_*)}\bigr)^{-1} \circ \bigl(\alpha_0(\widehat{\psi}_*)|_{\alpha(P_*),\alpha(P_*)}\bigr) \circ \bigl(\alpha_0(\widehat{\lambda}_{P_1})|_{\alpha(P_1),\alpha(P_*)}\bigr) \ .$$

Note that $\widehat{\lambda}_{P_1}$, $\widehat{\lambda}_{P_2}$, and $\widehat{\psi}_*$ are all in $\mathrm{Mor}(\mathcal{L}_0)$, since all subgroups strictly containing subgroups in $\mathcal{P}$ are in $\mathcal{H}_0 = \mathrm{Ob}(\mathcal{L}_0)$ by assumption. Also, the restrictions are well defined (for example, $\pi(\alpha_0(\widehat{\lambda}_{P_i}))(\alpha(P_i)) = \alpha(P^*)$) by (2).

Recall that $\mathcal{H}$ and $\mathcal{H}_0$ are both closed under overgroups and $\mathcal{F}$-conjugacy. Hence each morphism in $\mathcal{L}$ not in $\mathcal{L}_0$ factors uniquely as an isomorphism between subgroups in $\mathcal{P}$ followed by an inclusion (Proposition 1.11(b′)), and thus the above definitions extend to define α as a map from $\mathrm{Mor}(\mathcal{L})$ to itself. This clearly preserves composition of isomorphisms between subgroups in $\mathcal{P}$. To prove that α is a functor, it remains to show it preserves composites of inclusions followed by isomorphisms in $\mathcal{L}_0$. This means showing, for each $P_1, P_2 \in \mathcal{P}$, each $P_i \lneqq Q_i$, and each $\psi \in \mathrm{Iso}_{\mathcal{L}}(P_1,P_2)$ which extends to $\varphi \in \mathrm{Mor}_{\mathcal{L}_0}(Q_1,Q_2)$, that $\alpha(\psi) = \alpha_0(\varphi)|_{\alpha(P_1),\alpha(P_2)}$. Since $N_{Q_i}(P_i) \gneqq P_i$, we can assume $P_i \trianglelefteq Q_i$ for $i = 1,2$. Set $P_* = \nu(P_1) = \nu(P_2)$ again, and set $R_i = \pi(\widehat{\lambda}_{P_i})(Q_i) \gneqq P_*$. Then $P_* \trianglelefteq R_i$ since $P_i \trianglelefteq Q_i$. We saw that ψ factors in a unique way $\psi = \lambda_{P_2}^{-1} \circ \psi_* \circ \lambda_{P_1}$ for $\psi_* \in \mathrm{Aut}_{\mathcal{L}}(P_*)$. We also have $\varphi = \bar{\lambda}_{P_2}^{-1} \circ \varphi_* \circ \bar{\lambda}_{P_1}$, where $\bar{\lambda}_{P_i} = \widehat{\lambda}_{P_i}|_{Q_i,R_i}$ and $\varphi_* \in \mathrm{Mor}_{\mathcal{L}_0}(R_1,R_2)$. Thus $\alpha_0(\bar{\lambda}_{P_i})$ is a restriction of $\alpha_0(\widehat{\lambda}_{P_i})$ ($i = 1,2$), and hence an extension of $\alpha(\lambda_{P_i})$.

It remains to show $\alpha(\psi_*)$ is the restriction of $\alpha_0(\varphi_*)$. By definition, $\alpha(\psi_*)$ is the restriction to $\alpha(P_*)$ of $\alpha_0(\widehat{\psi}_*)$, where $\widehat{\psi}_* \in \mathrm{Aut}_{\mathcal{L}}(\widehat{P_*})$. Set $T_i = \langle \widehat{P_*}, R_i \rangle$. By Proposition 1.11(e), since $\psi_* \in \mathrm{Aut}_{\mathcal{L}}(P_*)$ extends to $\widehat{\psi}_* \in \mathrm{Aut}_{\mathcal{L}}(\widehat{P_*})$ and to $\varphi_* \in \mathrm{Mor}_{\mathcal{L}}(R_1,R_2)$, there is $\bar{\varphi}_* \in \mathrm{Mor}_{\mathcal{L}}(T_1,T_2)$ which extends both $\widehat{\psi}_*$ and φ_*. Hence $\alpha_0(\bar{\varphi}_*)$ extends both $\alpha_0(\widehat{\psi}_*)$ and $\alpha_0(\varphi_*)$ (all of these are in $\mathcal{L}_0$), and thus $\alpha(\psi_*)$ is a restriction of each of the latter. This finishes the proof that α is a functor. By construction, α is isotypical, sends inclusions to inclusions, and extends α_0; and thus Res is surjective.

We have now shown that restriction defines an isomorphism from $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ to $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_0)$. By Lemma 1.14(a), the outer automorphism groups of $\mathcal{L}$ and $\mathcal{L}_0$ are defined by dividing out by conjugation by elements of $\mathrm{Aut}_{\mathcal{L}}(S)$. Hence the induced homomorphism

$$\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \xrightarrow{\quad R \quad} \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_0)$$

is also an isomorphism. □

1.4. **Normal fusion subsystems.**

Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. By a *(saturated) fusion subsystem* of $\mathcal{F}$ over a subgroup $S_0 \le S$, we mean a subcategory $\mathcal{F}_0 \subseteq \mathcal{F}$ whose objects are the subgroups of S_0, and which is itself a (saturated) fusion system over S_0.

The following definition of a normal fusion subsystem is the same as that of a *weakly normal* fusion subsystem in [AKO, § I.6]. We have dropped the word "weakly" here, since the extra condition for being normal in the sense of Aschbacher ([AKO, Definition I.6.1]) will not be needed.

Definition 1.18. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathcal{F}_0 \subseteq \mathcal{F}$ be a saturated fusion subsystem over $S_0 \le S$. Then $\mathcal{F}_0$ is* normal *in $\mathcal{F}$ ($\mathcal{F}_0 \trianglelefteq \mathcal{F}$) if*

(i) *S_0 is strongly closed in $\mathcal{F}$;*

(ii) *for each $P, Q \le S_0$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, there are $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S_0)$ and $\varphi_0 \in \mathrm{Hom}_{\mathcal{F}_0}(\alpha(P),Q)$ such that $\varphi = \varphi_0 \circ \alpha|_{P,\alpha(P)}$; and*

(iii) *for each $P, Q \le S_0$, each $\varphi \in \mathrm{Hom}_{\mathcal{F}_0}(P,Q)$, and each $\beta \in \mathrm{Aut}_{\mathcal{F}}(S_0)$, $\beta\varphi\beta^{-1} \in \mathrm{Hom}_{\mathcal{F}_0}(\beta(P),\beta(Q))$.*

The above definition is equivalent to Puig's definition [Pg2, § 6.4], and also to Aschbacher's definition of an $\mathcal{F}$-invariant subsystem [Asch, § 3], except that they do not require the subsystem to be saturated. See [Pg2, Proposition 6.6], [Asch, Theorem 3.3], and [AKO, Proposition I.6.4] for proofs of the equivalence of these and other conditions.

We next list some of the basic properties of normal fusion subsystems, starting with the following technical result.

Lemma 1.19. *Fix a saturated fusion system $\mathcal{F}$ over a finite p-group S. Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be a fusion subsystem (not necessarily saturated) over the subgroup $S_0 \trianglelefteq S$, which satisfies conditions (i–iii) in Definition 1.18. Assume $P_0 \le S_0$ is $\mathcal{F}_0$-centric and fully normalized in $\mathcal{F}$, and $\mathrm{Out}_{S_0}(P_0) \cap O_p(\mathrm{Out}_{\mathcal{F}_0}(P_0)) = 1$. Then there is $P \le S$ which is $\mathcal{F}$-centric and $\mathcal{F}$-radical and such that $P \cap S_0 = P_0$.*

Proof. Set

$$P = \{x \in N_S(P_0) \,|\, c_x \in O_p(\mathrm{Aut}_{\mathcal{F}}(P_0))\} \ .$$

If $x \in P \cap S_0$, then $c_x \in O_p(\mathrm{Aut}_{\mathcal{F}}(P_0)) \cap \mathrm{Aut}_{\mathcal{F}_0}(P_0) = O_p(\mathrm{Aut}_{\mathcal{F}_0}(P_0))$ ($\mathrm{Aut}_{\mathcal{F}_0}(P_0)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(P_0)$ by 1.18(ii–iii)), so $c_x \in \mathrm{Aut}_{S_0}(P_0) \cap O_p(\mathrm{Aut}_{\mathcal{F}_0}(P_0)) = \mathrm{Inn}(P_0)$, and $x \in P_0$ since P_0 is $\mathcal{F}_0$-centric. Thus $P \cap S_0 = P_0$.

By construction, $N_S(P) = N_S(P_0)$. So if Q if $\mathcal{F}$-conjugate to P and $Q_0 = Q \cap S_0$, then $|N_S(Q)| \le |N_S(Q_0)| \le |N_S(P_0)| = |N_S(P)|$ since P_0 is fully normalized in $\mathcal{F}$ and $\mathcal{F}$-conjugate to Q_0. This proves that P is fully normalized in $\mathcal{F}$.

Now, $\mathrm{Aut}_S(P_0) \ge O_p(\mathrm{Aut}_{\mathcal{F}}(P_0))$ since P_0 is fully normalized in $\mathcal{F}$. So $\mathrm{Aut}_P(P_0) = O_p(\mathrm{Aut}_{\mathcal{F}}(P_0))$, and hence this is normal in $\mathrm{Aut}_{\mathcal{F}}(P_0)$. By the extension axiom, the restriction homomorphism $\mathrm{Aut}_{\mathcal{F}}(P) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(P_0)$ is surjective, and thus sends $O_p(\mathrm{Aut}_{\mathcal{F}}(P))$

into $O_p(\mathrm{Aut}_{\mathcal{F}}(P_0))$. So for all $x \in N_S(P)$ such that $c_x \in O_p(\mathrm{Aut}_{\mathcal{F}}(P))$, $c_x \in O_p(\mathrm{Aut}_{\mathcal{F}}(P_0))$, and hence $x \in P$. Since P is fully normalized, it is $\mathcal{F}$-centric and $\mathcal{F}$-radical by Lemma 1.4. □

The following is our main lemma listing properties of normal pairs of fusion systems. Recall that $O_p(\mathcal{F})$ is the largest normal p-subgroup of the fusion system $\mathcal{F}$.

Lemma 1.20. *Fix a saturated fusion system $\mathcal{F}$ over a finite p-group S, and let $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ be a normal fusion subsystem over the normal subgroup $S_0 \trianglelefteq S$. Then the following hold.*

(a) *Each $\mathcal{F}$-conjugacy class contains a subgroup $P \le S$ such that P and $P \cap S_0$ are both fully normalized in $\mathcal{F}$, and $P \cap S_0$ is fully normalized in $\mathcal{F}_0$.*

(b) *For each $P, Q \le S_0$ and each $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P,Q)$, $\varphi \mathrm{Aut}_{\mathcal{F}_0}(P)\varphi^{-1} = \mathrm{Aut}_{\mathcal{F}_0}(Q)$.*

(c) *The set of $\mathcal{F}_0$-centric subgroups of S_0, and the set of $\mathcal{F}_0$-radical subgroups of S_0, are both invariant under $\mathcal{F}$-conjugacy.*

(d) *If $P \le S$ is $\mathcal{F}$-centric and $\mathcal{F}$-radical, then $P \cap S_0$ is $\mathcal{F}_0$-centric and $\mathcal{F}_0$-radical. Conversely, if $P_0 \le S_0$ is $\mathcal{F}_0$-centric, $\mathcal{F}_0$-radical, and fully normalized in $\mathcal{F}$, then there is $P \le S$ which is $\mathcal{F}$-centric and $\mathcal{F}$-radical, and such that $P \cap S_0 = P_0$.*

(e) *$O_p(\mathcal{F}_0)$ is normal in $\mathcal{F}$, and $O_p(\mathcal{F}_0) = O_p(\mathcal{F}) \cap S_0$.*

Proof. Throughout the proof, whenever $P \le S$, we write $P_0 = P \cap S_0$ for short.

(a) Fix $Q \le S$. By [BLO2, Proposition A.2(b)], there are subgroups $R \le S$ and $P_0 \le S_0$ which are fully normalized in $\mathcal{F}$, and morphisms $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), N_S(R))$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_S(R_0), N_S(P_0))$ such that $\varphi(Q) = R$ and $\psi(R_0) = P_0$. Set $P = \psi(R)$ (note that $P \cap S_0 = P_0$ since S_0 is strongly closed). Since $N_S(R) \le N_S(R_0)$, P is also fully normalized in $\mathcal{F}$. Also, $P = \psi \circ \varphi(Q)$ is $\mathcal{F}$-conjugate to Q.

By [BLO2, Proposition A.2(b)] again, if P_0^* is $\mathcal{F}$-conjugate to P_0, then there is a morphism in $\mathcal{F}$ from $N_S(P_0^*)$ to $N_S(P_0)$ which sends P_0^* to P_0. In particular, $|N_{S_0}(P_0^*)| \le |N_{S_0}(P_0)|$, and hence P_0 is also fully normalized in $\mathcal{F}_0$.

(b) Fix $P, Q \le S_0$ and $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P,Q)$. By condition (ii) in Definition 1.18, there are $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S_0)$ and $\varphi_0 \in \mathrm{Iso}_{\mathcal{F}_0}(\alpha(P), Q)$ such that $\varphi = \varphi_0 \circ \alpha|_{P,\alpha(P)}$. Hence

$$\varphi \mathrm{Aut}_{\mathcal{F}_0}(P)\varphi^{-1} = \varphi_0 \mathrm{Aut}_{\mathcal{F}_0}(\alpha(P))\varphi_0^{-1} = \mathrm{Aut}_{\mathcal{F}_0}(Q) ,$$

where the first equality holds by condition (iii) in Definition 1.18.

(c) Fix $P \le S_0$, let $\mathcal{P}$ be the $\mathcal{F}$-conjugacy class of P, and let $\mathcal{P}_0$ be its $\mathcal{F}_0$-conjugacy class.

If P is $\mathcal{F}_0$-centric, then $C_{S_0}(P^*) = Z(P^*)$ for all $P^* \in \mathcal{P}_0$. For all $R \in \mathcal{P}$, there is $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S_0)$ such that $\alpha(R) \in \mathcal{P}_0$ (condition (ii) in Definition 1.18), and hence $C_{S_0}(R) = Z(R)$. Since this holds for all subgroups in $\mathcal{P}$, all of these subgroups are $\mathcal{F}_0$-centric.

Now assume P is $\mathcal{F}_0$-radical; then $O_p(\mathrm{Out}_{\mathcal{F}_0}(P^*)) = 1$ for all $P^* \in \mathcal{P}_0$. If $R \in \mathcal{P}$, and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S_0)$ is such that $\alpha(R) \in \mathcal{P}_0$, then by condition (iii) in Definition 1.18, conjugation by α sends $\mathrm{Out}_{\mathcal{F}_0}(R) \le \mathrm{Out}_{\mathcal{F}}(R)$ isomorphically to $\mathrm{Out}_{\mathcal{F}_0}(\alpha(R))$. Since $O_p(\mathrm{Out}_{\mathcal{F}_0}(\alpha(R))) = 1$, $O_p(\mathrm{Out}_{\mathcal{F}_0}(R)) = 1$. So all subgroups in $\mathcal{P}$ are $\mathcal{F}_0$-radical.

(d) The second statement was shown in Lemma 1.19. It remains to prove the first.

Assume P is $\mathcal{F}$-centric and $\mathcal{F}$-radical. We must show that P_0 is $\mathcal{F}_0$-centric and $\mathcal{F}_0$-radical. By (c), this is independent of the choice of P in its $\mathcal{F}$-conjugacy class, and hence by (a), it suffices to prove it when P_0 is fully normalized in $\mathcal{F}_0$. By (b), $\mathrm{Aut}_{\mathcal{F}_0}(P_0)$ is normal in

$\mathrm{Aut}_{\mathcal{F}}(P_0)$, and hence
$$O_p(\mathrm{Aut}_{\mathcal{F}_0}(P_0)) \le O_p(\mathrm{Aut}_{\mathcal{F}}(P_0)) \ .$$

Let T be the subgroup of all $x \in N_{S_0}(P_0)$ such that $c_x \in O_p(\mathrm{Aut}_{\mathcal{F}_0}(P_0))$. If $x \in T \cap N_S(P)$, then $c_x \in O_p(\mathrm{Aut}_{\mathcal{F}}(P_0))$, and c_x induces the identity on P/P_0 since $[x,P] \le P \cap [S_0,P] \le P_0$. Thus $c_x \in O_p(\mathrm{Aut}_{\mathcal{F}}(P))$ by Lemma 1.6, and $x \in P$ by Lemma 1.4 since P is centric and radical in $\mathcal{F}$.

Thus $T \cap N_S(P) \le P_0$. Also, P normalizes T by construction, so $T \le P_0$ by Lemma 1.7. Hence P_0 is centric and radical in $\mathcal{F}_0$ by Lemma 1.4 again.

(e) Set $Q = O_p(\mathcal{F}_0)$ and $R = O_p(\mathcal{F})$ for short. To prove that $Q \trianglelefteq \mathcal{F}$ and $R_0 = Q$, it suffices to show that $Q \trianglelefteq \mathcal{F}$ and $R_0 \trianglelefteq \mathcal{F}_0$. We apply Proposition 1.5, which says that a subgroup is normal in a saturated fusion system if and only if it is strongly closed and contained in all subgroups which are centric and radical. Since an intersection of strongly closed subgroups is strongly closed, R_0 is strongly closed in $\mathcal{F}$ and hence in $\mathcal{F}_0$.

If $P \le S$ is $\mathcal{F}$-centric and $\mathcal{F}$-radical, then P_0 is $\mathcal{F}_0$-centric and $\mathcal{F}_0$-radical by (d), so $P \ge P_0 \ge Q$. If P_0 is $\mathcal{F}_0$-centric and $\mathcal{F}_0$-radical, then the same holds for each subgroup in its $\mathcal{F}$-conjugacy class by (c). So by (d), there is $P^* \le S$ which is $\mathcal{F}$-centric and $\mathcal{F}$-radical with P^*_0 $\mathcal{F}$-conjugate to P_0; $P^* \ge R$, and hence P^*_0 and P_0 both contain R_0.

It remains to prove that Q is strongly closed in $\mathcal{F}$. Fix $\mathcal{F}$-conjugate elements $g,h \in S$ such that $g \in Q$; we must show $h \in Q$. Since S_0 is strongly closed in $\mathcal{F}$ (since $\mathcal{F}_0 \trianglelefteq \mathcal{F}$), $h \in S_0$. Fix $\varphi \in \mathrm{Iso}_{\mathcal{F}}(\langle g\rangle, \langle h\rangle)$ with $\varphi(g) = h$. Since $\mathcal{F}_0 \trianglelefteq \mathcal{F}$, there are morphisms $\chi \in \mathrm{Aut}_{\mathcal{F}}(S_0)$ and $\varphi_0 \in \mathrm{Iso}_{\mathcal{F}_0}(\langle g\rangle, \langle \chi^{-1}(h)\rangle)$ such that $\varphi = \chi \circ \varphi_0$. Then $g' \overset{\mathrm{def}}{=} \varphi_0(g) \in Q$, and $h = \chi(g')$. The invariance condition (iii) in Definition 1.18 implies that χ sends a normal subgroup of $\mathcal{F}_0$ to another normal subgroup. Thus $\chi(Q){\cdot}Q$ is also normal in $\mathcal{F}_0$, so $\chi(Q) = Q$ since Q is the largest subgroup of S_0 normal in $\mathcal{F}_0$, and thus $h = \chi(g') \in Q$. □

We now turn to the specific examples of normal fusion subsystems which we work with in this paper. We first look at those of p-power index and of index prime to p. Two other definitions are first needed. For any saturated fusion system $\mathcal{F}$, the *focal subgroup* $\mathfrak{foc}(\mathcal{F})$ and the *hyperfocal subgroup* $\mathfrak{hyp}(\mathcal{F})$ are defined by
$$\mathfrak{foc}(\mathcal{F}) = \langle s^{-1}t \,|\, s,t \in S \text{ are } \mathcal{F}\text{-conjugate}\rangle = \langle s^{-1}\alpha(s) \,|\, s \in P \le S,\ \alpha \in \mathrm{Aut}_{\mathcal{F}}(P)\rangle$$
$$\mathfrak{hyp}(\mathcal{F}) = \langle s^{-1}\alpha(s) \,|\, s \in P \le S,\ \alpha \in O^p(\mathrm{Aut}_{\mathcal{F}}(P))\rangle \ .$$
Note that in [BCGLO2], we wrote $O^p_{\mathcal{F}}(S) = \mathfrak{hyp}(\mathcal{F})$.

The following definition also includes many fusion subsystems which are not normal.

Definition 1.21 ([BCGLO2, Definition 3.1])**.** *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathcal{F}_0 \subseteq \mathcal{F}$ be a saturated fusion subsystem over a subgroup $S_0 \le S$.*

(a) $\mathcal{F}_0$ has p-power index *in $\mathcal{F}$ if $\mathfrak{hyp}(\mathcal{F}) \le S_0 \le S$, and* $\mathrm{Aut}_{\mathcal{F}_0}(P) \ge O^p(\mathrm{Aut}_{\mathcal{F}}(P))$ *for all $P \le S_0$.*

(b) $\mathcal{F}_0$ has index prime to p *in $\mathcal{F}$ if $S_0 = S$, and* $\mathrm{Aut}_{\mathcal{F}_0}(P) \ge O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ *for all $P \le S$.*

Recall that despite the terminology, these are not analogous to subgroups of a finite group of p-power index or index prime to p. Instead, they are analogous to subgroups which contain a normal subgroup having appropriate index.

The following theorem gives a complete description of all such fusion subsystems.

Theorem 1.22 ([BCGLO2, Theorems 4.3 & 5.4])**.** *The following hold for any saturated fusion system $\mathcal{F}$ over a finite p-group S.*

(a) *For each subgroup $S_0 \le S$ containing the hyperfocal subgroup $\mathfrak{hyp}(\mathcal{F})$, there is a unique fusion subsystem $\mathcal{F}_0$ over S_0 of p-power index in $\mathcal{F}$. Thus $\mathcal{F}$ contains a proper fusion subsystem of p-power index if and only if $\mathfrak{hyp}(\mathcal{F}) \lneqq S$, or equivalently $\mathfrak{foc}(\mathcal{F}) \lneqq S$.*

(b) *There is a subgroup $\Gamma \trianglelefteq \mathrm{Out}_{\mathcal{F}}(S)$ with the following properties. For each subsystem $\mathcal{F}_0 \subseteq \mathcal{F}$ of index prime to p, $\mathrm{Out}_{\mathcal{F}_0}(S) \ge \Gamma$. Conversely, for each $H \le \mathrm{Out}_{\mathcal{F}}(S)$ containing Γ, there is a unique subsystem $\mathcal{F}_0 \subseteq \mathcal{F}$ of index prime to p with $\mathrm{Out}_{\mathcal{F}_0}(S) = H$.*

Proof. The only part not shown in [BCGLO2] is that $\mathfrak{hyp}(\mathcal{F}) \lneqq S$ implies $\mathfrak{foc}(\mathcal{F}) \lneqq S$. By Theorem 1.3,

$$\mathfrak{foc}(\mathcal{F}) = \langle s^{-1}\alpha(s) \,|\, s \in P \le S,\ P \text{ fully normalized in } \mathcal{F},\ \alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \rangle \ .$$

Since $\mathrm{Aut}_{\mathcal{F}}(P) = O^p(\mathrm{Aut}_{\mathcal{F}}(P)) \cdot \mathrm{Aut}_S(P)$ when P is fully normalized, and since $s^{-1}\alpha(s) \in [S,S]$ when $s \in P$ and $\alpha \in \mathrm{Aut}_S(P)$, we have $\mathfrak{foc}(\mathcal{F}) = \mathfrak{hyp}(\mathcal{F}) \cdot [S,S]$. Also, $\mathfrak{hyp}(\mathcal{F}) \lneqq S$ implies there is $Q \trianglelefteq S$ such that $[S{:}Q] = p$ and $\mathfrak{hyp}(\mathcal{F}) \le Q$. Then $[S,S] \le Q$ since S/Q is abelian, and hence $\mathfrak{foc}(\mathcal{F}) \le Q \lneqq S$. □

In the situation of Theorem 1.22, a fusion subsystem of p-power index is normal in $\mathcal{F}$ exactly when its underlying p-group is normal in S, and a fusion subsystem $\mathcal{F}_0 \subseteq \mathcal{F}$ of index prime to p is normal in $\mathcal{F}$ exactly when $\mathrm{Aut}_{\mathcal{F}_0}(S)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(S)$ (cf. [AKO, Theorems I.7.4 and I.7.7(c)]). But in fact, we will only be concerned here (in Proposition 1.25(a,b)) with the minimal such fusion subsystems, defined as follows.

Definition 1.23. *For any saturated fusion system $\mathcal{F}$ over a finite p-group S, $O^p(\mathcal{F})$ and $O^{p'}(\mathcal{F})$ denote the unique minimal saturated fusion subsystems of p-power index over $\mathfrak{hyp}(\mathcal{F})$, or of index prime to p over S, respectively.*

We next recall the definitions of the normalizer fusion systems $N_{\mathcal{F}}^K(Q)$ (cf. [Pg2, § 2.8] or [BLO2, Definitions A.1, A.3]). For any group G, any subgroup $Q \le G$, and any $K \le \mathrm{Aut}(Q)$, define

$$N_G^K(Q) = \big\{ g \in N_G(Q) \,\big|\, c_g|_Q \in K \big\} \,.$$

For example, $N_G^{\mathrm{Aut}(Q)}(Q) = N_G(Q)$ is the usual normalizer, and $N_G^{\{\mathrm{Id}\}}(Q) = C_G(Q)$ is the centralizer.

Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and fix $Q \le S$ and $K \le \mathrm{Aut}(Q)$. We say Q is *fully K-normalized* if for each Q^* which is $\mathcal{F}$-conjugate to Q and each $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q,Q^*)$, $|N_S^K(Q)| \ge |N_S^{\varphi K \varphi^{-1}}(Q^*)|$. Let $N_{\mathcal{F}}^K(Q)$ be the fusion system over $N_S^K(Q)$ defined by setting, for all $P,R \le N_S^K(Q)$,

$$\mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(P,R) = \big\{ \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,R) \,\big|\, \exists\, \bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ,RQ),\ \bar\varphi|_P = \varphi,\ \bar\varphi(Q) = Q,\ \bar\varphi|_Q \in K \big\} \ .$$

As special cases, $C_{\mathcal{F}}(Q) = N_{\mathcal{F}}^{\{\mathrm{Id}\}}(Q)$ and $N_{\mathcal{F}}(Q) = N_{\mathcal{F}}^{\mathrm{Aut}(Q)}(Q)$. By [Pg2, Proposition 2.15] or [AKO, Theorem I.5.5], if Q is fully K-normalized in $\mathcal{F}$, then $N_{\mathcal{F}}^K(Q)$ is a saturated fusion system. If $K \ge \mathrm{Inn}(Q)$, then Q is normal in $N_{\mathcal{F}}^K(Q)$ by definition.

This construction is motivated by the following proposition.

Proposition 1.24 ([AKO, Proposition I.5.4])**.** *Fix a finite group G and $S \in \mathrm{Syl}_p(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. For $Q \le S$ and $K \le \mathrm{Aut}(Q)$, Q is fully K-normalized in $\mathcal{F}$ if and only if $N_S^K(Q) \in \mathrm{Syl}_p(N_G^K(Q))$. When this is the case, then $N_{\mathcal{F}}^K(Q) = \mathcal{F}_{N_S^K(Q)}(N_G^K(Q))$.*

We now give some examples of normal fusion subsystems: examples which will be important later in the paper. The most obvious example is the inclusion $\mathcal{F}_{S_0}(G_0) \subseteq \mathcal{F}_S(G)$ when $G_0 \trianglelefteq G$ are finite groups and $S_0 = S \cap G_0$, but this case will be handled later (Proposition 1.28).

Proposition 1.25. *The following hold for any saturated fusion system $\mathcal{F}$ over a finite p-group S.*

(a) $O^p(\mathcal{F}) \trianglelefteq \mathcal{F}$.

(b) $O^{p'}(\mathcal{F}) \trianglelefteq \mathcal{F}$.

(c) *For each $Q \trianglelefteq \mathcal{F}$ and each $K \trianglelefteq \mathrm{Aut}(Q)$, $N_{\mathcal{F}}^K(Q) \trianglelefteq \mathcal{F}$.*

Proof. **(a,b)** See [AKO, Theorems I.7.4 & I.7.7].

(c) Assume $g, h \in S$ are $\mathcal{F}$-conjugate. Since $Q \trianglelefteq \mathcal{F}$, there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle Q, g\rangle, \langle Q, h\rangle)$ such that $\varphi(g) = h$ and $\varphi(Q) = Q$. Set $\varphi_0 = \varphi|_Q \in \mathrm{Aut}_{\mathcal{F}}(Q)$. Then $c_h = \varphi_0 c_g \varphi_0^{-1}$ in $\mathrm{Aut}_{\mathcal{F}}(Q)$. Since $K \trianglelefteq \mathrm{Aut}(Q)$, $g \in N_S^K(Q)$ (i.e., $c_g \in K$) if and only if $h \in N_S^K(Q)$. This proves that $N_S^K(Q)$ is strongly closed in $\mathcal{F}$.

Set $\mathrm{Aut}_S^K(Q) = K \cap \mathrm{Aut}_S(Q)$ and $\mathrm{Aut}_{\mathcal{F}}^K(Q) = K \cap \mathrm{Aut}_{\mathcal{F}}(Q)$. Fix $P, R \le N_S^K(Q)$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, R)$. Since $Q \trianglelefteq \mathcal{F}$, there is $\widehat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(QP, QR)$ such that $\widehat{\varphi}|_P = \varphi$ and $\widehat{\varphi}(Q) = Q$. Set $\varphi_0 = \widehat{\varphi}|_Q \in \mathrm{Aut}_{\mathcal{F}}(Q)$. Since $K \trianglelefteq \mathrm{Aut}(Q)$ and $\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$, $\mathrm{Aut}_S^K(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}^K(Q))$. Since $\varphi_0 \mathrm{Aut}_S^K(Q) \varphi_0^{-1}$ is contained in $\mathrm{Aut}_{\mathcal{F}}^K(Q)$ (again since K is normal), there is $\chi \in \mathrm{Aut}_{\mathcal{F}}^K(Q)$ such that

$$(\chi\varphi_0)\mathrm{Aut}_S^K(Q)(\chi\varphi_0)^{-1} = \mathrm{Aut}_S^K(Q)\ .$$

By the extension axiom, there is $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_S^K(Q){\cdot}Q, S)$ such that $\bar{\varphi}|_Q = \chi\varphi_0$. Furthermore, $\bar{\varphi}(N_S^K(Q)) = N_S^K(Q)$ since $\chi\varphi_0$ normalizes $\mathrm{Aut}_S^K(Q)$.

Set $P_1 = \bar{\varphi}(P)$, $\widehat{\psi} = \widehat{\varphi} \circ (\bar{\varphi}|_{PQ,P_1Q})^{-1} \in \mathrm{Hom}_{\mathcal{F}}(P_1Q, RQ)$, and $\psi = \widehat{\psi}|_{P_1,R}$. Then $\widehat{\psi}|_Q = \chi^{-1}$, so $\psi \in \mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(P_1, R)$, and $\varphi = \psi \circ \bar{\varphi}|_{P,P_1}$. This proves condition (ii) in Definition 1.18. The last condition — the subsystem $N_{\mathcal{F}}^K(Q)$ is invariant under conjugation by elements of $\mathrm{Aut}_{\mathcal{F}}(N_S^K(Q))$ — is clear. □

We just showed that $O^{p'}(\mathcal{F})$ is normal in $\mathcal{F}$ for any $\mathcal{F}$. The following lemma can be thought of as a "converse" to this.

Lemma 1.26. *Assume $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ is a normal pair of fusion systems over the same finite p-group S. Then $\mathcal{F}_0$ has index prime to p in $\mathcal{F}$, and thus $\mathcal{F}_0 \supseteq O^{p'}(\mathcal{F})$.*

Proof. If $P, Q \le S$ are $\mathcal{F}$-conjugate, then by condition (ii) in Definition 1.18, P is $\mathcal{F}_0$-conjugate to $\alpha(Q)$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$. Since $|N_S(Q)| = |N_S(\alpha(Q))|$, this shows that P is fully normalized in $\mathcal{F}_0$ if and only if it is fully normalized in $\mathcal{F}$.

If $P \le S$ is fully normalized in $\mathcal{F}_0$ (and hence in $\mathcal{F}$), then $\mathrm{Aut}_{\mathcal{F}_0}(P)$ contains $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$. Also, since $\mathrm{Aut}_{\mathcal{F}_0}(P)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(P)$ by Lemma 1.20(b), $\mathrm{Aut}_{\mathcal{F}_0}(P)$ contains $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$. Since this property depends only on the isomorphism class of P in $\mathcal{F}_0$, it holds for all $P \le S$. So $\mathcal{F}_0$ has index prime to p in $\mathcal{F}$ by Definition 1.21(b). □

1.5. Normal linking subsystems.

The following definition of a normal linking subsystem seems to be the most appropriate one for our needs here; it is also the one used in [O3]. In the following definition (and elsewhere), whenever we say that $\mathcal{L}_0 \subseteq \mathcal{L}$ is a *pair of linking systems* associated to $\mathcal{F}_0 \subseteq \mathcal{F}$

(or $\mathcal{L}_0$ is a *linking subsystem*), it is understood not only that $\mathcal{L}_0$ is a subcategory of $\mathcal{L}$, but also that the structural functors for $\mathcal{L}_0$ are the restrictions of the structural functors $\mathcal{T}_{\mathrm{Ob}(\mathcal{L})}(S) \xrightarrow{\ \delta\ } \mathcal{L} \xrightarrow{\ \pi\ } \mathcal{F}$ for $\mathcal{L}$.

Definition 1.27. *Fix a pair of saturated fusion systems $\mathcal{F}_0 \subseteq \mathcal{F}$ over finite p-groups $S_0 \trianglelefteq S$ such that $\mathcal{F}_0 \trianglelefteq \mathcal{F}$, and let $\mathcal{L}_0 \subseteq \mathcal{L}$ be a pair of associated linking systems. Then $\mathcal{L}_0$ is* normal *in $\mathcal{L}$ ($\mathcal{L}_0 \trianglelefteq \mathcal{L}$) if*

(i) $\mathrm{Ob}(\mathcal{L}) = \{P \le S \mid P \cap S_0 \in \mathrm{Ob}(\mathcal{L}_0)\}$*;*

(ii) *for all $P, Q \in \mathrm{Ob}(\mathcal{L}_0)$ and $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$, there are morphisms $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S_0)$ and $\psi_0 \in \mathrm{Mor}_{\mathcal{L}_0}(\gamma(P), Q)$ such that $\psi = \psi_0 \circ \gamma|_{P,\gamma(P)}$; and*

(iii) *for all $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S_0)$ and $\psi \in \mathrm{Mor}(\mathcal{L}_0)$, $\gamma\psi\gamma^{-1} \in \mathrm{Mor}(\mathcal{L}_0)$.*

Here, for $P, Q \in \mathrm{Ob}(\mathcal{L}_0)$, and $\psi \in \mathrm{Mor}_{\mathcal{L}_0}(P,Q)$, we write $\gamma(P) = \pi(\gamma)(P)$, $\gamma(Q) = \pi(\gamma)(Q)$, and

$$\gamma\psi\gamma^{-1} = \gamma|_{Q,\gamma(Q)} \circ \psi \circ (\gamma|_{P,\gamma(P)})^{-1} \in \mathrm{Mor}_{\mathcal{L}}(\gamma(P), \gamma(Q))$$

for short. For any such pair $\mathcal{L}_0 \trianglelefteq \mathcal{L}$, the quotient group $\mathcal{L}/\mathcal{L}_0$ is defined by setting

$$\mathcal{L}/\mathcal{L}_0 = \mathrm{Aut}_{\mathcal{L}}(S_0)/\mathrm{Aut}_{\mathcal{L}_0}(S_0).$$

Also, $\mathcal{L}_0$ is centric *in $\mathcal{L}$ if for each $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S_0) \smallsetminus \mathrm{Aut}_{\mathcal{L}_0}(S_0)$, there is $\psi \in \mathrm{Mor}(\mathcal{L}_0)$ such that $\gamma\psi\gamma^{-1} \ne \psi$.*

In the situation of Definition 1.27, we will sometimes say that $\mathcal{L}_0 \trianglelefteq \mathcal{L}$ is a *normal pair of linking systems* associated to $\mathcal{F}_0 \trianglelefteq \mathcal{F}$, or just that $(S_0, \mathcal{F}_0, \mathcal{L}_0) \trianglelefteq (S, \mathcal{F}, \mathcal{L})$ is a *normal pair.*

One source of normal pairs of linking systems is a normal pair of finite groups; at least, under certain conditions.

Proposition 1.28. *Fix a pair $G_0 \trianglelefteq G$ of finite groups, choose $S \in \mathrm{Syl}_p(G)$, and set $S_0 = S \cap G_0 \in \mathrm{Syl}_p(G_0)$. Then $\mathcal{F}_{S_0}(G_0) \trianglelefteq \mathcal{F}_S(G)$. Assume in addition that $\mathcal{H}_0$ and $\mathcal{H}$ are sets of subgroups of S_0 and S, respectively, such that $\mathcal{L}_{S_0}^{\mathcal{H}_0}(G_0)$ and $\mathcal{L}_S^{\mathcal{H}}(G)$ are linking systems associated to $\mathcal{F}_{S_0}(G_0)$ and $\mathcal{F}_S(G)$, and such that $\mathcal{H} = \{P \le S \mid P \cap S_0 \in \mathcal{H}_0\}$. Then $\mathcal{L}_{S_0}^{\mathcal{H}_0}(G_0) \trianglelefteq \mathcal{L}_S^{\mathcal{H}}(G)$.*

Proof. Fix $P, Q \le S_0$ and $g \in N_G(P,Q)$. Then gS_0g^{-1} is another Sylow p-subgroup of G_0, so there is some $h \in G_0$ such that $(h^{-1}g)S_0(h^{-1}g)^{-1} = S_0$. Set $a = h^{-1}g$; thus $g = ha$ where $a \in N_G(S_0)$ and $h \in G_0$. Thus $c_g = c_h \circ c_a \in \mathrm{Hom}_G(P,Q)$, where $c_a \in \mathrm{Aut}_G(S_0)$ and $c_h \in \mathrm{Hom}_{G_0}(aPa^{-1}, Q)$. This proves condition (ii) in the definition of a normal fusion system; and condition (ii) in Definition 1.27 follows in a similar way. The other conditions clearly hold. □

When $(S_0, \mathcal{F}_0, \mathcal{L}_0) \trianglelefteq (S, \mathcal{F}, \mathcal{L})$ is a normal pair, then for each $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S_0)$, we let $c_\gamma \in \mathrm{Aut}(\mathcal{L}_0)$ denote the automorphism which sends P to $\gamma(P) = \pi(\gamma)(P)$ and sends $\psi \in \mathrm{Mor}_{\mathcal{L}_0}(P,Q)$ to $(\gamma|_{Q,\gamma(Q)}) \circ \psi \circ (\gamma|_{P,\gamma(P)})^{-1}$. The next lemma describes how to tell, in terms only of the fusion system $\mathcal{F}$, whether or not $c_{\delta(g)} = \mathrm{Id}_{\mathcal{L}_0}$ for $g \in S$ ($\delta = \delta_{S_0}$).

When $\mathcal{L}$ is a linking system associated to $\mathcal{F}$, and $A \trianglelefteq \mathcal{F}$, we say that an automorphism α of $\mathcal{L}$ is the *identity modulo A* if for each $P, Q \in \mathrm{Ob}(\mathcal{L})$ which contain A and each $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$, $\alpha(P) = P$, $\alpha(Q) = Q$, and $\alpha(\psi) = \psi \circ \delta_P(a)$ for some $a \in A$.

Lemma 1.29. *Let $(S_0, \mathcal{F}_0, \mathcal{L}_0) \trianglelefteq (S, \mathcal{F}, \mathcal{L})$ be a normal pair such that all objects in $\mathcal{L}$ are $\mathcal{F}$-centric. Fix $A \trianglelefteq \mathcal{F}_0$. Then for $g \in S$, $c_{\delta(g)} \in \mathrm{Aut}(\mathcal{L}_0)$ is the identity modulo A if and only if $[g, S_0] \le A$, and for each $P, Q \le S_0$ and $\varphi \in \mathrm{Mor}_{\mathcal{F}_0}(P,Q)$, φ extends to some $\bar\varphi \in \mathrm{Mor}_{\mathcal{F}}(\langle PA, g\rangle, \langle QA, g\rangle)$ such that $\bar\varphi(g) \in gA$.*

Proof. Fix $g \in S$. Set $\gamma = \delta_{S_0}(g)$ and $B = \langle g, A\rangle$ for short.

Assume $c_\gamma \in \mathrm{Aut}(\mathcal{L}_0)$ is the identity modulo A. Then $[g, S_0] \le A$ since $[\gamma, \delta_{S_0}(s)] \in \delta_{S_0}(A)$ for $s \in S_0$ (and δ_{S_0} is injective by Proposition 1.11(c)). Since $\mathcal{F}_0$ is generated by morphisms between objects of $\mathcal{L}_0$ which contain A (by Theorem 1.3 and Proposition 1.5, and since $A \trianglelefteq \mathcal{F}_0$), it suffices to prove the extension property for such morphisms. Fix $P, Q \in \mathrm{Ob}(\mathcal{L}_0)$ such that $A \le P$ and $A \le Q$, fix $\varphi \in \mathrm{Hom}_{\mathcal{F}_0}(P, Q)$, and choose a lifting of φ to $\psi \in \mathrm{Mor}_{\mathcal{L}_0}(P, Q)$. By assumption, $\delta_Q(g) \circ \psi \circ \delta_P(g)^{-1} = \psi \circ \delta_P(a)$ for some $a \in A$. So by Proposition 1.11(e), there is a unique morphism $\bar\psi \in \mathrm{Mor}_{\mathcal{L}}(PB, QB)$ such that $\bar\psi|_{P,Q} = \psi$, and $\delta_{QB}(g) \circ \bar\psi \circ \delta_{PB}(ag)^{-1} = \bar\psi$ by the uniqueness of the extension. Set $\bar\varphi = \pi(\bar\psi)$; then $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(PB, QB)$, $\bar\varphi|_P = \varphi$, and $\bar\varphi(ag) = g$ (so $\bar\varphi(g) \in gA$) by axiom (C).

Now assume $[g, S_0] \le A$, and g has the above extension property: each $\varphi \in \mathrm{Hom}_{\mathcal{F}_0}(P, Q)$ extends to $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(PB, QB)$ such that $\bar\varphi(g) \in gA$. We claim $c_\gamma \in \mathrm{Aut}(\mathcal{L}_0)$ is the identity modulo A. Since $[g, S_0] \le A$, $gPg^{-1} = P$ for all $P \in \mathrm{Ob}(\mathcal{L}_0)$ which contain A. Fix $\psi \in \mathrm{Mor}_{\mathcal{L}_0}(P, Q)$, where $A \le P, Q$. By assumption, $\pi(\psi)$ extends to some $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(PB, QB)$ such that $\bar\varphi(g) \in gA$, and this lifts to $\widehat{\psi} \in \mathrm{Mor}_{\mathcal{L}}(PB, QB)$. Since P is $\mathcal{F}$-centric, $\widehat{\psi}|_{P,Q} = \psi \circ \delta_P(x)$ for some $x \in Z(P)$. Upon replacing $\widehat{\psi}$ by $\widehat{\psi} \circ \delta_{PB}(x)^{-1}$ and $\bar\varphi$ by $\bar\varphi \circ c_x^{-1}$, we can assume $\widehat{\psi}|_{P,Q} = \psi$. By axiom (C), the conjugation action of $\delta_S(g)$ fixes $\widehat{\psi}$ modulo $\delta_{PB}(A)$, and hence $c_\gamma \in \mathrm{Aut}(\mathcal{L}_0)$ sends ψ into $\psi \circ \delta_P(A)$. □

The next lemma describes another way to construct normal pairs of linking systems.

Lemma 1.30. *Fix a normal pair of fusion systems $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ over p-groups $S_0 \trianglelefteq S$. Let $\mathcal{H}_0$ be a set of subgroups of S_0 such that*

- *$\mathcal{H}_0$ is closed under $\mathcal{F}$-conjugacy and overgroups, and contains all subgroups of S_0 which are $\mathcal{F}_0$-centric and $\mathcal{F}_0$-radical; and*
- *$\mathcal{H} \stackrel{\mathrm{def}}{=} \{P \le S \mid P \cap S_0 \in \mathcal{H}_0\}$ is contained in the set of $\mathcal{F}$-centric subgroups.*

Assume $\mathcal{F}$ has an associated centric linking system $\mathcal{L}^c$. Let $\mathcal{L} \subseteq \mathcal{L}^c$ be the full subcategory with object set $\mathcal{H}$. Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be the subcategory with object set $\mathcal{H}_0$, where for $P, Q \in \mathcal{H}_0$,

$$\mathrm{Mor}_{\mathcal{L}_0}(P, Q) = \{\psi \in \mathrm{Mor}_{\mathcal{L}}(P, Q) \mid \pi(\psi) \in \mathrm{Hom}_{\mathcal{F}_0}(P, Q)\}\ . \tag{3}$$

Then $\mathcal{L}_0 \trianglelefteq \mathcal{L}$ is a normal pair of linking systems associated to $\mathcal{F}_0 \trianglelefteq \mathcal{F}$. For any such pair $\mathcal{L}_0 \trianglelefteq \mathcal{L}$ with $\mathrm{Ob}(\mathcal{L}_0) = \mathcal{H}_0$ and $\mathrm{Ob}(\mathcal{L}) = \mathcal{H}$, $\mathcal{L}_0$ is centric in $\mathcal{L}$.

Proof. Since $\mathrm{Ob}(\mathcal{L})$ is closed under $\mathcal{F}$-conjugacy and under overgroups, and contains all subgroups which are $\mathcal{F}$-centric and $\mathcal{F}$-radical by Lemma 1.20(d) and the assumptions on $\mathcal{H}_0$, $\mathcal{L}$ is a linking system associated to $\mathcal{F}$. Since all objects in $\mathcal{L}_0$ are $\mathcal{F}$-centric, they are also $\mathcal{F}_0$-centric, and hence fully centralized in $\mathcal{F}_0$. Axiom (A) for $\mathcal{L}_0$ thus follows from axiom (A) for $\mathcal{L}$, together with the assumptions on $\mathcal{H}_0 = \mathrm{Ob}(\mathcal{L}_0)$. Axioms (B) and (C) for $\mathcal{L}_0$ follow immediately from those for $\mathcal{L}$, and $\mathcal{L}_0$ is thus a linking system associated to $\mathcal{F}_0$.

Condition (i) in Definition 1.27 holds by assumption, while conditions (ii) and (iii) follow from (3) and since $\mathcal{F}_0$ is normal in $\mathcal{F}$. Thus $\mathcal{L}_0 \trianglelefteq \mathcal{L}$.

Fix any such pair $\mathcal{L}_0 \trianglelefteq \mathcal{L}$ associated to $\mathcal{F}_0 \trianglelefteq \mathcal{F}$. Assume $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S_0)$ is such that $\gamma\psi\gamma^{-1} = \psi$ for each $\psi \in \mathrm{Mor}(\mathcal{L}_0)$. Since $\gamma(\delta_{S_0}(g))\gamma^{-1} = \delta_{S_0}(\pi(\gamma)(g))$ for $g \in S_0$ by axiom (C) for the linking system $\mathcal{L}$, $\pi(\gamma) = \mathrm{Id}_{S_0}$. Since $S_0 \in \mathcal{H}_0$ is $\mathcal{F}$-centric, this means that $\gamma = \delta_{S_0}(z)$ for some $z \in Z(S_0)$, and in particular, that $\gamma \in \mathrm{Aut}_{\mathcal{L}_0}(S_0)$. So $\mathcal{L}_0$ is centric in $\mathcal{L}$. □

We now list the examples of normal pairs of linking systems which motivated Definition 1.27, and which we need to refer to later.

Proposition 1.31. *Let $\mathcal{F}$ be a saturated fusion system over the finite p-group S, let $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ be a normal fusion subsystem over $S_0 \trianglelefteq S$, and let $\mathcal{H}_0$ be a set of subgroups of S_0. Assume that $\mathcal{F}$ has an associated centric linking system $\mathcal{L}^c$, and that one of the following three conditions holds.*

(a) $\mathcal{F}_0 = O^p(\mathcal{F})$, $S_0 = \mathfrak{hyp}(\mathcal{F})$, *and $\mathcal{H}_0$ is the set of $\mathcal{F}_0$-centric subgroups of S_0.*

(b) $\mathcal{F}_0 = O^{p'}(\mathcal{F})$, $S_0 = S$, *and $\mathcal{H}_0$ is the set of $\mathcal{F}_0$-centric subgroups of S_0.*

(c) *For some normal p-subgroup $Q \trianglelefteq \mathcal{F}$ and some normal subgroup $K \trianglelefteq \mathrm{Aut}(Q)$ containing $\mathrm{Inn}(Q)$, $\mathcal{F}_0 = N_{\mathcal{F}}^K(Q)$, $S_0 = N_S^K(Q)$, and $\mathcal{H}_0$ is the set of all $\mathcal{F}_0$-centric subgroups of S_0 which contain Q.*

Set $\mathcal{H} = \{P \le S \,|\, P \cap S_0 \in \mathcal{H}_0\}$. Then there is a normal pair of linking systems $\mathcal{L}_0 \trianglelefteq \mathcal{L}$ associated to $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ with $\mathrm{Ob}(\mathcal{L}_0) = \mathcal{H}_0$ and $\mathrm{Ob}(\mathcal{L}) = \mathcal{H}$. For any such normal pair $\mathcal{L}_0 \trianglelefteq \mathcal{L}$, $\mathcal{L}_0$ is centric in $\mathcal{L}$ in cases (b) and (c), and in case (a) if $Z(\mathcal{F}) = 1$. Furthermore, in cases (a) and (b), and in case (c) if $Q = O_p(\mathcal{F})$ and $K = \mathrm{Inn}(Q)$, $\mathcal{H}_0$ is $\mathrm{Aut}(S_0, \mathcal{F}_0)$-invariant, $\mathcal{H}$ is $\mathrm{Aut}(S, \mathcal{F})$-invariant, and $\mathcal{L}_0 \trianglelefteq \mathcal{L}$ can be chosen such that $\mathcal{L}_0$ is $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$-invariant.

Proof. In all cases, $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ by Proposition 1.25. Also, $\mathcal{H}_0$ is $\mathrm{Aut}(S_0, \mathcal{F}_0)$-invariant and $\mathcal{H}$ is $\mathrm{Aut}(S, \mathcal{F})$-invariant: this is clear in cases (a) and (b), and holds in case (c) when $Q = O_p(\mathcal{F})$ (since $Q = O_p(\mathcal{F}_0)$ by Lemma 1.20(e)).

(a) Set $S_0 = \mathfrak{hyp}(\mathcal{F})$, $\mathcal{F}_0 = O^p(\mathcal{F})$, and $\mathcal{H}_0 = \mathrm{Ob}(\mathcal{F}_0^c)$. By [BCGLO2, Theorem 4.3(a)], a subgroup of S_0 is $\mathcal{F}_0$-quasicentric if and only if it is $\mathcal{F}$-quasicentric. In particular, every $\mathcal{F}_0$-centric subgroup of S_0 is $\mathcal{F}$-quasicentric and hence all subgroups in $\mathcal{H}$ are $\mathcal{F}$-quasicentric. By Lemma 1.20(d), $\mathcal{H}$ contains all subgroups which are $\mathcal{F}$-centric and $\mathcal{F}$-radical. By Lemma 1.20(c), $\mathcal{H}_0$ is closed under $\mathcal{F}$-conjugacy, so $\mathcal{H}$ is closed under $\mathcal{F}$-conjugacy (and it is clearly closed under overgroups). Hence if $\mathcal{L}^q \supseteq \mathcal{L}^c$ is the quasicentric linking system which contains $\mathcal{L}^c$ constructed in [BCGLO1, Proposition 3.4], then the full subcategory $\mathcal{L} \subseteq \mathcal{L}^q$ with object set $\mathcal{H}$ is also a linking system associated to $\mathcal{F}$.

By [BCGLO2, Proposition 2.4], there is a unique map $\lambda\colon \mathrm{Mor}(\mathcal{L}^q) \longrightarrow S/S_0$ which sends composites to products and inclusions in $\mathcal{L}^q$ to the identity, and such that $\lambda(\delta_S(g)) = [g]$ for all $g \in S$. By [BCGLO2, Theorem 3.9], there is a p-local finite group $(S_0, \mathcal{F}_0', \mathcal{L}_0)$ where for $P, Q \in \mathrm{Ob}(\mathcal{L}_0)$,

$$\mathrm{Mor}_{\mathcal{L}_0}(P, Q) = \{\psi \in \mathrm{Mor}_{\mathcal{L}^q}(P, Q) \,|\, \lambda(\psi) = 1\}\ . \tag{4}$$

Furthermore, $\mathcal{F}_0'$ is constructed using [BCGLO2, Proposition 3.8] (cf. the proof of [BCGLO2, Theorem 3.9]), and hence (by part (b) of that proposition) it has p-power index in $\mathcal{F}$. Thus $\mathcal{F}_0' = \mathcal{F}_0$ by Theorem 1.22(a).

Now, $\mathrm{Ob}(\mathcal{L}_0) = \mathcal{H}_0$ since $\mathcal{L}_0$ is a centric linking system. Condition (i) in Definition 1.27 holds for $\mathcal{L}_0 \subseteq \mathcal{L}$ by definition of $\mathcal{H}$, condition (iii) ($\gamma \mathcal{L}_0 \gamma^{-1} = \mathcal{L}_0$ for $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S_0)$) holds by construction, and condition (ii) holds since $\lambda|_{\delta_{S_0}(S)}$ is surjective. So $\mathcal{L}_0 \trianglelefteq \mathcal{L}$.

We next check that $\mathcal{L}_0$ is $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$-invariant. Fix $\alpha \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ and set $\beta = \widetilde{\mu}_{\mathcal{L}}(\alpha) \in \mathrm{Aut}(S, \mathcal{F})$. Then $\beta(S_0) = S_0$ since $S_0 = \mathfrak{hyp}(\mathcal{F})$, and $\beta|_{S_0} \in \mathrm{Aut}(S_0, \mathcal{F}_0)$ by the uniqueness of $\mathcal{F}_0$ (Theorem 1.22(a) again). Since $\alpha(P) = \beta(P)$ for $P \in \mathrm{Ob}(\mathcal{L}_0)$ (Lemma 1.15), α sends $\mathrm{Ob}(\mathcal{L}_0) = \mathcal{H}_0$ to itself. By Lemma 1.17, $\alpha = \bar{\alpha}|_{\mathcal{L}}$ for some $\bar{\alpha} \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}^q)$, $\lambda \circ \bar{\alpha} = \bar{\beta} \circ \lambda$ (where $\bar{\beta} \in \mathrm{Aut}(S/S_0)$ is induced by β) by the uniqueness of λ, and hence $\alpha(\mathrm{Mor}(\mathcal{L}_0)) = \mathrm{Mor}(\mathcal{L}_0)$ by (4).

Now let $\mathcal{L}_0 \trianglelefteq \mathcal{L}$ be any normal pair of linking systems associated to $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ with these objects. Assume $Z(\mathcal{F}) = 1$; we must show $\mathcal{L}_0$ is centric in $\mathcal{L}$. Assume $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S_0)$ is such that $\gamma\psi\gamma^{-1} = \psi$ for each $\psi \in \mathrm{Mor}(\mathcal{L}_0)$. Since $\gamma(\delta_{S_0}(g))\gamma^{-1} = \delta_{S_0}(\pi(\gamma)(g))$ for $g \in S_0$ by axiom (C) for the linking system $\mathcal{L}$, $\pi(\gamma) = \mathrm{Id}_{S_0}$. So by axiom (A) (and since S_0 is fully centralized in $\mathcal{F}$), $\gamma = \delta_{S_0}(h)$ for some $h \in C_S(S_0)$.

Let $H \leq C_S(S_0)$ be the subgroup of all h such that the conjugation action of $\delta_{S_0}(h)$ on $\mathcal{L}_0$ is trivial. The p-group $\mathcal{L}/\mathcal{L}_0 = \mathrm{Aut}_{\mathcal{L}}(S_0)/\mathrm{Aut}_{\mathcal{L}_0}(S_0)$ acts on $\delta_{S_0}(H) \cong H$ by conjugation. Let H_0 be the fixed subgroup of this action. Note that $H_0 \leq Z(S)$ since H_0 is fixed by $\delta_{S_0}(S) \leq \mathrm{Aut}_{\mathcal{L}}(S_0)$. Fix $h \in H_0$, and set $\bar\gamma = \delta_S(h)$. Let $\bar\gamma(P)$ and $\bar\gamma\psi\bar\gamma^{-1}$ be as in Definition 1.27, but this time for all $P \in \mathrm{Ob}(\mathcal{L})$ and $\psi \in \mathrm{Mor}(\mathcal{L})$. For $P \in \mathrm{Ob}(\mathcal{L})$, $\bar\gamma(P) = hPh^{-1} = P$. Also, $\bar\gamma\psi\bar\gamma^{-1} = \psi$ for all $\psi \in \mathrm{Aut}_{\mathcal{L}}(S_0)$ by definition of H_0, $\bar\gamma\psi\bar\gamma^{-1} = \psi$ for $\psi \in \mathrm{Mor}(\mathcal{L}_0)$ by definition of H, and hence conjugation by $\bar\gamma$ is the identity on morphisms in $\mathcal{L}$ between subgroups in $\mathcal{H}_0$ by condition (ii) in Definition 1.27. By Proposition 1.11(f), for each $P, Q \in \mathcal{H}$, the restriction map from $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ to $\mathrm{Mor}_{\mathcal{L}}(P \cap S_0, Q \cap S_0)$ is injective, and hence $\bar\gamma\psi\bar\gamma^{-1} = \psi$ for all $\psi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$. Thus conjugation by $\bar\gamma = \delta_S(h)$ is the identity on $\mathcal{L}$, and so $h \in Z(\mathcal{F}) = 1$ by Lemma 1.14(a). Since $H_0 = 1$ is the fixed subgroup of an action of the p-group $\mathcal{L}/\mathcal{L}_0$ on the p-group H, $H = 1$, and so $\mathcal{L}_0$ is centric in $\mathcal{L}$.

(b) Set $\mathcal{F}_0 = O^{p'}(\mathcal{F})$. By [BCGLO2, Proposition 3.8(c)], a subgroup of S is $\mathcal{F}_0$-centric if and only if it is $\mathcal{F}$-centric. So upon letting $\mathcal{H}_0 = \mathcal{H}$ be the set of all $\mathcal{F}$-centric subgroups of S, the hypotheses of Lemma 1.30 are satisfied. By the lemma, there is a normal pair $\mathcal{L}_0 \trianglelefteq \mathcal{L}$ of linking systems associated to $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ with object set $\mathcal{H}_0 = \mathcal{H}$; and for any such pair, $\mathcal{L}_0$ is centric in $\mathcal{L}$. By the explicit description of $\mathcal{L}_0$ (formula (3) in Lemma 1.30), $\mathcal{L}_0$ is $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$-invariant.

(c) Fix $Q \trianglelefteq \mathcal{F}$ and $\mathrm{Inn}(Q) \leq K \trianglelefteq \mathrm{Aut}(Q)$, and set $S_0 = N_S^K(Q)$ and $\mathcal{F}_0 = N_{\mathcal{F}}^K(Q)$. Let $\mathcal{H}_0$ be the set of all $\mathcal{F}_0$-centric subgroups of S_0 which contain Q. We first check that all subgroups in $\mathcal{H} = \{P \leq S \,|\, P \cap S_0 \in \mathcal{H}_0\}$ are $\mathcal{F}$-centric; it suffices to show this for subgroups in $\mathcal{H}_0$. By Lemma 1.20(c) (and since $\mathcal{F}_0 \trianglelefteq \mathcal{F}$), the set of $\mathcal{F}_0$-centric subgroups, and hence also the set $\mathcal{H}_0$, are closed under $\mathcal{F}$-conjugacy. For each $P \in \mathcal{H}_0$, $C_S(P) \leq C_S(Q) \leq S_0$ since $P \geq Q$, and hence $C_S(P) = C_{S_0}(P) = Z(P)$ since P is $\mathcal{F}_0$-centric. Since this holds for all subgroups $\mathcal{F}$-conjugate to P, we conclude that P is $\mathcal{F}$-centric.

We just saw that $\mathcal{H}_0$ is closed under $\mathcal{F}$-conjugacy, and it is clearly closed under overgroups. Since $Q \trianglelefteq \mathcal{F}_0$, each subgroup of S_0 which is $\mathcal{F}_0$-centric and $\mathcal{F}_0$-radical contains Q by Proposition 1.5, and thus lies in $\mathcal{H}_0$. So by Lemma 1.30, there is a normal pair $\mathcal{L}_0 \trianglelefteq \mathcal{L}$ of linking systems associated to $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ with object sets $\mathcal{H}_0$ and $\mathcal{H}$, and for any such pair, $\mathcal{L}_0$ is centric in $\mathcal{L}$. If $Q = O_p(\mathcal{F})$ and $K = \mathrm{Inn}(Q)$, then $Q = O_p(\mathcal{F}_0)$ by Lemma 1.20(e), and so $\mathcal{F}_0$ is $\mathrm{Aut}(S, \mathcal{F})$-invariant. Hence $\mathcal{L}_0$ is $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$-invariant by the explicit description of $\mathcal{L}_0$ in Lemma 1.30. □

2. Reduced fusion systems and tame fusion systems

Throughout this section, p denotes a fixed prime, and we work with fusion systems over finite p-groups. We first define reduced fusion systems and the reduction of a fusion system. We then define tame fusion systems, and prove that a reduced fusion system is tame if every saturated fusion system which reduces to it is realizable (Theorem B). We then make a digression to look at the existence of linking systems in certain situations, before proving that all fusion systems whose reduction is tame are realizable (Theorem A). We thus end up with a way to "detect" exotic fusion systems in general while looking only at reduced fusion systems.

2.1. **Reduced fusion systems and reductions of fusion systems.**

We begin with the definition of a reduced fusion system, and the reduction of an (arbitrary) fusion system. See Proposition 1.8 and the discussion before that for the definition and properties of quotient fusion systems.

Definition 2.1. *A* reduced fusion system *is a saturated fusion system $\mathcal{F}$ such that*

- *$\mathcal{F}$ has no nontrivial normal p-subgroups,*
- *$\mathcal{F}$ has no proper normal subsystem of p-power index, and*
- *$\mathcal{F}$ has no proper normal subsystem of index prime to p.*

Equivalently, $\mathcal{F}$ is reduced if $O_p(\mathcal{F})=1$, $O^p(\mathcal{F})=\mathcal{F}$, and $O^{p'}(\mathcal{F})=\mathcal{F}$.

For any saturated fusion system $\mathcal{F}$, the reduction *of $\mathcal{F}$ is the fusion system $\mathfrak{red}(\mathcal{F})$ defined as follows. Set $\mathcal{F}_0=C_{\mathcal{F}}(O_p(\mathcal{F}))/Z(O_p(\mathcal{F}))$, and let $\mathcal{F}_0\supseteq\mathcal{F}_1\supseteq\mathcal{F}_2\supseteq\cdots\supseteq\mathcal{F}_m$ be such that $\mathcal{F}_i=O^p(\mathcal{F}_{i-1})$ if i is odd, $\mathcal{F}_i=O^{p'}(\mathcal{F}_{i-1})$ if i is even, and $O^p(\mathcal{F}_m)=O^{p'}(\mathcal{F}_m)=\mathcal{F}_m$. Then $\mathfrak{red}(\mathcal{F})=\mathcal{F}_m$.*

Fix any $\mathcal{F}$, and set $Q=O_p(\mathcal{F})$ for short. By definition of centralizer fusion systems, every morphism in $C_{\mathcal{F}}(Q)$ extends to a morphism in $\mathcal{F}$ which is the identity on Q, and hence to a morphism in $C_{\mathcal{F}}(Q)$ which is the identity on $Z(Q)$. This proves that $Z(Q)$ is always central in $C_{\mathcal{F}}(Q)$, and hence that $\mathcal{F}_0=C_{\mathcal{F}}(Q)/Z(Q)$ is well defined as a fusion system.

What is important in the last part of the definition of $\mathfrak{red}(\mathcal{F})$ is that we give an explicit procedure for successively applying $O^p(-)$ and $O^{p'}(-)$, starting with $\mathcal{F}_0$, until neither makes the fusion system any smaller. It seems likely that the final result $\mathfrak{red}(\mathcal{F})$ is independent of the order in which we apply these reductions, but we have not shown this, and do not need to know it when proving the results in this section.

Clearly, for these definitions to make sense, we want $\mathfrak{red}(\mathcal{F})$ to always be reduced.

Proposition 2.2. *The reduction of any saturated fusion system is reduced.*

For later reference, we also state the following, more technical result, which will be proven together with Proposition 2.2.

Lemma 2.3. *Let $\mathcal{F}$ be a saturated fusion system. Set $Q=O_p(\mathcal{F})$ and $\mathcal{F}_0=C_{\mathcal{F}}(Q)/Z(Q)$. Let $\mathcal{F}_0\supseteq\mathcal{F}_1\supseteq\cdots\supseteq\mathcal{F}_m=\mathfrak{red}(\mathcal{F})$ be such that for each i, $\mathcal{F}_i=O^p(\mathcal{F}_{i-1})$ or $\mathcal{F}_i=O^{p'}(\mathcal{F}_{i-1})$. Then $O_p(\mathcal{F}_i)=1$ for each $0\le i\le m$.*

Proof. Fix $\mathcal{F}$, and let $Q\trianglelefteq\mathcal{F}$ and the $\mathcal{F}_i$ be as above. Since $C_{\mathcal{F}}(Q)\trianglelefteq\mathcal{F}$ by Proposition 1.25(c), $O_p(C_{\mathcal{F}}(Q))\le O_p(\mathcal{F})=Q$ by Lemma 1.20(e). Hence $O_p(C_{\mathcal{F}}(Q))=Z(Q)$. We just saw that $Z(Q)$ is central in $C_{\mathcal{F}}(Q)$. So by Proposition 1.8, a subgroup $P/Z(Q)\le C_S(Q)/Z(Q)$ is normal in $C_{\mathcal{F}}(Q)/Z(Q)$ only if $P\trianglelefteq C_{\mathcal{F}}(Q)$. Thus $O_p(\mathcal{F}_0)=Z(Q)/Z(Q)=1$.

By definition, $O^p(\mathfrak{red}(\mathcal{F}))=O^{p'}(\mathfrak{red}(\mathcal{F}))=\mathfrak{red}(\mathcal{F})$. By Proposition 1.25(a,b), $\mathcal{F}_i\trianglelefteq\mathcal{F}_{i-1}$ for each $i\ge1$. So by Lemma 1.20(e) again, $O_p(\mathcal{F}_i)=1$ if $O_p(\mathcal{F}_{i-1})=1$. Since $O_p(\mathcal{F}_0)=1$, this proves that $O_p(\mathcal{F}_i)=1$ for each i. In particular, $O_p(\mathfrak{red}(\mathcal{F}))=1$, and hence $\mathfrak{red}(\mathcal{F})$ is reduced. □

A saturated fusion system $\mathcal{F}$ is *constrained* if there is a normal subgroup $Q\trianglelefteq\mathcal{F}$ which is $\mathcal{F}$-centric (cf. [BCGLO1, § 4]).

Proposition 2.4. *For any saturated fusion system $\mathcal{F}$, $\mathfrak{red}(\mathcal{F}) = 1$ (the fusion system over the trivial group) if and only if $\mathcal{F}$ is constrained.*

Proof. If $\mathcal{F}$ is constrained, then clearly $\mathfrak{red}(\mathcal{F}) = 1$. Conversely, assume $\mathcal{F}$ is a fusion system over a finite p-group S such that $\mathfrak{red}(\mathcal{F}) = 1$. Set $Q = O_p(\mathcal{F})$ and $\mathcal{F}_0 = C_{\mathcal{F}}(Q)/Z(Q)$. If $\mathcal{F}_0 = 1$, then $C_{\mathcal{F}}(Q)$ is a fusion system over $Z(Q)$, and hence $C_S(Q) = Z(Q)$. So Q is $\mathcal{F}$-centric, and hence $\mathcal{F}$ is constrained in this case.

If $\mathcal{F}_0 \neq 1$, then there is a sequence of fusion subsystems $1 = \mathcal{F}_m \subsetneqq \mathcal{F}_{m-1} \subsetneqq \cdots \subsetneqq \mathcal{F}_0$ such that for each i, $\mathcal{F}_{i+1} = O^p(\mathcal{F}_i)$ or $\mathcal{F}_{i+1} = O^{p'}(\mathcal{F}_i)$. By Lemma 2.3, $O_p(\mathcal{F}_i) = 1$ for each $0 \le i \le m$. Since $\mathcal{F}_{m-1} \neq 1$, it is a fusion system over a p-group $S_{m-1} \neq 1$, so $O^{p'}(\mathcal{F}_{m-1}) \neq 1$ (it is over the same p-group), which implies $O^p(\mathcal{F}_{m-1}) = 1$. Thus $\mathfrak{hyp}(\mathcal{F}_{m-1}) = 1$ by Definition 1.21(a), so there are no nontrivial automorphisms of order prime to p in $\mathcal{F}_{m-1}$, and $\mathcal{F}_{m-1}$ is the fusion system of the p-group S_{m-1}. This is impossible, since it would imply $O_p(\mathcal{F}_{m-1}) = S_{m-1} \neq 1$, and we conclude $\mathcal{F}_0 = 1$. □

2.2. **Tame fusion systems and the proof of Theorem B.**

Assume $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with $S \in \mathrm{Syl}_p(G)$. Let $\mathcal{H}$ be an $\mathrm{Aut}(G, S)$-invariant set of G-quasicentric subgroups of S such that $\mathcal{L} \stackrel{\mathrm{def}}{=} \mathcal{L}_S^{\mathcal{H}}(G)$ is a linking system associated to $\mathcal{F}$ (i.e. $\mathcal{H}$ is closed under overgroups and contains all $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups). Define the homomorphism

$$\widetilde{\kappa}_G^{\mathcal{H}}:\mathrm{Aut}(G, S) \longrightarrow \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L})$$

as follows. For $\beta \in \mathrm{Aut}(G, S)$, $\widetilde{\kappa}_G^{\mathcal{H}}(\beta)$ sends P to $\beta(P)$ and sends $[a] \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ (for $a \in N_G(P, Q)$) to $[\beta(a)]$.

For any $g \in N_G(S)$, $\widetilde{\kappa}_G^{\mathcal{H}}$ sends $c_g \in \mathrm{Aut}(G, S)$ to $c_{[g]} \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L})$, where $[g] \in \mathrm{Aut}_{\mathcal{L}}(S)$ is the class of g. Thus by Lemma 1.14, $\widetilde{\kappa}_G^{\mathcal{H}}$ induces a homomorphism

$$\kappa_G^{\mathcal{H}}: \mathrm{Out}(G) \longrightarrow \mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$$

by sending the class of β to the class of $\widetilde{\kappa}_G^{\mathcal{H}}(\beta)$. When $\mathcal{L} = \mathcal{L}_S^c(G)$ is the centric linking system of G, we write $\widetilde{\kappa}_G = \widetilde{\kappa}_G^{\mathcal{H}}$ and $\kappa_G = \kappa_G^{\mathcal{H}}$ for short.

Note that when $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{L} = \mathcal{L}_S^{\mathcal{H}}(G)$ as above, $\widetilde{\mu}_G^{\mathcal{H}} \circ \widetilde{\kappa}_G^{\mathcal{H}}: \mathrm{Aut}(G, S) \longrightarrow \mathrm{Aut}(S, \mathcal{F})$ is the restriction homomorphism.

Definition 2.5. *A saturated fusion system $\mathcal{F}$ over S is* tame *if there is a finite group G which satisfies:*

- *$S \in \mathrm{Syl}_p(G)$ and $\mathcal{F} \cong \mathcal{F}_S(G)$; and*
- *$\kappa_G: \mathrm{Out}(G) \longrightarrow \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G))$ is split surjective.*

In this situation, we say $\mathcal{F}$ is tamely realized *by G.*

The condition that κ_G be split surjective was chosen since, as we will see shortly, that is what is needed in the proof of Theorem B. In contrast, Theorem A would still be true (with essentially the same proof) if we replaced "split surjective" by "an isomorphism" in the above definition.

By Lemma 1.17, $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G)) \cong \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^{\mathcal{H}}(G))$ for any $\mathrm{Aut}(S, \mathcal{F})$-invariant set of objects $\mathcal{H}$ (which satisfies the conditions for $\mathcal{L}_S^{\mathcal{H}}(G)$ to be a linking system). Hence $\mathrm{Ker}(\kappa_G^{\mathcal{H}}) = \mathrm{Ker}(\kappa_G)$, and $\kappa_G^{\mathcal{H}}$ is (split) surjective if and only if κ_G is.

By [BLO1, Theorem B], κ_G is split surjective if and only if the natural map from $\mathrm{Out}(G)$ to $\mathrm{Out}(BG^\wedge_p)$ is split surjective, where $\mathrm{Out}(BG^\wedge_p)$ is the group of homotopy classes of self equivalences of $BG^\wedge_p$. So this gives another way to formulate the definition of tameness.

It is natural to ask whether a tame fusion system $\mathcal{F}$ can always be realized by a finite group G such that κ_G is an isomorphism. We know of no counterexamples to this, but do not know how to prove it either.

We are now ready to prove Theorem B: every reduced fusion system which is not tame is the reduction of some exotic fusion system. This is basically a consequence of the definition of tameness, together with [O3, Theorem 9] which gives a general procedure for constructing extensions of linking systems.

Theorem 2.6. *Let $\mathcal{F}$ be a reduced fusion system which is not tame. Then there is an exotic fusion system $\overline{\mathcal{F}}$ whose reduction is isomorphic to $\mathcal{F}$.*

Proof. If $\mathcal{F}$ is itself exotic, then we take $\overline{\mathcal{F}} = \mathcal{F}$. So assume $\mathcal{F}$ is the fusion system of a finite group, and hence that $\mathcal{F}$ has at least one associated centric linking system $\mathcal{L}$. Assume $\mathcal{L}$ is chosen such that $|\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})|$ is maximal among all centric linking systems associated to $\mathcal{F}$. (All such linking systems are isomorphic to each other by [BLO2, Proposition 3.1] and Theorem A in [O1, O2], but since those results use the classification of finite simple groups, we will not use them here.) Since $\mathcal{F}$ is not tame, it is not the fusion system of any finite group H for which κ_H is split surjective.

Since $Z(\mathcal{F}) = 1$ ($\mathcal{F}$ is reduced), we can identify $\mathrm{Aut}_{\mathcal{L}}(S)$ as a normal subgroup of $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ via its conjugation action on $\mathcal{L}$ (Lemma 1.14(a)). Thus $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) = \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})/\mathrm{Aut}_{\mathcal{L}}(S)$. Let A be any finite abelian p-group on which $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ acts faithfully, and let

$$\nu\colon \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}) \longrightarrow \mathrm{Aut}(A)$$

denote the given action. Thus $\mathrm{Ker}(\nu) = \mathrm{Aut}_{\mathcal{L}}(S)$.

Set $S_0 = A \times S$ and $\mathcal{F}_0 = A \times \mathcal{F}$ ($= \mathcal{F}_A(A) \times \mathcal{F}$). We refer to the beginning of Section 3, or to [BLO2, § 1], for the definition of the product of two fusion systems. Set $\mathcal{L}_0 = A \times \mathcal{L}$: the centric linking system associated to $\mathcal{F}_0$ whose objects are the subgroups $A \times P \le S_0$ for $P \in \mathrm{Ob}(\mathcal{L})$, and where $\mathrm{Mor}_{\mathcal{L}_0}(A \times P, A \times Q) = A \times \mathrm{Mor}_{\mathcal{L}}(P, Q)$.

Set $\Gamma_0 = \mathrm{Aut}_{\mathcal{L}_0}(S_0) = A \times \mathrm{Aut}_{\mathcal{L}}(S)$. Set $\Gamma = A \rtimes \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$: the semidirect product taken with respect to the action ν of $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ on A. Thus Γ_0 embeds as a normal subgroup of Γ, and $\Gamma/\Gamma_0 \cong \mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$. To avoid confusion, an element $\psi \in \mathrm{Aut}_{\mathcal{L}}(S)$ will be written c_ψ when regarded as an element of $\Gamma_0 \trianglelefteq \Gamma$.

We claim the given Γ-action on $\mathcal{L}_0$ satisfies the hypotheses of [O3, Theorem 9]. This means checking that the following diagram commutes:

$$\begin{array}{ccc}
A \times \mathrm{Aut}_{\mathcal{L}}(S) = \Gamma_0 & \xrightarrow{\ \mathrm{conj}\ } & \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_0) = \mathrm{Aut}^I_{\mathrm{typ}}(A \times \mathcal{L}) \\
\Big\downarrow{\scriptstyle \mathrm{incl}} & \nearrow{\scriptstyle \tau} & \Big\downarrow{\scriptstyle (\alpha \mapsto \alpha_{A\times S})} \\
A \rtimes \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}) = \Gamma & \xrightarrow{\ \mathrm{conj}\ } & \mathrm{Aut}(\Gamma_0)\ ,
\end{array}$$

where τ sends $(a, \gamma) \in A \rtimes \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ to $(\nu(\gamma), \gamma) \in \mathrm{Aut}^I_{\mathrm{typ}}(A \times \mathcal{L})$. For $\psi \in \mathrm{Aut}_{\mathcal{L}}(S)$, $\nu(c_\psi) = \mathrm{Id}_A$, so $\tau(a, c_\psi) = (\mathrm{Id}, c_\psi)$, which shows that the upper triangle commutes. As for the lower triangle, for $(a, \gamma) \in \Gamma$ and $(b, c_\psi) \in \Gamma_0$,

$$\tau(a,\gamma)(b, c_\psi) = (\nu(\gamma)(b), c_{\gamma(\psi)}) = (\nu(\gamma)(b)\cdot a, c_{\gamma(\psi)} \circ \gamma)(a,\gamma)^{-1} = (a,\gamma)(b,c_\psi)(a,\gamma)^{-1}$$

(since $c_{\gamma(\psi)} = \gamma \circ c_\psi \circ \gamma^{-1}$); and thus the lower triangle commutes.

Fix $\bar{S} \in \mathrm{Syl}_p(\Gamma)$. We identify $S_0 = O_p(\Gamma_0)$ (via δ_{S_0}), and hence $S_0 \trianglelefteq \bar{S}$. Since

$$C_\Gamma(S_0) = C_\Gamma(A \times S) = C_{\Gamma_0}(A \times S) = A \times C_{\mathrm{Aut}_{\mathcal{L}}(S)}(S) = A \times Z(S)$$

is a p-group, and since all objects in $\mathcal{L}_0$ are $\mathcal{F}_0$-centric by construction, [O3, Theorem 9] shows that there exists a saturated fusion system $\bar{\mathcal{F}}$ over $\bar{S}$ and an associated linking system $\bar{\mathcal{L}}$ such that $(S_0, \mathcal{F}_0, \mathcal{L}_0) \trianglelefteq (\bar{S}, \bar{\mathcal{F}}, \bar{\mathcal{L}})$ and $\mathrm{Aut}_{\bar{\mathcal{L}}}(S_0) = \Gamma$ with the action on $\mathcal{L}_0$ given by τ. In particular,

$$\mathrm{Aut}_{\bar{\mathcal{F}}}(S_0) = \mathrm{Aut}_\Gamma(S_0) = \left\{(\nu(\gamma), \widetilde{\mu}_{\mathcal{L}}(\gamma)) \,\middle|\, \gamma \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})\right\} . \tag{1}$$

Assume $\bar{\mathcal{F}}$ is realizable: the fusion system of a finite group $\bar{G}$. Since A is central in $\mathcal{F}_0$, $O_p(\mathcal{F}_0)/A$ is normal in $\mathcal{F}_0/A \cong \mathcal{F}$ by Proposition 1.8. Since $O_p(\mathcal{F}) = 1$ ($\mathcal{F}$ is reduced), this shows that $O_p(\mathcal{F}_0) = A$. By Lemma 1.20(e) we then get $A \trianglelefteq \bar{\mathcal{F}}$. By Proposition 1.24 we have $\bar{\mathcal{F}} \cong \mathcal{F}_{\bar{S}}(\bar{G}) = \mathcal{F}_{\bar{S}}(N_{\bar{G}}(A))$. Upon replacing $\bar{G}$ by $N_{\bar{G}}(A)$, we may assume $A \trianglelefteq \bar{G}$.

Set $G_0 = C_{\bar{G}}(A)$ and $G = G_0/A$. Assume the following two statements hold:

(i) $A = O_p(\bar{\mathcal{F}})$ and $C_{\bar{\mathcal{F}}}(A) = \mathcal{F}_0$.

(ii) The composite

$$\xi\colon \mathrm{Aut}_{\bar{G}}(A) \cong \bar{G}/G_0 \xrightarrow{\text{conj}} \mathrm{Out}(G) \xrightarrow{\kappa_G} \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}^c_S(G))$$

is injective.

We now finish the proof of the theorem, assuming (i) and (ii).

By (i), $C_{\bar{\mathcal{F}}}(O_p(\bar{\mathcal{F}}))/Z(O_p(\bar{\mathcal{F}})) = \mathcal{F}_0/A \cong \mathcal{F}$. Since $O^p(\mathcal{F}) = O^{p'}(\mathcal{F}) = \mathcal{F}$, this shows $\mathfrak{red}(\bar{\mathcal{F}}) \cong \mathcal{F}$. Also, $S_0 = C_{\bar{S}}(A) \in \mathrm{Syl}_p(C_{\bar{G}}(A))$ since $\mathcal{F}_0 = C_{\bar{\mathcal{F}}}(A)$. Hence by Proposition 1.24 (applied with $K = 1$), $\mathcal{F}_0 = C_{\bar{\mathcal{F}}}(A) = \mathcal{F}_{S_0}(C_{\bar{G}}(A))$, and so

$$\mathcal{F} \cong \mathcal{F}_0/A \cong \mathcal{F}_{S_0/A}(G_0/A) \cong \mathcal{F}_S(G) .$$

By condition (ii) in Definition 1.18 (applied to $\mathcal{F}_0 \trianglelefteq \bar{\mathcal{F}}$), and since $A \trianglelefteq \bar{\mathcal{F}}$ and $A \le S_0$, each $\varphi \in \mathrm{Aut}_{\bar{\mathcal{F}}}(A)$ has the form $\varphi = \varphi_0 \circ \alpha|_{A,A}$ for some $\alpha \in \mathrm{Aut}_{\bar{\mathcal{F}}}(S_0) = \mathrm{Aut}_\Gamma(S_0)$ and some $\varphi_0 \in \mathrm{Aut}_{\mathcal{F}_0}(A)$. Also, $\mathrm{Aut}_{\mathcal{F}_0}(A) = 1$ by definition, and thus

$$\mathrm{Aut}_{\bar{G}}(A) = \mathrm{Aut}_{\bar{\mathcal{F}}}(A) = \mathrm{Aut}_\Gamma(A) \cong \Gamma/\Gamma_0 \cong \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) .$$

So by (ii), there is a homomorphism s from $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ to $\mathrm{Out}(G)$ such that $\kappa_G \circ s$ is injective. Since $\mathcal{L}$ was chosen with $|\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})|$ maximal, $\kappa_G \circ s$ is an isomorphism, so κ_G is split surjective, contradicting the assumption that $\mathcal{F}$ is not tame. We conclude that $\bar{\mathcal{F}}$ is exotic (and $\mathfrak{red}(\bar{\mathcal{F}}) \cong \mathcal{F}$).

It remains to prove (i) and (ii).

Proof of (i): For each $P \in \mathrm{Ob}(\bar{\mathcal{L}})$, $P_0 = P \cap S_0 \in \mathrm{Ob}(\mathcal{L}_0)$ since $\mathcal{L}_0 \trianglelefteq \bar{\mathcal{L}}$, so $P_0 = A \times Q$ for some $Q \le S$ which is $\mathcal{F}$-centric. Then $C_{\bar{S}}(P) \le C_{\bar{S}}(A) = \bar{S} \cap C_\Gamma(A) = \bar{S} \cap \Gamma_0 = S_0$, so $C_{\bar{S}}(P) \le A \times Z(Q) \le P$. Since this holds for all subgroups $\bar{\mathcal{F}}$-conjugate to P, all objects in $\bar{\mathcal{L}}$ are $\bar{\mathcal{F}}$-centric.

Set $B = O_p(\bar{\mathcal{F}})$. We already saw that $A = O_p(\mathcal{F}_0)$. Hence $B \cap S_0 = A$ by Lemma 1.20(e). Since $B \trianglelefteq \bar{S}$ and $S_0 \trianglelefteq \bar{S}$, it follows that $[B, S_0] \le A$.

Since $A \trianglelefteq \mathcal{F}_0$ and $B \trianglelefteq \bar{\mathcal{F}}$, each $\varphi \in \mathrm{Hom}_{\mathcal{F}_0}(P, Q)$ can be extended to a morphism $\bar{\varphi} \in \mathrm{Hom}_{\bar{\mathcal{F}}}(PB, QB)$ such that $\bar{\varphi}|_{PA} \in \mathrm{Hom}_{\mathcal{F}_0}(PA, QA)$ and $\bar{\varphi}(B) = B$. Then $\bar{\varphi}|_A = \mathrm{Id}_A$. Hence for each $g \in B$, g and $\bar{\varphi}(g)$ have the same conjugation action on A, and $g^{-1}\bar{\varphi}(g) \in$

$C_B(A) = B \cap C_{\bar{S}}(A) = B \cap S_0 = A$. By Lemma 1.29, $c_{\delta(g)} = \tau(\delta_{S_0}(g)) \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_0)$ is the identity modulo A, and thus $g \in A$ by definition of τ. So $B = O_p(\bar{\mathcal{F}}) = A$.

Since $C_{\bar{S}}(A) = S_0$, $\mathcal{F}_0$ and $C_{\bar{\mathcal{F}}}(A)$ are both fusion systems over S_0. Also, $\mathcal{F}_0 \subseteq C_{\bar{\mathcal{F}}}(A)$ since A is central in $\mathcal{F}_0$. To see that $C_{\bar{\mathcal{F}}}(A) = \mathcal{F}_0$, fix $P, Q \le S_0$ and $\varphi \in \mathrm{Hom}_{C_{\bar{\mathcal{F}}}(A)}(P, Q)$. By definition, φ extends to $\bar{\varphi} \in \mathrm{Hom}_{\bar{\mathcal{F}}}(AP, AQ)$ with $\bar{\varphi}(A) = A$ and $\bar{\varphi}|_A = \mathrm{Id}_A$. Since $\mathcal{F}_0 \trianglelefteq \bar{\mathcal{F}}$, condition (ii) in Definition 1.18 shows that there are $\alpha \in \mathrm{Aut}_{\bar{\mathcal{F}}}(S_0)$ and $\varphi_0 \in \mathrm{Hom}_{\mathcal{F}_0}(\alpha(AP), AQ)$ such that $\bar{\varphi} = \varphi_0 \circ \alpha|_{AP,\alpha(AP)}$. For each $a \in A$, $\alpha(a) = \varphi_0^{-1}(\bar{\varphi}(a)) = a$, and thus $\alpha|_A = \mathrm{Id}_A$. Hence

$$\alpha \in \bigl\{\beta \in \mathrm{Aut}_{\bar{\mathcal{F}}}(S_0) \bigm| \beta|_A = \mathrm{Id}_A\bigr\} = \{\mathrm{Id}_A\} \times \mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_{\mathcal{F}_0}(S_0) ,$$

where the first equality holds by (1) (and since $\widetilde{\mu}_{\mathcal{L}}(\mathrm{Aut}_{\mathcal{L}}(S)) = \mathrm{Aut}_{\mathcal{F}}(S)$). Thus $\alpha \in \mathrm{Aut}_{\mathcal{F}_0}(S_0)$, so $\bar{\varphi} \in \mathrm{Mor}(\mathcal{F}_0)$, and hence also $\varphi \in \mathrm{Mor}(\mathcal{F}_0)$. This proves that $C_{\bar{\mathcal{F}}}(A) = \mathcal{F}_0$.

Proof of (ii): Set

$$\mathcal{L}^* = \mathcal{L}^c_S(G), \qquad \mathcal{L}^*_0 = \mathcal{L}^{\mathcal{H}_0}_{S_0}(G_0) = \mathcal{L}^c_{S_0}(G_0), \qquad \text{and} \qquad \bar{\mathcal{L}}^* = \mathcal{L}^{\bar{\mathcal{H}}}_{\bar{S}}(\bar{G}) ,$$

where $\mathcal{H}_0 = \mathrm{Ob}(\mathcal{L}_0)$ and $\bar{\mathcal{H}} = \mathrm{Ob}(\bar{\mathcal{L}})$. Note that $(S_0, \mathcal{F}_0, \mathcal{L}^*_0) \trianglelefteq (\bar{S}, \bar{\mathcal{F}}, \bar{\mathcal{L}}^*)$ by Proposition 1.28. Let $\mathrm{cj}\colon \bar{G} \longrightarrow \mathrm{Aut}(G)$ denote the conjugation action of $\bar{G}$ on G. Set

$$H = \bigl\{g \in N_{\bar{G}}(S_0) \bigm| \widetilde{\kappa}_G(\mathrm{cj}(g)) = \mathrm{Id}_{\mathcal{L}^*}\bigr\} \qquad \text{and} \qquad T = H \cap \bar{S} .$$

We first claim $T = A$. By [BCGLO2, Theorem 6.8], $\mathcal{L}^*_0/A$ is a centric linking system associated to $\mathcal{F}_0/A \cong \mathcal{F}$. Hence the natural functor $\mathcal{L}^*_0/A \longrightarrow \mathcal{L}^*$ (induced by the projection $G_0 \longrightarrow G$) is an isomorphism, since it commutes with the structure functors. So for $g \in \bar{S}$, $g \in T$ if and only if $c_{\delta(g)} \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}^*_0)$ is the identity modulo A, in the sense of Lemma 1.29. We showed in the proof of (i) that each $P \in \mathrm{Ob}(\bar{\mathcal{L}}^*) = \mathrm{Ob}(\bar{\mathcal{L}})$ is $\bar{\mathcal{F}}$-centric. Hence by Lemma 1.29, applied to both normal pairs $(S_0, \mathcal{F}_0, \mathcal{L}_0) \trianglelefteq (\bar{S}, \bar{\mathcal{F}}, \bar{\mathcal{L}})$ and $(S_0, \mathcal{F}_0, \mathcal{L}^*_0) \trianglelefteq (\bar{S}, \bar{\mathcal{F}}, \bar{\mathcal{L}}^*)$, $g \in T$ if and only if $c_{\delta(g)} \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_0)$ is the identity modulo A; i.e., induces the identity on $\mathcal{L}$. By definition of $\bar{S} \le \Gamma = A \rtimes \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$, this is the case exactly when $g \in A$.

Thus H is a normal subgroup of $N_{\bar{G}}(S_0)$ whose intersection with its Sylow p-subgroup $\bar{S}$ is A. It follows that H/A has order prime to p. We claim that $H \le G_0$. Fix $h \in H$ of order prime to p. Then $\mathrm{cj}(h) \in \mathrm{Aut}(G)$ acts via the identity on $S = S_0/A$, so $[h, S_0] \le A$. Hence by (1), $c_h = (\nu(\gamma), \mathrm{Id}_S) \in \mathrm{Aut}_{\bar{G}}(S_0)$ for some $\gamma \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ such that $\gamma \in \mathrm{Ker}(\widetilde{\mu}_{\mathcal{L}})$. Since $\mathrm{Ker}(\mu_{\mathcal{L}})$ is a p-group by Lemma 1.16 and h has order prime to p, $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S)$, so $\nu(\gamma) = 1 \in \mathrm{Aut}(A)$, and $h \in G_0$. Thus $H = O^p(H){\cdot}A \le G_0$.

Fix $g \in \bar{G}$ such that $c_g \in \mathrm{Ker}(\xi)$. Recall we are only interested in g modulo $C_{\bar{G}}(A) = G_0$. Since $\bar{G} = G_0{\cdot}N_{\bar{G}}(S_0)$ by the Frattini argument, we can assume g normalizes S_0. Thus $\kappa_G([\mathrm{cj}(g)]) = 1$ in $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}^*)$, so $\widetilde{\kappa}_G(\mathrm{cj}(g)) = c_\gamma$ for some $\gamma \in \mathrm{Aut}_{\mathcal{L}^*}(S)$. Let $h \in N_G(S)$ be such that $\gamma = [h]$ and lift h to $\widetilde{h} \in N_{G_0}(S_0)$. Upon replacing g by $\widetilde{h}^{-1}g$, we can assume $\widetilde{\kappa}_G(\mathrm{cj}(g)) = \mathrm{Id}_{\mathcal{L}^*}$, and thus $g \in H \le G_0$. Hence $c_g = \mathrm{Id}_A$, ξ is injective, and this finishes the proof of (ii). □

2.3. **Strongly tame fusion systems and linking systems for extensions.**

We are now ready to start working on the proof of Theorem A. As stated in the introduction, this proof uses the vanishing of certain higher limit groups, and through that depends on the classification of finite simple groups. In order to have a clean statement which does not depend on the classification (Theorem 2.20), we first define a certain class of finite groups which in fact (by the classification) includes all finite groups.

The obstruction groups for the existence and uniqueness of centric linking systems associated to a given saturated fusion system are higher derived functors for inverse limits taken over the centric orbit category of the fusion system. We begin by defining this category.

Definition 2.7. *Let $\mathcal{F}$ be a fusion system over a finite p-group S, and let $\mathcal{F}^c \subseteq \mathcal{F}$ be the full subcategory whose objects are the $\mathcal{F}$-centric subgroups of S. The* centric orbit category *$\mathcal{O}(\mathcal{F}^c)$ of $\mathcal{F}$ is the category with $\mathrm{Ob}(\mathcal{O}(\mathcal{F}^c)) = \mathrm{Ob}(\mathcal{F}^c)$, and where*

$$\mathrm{Mor}_{\mathcal{O}(\mathcal{F}^c)}(P,Q) = \mathrm{Inn}(Q)\backslash\mathrm{Hom}_{\mathcal{F}}(P,Q)$$

for any pair of objects $P, Q \le S$. In particular, $\mathrm{Aut}_{\mathcal{O}(\mathcal{F}^c)}(P) = \mathrm{Out}_{\mathcal{F}}(P)$ for each P. If $\mathcal{F}_0 \subseteq \mathcal{F}^c$ is any full subcategory, then $\mathcal{O}(\mathcal{F}_0)$ denotes the full subcategory of $\mathcal{O}(\mathcal{F}^c)$ with the same objects as $\mathcal{F}_0$.

We need the following technical result about higher limits over these orbit categories.

Lemma 2.8. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Let $\mathcal{H} \subseteq \mathrm{Ob}(\mathcal{F}^c)$ be any subset which is closed under $\mathcal{F}$-conjugacy and overgroups, and let $\mathcal{F}^{\mathcal{H}} \subseteq \mathcal{F}^c$ be the full subcategory with object set $\mathcal{H}$. Fix a functor $F\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$. Assume, for each $P \in \mathrm{Ob}(\mathcal{F}^c)\smallsetminus\mathcal{H}$, that either $O_p(\mathrm{Out}_{\mathcal{F}}(P)) \ne 1$, or some element of order p in $\mathrm{Out}_{\mathcal{F}}(P)$ acts trivially on $F(P)$. Let $F_0\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$ be the functor where $F_0(P) = F(P)$ if $P \in \mathcal{H}$ and $F_0(P) = 0$ otherwise. Then*

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^{*}(F) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^{*}(F_0) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^{\mathcal{H}})}{}^{*}(F|_{\mathcal{O}(\mathcal{F}^{\mathcal{H}})^{\mathrm{op}}}) .$$

Proof. Let $F_1 \subseteq F$ be the subfunctor defined by setting $F_1(P) = F(P)$ if $P \notin \mathcal{H}$ and $F_1(P) = 0$ otherwise. Thus $F_0 \cong F/F_1$. By [O1, Lemma 2.3], $\varprojlim^*_{\mathcal{O}(\mathcal{F}^c)}(F_1) = 0$ if certain graded groups $\Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); F(P))$ vanish for each $P \in \mathrm{Ob}(\mathcal{F}^c)\smallsetminus\mathcal{H}$. By [JMO, Proposition 6.1(ii)], this is the case whenever $O_p(\mathrm{Out}_{\mathcal{F}}(P)) \ne 1$, or some element of order p in $\mathrm{Out}_{\mathcal{F}}(P)$ acts trivially on $F(P)$. This proves the first isomorphism.

For any category $\mathcal{C}$, let $\mathcal{C}$-mod be the category of contravariant functors from $\mathcal{C}$ to abelian groups. Let E be the functor "extension by zero" from $\mathcal{O}(\mathcal{F}^{\mathcal{H}})$-$\mathsf{mod}$ to $\mathcal{O}(\mathcal{F}^c)$-$\mathsf{mod}$. Since $\mathcal{H} \subseteq \mathrm{Ob}(\mathcal{F}^c)$ is closed under $\mathcal{F}$-conjugacy and overgroups, E is right adjoint to the restriction functor. Thus E sends injectives to injectives. So for Φ in $\mathcal{O}(\mathcal{F}^{\mathcal{H}})$-$\mathsf{mod}$, $\varprojlim^*_{\mathcal{O}(\mathcal{F}^{\mathcal{H}})}(\Phi) \cong \varprojlim^*_{\mathcal{O}(\mathcal{F}^c)}(E(\Phi))$. Since $F_0 = E(F|_{\mathcal{O}(\mathcal{F}^{\mathcal{H}})^{\mathrm{op}}})$, the second isomorphism now follows. □

For any saturated fusion system $\mathcal{F}$ over a finite p-group S, let

$$\mathcal{Z}_{\mathcal{F}}\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$$

be the functor which sends an object P of $\mathcal{F}^c$ to $Z(P) = C_S(P)$. For each $\varphi \in \mathrm{Hom}_{\mathcal{F}^c}(P,Q)$,

$$\mathcal{Z}_{\mathcal{F}}([\varphi]) = \varphi^{-1}|_{Z(Q)}\colon Z(Q) \longrightarrow Z(P).$$

By [BLO2, Proposition 3.1], the obstruction to the existence of a centric linking system associated to $\mathcal{F}$ lies in $\varprojlim^3_{\mathcal{O}(\mathcal{F}^c)}(\mathcal{Z}_{\mathcal{F}})$, and the obstruction to its uniqueness lies in $\varprojlim^2_{\mathcal{O}(\mathcal{F}^c)}(\mathcal{Z}_{\mathcal{F}})$. The main results in [O1] and [O2] state that these groups vanish whenever $\mathcal{F}$ is the fusion system of a finite group G.

We also need to work with the following closely related categories and functors. For any finite group G, let $\mathcal{O}_p(G)$ be the p-subgroup orbit category of G as defined in [O2] and [O1]. Thus $\mathrm{Ob}(\mathcal{O}_p(G))$ is the set of p-subgroups of G, and $\mathrm{Mor}_{\mathcal{O}_p(G)}(P,Q) = Q\backslash N_G(P,Q)$. Let

$\mathcal{Z}_G\colon \mathcal{O}_p(G)^{\rm op} \longrightarrow \mathsf{Ab}$ be the functor $\mathcal{Z}_G(P) = Z(P)$ if $Z(P) \in \mathrm{Syl}_p(C_G(P))$ and $\mathcal{Z}_G(P) = 0$ otherwise. For $H \trianglelefteq G$, let $\mathcal{Z}_G^H \subseteq \mathcal{Z}_G$ be the subfunctor $\mathcal{Z}_G^H(P) = \mathcal{Z}_G(P) \cap H$.

Definition 2.9. *Fix a prime p.*

(a) *Let $\widehat{\mathfrak{L}}(p)$ be the class of all nonabelian finite simple groups L with the following property. For each finite group G, and each pair of subgroups $H \trianglelefteq K \trianglelefteq G$ both normal in G such that $K/H \cong L^m$ for some $m \geq 1$, $\varprojlim^i_{\mathcal{O}_p(G)}(\mathcal{Z}_G^K/\mathcal{Z}_G^H) = 0$ for $i \geq 2$.*

(b) *Let $\mathfrak{G}(p)$ be the class of all finite groups G all of whose nonabelian composition factors lie in $\widehat{\mathfrak{L}}(p)$.*

(c) *A saturated fusion system $\mathcal{F}$ over a finite p-group S is* strongly tame *if it is tamely realizable by a group $G \in \mathfrak{G}(p)$.*

In fact, by results in [O1, O2], all finite groups are in $\mathfrak{G}(p)$ for each p.

Theorem 2.10. *For each prime p, the class $\widehat{\mathfrak{L}}(p)$ contains all nonabelian finite simple groups, and the class $\mathfrak{G}(p)$ contains all finite groups. Hence all tame fusion systems are strongly tame.*

Proof. The last two statements follow immediately from the first one and the definitions. So we need only show that $\widehat{\mathfrak{L}}(p)$ contains all nonabelian finite simple groups.

Assume p is odd. By [O1, Proposition 4.1] (and its proof), a nonabelian finite simple group L with $S \in \mathrm{Syl}_p(L)$ lies in $\widehat{\mathfrak{L}}(p)$ if there is a subgroup $Q \leq \mathfrak{X}_L(S)$ which is centric in S (i.e., $C_S(Q) \leq Q$) and not $\mathrm{Aut}(L)$-conjugate to any other subgroup of S. Here, $\mathfrak{X}_L(S)$ is a certain subgroup of S defined in [O1, §§ 3–4]. By [O1, Propositions 4.2–4.4] (and the classification theorem), all nonabelian finite simple groups have this property, and thus they all lie in $\widehat{\mathfrak{L}}(p)$.

If $p = 2$, then by [O2, Proposition 2.7], a nonabelian finite simple group L is contained in $\widehat{\mathfrak{L}}(2)$ if it is contained in the class $\mathfrak{L}^{\geq 2}(2)$ defined in [O2, Definition 2.8]. By [O2, Theorems 5.1, 6.2, 7.5, 8.13, & 9.1] and the classification theorem, all nonabelian finite simple groups are contained in $\mathfrak{L}^{\geq 2}(2)$. □

Theorem 2.10 together with Theorem 2.20 will imply Theorem A. From now on, for the rest of the section, we avoid using the classification theorem by assuming whenever necessary that our groups are in $\mathfrak{G}(p)$ and applying the following lemma.

Lemma 2.11. *Fix a finite group G with Sylow subgroup $S \in \mathrm{Syl}_p(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. Assume $G \in \mathfrak{G}(p)$. Then the following hold.*

(a) *If $\widehat{G}$ is a finite group with $G \trianglelefteq \widehat{G}$, then $\varprojlim^i_{\mathcal{O}_p(\widehat{G})}(\mathcal{Z}_{\widehat{G}}^G) = 0$ for each $i \geq 2$.*

(b) *If $G = G_1 \times G_2$, then $G \in \mathfrak{G}(p)$ if and only if $G_1, G_2 \in \mathfrak{G}(p)$. If $H \trianglelefteq G$ and G/H is p-solvable, then $H \in \mathfrak{G}(p)$ if and only if $G \in \mathfrak{G}(p)$.*

(c) *Let $\mathcal{L}$ be a linking system associated to $\mathcal{F}$ such that all subgroups in $\mathcal{H} \overset{\rm def}{=} \mathrm{Ob}(\mathcal{L})$ are $\mathcal{F}$-centric. Then $\mathcal{L} \cong \mathcal{L}_S^{\mathcal{H}}(G)$.*

(d) *The homomorphism $\mu_G\colon \mathrm{Out}_{\rm typ}(\mathcal{L}_S^c(G)) \longrightarrow \mathrm{Out}(S, \mathcal{F})$ defined in Section 1.3 is surjective.*

Proof. **(a)** Let $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_m = G$ be a sequence of subgroups, all normal in $\widehat{G}$, such that for each r, G_{r+1}/G_r is a minimal nontrivial normal subgroup of $\widehat{G}/G_r$. By [G, Theorem 2.1.5], each quotient G_{r+1}/G_r is a product of simple groups isomorphic to each other. By [O2, Proposition 2.2], if G_{r+1}/G_r is abelian, then $\varprojlim^i_{\mathcal{O}_p(\widehat{G}^c)}(\mathcal{Z}_{\widehat{G}}^{G_{r+1}}/\mathcal{Z}_{\widehat{G}}^{G_r}) = 0$ for all $i \geq 1$. Thus (a) follows immediately from the definition of $\widehat{\mathfrak{L}}(p)$, together with the exact sequences

$$\varprojlim_{\mathcal{O}_p(\widehat{G})}{}^i(\mathcal{Z}_{\widehat{G}}^{G_s}/\mathcal{Z}_{\widehat{G}}^{G_r}) \longrightarrow \varprojlim_{\mathcal{O}_p(\widehat{G})}{}^i(\mathcal{Z}_{\widehat{G}}^{G_t}/\mathcal{Z}_{\widehat{G}}^{G_r}) \longrightarrow \varprojlim_{\mathcal{O}_p(\widehat{G})}{}^i(\mathcal{Z}_{\widehat{G}}^{G_t}/\mathcal{Z}_{\widehat{G}}^{G_s})$$

for all $0 \leq r < s < t \leq m$ and all $i \geq 2$.

(b) The first statement is immediate, since a simple group is a composition factor of $G = G_1 \times G_2$ if and only if it is a composition factor of G_1 or of G_2. When $H \trianglelefteq G$ and G/H is p-solvable, then the only simple groups which could be composition factors of G but not of H are C_p and simple groups of order prime to p. So we need only show that every nonabelian simple group of order prime to p lies in $\widehat{\mathfrak{L}}(p)$.

Fix such a simple group L, and assume $H \trianglelefteq K \trianglelefteq G$ (where $H \trianglelefteq G$), and $K/H \cong L^m$ for some m. Then $\mathcal{Z}_G^H = \mathcal{Z}_G^K$ since K/H has order prime to p; and thus $L \in \widehat{\mathfrak{L}}(p)$.

(c,d) By (a), applied with $\widehat{G} = G$, $\varprojlim^2_{\mathcal{O}_p(G)}(\mathcal{Z}_G) = 0$. So by [BLO1, Theorem E], μ_G is onto. This proves (d).

Now let $\mathcal{L}$ be a linking system associated to $\mathcal{F}$, set $\mathcal{H} = \mathrm{Ob}(\mathcal{L})$, and assume $\mathcal{H} \subseteq \mathrm{Ob}(\mathcal{F}^c)$. Since $\mathcal{H}$ contains all subgroups of S which are $\mathcal{F}$-centric and $\mathcal{F}$-radical, $O_p(\mathrm{Out}_{\mathcal{F}}(P)) \neq 1$ for $P \in \mathrm{Ob}(\mathcal{F}^c) \smallsetminus \mathcal{H}$. Hence

$$\varprojlim^2_{\mathcal{O}(\mathcal{F}^{\mathcal{H}})}(\mathcal{Z}_{\mathcal{F}}|_{\mathcal{O}(\mathcal{F}^{\mathcal{H}})^{\mathrm{op}}}) \cong \varprojlim^2_{\mathcal{O}(\mathcal{F}^c)}(\mathcal{Z}_{\mathcal{F}}) \cong \varprojlim^2_{\mathcal{O}_p(G)}(\mathcal{Z}_G) = 0,$$

where the first isomorphism holds by Lemma 2.8, the second by [O1, Lemma 2.1], and the third group was just shown to vanish in the proof of (d). So by the same argument as that used in the proof of [BLO2, Proposition 3.1], all linking systems associated to $\mathcal{F}$ with object set $\mathcal{H}$ are isomorphic. In particular, $\mathcal{L} \cong \mathcal{L}_S^{\mathcal{H}}(G)$, and this proves (c). □

The following is the main technical result in this subsection, and will be needed in the proof of Theorem A. Given $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ satisfying certain technical assumptions, and given a linking system $\mathcal{L}_0$ associated to $\mathcal{F}_0$, we want to find $\mathcal{L}$ associated to $\mathcal{F}$ such that $\mathcal{L}_0 \trianglelefteq \mathcal{L}$. It is natural to ask why this cannot be done using [O3, Theorem 9], where conditions are explicitly set up to construct extensions of fusion and linking systems. There seem to be two difficulties with that approach. First, the hypotheses of Proposition 2.12 are very different from those in [O3], and it is not clear how to convert from the one to the other. But more seriously, even if one does manage to do that and construct an extension $(\mathcal{F}', \mathcal{L}')$ of $(\mathcal{F}_0, \mathcal{L}_0)$, it is not clear how to prove that $\mathcal{F}' \cong \mathcal{F}$; i.e., that $\mathcal{L}'$ really is a linking system associated to $\mathcal{F}$.

Proposition 2.12. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathcal{F}_0 \subseteq \mathcal{F}$ be one of the following saturated fusion subsystems over $S_0 \trianglelefteq S$:*

(a) $\mathcal{F}_0 = O^p(\mathcal{F})$, *or*

(b) $\mathcal{F}_0 = O^{p'}(\mathcal{F})$, *or*

(c) $\mathcal{F}_0 = N_{\mathcal{F}}^K(Q)$ *for some $Q \trianglelefteq \mathcal{F}$ and some $K \trianglelefteq \mathrm{Aut}(Q)$ containing* $\mathrm{Inn}(Q)$.

Assume $\mathcal{F}_0$ is strongly tame. Then there is a centric linking system associated to $\mathcal{F}$.

Proof. In all cases (a), (b), and (c), $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ by Proposition 1.25.

Since $\mathcal{F}_0$ is strongly tame, we can choose a finite group $G_0 \in \mathfrak{G}(p)$ such that $S_0 \in \mathrm{Syl}_p(G_0)$, $\mathcal{F}_0 = \mathcal{F}_{S_0}(G_0)$, and κ_{G_0} is split surjective. We first claim that

$$\widetilde{\mu}_{G_0} \circ \widetilde{\kappa}_{G_0} \colon \mathrm{Aut}(G_0, S_0) \longrightarrow \mathrm{Aut}(S_0, \mathcal{F}_0) \quad \text{is onto.} \tag{2}$$

As noted in Section 2.2, this composite is defined by restriction. Since $G_0 \in \mathfrak{G}(p)$, μ_{G_0} is surjective by Lemma 2.11(d). Also, κ_{G_0} is split surjective by assumption. Thus every element of $\mathrm{Out}(S_0, \mathcal{F}_0) = \mathrm{Aut}(S_0, \mathcal{F}_0)/\mathrm{Aut}_{G_0}(S_0)$ extends to an element of $\mathrm{Out}(G_0) = \mathrm{Aut}(G_0, S_0)/\mathrm{Aut}_{N_{G_0}(S_0)}(G_0)$, and hence the map in (2) is onto.

Define

$$\Delta = \bigl\{\alpha \in \mathrm{Aut}(G_0, S_0) \,\big|\, \alpha|_{S_0} \in \mathrm{Aut}_{\mathcal{F}}(S_0)\bigr\} .$$

We just showed that every element in $\mathrm{Aut}_{\mathcal{F}}(S_0)$ is the restriction of some element in Δ. Fix $S_\Delta \in \mathrm{Syl}_p(\Delta)$ which surjects onto $\mathrm{Aut}_S(S_0)$ under the restriction map to $\mathrm{Aut}_{\mathcal{F}}(S_0)$. Set

$$\widehat{G} = G_0 \rtimes \Delta \qquad \text{and} \qquad \widehat{S} = S_0 \rtimes S_\Delta.$$

Thus $\widehat{S} \in \mathrm{Syl}_p(\widehat{G})$.

Now set $S_1 = S$, $\mathcal{F}_1 = \mathcal{F}$, $S_2 = \widehat{S}$, and $\mathcal{F}_2 = \mathcal{F}_{\widehat{S}}(\widehat{G})$. We claim, for each $P_0, Q_0 \le S_0$, that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{F}_2}(P_0, Q_0) &= \mathrm{Hom}_{\mathcal{F}_1}(P_0, Q_0) \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{F}}(P_0, Q_0) \\ \mathrm{Hom}_{S_2}(P_0, Q_0) &= \mathrm{Hom}_{S_1}(P_0, Q_0) \overset{\text{def}}{=} \mathrm{Hom}_{S}(P_0, Q_0) . \end{aligned} \tag{3}$$

We have already remarked that $\mathcal{F}_0 \trianglelefteq \mathcal{F} = \mathcal{F}_1$, and $\mathcal{F}_0 \trianglelefteq \mathcal{F}_2$ by Proposition 1.28 since they are the fusion systems of $G_0 \trianglelefteq \widehat{G}$. Hence by condition (ii) in Definition 1.18, each $\varphi \in \mathrm{Hom}_{\mathcal{F}_i}(P, Q)$ (for $i = 1, 2$ and $P, Q \le S_0$) is the composite of a morphism in $\mathcal{F}_0$ and the restriction of a morphism in $\mathrm{Aut}_{\mathcal{F}_i}(S_0)$. Furthermore,

$$\mathrm{Aut}_{\mathcal{F}_2}(S_0) = \mathrm{Aut}_{\widehat{G}}(S_0) = \bigl\langle \mathrm{Aut}_{G_0}(S_0)\,,\, \mathrm{Res}^{G_0}_{S_0}(\Delta)\bigr\rangle = \mathrm{Aut}_{\mathcal{F}_1}(S_0)$$

by (2), and the first line in (3) now follows. The second holds since $\mathrm{Aut}_{S_2}(S_0) = \mathrm{Aut}_S(S_0)$ by definition of $S_2 = S_0 \rtimes S_\Delta$.

We next claim that for all $P_0 \le S_0$,

$$P_0 \text{ is fully centralized in } \mathcal{F}_2 \quad \Longleftrightarrow \quad P_0 \text{ is fully centralized in } \mathcal{F}_1 = \mathcal{F} . \tag{4}$$

By (3), the $\mathcal{F}_1$- and $\mathcal{F}_2$-conjugacy classes of P_0 are the same, and $\mathrm{Aut}_{S_2}(S_0) = \mathrm{Aut}_S(S_0)$. Hence for each Q_0 which is $\mathcal{F}_i$-conjugate to P_0,

$$\frac{|C_{S_1}(Q_0)|}{|C_{S_1}(S_0)|} = \bigl|\bigl\{\alpha \in \mathrm{Aut}_S(S_0) \,\big|\, \alpha|_{Q_0} = \mathrm{Id}\bigr\}\bigr| = \frac{|C_{S_2}(Q_0)|}{|C_{S_2}(S_0)|} ,$$

and so $|C_{S_1}(P_0)|$ is maximal if and only if $|C_{S_2}(P_0)|$ is maximal.

We want to compute $\varprojlim^*_{\mathcal{O}(\mathcal{F}_1^c)}(\mathcal{Z}_{\mathcal{F}_1})$ by comparing it with $\varprojlim^*_{\mathcal{O}(\mathcal{F}_2^c)}(\mathcal{Z}_{\mathcal{F}_2})$. To do this, we first define in Step 1 certain full subcategories $\mathcal{F}_i^* \subseteq \mathcal{F}_i^c$, and an intermediate category $\mathcal{C}$ which can be used to compare $\mathcal{O}(\mathcal{F}_1^*)$ with $\mathcal{O}(\mathcal{F}_2^*)$. Certain properties of the "comparison functors" $\Phi_i \colon \mathcal{O}(\mathcal{F}_i^*) \longrightarrow \mathcal{C}$ are stated and proven in Step 2. In Step 3, we define certain subfunctors $\mathcal{Z}_i \subseteq \mathcal{Z}_{\mathcal{F}_i}$ on $\mathcal{O}(\mathcal{F}_i^c)$, and prove that $\varprojlim^*_{\mathcal{O}(\mathcal{F}_1^c)}(\mathcal{Z}_1) \cong \varprojlim^*_{\mathcal{O}(\mathcal{F}_2^c)}(\mathcal{Z}_2)$ using the intermediate categories $\mathcal{O}(\mathcal{F}_i^*)$ and $\mathcal{C}$ to compare them. Finally, in Step 4, we prove that $\varprojlim^*_{\mathcal{O}(\mathcal{F}_2^c)}(\mathcal{Z}_2) = 0$ for $* \ge 2$, and then show that $\varprojlim^*_{\mathcal{O}(\mathcal{F}^c)}(\mathcal{Z}_{\mathcal{F}}) \cong \varprojlim^*_{\mathcal{O}(\mathcal{F}_1^c)}(\mathcal{Z}_1)$ for $* \ge 1$ by analyzing individually the three cases (a)–(c).

Throughout the rest of the proof, whenever $P \le S_1$ or $P \le S_2$, we write $P_0 = P \cap S_0$.

Step 1: Let $\mathcal{F}_i^* \subseteq \mathcal{F}_i$ $(i = 1, 2)$ be the full subcategories with objects

$$\mathrm{Ob}(\mathcal{F}_i^*) = \bigl\{P \le S_i \,\big|\, C_{S_i}(Q_0) \le Q \text{ for all } Q\ \mathcal{F}_i\text{-conjugate to } P\bigr\}.$$

All objects in $\mathcal{F}_i^*$ are $\mathcal{F}_i$-centric; i.e., $\mathcal{F}_i^* \subseteq \mathcal{F}_i^c$. Also, if $P \le S_i$ is $\mathcal{F}_i$-conjugate to an object in $\mathcal{F}_i^*$, then $P \in \mathrm{Ob}(\mathcal{F}_i^*)$.

We next construct a category $\mathcal{C}$ which acts as intermediary between the orbit categories of $\mathcal{F}_1^*$ and $\mathcal{F}_2^*$. It will be a subcategory of a larger category $\widehat{\mathcal{C}}$, defined by setting

$$\mathrm{Ob}(\widehat{\mathcal{C}}) = \bigl\{(P_0,K) \,\big|\, P_0 \le S_0 \text{ is } \mathcal{F}_0\text{-centric and fully centralized in } \mathcal{F}, \\ \mathrm{Inn}(P_0) \le K \le \mathrm{Aut}_S(P_0)\bigr\}$$

and

$$\mathrm{Mor}_{\widehat{\mathcal{C}}}\bigl((P_0,K),(Q_0,L)\bigr) = L\backslash\bigl\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(P_0,Q_0) \,\big|\, \varphi K \subseteq L\varphi\bigr\}.$$

Here, we regard φK and $L\varphi$ as subsets of $\mathrm{Hom}_{\mathcal{F}}(P_0,Q_0)$.

Define functors

$$\mathcal{O}(\mathcal{F}_1^*) \xrightarrow{\ \Phi_1\ } \widehat{\mathcal{C}} \xleftarrow{\ \Phi_2\ } \mathcal{O}(\mathcal{F}_2^*)\ ,$$

by setting $\Phi_i(P) = (P_0, \mathrm{Aut}_P(P_0))$ and $\Phi_i([\varphi]) = [\varphi|_{P_0}]$ for $P,Q \in \mathrm{Ob}(\mathcal{F}_i^*)$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}_i^*}(P,Q)$. Since restriction sends $\mathrm{Inn}(Q)$ to $\mathrm{Aut}_Q(Q_0)$ for $Q \in \mathrm{Ob}(\mathcal{F}_i^*)$, Φ_i is well defined on morphisms if it is defined on objects.

To see that Φ_i is well defined on objects, fix $P \in \mathrm{Ob}(\mathcal{F}_i^*)$, and set $K = \mathrm{Aut}_P(P_0)$. Then $\mathrm{Inn}(P_0) \le \mathrm{Aut}_P(P_0) \le \mathrm{Aut}_S(P_0)$, since $P \ge P_0$, and since $\mathrm{Aut}_S(P_0) = \mathrm{Aut}_{S_i}(P_0)$ by (3). By Lemma 1.20(a), P is $\mathcal{F}_i$-conjugate to some Q such that Q_0 is fully normalized in $\mathcal{F}_i$ and in $\mathcal{F}_0$. Hence P_0 is $\mathcal{F}_0$-centric by Lemma 1.20(c). Also, $|C_{S_i}(P_0)| = |C_P(P_0)| = |C_Q(Q_0)| = |C_{S_i}(Q_0)|$ by definition of $\mathrm{Ob}(\mathcal{F}_i^*)$, and so P_0 is fully centralized in $\mathcal{F}_i$ (hence in $\mathcal{F}$ by (4)) since Q_0 is fully centralized in $\mathcal{F}_i$. Thus $(P_0,K) \in \mathrm{Ob}(\widehat{\mathcal{C}})$.

We claim that $\mathrm{Im}(\Phi_1) = \mathrm{Im}(\Phi_2)$. In what follows, when $P_0 \le S_0$ and $K \le \mathrm{Aut}(P_0)$, we set $N_S^K(P_0) = \{g \in N_S(P_0) \,|\, c_g \in K\}$. Then

$$P \in \mathrm{Ob}(\mathcal{F}_i^*) \text{ and } \Phi_i(P) = (P_0,K) \quad\Longrightarrow\quad P = N_{S_i}^K(P_0) \tag{5}$$

since $P \ge C_{S_i}(P_0)$ by definition of $\mathrm{Ob}(\mathcal{F}_i^*)$.

Assume $(P_0,K) \in \mathrm{Ob}(\widehat{\mathcal{C}})$, and set $P_i = N_{S_i}^K(P_0)$ for $i = 1,2$. Then $P_i \ge P_0$ and $K = \mathrm{Aut}_{P_i}(P_0)$, since $\mathrm{Inn}(P_0) \le K \le \mathrm{Aut}_S(P_0)$ by assumption. Also, $P_1 \cap S_0 = N_{S_0}^K(P_0) = P_2 \cap S_0$, so $P_1 \cap S_0 = P_0$ if and only if $P_2 \cap S_0 = P_0$, and we assume this is the case since otherwise (P_0,K) is in the image of neither functor Φ_i by (5). By assumption, P_0 is fully centralized in $\mathcal{F}$, and hence in $\mathcal{F}_i$ by (4). So for each Q which is $\mathcal{F}_i$-conjugate to P_i, $|C_{S_i}(Q_0)| \le |C_{S_i}(P_0)| = |C_{P_i}(P_0)| = |C_Q(Q_0)|$, where the last equality holds since any $\varphi \in \mathrm{Iso}_{\mathcal{F}_i}(P_i,Q)$ induces an isomorphism of pairs $(P_i,P_0) \cong (Q,Q_0)$. Thus $C_{S_i}(Q_0) \le Q$. This proves that $P_i \in \mathrm{Ob}(\mathcal{F}_i^*)$, and hence that $\Phi_i(P_i) = (P_0,K)$ for $i = 1,2$.

Now fix objects (P_0,K) and (Q_0,L) in $\mathrm{Im}(\Phi_i)$, and choose $\varphi_0 \in \mathrm{Hom}_{\mathcal{F}}(P_0,Q_0)$ such that $\varphi_0 K \subseteq L\varphi_0$. Thus $[\varphi_0] \in \mathrm{Mor}_{\widehat{\mathcal{C}}}((P_0,K),(Q_0,L))$. If $[\varphi_0] = \Phi_i([\varphi])$ for some $\varphi \in \mathrm{Hom}_{\mathcal{F}_i^*}(P,Q)$ ($i = 1$ or 2), then $\varphi(P) \in \mathrm{Ob}(\mathcal{F}_i^*)$, so $\varphi_0(P_0)$ is fully centralized in $\mathcal{F}$ (since $\Phi_i(\varphi(P)) = (\varphi_0(P_0), \mathrm{Aut}_{\varphi(P)}(\varphi_0(P_0))) \in \mathrm{Ob}(\widehat{\mathcal{C}})$), and hence in $\mathcal{F}_1$ and $\mathcal{F}_2$ by (4). So we assume this from now on.

Set $P_i = N_{S_i}^K(P_0)$ and $Q_i = N_{S_i}^L(Q_0)$: these are both in $\mathrm{Ob}(\mathcal{F}_i^*)$ by (5). Set $R_0 = \varphi_0(P_0)$, let $\dot\varphi_0 \in \mathrm{Iso}_{\mathcal{F}}(P_0,R_0)$ be the restriction of φ_0, and set $M = \dot\varphi_0 K \dot\varphi_0^{-1} \le \mathrm{Aut}_{\mathcal{F}}(R_0)$. Then $\varphi_0 K \dot\varphi_0^{-1} \subseteq L|_{R_0} \subseteq \mathrm{Hom}_S(R_0,Q_0)$, and so $M \le \mathrm{Aut}_S(R_0) = \mathrm{Aut}_{S_i}(R_0)$. By the extension axiom for $\mathcal{F}_i$, φ_0 extends to some $\varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i,S_i)$, and $[\varphi_0] \in \mathrm{Im}(\Phi_i)$ if and only if φ_i can be chosen with $\varphi_i(P_i) \le Q_i$. Now, $M = \mathrm{Aut}_{\varphi_i(P_i)}(R_0)$ since $K = \mathrm{Aut}_{P_i}(P_0)$, so $\Phi_i(\varphi_i(P_i)) = (R_0,M)$, and $\varphi_i(P_i) = N_{S_i}^M(R_0)$ by (5). Hence $\varphi_i(P_i) \le Q_i$ if and only if for all

$\alpha \in \mathrm{Aut}_{S_i}(S_0)$,

$$\alpha(R_0) = R_0 \text{ and } \alpha|_{R_0} \in M \quad \Longrightarrow \quad \alpha(Q_0) = Q_0 \text{ and } \alpha|_{Q_0} \in L\ .$$

Since $\mathrm{Aut}_{S_1}(S_0) = \mathrm{Aut}_{S_2}(S_0)$ by (3), $[\varphi_0] \in \mathrm{Im}(\Phi_1)$ if and only if $[\varphi_0] \in \mathrm{Im}(\Phi_2)$.

Now set $\mathcal{C} = \mathrm{Im}(\Phi_1) = \mathrm{Im}(\Phi_2) \subseteq \widehat{\mathcal{C}}$. Since the Φ_i are injective on objects by (5), this is a subcategory of $\widehat{\mathcal{C}}$. From now on, we regard the Φ_i as functors to $\mathcal{C}$.

Step 2: For each $i = 1, 2$, and each $P \in \mathrm{Ob}(\mathcal{F}_i^*)$, set

$$\begin{aligned}\Gamma_i(P) &= \mathrm{Ker}\big[\mathrm{Out}_{\mathcal{F}_i}(P) \xrightarrow{\ \Phi_i\ } \mathrm{Aut}_{\mathcal{C}}(P_0, \mathrm{Aut}_P(P_0))\big] \\ &= \mathrm{Ker}\big[\mathrm{Out}_{\mathcal{F}_i}(P) \xrightarrow{\ R\ } N_{\mathrm{Aut}_{\mathcal{F}}(P_0)}(\mathrm{Aut}_P(P_0))/\mathrm{Aut}_P(P_0)\big]\end{aligned}$$

where R is induced by restriction. We claim that, for each $i = 1, 2$,

(i) $\Phi_i \colon \mathcal{O}(\mathcal{F}_i^*) \longrightarrow \mathcal{C}$ is bijective on objects and surjective on morphism sets;

(ii) $\Gamma_i(P)$ has order prime to p for all P; and

(iii) whenever $\psi, \psi' \in \mathrm{Mor}_{\mathcal{O}(\mathcal{F}_i^*)}(P, Q)$ are such that $\Phi_i(\psi) = \Phi_i(\psi')$, there is $\chi \in \Gamma_i(P)$ such that $\psi' = \psi \circ \chi$.

Point (i) follows from (5) and the definition of $\mathcal{C}$ in Step 1.

When proving (ii), it suffices to consider the case where P is fully normalized in $\mathcal{F}_i$. If $g \in N_{S_i}(P)$ is such that $[c_g] \in \Gamma_i(P)$, then $c_g|_{P_0} \in \mathrm{Aut}_P(P_0)$; and since $C_{S_i}(P_0) \le P$, this implies $g \in P$ and $[c_g] = 1 \in \Gamma_i(P) \le \mathrm{Out}_{\mathcal{F}_i}(P)$. Thus $\mathrm{Out}_{S_i}(P)$ is a Sylow p-subgroup of $\mathrm{Out}_{\mathcal{F}_i}(P)$ and intersects trivially with $\Gamma_i(P) \trianglelefteq \mathrm{Out}_{\mathcal{F}_i}(P)$, so $|\Gamma_i(P)|$ is prime to p.

It remains to prove (iii). Assume $\psi, \psi' \in \mathrm{Mor}_{\mathcal{O}(\mathcal{F}_i^*)}(P, Q)$ are such that $\Phi_i(\psi) = \Phi_i(\psi')$. Fix $\varphi, \varphi' \in \mathrm{Hom}_{\mathcal{F}_i^*}(P, Q)$ such that $\psi = [\varphi]$ and $\psi' = [\varphi']$. Then $\varphi|_{P_0} = c_g \circ \varphi'|_{P_0}$ for some $c_g \in \mathrm{Aut}_Q(Q_0)$; i.e., for some $g \in Q$. So upon replacing φ' by $c_g \circ \varphi'$ (this time with $c_g \in \mathrm{Inn}(Q)$), we can assume $\varphi_0 \overset{\text{def}}{=} \varphi|_{P_0} = \varphi'|_{P_0}$. Since $P \in \mathrm{Ob}(\mathcal{F}_i^*)$, we have $C_{S_i}(\varphi_0(P_0)) \le \varphi(P)$, so

$$\varphi(P) = \big\{ g \in N_{S_i}(\varphi_0(P_0)) \,\big|\, c_g \in \varphi_0 \mathrm{Aut}_P(P_0) \varphi_0^{-1} \big\} = \varphi'(P).$$

Hence there is a unique $\beta \in \mathrm{Aut}_{\mathcal{F}_i}(P)$ such that $\varphi' = \varphi \circ \beta$; and also $\beta|_{P_0} = \mathrm{Id}$. So $\psi' = \psi \circ [\beta]$ in $\mathrm{Mor}_{\mathcal{O}(\mathcal{F}_i^*)}(P, Q)$, where $[\beta] \in \Gamma_i(P)$.

Step 3: Define functors

$$\mathcal{Z}_i \colon \mathcal{O}(\mathcal{F}_i^c)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod} \qquad \text{and} \qquad \mathcal{Z}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$$

by setting $\mathcal{Z}_i(P) = Z(P) \cap S_0 = C_{Z(P_0)}(P)$ and $\mathcal{Z}_{\mathcal{C}}(Q, K) = C_{Z(Q)}(K)$ (the subgroup of elements of $Z(Q)$ fixed pointwise by K). Morphisms are sent in the obvious way. Set $\mathcal{Z}_{i*} = \mathcal{Z}_i|_{\mathcal{O}(\mathcal{F}_i^*)^{\mathrm{op}}}$.

We claim that

$$\varprojlim_{\mathcal{O}(\mathcal{F}_1^c)}{}^*(\mathcal{Z}_1) \cong \varprojlim_{\mathcal{O}(\mathcal{F}_1^*)}{}^*(\mathcal{Z}_{1*}) \cong \varprojlim_{\mathcal{C}}{}^*(\mathcal{Z}_{\mathcal{C}}) \cong \varprojlim_{\mathcal{O}(\mathcal{F}_2^*)}{}^*(\mathcal{Z}_{2*}) \cong \varprojlim_{\mathcal{O}(\mathcal{F}_2^c)}{}^*(\mathcal{Z}_2). \tag{6}$$

Since $\mathcal{Z}_{i*} = \mathcal{Z}_{\mathcal{C}} \circ \Phi_i$ by definition, the second and third isomorphisms follow from points (i–iii) in Step 2 and [BLO1, Lemma 1.3].

We prove the other isomorphisms in (6) using Lemma 2.8. Fix $i = 1, 2$. We already saw in Step 1 that $\mathrm{Ob}(\mathcal{F}_i^*)$ is closed (inside $\mathrm{Ob}(\mathcal{F}_i^c)$) with respect to $\mathcal{F}_i$-conjugacy. If $P \le Q \le S_i$ and $P \in \mathrm{Ob}(\mathcal{F}_i^*)$, then for each Q^* which is $\mathcal{F}_i$-conjugate to Q, if we set $P^* = \varphi(P) \le Q^*$ for some $\varphi \in \mathrm{Iso}_{\mathcal{F}_i}(Q, Q^*)$, then $C_{S_i}(P_0^*) \le P^*$ implies $C_{S_i}(Q_0^*) \le Q^*$, and so $Q \in \mathrm{Ob}(\mathcal{F}_i^*)$. Thus $\mathrm{Ob}(\mathcal{F}_i^*)$ is closed with respect to overgroups.

For each object P in $\mathcal{F}_i^c$ not in $\mathcal{F}_i^*$, there is P^* $\mathcal{F}_i$-conjugate to P such that $C_{S_i}(P_0^*) \nleq P^*$. By Lemma 1.7, there is $g \in N_{S_i}(P^*) \smallsetminus P^*$ which centralizes P_0^*. Thus $[c_g] \in \mathrm{Out}_{\mathcal{F}_i}(P^*)$ is a nontrivial element of p-power order which acts trivially on $\mathcal{Z}_i(P^*)$. So by Lemma 2.8, $\varprojlim^*_{\mathcal{O}(\mathcal{F}_i^c)}(\mathcal{Z}_i) \cong \varprojlim^*_{\mathcal{O}(\mathcal{F}_i^*)}(\mathcal{Z}_{i*})$ for each $i = 1, 2$; and this finishes the proof of (6).

Step 4: By [O1, Lemma 2.1] and Lemma 2.11(a),

$$\varprojlim^j_{\mathcal{O}(\mathcal{F}_2^c)}(\mathcal{Z}_2) \overset{\text{def}}{=} \varprojlim^j_{\mathcal{O}(\mathcal{F}^c_{\widehat{S}}(\widehat{G}))}\big(Z(-) \cap S_0\big) \cong \varprojlim^j_{\mathcal{O}_p(\widehat{G})}(\mathcal{Z}_{\widehat{G}}^{G_0}) = 0$$

for $j \geq 2$. Hence by (6), $\varprojlim^j_{\mathcal{O}(\mathcal{F}^c)}(\mathcal{Z}_1) = 0$ for $j \geq 2$ (recall $\mathcal{F} = \mathcal{F}_1$).

We claim that for $j \geq 2$,

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^j(\mathcal{Z}_\mathcal{F}) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^j(\mathcal{Z}_1). \tag{7}$$

Set $\widehat{\mathcal{Z}} = \mathcal{Z}_\mathcal{F}/\mathcal{Z}_1$ for short; thus $\widehat{\mathcal{Z}}(P) = Z(P)/(Z(P) \cap S_0)$ for each P. If $\mathcal{F}_0 = O^{p'}(\mathcal{F})$, then (7) holds since $S_0 = S$ and hence $\mathcal{Z}_\mathcal{F} = \mathcal{Z}_1$. If $\mathcal{F}_0 = N_\mathcal{F}^K(Q)$ for some $Q \trianglelefteq \mathcal{F}$ and some $K \trianglelefteq \mathrm{Aut}(Q)$, then $\widehat{\mathcal{Z}}(P) = 0$ for each $P \in \mathrm{Ob}(\mathcal{F}^c)$ which contains Q, in particular for each subgroup which is $\mathcal{F}$-centric and $\mathcal{F}$-radical (Proposition 1.5); and (7) holds by Lemma 2.8.

Assume $\mathcal{F}_0 = O^p(\mathcal{F})$. For each $P \in \mathrm{Ob}(\mathcal{F}^c)$, let $H_P \trianglelefteq \mathrm{Out}_\mathcal{F}(P)$ be the kernel of the $\mathrm{Out}_\mathcal{F}(P)$-action on $\widehat{\mathcal{Z}}(P) = Z(P)/(Z(P) \cap S_0)$. By definition of $S_0 = \mathfrak{hyp}(\mathcal{F})$, H_P contains $O^p(\mathrm{Out}_\mathcal{F}(P))$, and thus $\mathrm{Out}_\mathcal{F}(P)/H_P$ is a p-group. So for $j \geq 1$,

$$\Lambda^j(\mathrm{Out}_\mathcal{F}(P); \widehat{\mathcal{Z}}(P)) \cong \begin{cases} 0 & \text{if } p \big| |H_P| \\ 0 & \text{if } p \nmid |\mathrm{Out}_\mathcal{F}(P)| \\ \Lambda^j(\mathrm{Out}_\mathcal{F}(P)/H_P; \widehat{\mathcal{Z}}(P)) = 0 & \text{otherwise} \end{cases}$$

by [JMO, Proposition 6.1]: by point (ii) of the proposition in the first case, by point (i) in the second, and by points (iii) and (ii) in the third. So by [O1, Lemma 2.3], $\varprojlim^j_{\mathcal{O}(\mathcal{F}^c)}(\widehat{\mathcal{Z}}) = 0$ for all $j \geq 1$, and (7) also holds in this case.

We now conclude that $\varprojlim^j_{\mathcal{O}(\mathcal{F}^c)}(\mathcal{Z}_\mathcal{F}) = 0$ for all $j \geq 2$. So by [BLO2, Proposition 3.1], there is a (unique) centric linking system $\mathcal{L}^c$ associated to $\mathcal{F}$. □

2.4. **Proof of Theorem A.**

We want to show that if $\mathfrak{red}(\mathcal{F})$ is tame, then so is $\mathcal{F}$. The proof splits naturally into two parts. We first show, under certain additional hypotheses, that if $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ and $\mathcal{F}_0$ is tame, then $\mathcal{F}$ is tame. Afterwards, we show (again under additional hypotheses) that $\mathcal{F}$ is tame if $\mathcal{F}/Z(\mathcal{F})$ is tame. In both cases, this means proving that certain homomorphisms are split surjective, by first constructing an appropriate pullback square of automorphism groups, and then applying the following elementary lemma.

Lemma 2.13. *If the following square of groups and homomorphisms*

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha} & A_2 \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow{\beta} & B_2 \end{array}$$

is a pullback square, and β is split surjective, then α is split surjective. □

We first work with normal subsystems. We first recall some convenient notation. When P is a p-centric subgroup of a finite group G (i.e., an $\mathcal{F}_S(G)$-centric subgroup when $P \leq$

$S \in \mathrm{Syl}_p(G)$), we set $C'_G(P) = O^p(C_G(P))$. Thus $C'_G(P)$ has order prime to p, and $C_G(P) = Z(P) \times C'_G(P)$.

For any normal pair $(S_0, \mathcal{F}_0, \mathcal{L}_0) \trianglelefteq (S, \mathcal{F}, \mathcal{L})$, let

$$\widetilde{\rho} = \widetilde{\rho}^{\mathcal{L}}_{\mathcal{L}_0} \colon \mathrm{Aut}_{\mathcal{L}}(S_0) \longrightarrow \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_0)$$

be the homomorphism which sends $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S_0)$ to c_γ. Here, $c_\gamma \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_0)$ sends an object P to $\pi(\gamma)(P)$ and sends $\psi \in \mathrm{Mor}_{\mathcal{L}_0}(P, Q)$ to $(\gamma|_{Q,\pi(\gamma)(Q)}) \circ \psi \circ (\gamma|_{P,\pi(\gamma)(P)})^{-1}$ (well defined by Definition 1.27). Let

$$\rho = \rho^{\mathcal{L}}_{\mathcal{L}_0} \colon \underset{=\mathrm{Aut}_{\mathcal{L}}(S_0)/\mathrm{Aut}_{\mathcal{L}_0}(S_0)}{\mathcal{L}/\mathcal{L}_0} \longrightarrow \underset{=\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_0)/\{c_\gamma \,|\, \gamma \in \mathrm{Aut}_{\mathcal{L}_0}(S_0)\}}{\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_0)}$$

be the homomorphism induced by $\widetilde{\rho}$, which sends $[\gamma]$ to the class of c_γ. This is analogous to the conjugation homomorphism $G/G_0 \longrightarrow \mathrm{Out}(G_0)$ for a pair of groups $G_0 \trianglelefteq G$. For example, $\mathcal{L}_0$ is centric in $\mathcal{L}$ (see Definition 1.27) if and only if $\rho^{\mathcal{L}}_{\mathcal{L}_0}$ is injective.

We next show that when $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ have associated linking systems $\mathcal{L}_0 \trianglelefteq \mathcal{L}$, where $\mathcal{L}_0$ is centric in $\mathcal{L}$ and $\mathcal{F}_0$ is realizable, then under some extra conditions, $\mathcal{F}$ is also realizable.

Lemma 2.14. *Fix a normal pair $(S_0, \mathcal{F}_0, \mathcal{L}_0) \trianglelefteq (S, \mathcal{F}, \mathcal{L})$ such that $\mathcal{L}_0$ is centric in $\mathcal{L}$. Set $\mathcal{H}_0 = \mathrm{Ob}(\mathcal{L}_0)$ and $\mathcal{H} = \mathrm{Ob}(\mathcal{L})$, and assume $\mathcal{H}_0$ is $\mathrm{Aut}(S_0, \mathcal{F}_0)$-invariant. Assume there is a finite group G_0 such that*

(a) *$S_0 \in \mathrm{Syl}_p(G_0)$, $\mathcal{F}_0 = \mathcal{F}_{S_0}(G_0)$, and $\mathcal{L}_0 \cong \mathcal{L}^{\mathcal{H}_0}_{S_0}(G_0)$;*

(b) *$Z(G_0) = Z(\mathcal{F}_0)$; and*

(c) *there is a homomorphism $\widehat{\rho} \colon \mathcal{L}/\mathcal{L}_0 \longrightarrow \mathrm{Out}(G_0)$ such that*

$$\kappa^{\mathcal{H}_0}_{G_0} \circ \widehat{\rho} = \rho^{\mathcal{L}}_{\mathcal{L}_0} \colon \mathcal{L}/\mathcal{L}_0 \longrightarrow \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_0)\ .$$

Then $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{L} \cong \mathcal{L}^{\mathcal{H}}_S(G)$ for some finite group G such that $S \in \mathrm{Syl}_p(G)$, $G_0 \trianglelefteq G$, $G/G_0 \cong \mathcal{L}/\mathcal{L}_0$, and such that the extension realizes the given outer action $\widehat{\rho}$ of $G/G_0 \cong \mathcal{L}/\mathcal{L}_0$ on G_0.

Proof. We construct the group G in Step 1, and prove that $\mathcal{L}^{\mathcal{H}}_S(G) \cong \mathcal{L}$ and $\mathcal{F}_S(G) = \mathcal{F}$ in Step 2. Throughout the proof, we identify $\mathcal{L}_0$ with $\mathcal{L}^{\mathcal{H}_0}_{S_0}(G_0)$.

Step 1: Consider the following diagram whose rows are exact by Lemma 1.14:

$$\begin{array}{ccccccccc}
1 \longrightarrow & Z(G_0) & \longrightarrow & N_{G_0}(S_0) & \xrightarrow{\mathrm{conj}} & \mathrm{Aut}(G_0, S_0) & \xrightarrow{\mathrm{pr}_1} & \mathrm{Out}(G_0) & \longrightarrow 1 \\
& \Big\| & & \lambda_0 \downarrow & & \widetilde{\kappa} \downarrow {=\widetilde{\kappa}^{\mathcal{H}_0}_{G_0}} & & \kappa \downarrow {=\kappa^{\mathcal{H}_0}_{G_0}} & \\
1 \longrightarrow & Z(\mathcal{F}_0) & \longrightarrow & \mathrm{Aut}_{\mathcal{L}_0}(S_0) & \xrightarrow{\mathrm{conj}} & \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_0) & \xrightarrow{\mathrm{pr}_2} & \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_0) & \longrightarrow 1\ .
\end{array} \qquad (8)$$

Here, λ_0 sends $g \in N_{G_0}(S_0)$ to its class in $\mathrm{Aut}_{\mathcal{L}_0}(S_0) = N_{G_0}(S_0)/C'_{G_0}(S_0)$. The first and third squares clearly commute. The second square commutes since for $g \in N_{G_0}(S_0)$, $\widetilde{\kappa}$ sends c_g to the automorphism $[a] \mapsto [gag^{-1}] = c_{\lambda_0(g)}(a)$. By definition of $\widetilde{\kappa} = \widetilde{\kappa}^{\mathcal{H}_0}_{G_0}$,

$$\widetilde{\kappa}(\beta)(\lambda_0(g)) = \lambda_0(\beta(g)) \qquad \text{for all } \beta \in \mathrm{Aut}(G_0, S_0),\ g \in N_{G_0}(S_0)\ . \qquad (9)$$

Set $\mathrm{Aut}(G_0, S_0)_{\widehat{\rho}} = \mathrm{pr}_1^{-1}(\widehat{\rho}(\mathcal{L}/\mathcal{L}_0))$. Since $\mathcal{L}_0$ is centric in $\mathcal{L}$, $\rho = \rho^{\mathcal{L}}_{\mathcal{L}_0}$ sends $\mathcal{L}/\mathcal{L}_0$ injectively into $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_0)$. Hence κ sends $\widehat{\rho}(\mathcal{L}/\mathcal{L}_0)$ injectively into $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_0)$. So by a diagram chase in (8), $N_{G_0}(S_0)$ is the pullback of $\mathrm{Aut}(G_0, S_0)_{\widehat{\rho}}$ and $\mathrm{Aut}_{\mathcal{L}_0}(S_0)$ over $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_0)$.

Let H be the group which makes the following square a pullback:

$$\begin{array}{ccc} H & \xrightarrow{\ \varphi\ } & \mathrm{Aut}(G_0, S_0)_{\widehat{\rho}} \\ \Big\downarrow{\scriptstyle \lambda} & & \Big\downarrow{\scriptstyle \widetilde{\kappa}} \\ \mathrm{Aut}_{\mathcal{L}}(S_0) & \xrightarrow{\ \widetilde{\rho}\ } & \mathrm{Aut}^{I}_{\mathrm{typ}}(\mathcal{L}_0)\ . \end{array} \tag{10}$$

For each $\alpha \in \mathrm{Aut}_{\mathcal{L}}(S_0)$, $\widetilde{\rho}(\alpha) \in \mathrm{pr}_2^{-1}(\rho(\mathcal{L}/\mathcal{L}_0))$ by definition, and hence lifts to an element of $\mathrm{Aut}(G_0, S_0)_{\widehat{\rho}}$. This proves that λ is onto. By comparison with the middle square in (8), we can identify $N_{G_0}(S_0)$ with $\lambda^{-1}(\mathrm{Aut}_{\mathcal{L}_0}(S_0)) \trianglelefteq H$. Thus

$$H/N_{G_0}(S_0) = H/H_0 \cong \mathrm{Aut}_{\mathcal{L}}(S_0)/\mathrm{Aut}_{\mathcal{L}_0}(S_0) = \mathcal{L}/\mathcal{L}_0\ , \tag{11}$$

where we set $H_0 = N_{G_0}(S_0)$, regarded as a subgroup of G_0 and of H.

We claim that for all $h \in H$ and $a \in H_0$,

$$\varphi(h)(a) = hah^{-1} \in H_0\ . \tag{12}$$

Since (10) is a pullback, it suffices to prove (12) after applying φ and after applying λ. It holds after applying λ (or λ_0) since

$$\lambda_0(\varphi(h)(a)) = \widetilde{\kappa}(\varphi(h))(\lambda_0(a)) = \widetilde{\rho}(\lambda(h))(\lambda_0(a)) = \lambda(h)\lambda_0(a)\lambda(h)^{-1} = \lambda_0(hah^{-1})\ :$$

the first equality by (9), the second by the commutativity of (10), and the third since $\widetilde{\rho}$ is defined by conjugation in $\mathcal{L}$. Since $\varphi|_{H_0}$ is also defined to be conjugation,

$$\varphi(\varphi(h)(a)) = c_{\varphi(h)(a)} = \varphi(h) \circ c_a \circ \varphi(h)^{-1} = \varphi(h) \circ \varphi(a) \circ \varphi(h)^{-1} = \varphi(hah^{-1})\ .$$

This finishes the proof of (12).

We want to construct a group G with $G_0 \trianglelefteq G$, $G/G_0 \cong \mathcal{L}/\mathcal{L}_0$, and $N_G(S_0) = H$. To do this, first set $\Gamma = G_0 \rtimes H$: the semidirect product with the action of H on G_0 given by φ as defined in (10). Elements of Γ are written as pairs (g, h) for $g \in G_0$ and $h \in H$. Thus $(g, h)(g', h') = (g{\cdot}\varphi(h)(g'), hh')$. Set $N = \{(a, a^{-1}) \,|\, a \in H_0\}$. For $a, b \in H_0$,

$$(a, a^{-1})(b, b^{-1}) = (a{\cdot}\varphi(a^{-1})(b), a^{-1}b^{-1}) = (a{\cdot}a^{-1}ba, a^{-1}b^{-1}) = (ba, (ba)^{-1}) \in N,$$

where the second equality holds by (12). Thus N is a subgroup. For $g \in G_0$ and $a \in H_0$,

$$\begin{aligned} (g, 1)(a, a^{-1})(g, 1)^{-1} &= (ga, a^{-1})(g^{-1}, 1) = (ga{\cdot}\varphi(a^{-1})(g^{-1}), a^{-1}) \\ &= (ga{\cdot}a^{-1}g^{-1}a, a^{-1}) = (a, a^{-1})\ ; \end{aligned}$$

where $\varphi(a^{-1})(g^{-1}) = a^{-1}g^{-1}a$ since by construction, $\varphi|_{H_0}$ is the conjugation homomorphism of (8). Thus $(g, 1)$ normalizes (centralizes) N. For $h \in H$ and $a \in H_0$,

$$(1, h)(a, a^{-1})(1, h)^{-1} = (\varphi(h)(a), ha^{-1})(1, h^{-1}) = (\varphi(h)(a), (hah^{-1})^{-1}) \in N$$

by (12), and thus $(1, h)$ also normalizes N. This proves that $N \trianglelefteq \Gamma$.

Now set $G = \Gamma/N$, and regard G_0 and H as subgroups of G. By construction, $G = G_0H$, $G_0 \cap H = H_0 = N_{G_0}(S_0)$, $G_0 \trianglelefteq G$, and $G/G_0 \cong H/H_0 \cong \mathcal{L}/\mathcal{L}_0$ (the last isomorphism by (11)). Also, $H \le N_G(S_0)$, and since $[H{:}N_{G_0}(S_0)] = [G{:}G_0] \ge [N_G(S_0){:}N_{G_0}(S_0)]$, we have $H = N_G(S_0)$. The outer conjugation action of G/G_0 on G_0 is induced by φ. Consider the following diagram

$$\begin{array}{ccc} G/G_0 \cong H/H_0 & \xrightarrow{\ \bar{\varphi}\ } & \mathrm{Out}(G_0) \\ {\scriptstyle \cong}\Big\downarrow{\scriptstyle \bar{\lambda}} & \nearrow{\scriptstyle \widehat{\rho}} & \Big\downarrow{\scriptstyle \kappa} \\ \mathcal{L}/\mathcal{L}_0 \overset{\mathrm{def}}{=} \mathrm{Aut}_{\mathcal{L}}(S_0)/\mathrm{Aut}_{\mathcal{L}_0}(S_0) & \xrightarrow{\ \rho^{\mathcal{L}}_{\mathcal{L}_0}\ } & \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_0) \end{array} \tag{13}$$

where $\bar{\varphi}$ and $\bar{\lambda}$ are induced by φ and λ and the square commutes by (10), and where the lower triangle commutes by condition (c). Then $\mathrm{Im}(\bar{\varphi}) \le \mathrm{Im}(\widehat{\rho})$ by definition of $\mathrm{Aut}(G_0, S_0)_{\widehat{\rho}}$, and κ sends $\mathrm{Im}(\widehat{\rho})$ isomorphically to $\mathrm{Im}(\rho^{\mathcal{L}}_{\mathcal{L}_0})$ since $\rho^{\mathcal{L}}_{\mathcal{L}_0}$ is injective. Thus the upper triangle in (13) commutes, so the outer conjugation action of G/G_0 on G_0 is equal to $\widehat{\rho}$ via our identification $G/G_0 \cong \mathcal{L}/\mathcal{L}_0$.

By comparison of (8) and (10), we see that

$$\mathrm{Ker}(\lambda) = \mathrm{Ker}(\lambda_0) = C'_{G_0}(S_0)\ .$$

In particular, $\mathrm{Ker}(\lambda)$ has order prime to p. Also, $\delta_{S_0}(S)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{L}}(S_0)$ by Proposition 1.11(d). Fix any Sylow p-subgroup of $\lambda^{-1}(\delta_{S_0}(S))$, and identify it with S via $\delta_{S_0}^{-1} \circ \lambda$. Since $[G{:}H] = [G_0{:}H_0]$ is prime to p, we also have $S \in \mathrm{Syl}_p(G)$.

Step 2: Set $\mathcal{F}' = \mathcal{F}_S(G)$ for short. By Proposition 1.28, $\mathcal{F}_0 = \mathcal{F}_{S_0}(G_0)$ is normal in $\mathcal{F}'$. So by Lemma 1.20(d), $\mathcal{H} = \mathrm{Ob}(\mathcal{L})$ contains all subgroups of S which are $\mathcal{F}'$-centric and $\mathcal{F}'$-radical.

We next show that all subgroups in $\mathcal{H}$ are G-quasicentric. Since overgroups of G-quasicentric subgroups are G-quasicentric, it suffices to prove this for $P \in \mathcal{H}_0$. Fix such P, and assume it is fully centralized in $\mathcal{F}'$. We must show that $O_{p'}(C_G(P)) = O^p(C_G(P))$; i.e., that $C_G(P)$ contains a normal subgroup of order prime to p and of p-power index. Define

$$\Phi_P\colon N_G(P) \longrightarrow \mathrm{Aut}_{\mathcal{L}}(P)$$

as follows. Fix $g \in N_G(P)$, write $g = g_0h$ for some $g_0 \in G_0$ and $h \in H = N_G(S_0)$, and set $\Phi_P(g) = [g_0] \circ \lambda(h)|_{P,hPh^{-1}}$, where $[g_0] \in \mathrm{Mor}_{\mathcal{L}_0}(hPh^{-1}, P)$ is induced by the identification $\mathcal{L}_0 = \mathcal{L}^{\mathcal{H}_0}_{S_0}(G_0)$. If $g = g_0h = g_0'h'$ where $g_0, g_0' \in G_0$ and $h, h' \in H$, then $a \overset{\text{def}}{=} g_0^{-1}g_0' = hh'^{-1} \in H_0$, so $g_0' = g_0a$, $h = ah'$, and

$$[g_0] \circ \lambda(h)|_{P,hPh^{-1}} = [g_0] \circ \lambda(a)|_{h'Ph'^{-1},hPh^{-1}} \circ \lambda(h')|_{P,h'Ph'^{-1}} = [g_0a] \circ \lambda(h')|_{P,h'Ph'^{-1}}.$$

Thus $\Phi_P(g)$ is well defined, independently of the choice of g_0 and h, and $\Phi_{S_0} = \lambda$. Moreover, $\Phi_P|_{N_S(P)} = \delta_P$, since $\lambda|_S = \delta_{S_0}\colon S \longrightarrow \mathrm{Aut}_{\mathcal{L}}(P)$ by the identification of S as a subgroup of H. To see that Φ_P is a homomorphism, it suffices to check that

$$[hg_0h^{-1}] = \lambda(h) \circ [g_0] \circ \lambda(h)^{-1} \tag{14}$$

for each $g_0 \in G_0$ and $h \in H$, and this follows from the commutativity of (10).

We next claim that the composite $\pi_P \circ \Phi_P\colon N_G(P) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(P)$ sends $g \in N_G(P)$ to $c_g \in \mathrm{Aut}(P)$. Set $g = g_0h$ as above. By definition of the linking system $\mathcal{L}^{\mathcal{H}_0}_{S_0}(G_0)$, $\pi_P(\Phi_P(g_0)) = \pi_P([g_0]) = c_{g_0}$. By (14) and axiom (C) for the linking system $\mathcal{L}$, $\pi_{S_0}(\lambda(h)) \in \mathrm{Aut}_{\mathcal{F}}(S_0)$ is conjugation by h, and hence it is also conjugation by h on $P \le S_0 \trianglelefteq H$. This proves the claim.

Since $\mathcal{F}_0 \trianglelefteq \mathcal{F}$, $\mathcal{F}_0 \trianglelefteq \mathcal{F}'$, and $\mathrm{Aut}_{\mathcal{F}}(S_0) = \mathrm{Aut}_{\mathcal{F}'}(S_0)$, the $\mathcal{F}$- and $\mathcal{F}'$-conjugacy classes of any subgroup $Q \le S_0$ are the same. It follows that $\mathcal{H}$ is closed under $\mathcal{F}'$-conjugacy, and that P is fully centralized in $\mathcal{F}$. Hence

- $\mathrm{Ker}[\mathrm{Aut}_{\mathcal{L}}(P) \xrightarrow{\ \pi_P\ } \mathrm{Aut}_{\mathcal{F}}(P)] = \delta_P(C_S(P))$;
- $\Phi_P|_{N_S(P)} = \delta_P$ is injective by Proposition 1.11(c);
- $\mathrm{Ker}(\pi_P \circ \Phi_P) = C_G(P)$ since $\pi_P \circ \Phi_P(g) = c_g$; and
- $C_S(P) \in \mathrm{Syl}_p(C_G(P))$ by [BLO2, Proposition 1.3].

Hence $\mathrm{Ker}(\Phi_P)$ is a normal subgroup of $C_G(P)$ of order prime to p, and $C_G(P)/\mathrm{Ker}(\Phi_P) \cong C_S(P)$ is a p-group. It follows that $\mathrm{Ker}(\Phi_P) = O^p(C_G(P))$, and thus that P is G-quasicentric.

Set $\mathcal{L}' = \mathcal{L}_S^{\mathcal{H}}(G)$. We have now shown that $\mathcal{H}$ satisfies the conditions which ensure that $\mathcal{L}'$ is a linking system associated to $\mathcal{F}_S(G)$. By Proposition 1.28 again, $\mathcal{L}'$ contains $\mathcal{L}_0$ as a normal linking subsystem. Also, $\mathrm{Aut}_{\mathcal{L}'}(S_0) = H/\mathrm{Ker}(\lambda) \cong \mathrm{Aut}_{\mathcal{L}}(S_0)$ since $O^p(C_G(S_0)) = \mathrm{Ker}(\Phi_{S_0}) = \mathrm{Ker}(\lambda)$, and they have the same action on $\mathcal{L}_0$ (under this identification) by the commutativity of (10).

Now, $C_{\mathrm{Aut}_{\mathcal{L}}(S_0)}(S_0) = \delta_{S_0}(C_S(S_0))$ by axioms (C) and (A) (and since S_0 is fully centralized in $\mathcal{F}$). Each $P \in \mathcal{H}_0$ is $\mathcal{F}$-quasicentric by Proposition 1.11(g), and hence satisfies the second condition in Definition 1.10(b). (A priori, this condition only holds when $P \in \mathcal{H}_0$ is fully centralized in $\mathcal{F}$, but it is easily extended to arbitrary subgroups in $\mathcal{H}_0$.) Thus conditions (2) and (3) in the statement of [O3, Theorem 9] hold, where $\Gamma = \mathrm{Aut}_{\mathcal{L}}(S_0)$ and $\tau = \widetilde{\rho}$. So by the uniqueness statement in that theorem, $\mathcal{F} = \mathcal{F}'$ and $\mathcal{L} \cong \mathcal{L}'$. □

In order to compare tameness of $\mathcal{F}_0$ and of $\mathcal{F}$ when $(S_0, \mathcal{F}_0, \mathcal{L}_0) \trianglelefteq (S, \mathcal{F}, \mathcal{L})$, we need to compare the automorphisms of $\mathcal{L}_0$ with those of $\mathcal{L}$. This is done in the following lemma. For any normal pair $\mathcal{L}_0 \trianglelefteq \mathcal{L}$ of linking systems, we set

$$\begin{aligned} \mathrm{Aut}_{\mathcal{L}}(\mathcal{L}_0) &= \widetilde{\rho}^{\mathcal{L}}_{\mathcal{L}_0}(\mathrm{Aut}_{\mathcal{L}}(S_0)) = \{c_\gamma \mid \gamma \in \mathrm{Aut}_{\mathcal{L}}(S_0)\} \le \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_0) \\ \mathrm{Out}_{\mathcal{L}}(\mathcal{L}_0) &= \rho^{\mathcal{L}}_{\mathcal{L}_0}(\mathcal{L}/\mathcal{L}_0) = \mathrm{Aut}_{\mathcal{L}}(\mathcal{L}_0)/\mathrm{Aut}_{\mathcal{L}_0}(\mathcal{L}_0) \le \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_0)\ . \end{aligned}$$

Lemma 2.15. *Fix a pair of finite groups $G_0 \trianglelefteq G$, let $S_0 \trianglelefteq S$ be Sylow p-subgroups of $G_0 \trianglelefteq G$, and set $\mathcal{F}_0 = \mathcal{F}_{S_0}(G_0)$ and $\mathcal{F} = \mathcal{F}_S(G)$. Assume $Z(G_0) = Z(\mathcal{F}_0)$. Let $\mathcal{H}_0$ and $\mathcal{H}$ be sets of subgroups such that*

$$\mathcal{L}_0 \overset{\mathrm{def}}{=} \mathcal{L}_{S_0}^{\mathcal{H}_0}(G_0) \qquad \text{and} \qquad \mathcal{L} \overset{\mathrm{def}}{=} \mathcal{L}_S^{\mathcal{H}}(G)$$

are linking systems associated to $\mathcal{F}_0$ and $\mathcal{F}$, respectively. Assume

$$\mathcal{L}_0 \trianglelefteq \mathcal{L}\ , \qquad \mathcal{L}_0 \text{ is centric in } \mathcal{L}\ , \qquad \text{and} \qquad \mathcal{L}/\mathcal{L}_0 \cong G/G_0\ .$$

Assume also $\mathcal{H}_0$ is $\mathrm{Aut}(S_0, \mathcal{F}_0)$-invariant, and $\mathcal{H}$ is $\mathrm{Aut}(S, \mathcal{F})$-invariant. Then the following square

$$\begin{array}{ccc} \mathrm{Out}(G, G_0) & \xrightarrow{\quad\kappa\quad} & \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}, \mathcal{L}_0) \\ \Big\downarrow{\scriptstyle R_1} & & \Big\downarrow{\scriptstyle R_2} \\ N_{\mathrm{Out}(G_0)}(\mathrm{Out}_G(G_0))/\mathrm{Out}_G(G_0) & \xrightarrow{\quad\kappa^*\quad} & N_{\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_0)}(\mathrm{Out}_{\mathcal{L}}(\mathcal{L}_0))/\mathrm{Out}_{\mathcal{L}}(\mathcal{L}_0) \end{array} \tag{15}$$

is a pullback. Here, $\mathrm{Out}(G, G_0) \le \mathrm{Out}(G)$ and $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}, \mathcal{L}_0) \le \mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ are the subgroups of classes of automorphisms which leave G_0 and $\mathcal{L}_0$ invariant, respectively, κ is the restriction of $\kappa_G^{\mathcal{H}}$, κ^* is induced by $\kappa_{G_0}^{\mathcal{H}_0}$, and R_1 and R_2 are induced by restriction.

Proof. By the Frattini argument, $G = G_0{\cdot}N_G(S_0)$ (all subgroups G-conjugate to S_0 are G_0-conjugate to S_0). Hence $G/G_0 \cong N_G(S_0)/N_{G_0}(S_0)$, while

$$\mathcal{L}/\mathcal{L}_0 \overset{\mathrm{def}}{=} \mathrm{Aut}_{\mathcal{L}}(S_0)/\mathrm{Aut}_{\mathcal{L}_0}(S_0) = \big(N_G(S_0)/O^p(C_G(S_0))\big)\big/\big(N_{G_0}(S_0)/C'_{G_0}(S_0)\big)$$

(and S_0 is G-quasicentric since it is an object of the linking system $\mathcal{L} = \mathcal{L}_S^{\mathcal{H}}(G)$). Since $G/G_0 \cong \mathcal{L}/\mathcal{L}_0$, it follows that $O^p(C_G(S_0)) = C'_{G_0}(S_0)$. Also, for each $g \in C_G(G_0) \le N_G(S_0)$, $[g] \in \mathrm{Aut}_{\mathcal{L}}(S_0)$ acts trivially on $\mathcal{L}_0$ under conjugation, so $[g] \in \mathrm{Aut}_{\mathcal{L}_0}(S_0)$ since $\mathcal{L}_0$ is centric in $\mathcal{L}$, and hence $g \in G_0$. We have now shown that

$$O^p(C_G(S_0)) = C'_{G_0}(S_0) \qquad \text{and} \qquad C_G(G_0) = Z(G_0)\ . \tag{16}$$

Step 1: We first show the following square is a pullback:

$$\begin{array}{ccc} \mathrm{Aut}(G,G_0,S{\cdot}C'_{G_0}(S_0)) & \xrightarrow{\ \widetilde{\kappa}\ } & \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L},\mathcal{L}_0) \\ {\scriptstyle \mathrm{Res}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_2} \\ N_{\mathrm{Aut}(G_0,S_0)}(\mathrm{Aut}_G(G_0,S_0)) & \xrightarrow{\ \widetilde{\kappa}_0\ } & N_{\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_0)}(\mathrm{Aut}_{\mathcal{L}}(\mathcal{L}_0)) \end{array} \quad (17)$$

Here, $\mathrm{Aut}(G,G_0,S{\cdot}C'_{G_0}(S_0))$ is the group of automorphisms of G which send both G_0 and $S{\cdot}C'_{G_0}(S_0)$ to themselves and $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L},\mathcal{L}_0) \le \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ is the subgroup of elements which leave $\mathcal{L}_0$ invariant.

Both Res_1 and Res_2 are defined by restriction. Each $\alpha \in \mathrm{Aut}(G,G_0,S{\cdot}C'_{G_0}(S_0))$ leaves $S_0 \times C'_{G_0}(S_0) = G_0 \cap (S{\cdot}C'_{G_0}(S_0))$ invariant, and hence also leaves S_0 invariant. Clearly, $\alpha|_{\mathrm{Aut}(G_0,S_0)}$ normalizes $\mathrm{Aut}_G(G_0,S_0)$. To see that Res_2 maps to the normalizer, fix $\sigma \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L},\mathcal{L}_0)$ and $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S_0)$, and set $\sigma_0 = \sigma|_{\mathcal{L}_0} \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_0)$. Then

$$\sigma_0 c_\gamma \sigma_0^{-1} = c_{\sigma(\gamma)}, \quad (18)$$

(using Lemma 1.15 to show this holds on objects), and thus σ_0 normalizes $\mathrm{Aut}_{\mathcal{L}}(\mathcal{L}_0)$.

The homomorphism $\widetilde{\kappa}_0$ is the restriction of $\widetilde{\kappa}^{\mathcal{H}_0}_{G_0}$, which is defined since $\mathcal{H}_0$ is $\mathrm{Aut}(S_0,\mathcal{F}_0)$-invariant. Since $\widetilde{\kappa}^{\mathcal{H}_0}_{G_0}$ maps $\mathrm{Aut}_G(G_0,S_0)$ onto $\mathrm{Aut}_{\mathcal{L}}(\mathcal{L}_0)$, it sends the normalizer of $\mathrm{Aut}_G(G_0,S_0)$ into the normalizer of $\mathrm{Aut}_{\mathcal{L}}(\mathcal{L}_0)$.

Defining $\widetilde{\kappa}$ requires more explanation. For $\alpha \in \mathrm{Aut}(G,G_0,S{\cdot}C'_{G_0}(S_0))$, $\alpha(S)$ is a Sylow p-subgroup of $S{\cdot}C'_{G_0}(S_0)$, so $\alpha(S) = hSh^{-1}$ for some $h \in C'_{G_0}(S_0)$. Hence $c_h^{-1} \circ \alpha \in \mathrm{Aut}(G,G_0,S)$ and we define $\widetilde{\kappa}(\alpha) = \widetilde{\kappa}^{\mathcal{H}}_G(c_h^{-1} \circ \alpha) \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L},\mathcal{L}_0)$. If $h' \in C'_{G_0}(S_0)$ with $\alpha(S) = h'Sh'^{-1}$, then $h^{-1}h' \in C'_{G_0}(S_0) \cap N_G(S)$. Since S_0 is strongly closed in $\mathcal{F}$, the restriction homomorphism

$$N_G(S)/C'_G(S) = \mathrm{Aut}_{\mathcal{L}}(S) \longrightarrow \mathrm{Aut}_{\mathcal{L}}(S_0) = N_G(S_0)/C'_{G_0}(S_0)$$

is injective by Proposition 1.11(f). It follows that $h^{-1}h' \in C'_G(S)$, so $\widetilde{\kappa}^{\mathcal{H}}_G(c_{h^{-1}h'}) = 1$ since $C'_G(S) \le O^p(C_G(P))$ for each $P \le S$. Thus $\widetilde{\kappa}$ is well defined, and it is easily seen to be a homomorphism. Since conjugation by any element of $C'_{G_0}(S_0)$ induces the identity in $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_0)$ (and since $\mathrm{Res}_2 \circ \widetilde{\kappa}^{\mathcal{H}}_G = \widetilde{\kappa}^{\mathcal{H}_0}_{G_0} \circ \mathrm{Res}_1$ as maps from $\mathrm{Aut}(G,G_0,S)$ to $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_0)$), square (17) commutes.

Next consider the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(G_0) & \longrightarrow & N_G(S_0) & \xrightarrow{\ \mathrm{cj}_1\ } & \mathrm{Aut}_G(G_0,S_0) & \longrightarrow & 1 \\ & & \| & & {\scriptstyle \lambda_0}\downarrow{\scriptstyle g\mapsto[g]} & & \downarrow{\scriptstyle \widetilde{\kappa}_1} & & \\ 1 & \longrightarrow & Z(\mathcal{F}_0) & \xrightarrow{\ \delta_{S_0}\ } & \mathrm{Aut}_{\mathcal{L}}(S_0) & \xrightarrow{\ \mathrm{cj}_2\ } & \mathrm{Aut}_{\mathcal{L}}(\mathcal{L}_0) & \longrightarrow & 1\,. \end{array} \quad (19)$$

Here, cj_1 and cj_2 are induced by conjugation, and $\widetilde{\kappa}_1$ is the restriction of $\widetilde{\kappa}_0$. Both rows in (19) are exact: the first since $\mathrm{Ker}(\mathrm{cj}_1) = C_G(G_0) = Z(G_0)$ by (16); and the second since $\mathrm{Ker}(\mathrm{cj}_2) \le \mathrm{Aut}_{\mathcal{L}_0}(S_0)$ ($\mathcal{L}_0$ is centric in $\mathcal{L}$) and hence $\mathrm{Ker}(\mathrm{cj}_2) = Z(\mathcal{F}_0)$ by Lemma 1.14(a). Thus the right hand square in (19) is a pullback square.

Fix automorphisms

$$\alpha \in N_{\mathrm{Aut}(G_0,S_0)}(\mathrm{Aut}_G(G_0,S_0)) \qquad \text{and} \qquad \chi \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L},\mathcal{L}_0)$$

such that $\chi|_{\mathcal{L}_0} = \widetilde{\kappa}_0(\alpha)$. Then $\chi(S_0) = S_0$, so χ_{S_0} is an automorphism of $\mathrm{Aut}_{\mathcal{L}}(S_0) = N_G(S_0)/C'_{G_0}(S_0)$ by (16).

We first construct $\beta \in \mathrm{Aut}(N_G(S_0))$ such that for each $g \in N_G(S_0)$, $c_{\beta(g)} = \alpha c_g \alpha^{-1}$ in $\mathrm{Aut}(G_0)$ and $\chi_{S_0}([g]) = [\beta(g)]$ in $\mathrm{Aut}_{\mathcal{L}}(S_0)$. Consider the following automorphisms

$$c_\alpha \in \mathrm{Aut}\big(\mathrm{Aut}_G(G_0, S_0)\big), \qquad \chi_{S_0} \in \mathrm{Aut}\big(\mathrm{Aut}_{\mathcal{L}}(S_0)\big), \qquad c_{\widetilde{\kappa}_0(\alpha)} = c_\chi \in \mathrm{Aut}\big(\mathrm{Aut}_{\mathcal{L}}(\mathcal{L}_0)\big)$$

of groups in the pullback square in (19). We want to define β as the pullback of c_α and χ_{S_0} over c_χ. For $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S_0)$, $c_\chi(\mathrm{cj}_2(\gamma)) = \chi c_\gamma \chi^{-1} = c_{\chi(\gamma)} = \mathrm{cj}_2(\chi_{S_0}(\gamma))$ (using (18)) and thus $\mathrm{cj}_2 \circ \chi_{S_0} = c_\chi \circ \mathrm{cj}_2$. By a similar (but simpler) computation, $\widetilde{\kappa}_1 \circ c_\alpha = c_{\widetilde{\kappa}_0(\alpha)} \circ \widetilde{\kappa}_1$; and hence these three automorphisms pull back (via the pullback square in (19)) to a unique $\beta \in \mathrm{Aut}(N_G(S_0))$. Thus for $g \in N_G(S_0)$,

$$[\beta(g)] = \chi_{S_0}([g]) \in \mathrm{Aut}_{\mathcal{L}}(S_0) \quad \text{and} \quad \mathrm{cj}_1(\beta(g)) = c_\alpha \circ \mathrm{cj}_1(g) = \alpha c_g \alpha^{-1} \in \mathrm{Aut}(G_0)\ . \tag{20}$$

Now, $\chi_{S_0}(\delta_{S_0}(S_0)) = \delta_{S_0}(S_0)$ and $\chi_{S_0}(\delta_{S_0}(S)) = \delta_{S_0}(S)$ since χ is isotypical and sends inclusions to inclusions (and hence restrictions to restrictions). Since $\mathrm{Aut}_{\mathcal{L}}(S_0) = N_G(S_0)/C'_{G_0}(S_0)$ by (16), (20) implies that β sends $S_0 \times C'_{G_0}(S_0)$ to itself and sends $S{\cdot}C'_{G_0}(S_0)$ to itself. In particular, $\beta(S_0) = S_0$.

Now, for all $g \in N_{G_0}(S_0)$,

$$\lambda_0(\alpha(g)) = [\alpha(g)] = \widetilde{\kappa}_0(\alpha)([g]) = \chi_{S_0}([g]) \in \mathrm{Aut}_{\mathcal{L}}(S_0) \qquad (\widetilde{\kappa}_0(\alpha) = \chi|_{\mathcal{L}_0})$$

and

$$\mathrm{cj}_1 \circ \alpha(g) = c_{\alpha(g)} = \alpha c_g \alpha^{-1} \in \mathrm{Aut}_G(G_0, S_0)\ .$$

Thus $\lambda_0(\alpha(g)) = \lambda_0(\beta(g))$ and $\mathrm{cj}_1(\alpha(g)) = \mathrm{cj}_1(\beta(g))$ by comparison with (20); and hence $\alpha(g) = \beta(g)$ by the pullback square in (19). This proves that $\alpha|_{N_{G_0}(S_0)} = \beta|_{N_{G_0}(S_0)}$.

We already saw that $G = G_0{\cdot}N_G(S_0)$. Define $\widehat{\alpha} \in \mathrm{Aut}(G, G_0, S{\cdot}C'_{G_0}(S_0))$ by setting $\widehat{\alpha}(g_0h) = \alpha(g_0)\beta(h)$ for $g_0 \in G_0$ and $h \in N_G(S_0)$. Since $\alpha|_{N_{G_0}(S_0)} = \beta|_{N_{G_0}(S_0)}$, this is well defined as a bijective map of sets. For all $g_0, g_0' \in G_0$ and $h, h' \in N_G(S_0)$,

$$\begin{aligned}\widehat{\alpha}(g_0h{\cdot}g_0'h') &= \widehat{\alpha}(g_0{\cdot}c_h(g_0'){\cdot}hh') = \alpha(g_0)\alpha(c_h(g_0'))\beta(hh')\\ &= \alpha(g_0)c_{\beta(h)}(\alpha(g_0'))\beta(hh') = \alpha(g_0)\beta(h)\alpha(g_0')\beta(h') = \widehat{\alpha}(g_0h)\widehat{\alpha}(g_0'h'),\end{aligned}$$

where the third equality follows from the condition $c_{\beta(h)} = \alpha c_h \alpha^{-1}$. It now follows that $\widehat{\alpha} \in \mathrm{Aut}(G, G_0)$. Also, $\widehat{\alpha}$ sends $S{\cdot}C'_{G_0}(S_0)$ to itself since β does.

By construction, $\mathrm{Res}_1(\widehat{\alpha}) = \widehat{\alpha}|_{G_0} = \alpha$. We claim that $\widetilde{\kappa}(\widehat{\alpha}) = \chi$. Since $\widehat{\alpha}|_{G_0} = \alpha$ and $\chi|_{\mathcal{L}_0} = \widetilde{\kappa}_0(\alpha)$, $\widetilde{\kappa}(\widehat{\alpha})$ and χ define the same action on $\mathcal{L}_0$ (by the commutativity of (17)). Choose $h \in C'_{G_0}(S_0) = O^p(C_G(S_0))$ with $\widehat{\alpha}(S) = hSh^{-1}$. For $g \in N_G(S_0)$,

$$\widetilde{\kappa}(\widehat{\alpha})([g]) = \widetilde{\kappa}_G^{\mathcal{H}}(c_h^{-1} \circ \widehat{\alpha})([g]) = [h^{-1}\widehat{\alpha}(g)h] = [\widehat{\alpha}(g)] = [\beta(g)] = \chi([g]) \in \mathrm{Aut}_{\mathcal{L}}(S_0)$$

by (20) and since $\widehat{\alpha}|_{N_G(S_0)} = \beta$. Hence $\widetilde{\kappa}(\widehat{\alpha})$ and χ define the same action on $\mathrm{Aut}_{\mathcal{L}}(S_0)$. Since $\mathcal{L}_0$ and $\mathrm{Aut}_{\mathcal{L}}(S_0)$ generate the full subcategory $\mathcal{L}|_{\le S_0}$, $\widetilde{\kappa}(\widehat{\alpha})$ and χ are equal after restriction to this subcategory.

We just showed that $\chi_{S_0}([s]) = \widetilde{\kappa}(\widehat{\alpha})_{S_0}([s])$ for $s \in S$. So by Proposition 1.11(f), $\chi_S([s]) = \widetilde{\kappa}(\widehat{\alpha})_S([s])$ in $\mathrm{Aut}_{\mathcal{L}}(S)$. Lemma 1.15 now implies that $\chi(P) = \widetilde{\kappa}(\widehat{\alpha})(P)$ for $P \in \mathrm{Ob}(\mathcal{L})$. Since both $\widetilde{\kappa}(\widehat{\alpha})$ and χ send inclusions to inclusions, and since the restriction map from $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ to $\mathrm{Mor}_{\mathcal{L}}(P \cap S_0, Q \cap S_0)$ is injective for all $P, Q \in \mathcal{H}$ by Proposition 1.11(f) again, it now follows that $\widetilde{\kappa}(\widehat{\alpha}) = \chi$.

To prove (17) is a pullback, it remains to show $\widetilde{\kappa} \times \mathrm{Res}_1$ is injective. So assume $\widehat{\alpha} \in \mathrm{Aut}(G, G_0, S{\cdot}C'_{G_0}(S_0))$ is such that $\widehat{\alpha}|_{G_0} = \mathrm{Id}_{G_0}$ and $\widetilde{\kappa}(\widehat{\alpha}) = \mathrm{Id}_{\mathcal{L}}$. For each $g \in G$, $c_{\widehat{\alpha}(g)} = c_g \in \mathrm{Aut}(G_0)$, and hence $g^{-1}\widehat{\alpha}(g) \in C_G(G_0) = Z(G_0)$ by (16). Since $\widetilde{\kappa}(\widehat{\alpha}) = \mathrm{Id}_{\mathcal{L}}$, $\widehat{\alpha}$ induces the identity on $\mathrm{Aut}_{\mathcal{L}}(S_0) = N_G(S_0)/C'_{G_0}(S_0)$ (see (16) again). Since $G = G_0{\cdot}N_G(S_0)$ and $\widehat{\alpha}|_{G_0} = \mathrm{Id}$, $g^{-1}\widehat{\alpha}(g) \in C'_{G_0}(S_0)$ for all $g \in G$. Finally, $C'_{G_0}(S_0) \cap Z(G_0) = 1$ because $Z(G_0) = Z(\mathcal{F}_0) \le S_0$ is a p-group, and we conclude that $\widehat{\alpha} = \mathrm{Id}_G$.

Step 2: We are now ready to prove (15) is a pullback. Fix elements

$$[\alpha] \in N_{\mathrm{Out}(G_0)}(\mathrm{Out}_G(G_0))/\mathrm{Out}_G(G_0) \qquad \text{and} \qquad [\chi] \in \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}, \mathcal{L}_0)$$

such that $\kappa^*([\alpha]) = R_2([\chi])$, and choose liftings $\alpha \in \mathrm{Aut}(G_0, S_0)$ and $\chi \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}, \mathcal{L}_0)$. Then α normalizes $\mathrm{Aut}_G(G_0)$, and hence also normalizes $\mathrm{Aut}_G(G_0, S_0)$.

Since $\kappa^*([\alpha]) = R_2([\chi])$, $\chi|_{\mathcal{L}_0} = \widetilde{\kappa}_0(\alpha) \circ c_{[x]}$ for some element $x \in N_G(S_0)$ (where $[x] \in \mathrm{Aut}_{\mathcal{L}}(S_0)$ is the class of x). Upon replacing α by $\alpha \circ c_x \in \mathrm{Aut}(G_0)$, we can arrange that $\chi|_{\mathcal{L}_0} = \widetilde{\kappa}_0(\alpha)$. Hence α and χ pull back to an element of $\mathrm{Aut}(G, G_0, S{\cdot}C'_{G_0}(S_0))$ by Step 1, and so $[\alpha]$ and $[\chi]$ pull back to an element of $\mathrm{Out}(G, G_0)$.

To see that this pullback is unique, fix $[\gamma] \in \mathrm{Out}(G, G_0)$ such that $R_1([\gamma]) = 1$ and $\kappa([\gamma]) = 1$, and choose $\gamma \in \mathrm{Aut}(G, G_0)$ which represents $[\gamma]$. Then $\gamma(S) = gSg^{-1}$ for some $g \in G$, and upon replacing γ by $c_g^{-1} \circ \gamma$, we can assume $\gamma(S) = S$. Also, $\widetilde{\kappa}(\gamma) = c_{[y]}$ for some $y \in N_G(S)$; and upon replacing γ by $\gamma \circ c_y^{-1}$, we can assume $\widetilde{\kappa}(\gamma) = \mathrm{Id}_{\mathcal{L}}$. Now, $\gamma|_{G_0} = c_h$ for some $h \in N_G(S_0)$, and $c_{[h]} = \mathrm{Id}_{\mathcal{L}_0}$. Hence $h \in G_0$ since $\mathcal{L}_0$ is centric in $\mathcal{L}$, and so $h \in C_{G_0}(S_0) = Z(S_0) \times C'_{G_0}(S_0)$.

Write $h = h_1 h_2$, where $h_1 \in Z(S_0)$ and $h_2 \in C'_{G_0}(S_0)$. Thus $[h] = [h_1] \in \mathrm{Aut}_{\mathcal{L}_0}(S_0)$, and $h_1 \in Z(\mathcal{F}_0) = Z(G_0)$ since $c_{[h]} = \mathrm{Id}_{\mathcal{L}_0}$ (see Lemma 1.14(a)). Thus $\gamma|_{G_0} = c_h = c_{h_2}$ in $\mathrm{Aut}(G_0)$. Since $[S, h_2] \le [S, C'_{G_0}(S_0)] \le C'_{G_0}(S_0)$, $c_{h_2} \in \mathrm{Aut}(G, G_0, S{\cdot}C'_{G_0}(S_0))$. Also, $\widetilde{\kappa}(c_{h_2}) = \mathrm{Id}$ by definition of $\widetilde{\kappa}$ (and since $h_2 \in C'_{G_0}(S_0)$). Thus $\gamma = c_{h_2}$ since (17) is a pullback, and so $[\gamma] = 1$ in $\mathrm{Out}(G, G_0)$. □

We are finally ready to prove:

Proposition 2.16. *Let $(S_0, \mathcal{F}_0, \mathcal{L}_0) \trianglelefteq (S, \mathcal{F}, \mathcal{L})$ be a normal pair such that $\mathcal{L}_0$ is centric in $\mathcal{L}$, $\mathrm{Ob}(\mathcal{L}_0)$ and $\mathrm{Ob}(\mathcal{L})$ are $\mathrm{Aut}(S_0, \mathcal{F}_0)$- and $\mathrm{Aut}(S, \mathcal{F})$-invariant, respectively, and $\mathcal{L}_0$ is $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$-invariant. Assume $\mathcal{F}_0$ is tamely realized by some finite group G_0 such that $S_0 \in \mathrm{Syl}_p(G_0)$, $Z(G_0) = Z(\mathcal{F}_0)$, and $\mathcal{L}_0 \cong \mathcal{L}_{S_0}^{\mathrm{Ob}(\mathcal{L}_0)}(G_0)$. Then $\mathcal{F}$ is tamely realized by a finite group G such that $S \in \mathrm{Syl}_p(G)$, $G_0 \trianglelefteq G$ and $G/G_0 \cong \mathcal{L}/\mathcal{L}_0$.*

Proof. Set $\mathcal{H} = \mathrm{Ob}(\mathcal{L})$ and $\mathcal{H}_0 = \mathrm{Ob}(\mathcal{L}_0)$. By assumption, $\mathcal{F}_0 = \mathcal{F}_{S_0}(G_0)$, and κ_{G_0} is split surjective. Also, $\mathcal{L}_0 \cong \mathcal{L}_{S_0}^{\mathcal{H}_0}(G_0)$ by assumption, and we identify these two linking systems. By Lemma 1.17, $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_0) \cong \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_{S_0}^{\mathcal{H}_0^c}(G_0)) \cong \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_{S_0}^c(G_0))$, where $\mathcal{H}_0^c$ is the set of $\mathcal{F}_0$-centric subgroups in $\mathcal{H}_0$. Choose a splitting

$$s\colon \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_0) \cong \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_{S_0}^c(G_0)) \longrightarrow \mathrm{Out}(G_0)$$

for $\kappa_{G_0}^{\mathcal{H}_0}$, and set

$$\widehat{\rho} = s \circ \rho_{\mathcal{L}_0}^{\mathcal{L}}\colon \mathcal{L}/\mathcal{L}_0 \longrightarrow \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_0) \longrightarrow \mathrm{Out}(G_0)\ .$$

By Lemma 2.14, there is a finite group G such that $S \in \mathrm{Syl}_p(G)$, $G_0 \trianglelefteq G$, $\mathcal{F} = \mathcal{F}_S(G)$, $\mathcal{L} \cong \mathcal{L}_S^{\mathcal{H}}(G)$, $G/G_0 \cong \mathcal{L}/\mathcal{L}_0$, and such that the outer action of G/G_0 on G_0 is equal to $\widehat{\rho}$ via this last isomorphism. In particular, s sends $\mathrm{Out}_{\mathcal{L}}(\mathcal{L}_0) = \mathrm{Im}(\rho_{\mathcal{L}_0}^{\mathcal{L}})$ isomorphically to $\mathrm{Out}_G(G_0) = \mathrm{Im}(\widehat{\rho})$.

Since $\mathcal{L}_0$ is $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$-invariant by assumption, $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}, \mathcal{L}_0) = \mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$. So by Lemma 2.15, the following is a pullback square:

$$\begin{array}{ccc}
\mathrm{Out}(G, G_0) & \xrightarrow{\quad\kappa\quad} & \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \\
{\scriptstyle R_1}\big\downarrow & & {\scriptstyle R_2}\big\downarrow \\
N_{\mathrm{Out}(G_0)}(\mathrm{Out}_G(G_0))/\mathrm{Out}_G(G_0) & \xrightarrow{\quad\kappa^*\quad} & N_{\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_0)}(\mathrm{Out}_{\mathcal{L}}(\mathcal{L}_0))/\mathrm{Out}_{\mathcal{L}}(\mathcal{L}_0)
\end{array} \tag{21}$$

where κ^* is induced by $\kappa_{G_0}^{\mathcal{H}_0}$. Since the splitting s of $\kappa_{G_0}^{\mathcal{H}_0}$ sends $\mathrm{Out}_{\mathcal{L}}(\mathcal{L}_0)$ isomorphically to $\mathrm{Out}_G(G_0)$, it induces a splitting s^* of κ^*. Since (21) is a pullback, s^* induces a splitting of $\kappa = \kappa_G^{\mathcal{H}}|_{\mathrm{Out}(G,G_0)}$ (Lemma 2.13). By Lemma 1.17, $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \cong \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G))$, and so $\mathcal{F}$ is tamely realized by G. □

We next turn to central extensions of fusion and linking systems. In the following lemma, when $\mathcal{L}$ is a linking system associated to $\mathcal{F}$ over the p-group S, and $A \le S$, we set

$$\mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L}, A) = \{\alpha \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L}) \mid \alpha_S(\delta_S(A)) = \delta_S(A)\}\ ,$$

and let $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}, A)$ be its image in $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$.

Lemma 2.17. *Fix a finite group G and a central p-subgroup $A \le Z(G)$. Choose $S \in \mathrm{Syl}_p(G)$, and set $\overline{G} = G/A$ and $\overline{S} = S/A \in \mathrm{Syl}_p(\overline{G})$. Set $\mathcal{F} = \mathcal{F}_S(G)$, $\overline{\mathcal{F}} = \mathcal{F}_{\overline{S}}(\overline{G})$ and*

$$\mathcal{H} = \{P \le S \mid P \ge A,\ P/A \text{ is } \overline{\mathcal{F}}\text{-centric}\}\ .$$

Then $\mathcal{H}$ contains all subgroups of S which are $\mathcal{F}$-centric and $\mathcal{F}$-radical, all subgroups in $\mathcal{H}$ are $\mathcal{F}$-centric, and hence $\mathcal{L} \overset{\text{def}}{=} \mathcal{L}_S^{\mathcal{H}}(G)$ is a linking system associated to $\mathcal{F}$. If, furthermore, $Z(\overline{G}) = Z(\overline{\mathcal{F}})$, then the following square is a pullback:

$$\begin{array}{ccc} \mathrm{Out}(G,A) & \xrightarrow{\kappa_{G,A}^{\mathcal{H}}} & \mathrm{Out}_{\mathrm{typ}}(\mathcal{L},A) \\ \Big\downarrow{\scriptstyle \nu_1} & & \Big\downarrow{\scriptstyle \nu_2} \\ \mathrm{Out}(\overline{G}) & \xrightarrow{\kappa_{\overline{G}}} & \mathrm{Out}_{\mathrm{typ}}(\overline{\mathcal{L}}) \end{array}\ , \tag{22}$$

where $\overline{\mathcal{L}} = \mathcal{L}_{\overline{S}}^c(\overline{G})$, $\kappa_{G,A}^{\mathcal{H}}$ is defined analogously to κ_G, and ν_1 and ν_2 are induced by the projections $G \twoheadrightarrow \overline{G}$ and $\mathcal{L} \twoheadrightarrow \overline{\mathcal{L}}$.

Proof. We first prove the statements about $\mathcal{H} = \mathrm{Ob}(\mathcal{L})$. If $P \in \mathcal{H}$, then P is $\mathcal{F}$-centric since $A \le P$ and P/A is $\overline{\mathcal{F}}$-centric (cf. [BCGLO2, Lemma 6.4(a)]). Now assume $P \le S$ is $\mathcal{F}$-centric and $\mathcal{F}$-radical; we must show $P \in \mathcal{H}$. Since P is $\mathcal{F}$-centric, $A \le C_S(P) \le P$. For $x \in S$ with $xA \in C_{\overline{S}}(P/A)$, c_x induces the identity on A and on P/A. Hence $c_x \in O_p(\mathrm{Aut}_{\mathcal{F}}(P))$ by Lemma 1.6, so $x \in P$ by Lemma 1.4. This proves $C_{\overline{S}}(P/A) \le P/A$. Since this argument applies to all subgroups $\mathcal{F}$-conjugate to P, we conclude that P/A is $\overline{\mathcal{F}}$-centric, so $P \in \mathcal{H}$.

Consider the following diagram (with homomorphisms defined below):

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{Hom}(\overline{G},A) & \xrightarrow{\lambda_1} & \mathrm{Aut}(G,S,A) & \xrightarrow{(\widetilde{\nu}_1,r_1)} & \mathrm{Aut}(\overline{G},\overline{S}) \times \mathrm{Aut}(A) \\ & & \cong\Big\downarrow{\scriptstyle \tau} & & \Big\downarrow{\scriptstyle \widetilde{\kappa}_1} & & \Big\downarrow{\scriptstyle \widetilde{\kappa}_2\times\mathrm{Id}} \\ 1 & \longrightarrow & \mathrm{Hom}(\pi_1(|\overline{\mathcal{L}}|),A) & \xrightarrow{\lambda_2} & \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L},A) & \xrightarrow{(\widetilde{\nu}_2,r_2)} & \mathrm{Aut}_{\mathrm{typ}}^I(\overline{\mathcal{L}}) \times \mathrm{Aut}(A) \end{array}\ . \tag{23}$$

Here, $\widetilde{\nu}_1$ and $\widetilde{\nu}_2$ are induced by the projection $G \twoheadrightarrow \overline{G}$ and r_1 and r_2 by restriction to A, and $\mathrm{Aut}(G,S,A) \le \mathrm{Aut}(G)$ is the subgroup of automorphisms which leave both S and A invariant. Also, $\widetilde{\kappa}_1 = \widetilde{\kappa}_{G,A}^{\mathcal{H}}$ (defined analogously to $\widetilde{\kappa}_G$), and $\widetilde{\kappa}_2 = \widetilde{\kappa}_{\overline{G}}$. The right hand square clearly commutes.

For $\beta \in \mathrm{Hom}(\overline{G},A)$ and $g \in G$, $\lambda_1(\beta)(g) = g{\cdot}\beta(gA)$. For any morphism $\overline{\psi} \in \mathrm{Mor}_{\overline{\mathcal{L}}}(P,Q)$, let $[\overline{\psi}] \in \pi_1(|\overline{\mathcal{L}}|)$ be the class of the loop based at the vertex $\overline{S}$, formed by the edges $\iota_P^{\overline{S}}$, $\overline{\psi}$, and $\iota_Q^{\overline{S}}$ (in that order). For $\beta \in \mathrm{Hom}(\pi_1(|\overline{\mathcal{L}}|),A)$, $\lambda_2(\beta)$ is the automorphism of $\mathcal{L}$ which is the identity on objects, and sends $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ (with image $\overline{\psi} \in \mathrm{Mor}_{\overline{\mathcal{L}}}(P/A,Q/A)$) to

$\psi \circ \delta_P(\beta([\bar{\psi}]))$. It follows immediately from these definitions that for $i = 1, 2$, λ_i is injective and $(\widetilde{\nu}_i, r_i) \circ \lambda_i$ is trivial.

Since A is a finite abelian p-group, $\mathrm{Hom}(\pi_1(X), A) \cong H^1(X; A) \cong H^1(X^\wedge_p; A)$ for any "p-good" space X (the second isomorphism by [BK, Definition I.5.1]). Also, $|\bar{\mathcal{L}}|$ is p-good by [BLO2, Proposition 1.12], $B\bar{G}$ is p-good since it has finite fundamental group (cf. [BK, Proposition VII.5.1]), and $B\bar{G}^\wedge_p \simeq |\bar{\mathcal{L}}|^\wedge_p$ by [BLO1, Proposition 1.1]. We thus get an isomorphism

$$\tau\colon \mathrm{Hom}(\bar{G}, A) \xrightarrow{\ \cong\ } H^1(B\bar{G}^\wedge_p; A) \xrightarrow{\ \cong\ } H^1(|\bar{\mathcal{L}}|^\wedge_p; A) \xrightarrow{\ \cong\ } \mathrm{Hom}(\pi_1(|\bar{\mathcal{L}}|), A)\ .$$

Alternatively, by [BCGLO2, Theorem B], $\pi_1(|\bar{\mathcal{L}}|)/O^p(\pi_1(|\bar{\mathcal{L}}|)) \cong \bar{S}/\mathfrak{hyp}(\bar{\mathcal{F}})$, where for an infinite group Γ, $O^p(\Gamma)$ denotes the intersection of all normal subgroups of p-power index. By the hyperfocal subgroup theorem for groups [Pg1, § 1.1], $\bar{G}/O^p(\bar{G}) \cong \bar{S}/\mathfrak{hyp}(\bar{\mathcal{F}})$; and these isomorphisms induce an isomorphism

$$\tau\colon \mathrm{Hom}(\bar{G}, A) \xrightarrow{\ \cong\ } \mathrm{Hom}(\bar{S}/\mathfrak{hyp}(\bar{\mathcal{F}}), A) \xrightarrow{\ \cong\ } \mathrm{Hom}(\pi_1(|\bar{\mathcal{L}}|), A)\ .$$

By either construction, τ makes the left hand square in (23) commute.

An element $\alpha \in \mathrm{Ker}(\widetilde{\nu}_1, r_1)$ is an automorphism of G which induces the identity on A and on $\bar{G} = G/A$, and since $A \le Z(G)$, any such automorphism has the form $\alpha(g) = g{\cdot}\beta(gA)$ for some unique $\beta \in \mathrm{Hom}(\bar{G}, A)$. Thus the top row in (23) is exact.

Similarly, an element $\alpha \in \mathrm{Ker}(\widetilde{\nu}_2, r_2)$ is an isotypical automorphism of $\mathcal{L}$ which sends inclusions to inclusions and induces the identity on $\bar{\mathcal{L}}$ and on A. Since $\mathcal{L} \longrightarrow \bar{\mathcal{L}}$ is bijective on objects (by definition), α induces the identity on objects in $\mathcal{L}$, and on morphisms it has the form $\alpha(\psi) = \psi \circ \beta(\bar{\psi})$ for some $\beta\colon \mathrm{Mor}(\bar{\mathcal{L}}) \longrightarrow A$ which preserves composition and sends inclusions to the identity. Such a β is equivalent to a homomorphism from $\pi_1(|\bar{\mathcal{L}}|)$ to A (cf. [OV1, Proposition A.3(a)]), so $\alpha = \lambda_2(\beta)$, and thus the second row in (23) is exact.

We are now ready to prove that (22) is a pullback. Fix automorphisms $\alpha \in \mathrm{Aut}(\bar{G}, \bar{S})$ and $\beta \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}, A)$ such that $\kappa_{\bar{G}}([\alpha]) = \nu_2([\beta])$. Then $\widetilde{\nu}_2(\beta) = \widetilde{\kappa}_2(\alpha) \circ c_{[x]}$ for some $x \in N_{\bar{G}}(\bar{S})$ which induces $[x] \in \mathrm{Aut}_{\bar{\mathcal{L}}}(\bar{S})$. So upon replacing α by $\alpha \circ c_x$, we can assume $\widetilde{\kappa}_2(\alpha) = \widetilde{\nu}_2(\beta)$. Consider the following diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & \bar{G} & \longrightarrow & 1 \\
 & & \cong\big\downarrow r_2(\beta) & & \big\downarrow \widehat{\alpha} & & \cong\big\downarrow \alpha & & \\
1 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & \bar{G} & \longrightarrow & 1
\end{array}$$

We want to find $\widehat{\alpha} \in \mathrm{Aut}(G)$ which makes the two squares commute. This means showing that the class $[G] \in H^2(\bar{G}; A)$ is invariant under the automorphism of $H^2(\bar{G}; A)$ induced by $r_2(\beta)$ and α. But $\beta \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ induces an automorphism $\gamma = \beta_S|_{\delta_S(S)} \in \mathrm{Aut}(S, \mathcal{F})$ (see Lemma 1.15). Also, $\gamma|_A = \beta_S|_A = r_2(\beta)$, γ induces the automorphism $(\widetilde{\nu}_2(\beta))_{\bar{S}}|_{\bar{S}} = \alpha|_{\bar{S}}$ on $\bar{S}$, and thus $[S] \in H^2(\bar{S}; A)$ is invariant under these automorphisms of $\bar{S}$ and A. Since $H^2(\bar{G}; A)$ injects into $H^2(\bar{S}; A)$ under restriction, this proves that $[G]$ is also invariant, and hence that there is an automorphism $\widehat{\alpha} \in \mathrm{Aut}(G, S, A)$ as desired.

Thus $(\widetilde{\nu}_1, r_1)(\widehat{\alpha}) = (\alpha, r_2(\beta))$. By the commutativity of (23),

$$(\widetilde{\nu}_2, r_2)(\widetilde{\kappa}_1(\widehat{\alpha})) = (\widetilde{\kappa}_2(\alpha), r_2(\beta)) = (\widetilde{\nu}_2, r_2)(\beta).$$

Hence there is $\chi \in \operatorname{Hom}(\overline{G}, A)$ such that $\lambda_2(\tau(\chi)) = \widetilde{\kappa}_1(\widehat{\alpha})^{-1} \circ \beta$, and the element $\widehat{\alpha} \circ \lambda_1(\chi) \in \operatorname{Aut}(G, S, A)$ pulls back $\alpha \in \operatorname{Aut}(\overline{G}, \overline{S})$ and $\beta \in \operatorname{Aut}^I_{\mathrm{typ}}(\mathcal{L}, A)$.

This proves that $\operatorname{Out}(G, A)$ surjects onto the pullback in square (22). To prove that it injects into the pullback, fix $\widehat{\alpha} \in \operatorname{Aut}(G, S, A)$ such that $\kappa^{\mathcal{H}}_G([\widehat{\alpha}]) = 1$ and $\nu_1([\widehat{\alpha}]) = 1$. Upon composing $\widehat{\alpha}$ by an appropriate inner automorphism, we can assume it induces the identity on $\overline{G}$. Thus $\widetilde{\kappa}_1(\widehat{\alpha}) = c_{[x]} \in \operatorname{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ for some $x \in N_G(S)$ inducing $[x] \in \operatorname{Aut}_{\mathcal{L}}(S)$, where $c_{[x]}$ induces the identity on $\overline{\mathcal{L}}$. This means that $xA \in Z(\overline{\mathcal{F}})$ (Lemma 1.14(a)), and hence $xA \in Z(\overline{G})$ by assumption. So upon replacing $\widehat{\alpha}$ by $\widehat{\alpha} \circ c_x^{-1} \in \operatorname{Aut}(G)$ we have an automorphism which induces the identity on $\mathcal{L}$ and on $\overline{G}$. By the exactness of the rows in (23) again, $\widehat{\alpha} = \mathrm{Id}$, and this finishes the proof. □

Lemma 2.17 now implies the result we need about tameness.

Proposition 2.18. *Fix a saturated fusion system $\mathcal{F}$ over a finite p-group S. Assume $\mathcal{F}/Z(\mathcal{F})$ is tamely realized by the finite group $\overline{G}$ such that $O_{p'}(\overline{G}) = 1$ and $Z(\overline{G}) = Z(\mathcal{F}/Z(\mathcal{F}))$. Then $\mathcal{F}$ is tamely realized by a finite group G such that $Z(G) = Z(\mathcal{F})$ and $G/Z(G) \cong \overline{G}$, and hence $O_{p'}(G) = 1$. If $\overline{G} \in \mathfrak{G}(p)$, then $G \in \mathfrak{G}(p)$.*

Proof. Set $A = Z(\mathcal{F})$ and $\overline{S} = S/A$ for short. By assumption, $\overline{S} \in \operatorname{Syl}_p(\overline{G})$, $\mathcal{F}/A \cong \mathcal{F}_{\overline{S}}(\overline{G})$, $\kappa_{\overline{G}}$ is split surjective, $O_{p'}(\overline{G}) = 1$, and $Z(\overline{G}) = Z(\mathcal{F}/A)$.

By [BCGLO2, Corollary 6.14], the fusion system $\mathcal{F}$ is realizable, and by the proof of that corollary, it is realizable by a finite group G such that $S \in \operatorname{Syl}_p(G)$, $A \le Z(G)$, and $G/A \cong \overline{G}$. Hence $O_{p'}(G) = 1$, so $Z(G)$ is a p-group which is central in $\mathcal{F}$. Thus $Z(G) = Z(\mathcal{F})$.

Let $\mathcal{L} \subseteq \mathcal{L}^c_S(G)$ be the full subcategory whose objects are the subgroups $P \le S$ such that $P \ge A$ and P/A is $\mathcal{F}/A$-centric, and set $\overline{\mathcal{L}} = \mathcal{L}^c_{\overline{S}}(\overline{G})$. Then $\mathcal{L}$ is a linking system associated to $\mathcal{F}$ by Lemma 2.17, and $A = Z(\mathcal{F})$ is invariant under all automorphisms in $\operatorname{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ by Lemma 1.15. Lemma 2.17 now implies that the following is a pullback square:

$$\begin{array}{ccc} \operatorname{Out}(G, A) & \xrightarrow{\ \kappa\ } & \operatorname{Out}_{\mathrm{typ}}(\mathcal{L}) \\ \big\downarrow & & \big\downarrow \\ \operatorname{Out}(\overline{G}) & \xrightarrow{\ \kappa_{\overline{G}}\ } & \operatorname{Out}_{\mathrm{typ}}(\overline{\mathcal{L}}) . \end{array}$$

By assumption, $\kappa_{\overline{G}}$ is split surjective. Hence $\kappa = \kappa^{\mathcal{H}}_G|_{\operatorname{Out}(G,A)}$ ($\mathcal{H} = \operatorname{Ob}(\mathcal{L})$) is also split surjective by Lemma 2.13, so $\kappa^{\mathcal{H}}_G$ is split surjective. Since $\operatorname{Out}_{\mathrm{typ}}(\mathcal{L}) \cong \operatorname{Out}_{\mathrm{typ}}(\mathcal{L}^c_S(G))$ by Lemma 1.17, this finishes the proof that $\mathcal{F}$ is tame.

By construction, G and $\overline{G}$ have the same nonabelian composition factors. Hence $G \in \mathfrak{G}(p)$ if $\overline{G} \in \mathfrak{G}(p)$. □

One more technical lemma is needed before we can prove Theorem A.

Lemma 2.19. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. If $\mathcal{F}$ is tame, then there is a finite group G such that $O_{p'}(G) = 1$ and $\mathcal{F}$ is tamely realized by G. If $\mathcal{F}$ is strongly tame, then G can be chosen such that in addition, $G \in \mathfrak{G}(p)$.*

Proof. Fix any $\widehat{G}$ which tamely realizes $\mathcal{F}$. If $\mathcal{F}$ is strongly tame, we assume $\widehat{G} \in \mathfrak{G}(p)$. Thus $S \in \operatorname{Syl}_p(\widehat{G})$, $\mathcal{F} \cong \mathcal{F}_S(\widehat{G})$, and $\kappa_{\widehat{G}}$ is split surjective. Set $G = \widehat{G}/O_{p'}(\widehat{G})$, and identify S with its image in G. Since G is a quotient group of $\widehat{G}$, $G \in \mathfrak{G}(p)$ if $\widehat{G} \in \mathfrak{G}(p)$.

By construction, $\mathcal{F}_S(G) \cong \mathcal{F}_S(\widehat{G}) \cong \mathcal{F}$, and $O_{p'}(G) = 1$. The natural homomorphism from $\widehat{G}$ onto G induces a homomorphism between their outer automorphism groups and an isomorphism between their linking systems, and the resulting square

$$\begin{array}{ccc} \mathrm{Out}(\widehat{G}) & \longrightarrow & \mathrm{Out}(G) \\ \Big\downarrow{\scriptstyle \kappa_{\widehat{G}}} & & \Big\downarrow{\scriptstyle \kappa_G} \\ \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(\widehat{G})) & \xrightarrow{\ \cong\ } & \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G)) \end{array}$$

commutes. Since $\kappa_{\widehat{G}}$ is split surjective, so is κ_G. □

We are now ready to prove Theorem A. Recall that $\mathfrak{red}(\mathcal{F})$ denotes the reduction of a fusion system $\mathcal{F}$ (see Definition 2.1).

Theorem 2.20. *For any saturated fusion system $\mathcal{F}$ over a finite p-group S, if $\mathfrak{red}(\mathcal{F})$ is strongly tame, then $\mathcal{F}$ is tame.*

Proof. Set $Q = O_p(\mathcal{F})$, $S_0 = C_S(Q)/Z(Q)$, and $\mathcal{F}_0 = C_{\mathcal{F}}(Q)/Z(Q)$. Let $\mathfrak{red}(\mathcal{F}) = \mathcal{F}_m \subseteq \mathcal{F}_{m-1} \subseteq \cdots \subseteq \mathcal{F}_0$ be a sequence of fusion subsystems, where for each i, $\mathcal{F}_i = O^p(\mathcal{F}_{i-1})$ or $\mathcal{F}_i = O^{p'}(\mathcal{F}_{i-1})$. Let $S_m \trianglelefteq \cdots \trianglelefteq S_0$ be the corresponding sequence of p-groups: each $\mathcal{F}_i$ is a fusion system over S_i. By Lemma 2.3, $O_p(\mathcal{F}_i) = 1$ for each i, and hence $Z(\mathcal{F}_i) = 1$ for each i.

We first show inductively that each of the $\mathcal{F}_i$ is strongly tame. Fix $1 \le i \le m$, and assume $\mathcal{F}_i$ is tamely realized by $G_i \in \mathfrak{G}(p)$. By Lemma 2.19, we can assume $O_{p'}(G_i) = 1$. Thus $Z(G_i)$ is a p-group central in the fusion system $\mathcal{F}_i$, and hence $Z(G_i) = 1$ since $Z(\mathcal{F}_i) = 1$. By Proposition 2.12(a,b), there is a centric linking system associated to $\mathcal{F}_{i-1}$. Hence by Proposition 1.31(a,b), there are linking systems $\mathcal{L}_i \trianglelefteq \mathcal{L}_{i-1}$ associated to $\mathcal{F}_i \trianglelefteq \mathcal{F}_{i-1}$ such that $\mathcal{L}_i$ is a centric linking system (so $\mathrm{Ob}(\mathcal{L}_i)$ is $\mathrm{Aut}(S_i, \mathcal{F}_i)$-invariant), $\mathrm{Ob}(\mathcal{L}_{i-1})$ is $\mathrm{Aut}(S_{i-1}, \mathcal{F}_{i-1})$-invariant, and $\mathcal{L}_i$ is $\mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L}_{i-1})$-invariant. Also, $\mathcal{L}_i$ is centric in $\mathcal{L}_{i-1}$ by Proposition 1.31(a,b) again (and since $Z(\mathcal{F}_{i-1}) = 1$). By Lemma 2.11(c), $\mathcal{L}_i \cong \mathcal{L}_{S_i}^c(G_i)$. The hypotheses of Proposition 2.16 are thus satisfied, and hence $\mathcal{F}_{i-1}$ is tamely realized by some G_{i-1} such that $G_i \trianglelefteq G_{i-1}$ and $G_{i-1}/G_i \cong \mathcal{L}_{i-1}/\mathcal{L}_i$. In particular, G_{i-1}/G_i is p-solvable, and so $G_{i-1} \in \mathfrak{G}(p)$ by Lemma 2.11(b).

Since $\mathcal{F}_m$ was assumed to be tamely realized by some $G_m \in \mathfrak{G}(p)$, we now conclude that $\mathcal{F}_0$ is tamely realized by $G_0 \in \mathfrak{G}(p)$. By Lemma 2.19 again, we can assume $O_{p'}(G_0) = 1$, and $Z(G_0) = 1$ since $Z(\mathcal{F}_0) = 1$. Next consider the saturated fusion system $\mathcal{F}^* \stackrel{\text{def}}{=} N_{\mathcal{F}}^{\mathrm{Inn}(Q)}(Q)$ over $S^* \stackrel{\text{def}}{=} Q{\cdot}C_S(Q)$. Since $\mathcal{F}^* \trianglelefteq \mathcal{F}$ by Proposition 1.25(c), $O_p(\mathcal{F}^*) = Q$ by Lemma 1.20(e). Let $Z(Q) = Z_1(Q) \le Z_2(Q) \le \cdots \le Q$ be the upper central series for Q. Since $\mathrm{Aut}_{\mathcal{F}^*}(Q) = \mathrm{Inn}(Q)$, $Z_{i+1}(Q)/Z_i(Q)$ is central in $\mathcal{F}^*/Z_i(Q)$ for each i. Also, by repeated application of Proposition 1.8, if $P/Z_i(Q) = Z(\mathcal{F}^*/Z_i(Q))$, then $P \trianglelefteq \mathcal{F}^*$, and hence $P \le Q$. Thus $Z(\mathcal{F}^*/Z_i(Q)) \le Z(Q/Z_i(Q)) = Z_{i+1}(Q)/Z_i(Q)$, and these two subgroups are equal.

In other words, $\mathcal{F}^*/Q$ is obtained from $\mathcal{F}^*$ by sequentially dividing out by its center until the fusion system is centerfree. Now identify $C_S(Q)/Z(Q)$ with $N_S^{\mathrm{Inn}(Q)}(Q)/Q$ in the canonical way. By definition, each morphism in $C_{\mathcal{F}}(Q)$ extends to a morphism between subgroups containing Q which is the identity on Q and hence lies in $\mathcal{F}^*$. Thus $C_{\mathcal{F}}(Q)/Z(Q) \subseteq \mathcal{F}^*/Q$, and the opposite inclusion holds by a similar argument. Hence $\mathcal{F}_0 = C_{\mathcal{F}}(Q)/Z(Q)$ is obtained from $\mathcal{F}^*$ by sequentially dividing out by its center. By repeated application of Proposition 2.18, $\mathcal{F}^*$ is tamely realizable by some finite group $G^* \in \mathfrak{G}(p)$ such that $O_{p'}(G^*) = 1$ and $Z(G^*) = Z(\mathcal{F}^*)$.

By Proposition 2.12(c), there is a centric linking system associated to $\mathcal{F}$. Hence by Proposition 1.31(c), there are linking systems $\mathcal{L}^* \trianglelefteq \mathcal{L}$ associated to $\mathcal{F}^* \trianglelefteq \mathcal{F}$, where all objects in $\mathcal{L}^*$ are $\mathcal{F}^*$-centric, $\mathrm{Ob}(\mathcal{L}^*)$ is $\mathrm{Aut}(S^*, \mathcal{F}^*)$-invariant, $\mathrm{Ob}(\mathcal{L})$ is $\mathrm{Aut}(S, \mathcal{F})$-invariant, $\mathcal{L}^*$ is $\mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L})$-invariant, and $\mathcal{L}^*$ is centric in $\mathcal{L}$. By Lemma 2.11(c) (and since $G^* \in \mathfrak{G}(p)$), $\mathcal{L}^* \cong \mathcal{L}_{S^*}^{\mathrm{Ob}(\mathcal{L}^*)}(G^*)$. Hence by Proposition 2.16, $\mathcal{F}$ is tamely realized by a finite group G. □

3. Decomposing reduced fusion systems as products

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are fusion systems over finite p-groups S_1 and S_2, respectively, then $\mathcal{F}_1 \times \mathcal{F}_2$ is the fusion system over $S_1 \times S_2$ defined as follows. For all $P, Q \le S_1 \times S_2$, if $P_i, Q_i \le S_i$ denote the images of P and Q under projection to S_i, then

$$\mathrm{Hom}_{\mathcal{F}_1 \times \mathcal{F}_2}(P, Q) = \bigl\{ (\varphi_1, \varphi_2)|_P \bigm| \varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, Q_i),\ (\varphi_1, \varphi_2)(P) \le Q \bigr\} .$$

Here, we regard P and Q as subgroups of $P_1 \times P_2$ and $Q_1 \times Q_2$, respectively. Thus $\mathcal{F}_1 \times \mathcal{F}_2$ is the smallest fusion system over $S_1 \times S_2$ for which

$$\mathrm{Hom}_{\mathcal{F}_1 \times \mathcal{F}_2}(P_1 \times P_2, Q_1 \times Q_2) = \mathrm{Hom}_{\mathcal{F}_1}(P_1, Q_1) \times \mathrm{Hom}_{\mathcal{F}_2}(P_2, Q_2)$$

for each $P_i, Q_i \le S_i$. By [BLO2, Lemma 1.5], $\mathcal{F}_1 \times \mathcal{F}_2$ is saturated if $\mathcal{F}_1$ and $\mathcal{F}_2$ are saturated. We leave it as an easy exercise to check, for any pair of finite groups G_1, G_2 with Sylow subgroups $S_i \in \mathrm{Syl}_p(G_i)$, that $\mathcal{F}_{S_1 \times S_2}(G_1 \times G_2) = \mathcal{F}_{S_1}(G_1) \times \mathcal{F}_{S_2}(G_2)$.

We say that a nontrivial fusion system $\mathcal{F}$ is *indecomposable* if it has no decomposition as a product of fusion systems over nontrivial p-groups. The main result in this section is Theorem C: every reduced fusion system has a unique decomposition as a product of reduced indecomposable fusion systems, and the product is tame if each of the indecomposable factors is tame. The first statement will be proven as Proposition 3.6, and the second as Theorem 3.7.

We first prove the following easy lemma about fusion systems over products of finite p-groups.

Lemma 3.1. *Let S_1, S_2 be a pair of finite p-groups, and set $S = S_1 \times S_2$. For each subgroup $P \le S$ which does* not *split as a product $P = P_1 \times P_2$ for $P_i \le S_i$, there is $x \in N_S(P) \smallsetminus P$ such that $c_x \in O_p(\mathrm{Aut}(P))$. Hence for each saturated fusion system $\mathcal{F}$ over S, and each subgroup $P \le S$ which is $\mathcal{F}$-centric and $\mathcal{F}$-radical, $P = P_1 \times P_2$ for some pair of subgroups $P_i \le S_i$.*

Proof. We prove the first statement; the last then follows by Lemma 1.4.

Fix $P \le S$. For $i = 1, 2$, let $P_i \le S_i$ be the image of P under projection. Thus $P \le P_1 \times P_2$. Let $Z_k(P)$ and $Z_k(P_i)$ be the k-th terms in the upper central series for P and P_i; i.e., $Z_1(P) = Z(P)$ and $Z_{k+1}(P)/Z_k(P) = Z(P/Z_k(P))$. We claim that for each k,

$$Z_k(P) = P \cap (Z_k(P_1) \times Z_k(P_2)) . \tag{1}$$

This is clear for $k = 1$: an element of P is central only if it commutes with all elements in P_1 and all elements in P_2. If (1) holds for k, then $P/Z_k(P)$ can be identified as a subgroup of $(P_1/Z_k(P_1)) \times (P_2/Z_k(P_2))$ (a subgroup which projects onto each factor), and the result for $Z_{k+1}(P)$ then follows immediately.

If $P \lneqq P_1 \times P_2$, then choose $x \in N_{P_1 \times P_2}(P) \smallsetminus P$ (see [Sz1, Theorem 2.1.6]). By (1), conjugation by x acts via the identity on each quotient $Z_{k+1}(P)/Z_k(P)$. So $c_x \in O_p(\mathrm{Aut}(P))$ by Lemma 1.6. □

The next lemma gives some basic properties of product fusion systems.

Lemma 3.2. *Assume $\mathcal{F}_1$ and $\mathcal{F}_2$ are saturated fusion systems over finite p-groups S_1 and S_2. For each $i = 1, 2$, let $\mathcal{F}'_i \subseteq \mathcal{F}_i$ be a saturated fusion subsystem over $S'_i \leq S_i$.*

(a) *If $\mathcal{F}'_i \trianglelefteq \mathcal{F}_i$ for $i = 1, 2$, then $\mathcal{F}'_1 \times \mathcal{F}'_2$ is normal in $\mathcal{F}_1 \times \mathcal{F}_2$.*

(b) *If $\mathcal{F}'_i$ has index prime to p in $\mathcal{F}_i$ for $i = 1, 2$, then $\mathcal{F}'_1 \times \mathcal{F}'_2$ has index prime to p in $\mathcal{F}_1 \times \mathcal{F}_2$.*

Proof. Set $S = S_1 \times S_2$, $S' = S'_1 \times S'_2$, $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$, and $\mathcal{F}' = \mathcal{F}'_1 \times \mathcal{F}'_2$.

(a) Since S'_i is strongly closed in $\mathcal{F}_i$, S' is strongly closed in $\mathcal{F}$.

Fix $P, Q \leq S'$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$. Let $P_i, Q_i \leq S'_i$ be the images of P and Q under projection to S'_i. Then $\varphi = (\varphi_1, \varphi_2)|_P$ for some $\varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, Q_i)$. By condition (ii) in Definition 1.18, there are morphisms $\alpha_i \in \mathrm{Aut}_{\mathcal{F}_i}(S'_i)$ and $\varphi'_i \in \mathrm{Hom}_{\mathcal{F}'_i}(\alpha_i(P_i), Q_i)$ such that $\varphi_i = \varphi'_i \circ \alpha_i|_{P_i,\alpha_i(P_i)}$. Set $\alpha = (\alpha_1, \alpha_2) \in \mathrm{Aut}_{\mathcal{F}}(S')$, and set $\varphi' = (\varphi'_1, \varphi'_2)|_{\alpha(P)}$. Then $\varphi'(\alpha(P)) \leq Q$, so $\varphi' \in \mathrm{Hom}_{\mathcal{F}'}(\alpha(P), Q)$ and $\varphi = \varphi' \circ \alpha|_{P,\alpha(P)}$. This proves condition (ii) for the pair $\mathcal{F}' \subseteq \mathcal{F}$.

Let $P, Q \leq S'$ and $P_i, Q_i \leq S'_i$ be as above, and fix $\varphi = (\varphi_1, \varphi_2)|_P \in \mathrm{Hom}_{\mathcal{F}'}(P, Q)$ and $\beta = (\beta_1, \beta_2) \in \mathrm{Aut}_{\mathcal{F}}(S')$. Then $\beta_i\varphi_i\beta_i^{-1} \in \mathrm{Hom}_{\mathcal{F}'_i}(\beta_i(P_i), \beta_i(Q_i))$ by condition (iii) for the normal pair $\mathcal{F}'_i \trianglelefteq \mathcal{F}_i$. Also, $\beta\varphi\beta^{-1}(\beta(P)) \leq \beta(Q)$, and hence $\beta\varphi\beta^{-1} \in \mathrm{Hom}_{\mathcal{F}'}(\beta(P), \beta(Q))$. This proves condition (iii) for the pair $\mathcal{F}' \subseteq \mathcal{F}$, and finishes the proof that $\mathcal{F}'$ is normal in $\mathcal{F}$.

(b) Note that $S'_i = S_i$, since $\mathcal{F}'_i$ has index prime to p in $\mathcal{F}_i$. Since $\mathcal{F}'_i \supseteq O^{p'}(\mathcal{F}_i)$, it suffices to prove this point when $\mathcal{F}'_i = O^{p'}(\mathcal{F}_i)$, and thus when $\mathcal{F}'_i \trianglelefteq \mathcal{F}_i$ (Proposition 1.25(b)). Hence $\mathcal{F}'_1 \times \mathcal{F}'_2$ is normal in $\mathcal{F}_1 \times \mathcal{F}_2$ by (a). Since they are fusion systems over the same p-group, the result now follows by Lemma 1.26. □

We next prove the following criterion for a reduced fusion system to decompose: $\mathcal{F}$ factors as a product of fusion subsystems whenever S factors as a product of subgroups which are strongly closed in $\mathcal{F}$.

Proposition 3.3. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group $S = S_1 \times \cdots \times S_m$, where $S_1, \dots, S_m$ are all strongly closed in $\mathcal{F}$. Set $\mathcal{F}_i = \mathcal{F}|_{S_i}$ ($i = 1, \dots, m$): the full subcategory of $\mathcal{F}$ with objects the subgroups of S_i, regarded as a fusion system over S_i. For each i, let $S_i^* = \prod_{j \neq i} S_j$, identify $S = S_i \times S_i^*$, and let $\mathcal{F}'_i \subseteq \mathcal{F}_i$ be the fusion subsystem over S_i where for $P, Q \leq S_i$,*

$$\mathrm{Hom}_{\mathcal{F}'_i}(P, Q) = \left\{ \varphi \in \mathrm{Hom}_{\mathcal{F}_i}(P, Q) \,\middle|\, (\varphi, \mathrm{Id}_{S_i^*}) \in \mathrm{Hom}_{\mathcal{F}}(P \times S_i^*, Q \times S_i^*) \right\} .$$

Then $\mathcal{F}'_i$ and $\mathcal{F}_i$ are saturated fusion systems for each i, $O^{p'}(\mathcal{F}_i) \subseteq \mathcal{F}'_i$, and

$$\mathcal{F}'_1 \times \cdots \times \mathcal{F}'_m \subseteq \mathcal{F} \subseteq \mathcal{F}_1 \times \cdots \times \mathcal{F}_m .$$

If $O^{p'}(\mathcal{F}) = \mathcal{F}$, then $\mathcal{F}'_i = \mathcal{F}_i$ for each i, and hence $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_m$.

Proof. Fix $i \in \{1, \dots, m\}$. We first claim that

$$\begin{aligned} &\forall\ P, Q \leq S_i \text{ and } \varphi \in \mathrm{Hom}_{\mathcal{F}_i}(P, Q), \text{ there are } \psi \in \mathrm{Aut}_{\mathcal{F}}(S_i^*) \text{ and } \chi \in \mathrm{Aut}_{\mathcal{F}_i}(S_i) \\ &\quad \text{such that} \quad (\varphi, \psi) \in \mathrm{Hom}_{\mathcal{F}}(P \times S_i^*, Q \times S_i^*) \quad \text{and} \quad \chi|_Q \circ \varphi \in \mathrm{Hom}_{\mathcal{F}'_i}(P, S_i) . \end{aligned} \tag{2}$$

If $\varphi(P)$ is fully centralized in $\mathcal{F}$, the existence of ψ follows by the extension axiom, and since the S_i are all strongly closed in $\mathcal{F}$. The general case then follows upon choosing $\alpha \in \mathrm{Iso}_{\mathcal{F}}(\varphi(P), R)$ where $R \leq S_i$ is fully centralized in $\mathcal{F}$, and applying the extension axiom to $\alpha\circ\varphi$ and to α. By the extension axiom again, this time applied to ψ, there is χ such that $(\chi^{-1},\psi)\in \mathrm{Aut}_{\mathcal{F}}(S)$, and hence $\chi|_Q\circ\varphi\in \mathrm{Hom}_{\mathcal{F}'_i}(P,S_i)$. This finishes the proof of (2).

Two subgroups of S_i are $\mathcal{F}_i$-conjugate if and only if they are $\mathcal{F}$-conjugate; and they cannot be $\mathcal{F}$-conjugate to any other subgroups of S since S_i is strongly closed. Also, for $P\le S_i$, $|N_S(P)|=|N_{S_i}(P)|\cdot|S_i^*|$ and $|C_S(P)|=|C_{S_i}(P)|\cdot|S_i^*|$. Hence P is fully normalized (centralized) in $\mathcal{F}_i$ if and only if it is fully normalized (centralized) in $\mathcal{F}$. By (2), $P,Q\le S_i$ are $\mathcal{F}_i$-conjugate only if P is $\mathcal{F}'_i$-conjugate to a subgroup in the $\mathrm{Aut}_{\mathcal{F}_i}(S_i)$-orbit of Q, and hence P is fully normalized (centralized) in $\mathcal{F}_i$ if and only if it is fully normalized (centralized) in $\mathcal{F}'_i$. Also, in the context of axiom (II), $N_\varphi^{\mathcal{F}}=N_\varphi^{\mathcal{F}_i}\times S_i^*$ for all $\varphi\in\mathrm{Mor}(\mathcal{F}_i)$, and $N^{\mathcal{F}}_{(\varphi,\mathrm{Id}_{S_i^*})}=N_\varphi^{\mathcal{F}'_i}\times S_i^*$ for all $\varphi\in\mathrm{Mor}(\mathcal{F}'_i)$. Axioms (I) and (II) for $\mathcal{F}_i$ and for $\mathcal{F}'_i$ now follow easily from the same axioms applied to $\mathcal{F}$; and thus $\mathcal{F}_i$ and $\mathcal{F}'_i$ are saturated.

Fix $P\le S_i$, and choose $\varphi\in\mathrm{Aut}_{\mathcal{F}_i}(P)$ and $\alpha\in\mathrm{Aut}_{\mathcal{F}'_i}(P)$. By (2), there is $\psi\in\mathrm{Aut}_{\mathcal{F}}(S_i^*)$ such that $(\varphi,\psi),(\alpha,\mathrm{Id})\in\mathrm{Aut}_{\mathcal{F}}(P\times S_i^*)$. Hence $(\varphi\alpha\varphi^{-1},\mathrm{Id})\in\mathrm{Aut}_{\mathcal{F}}(P\times S_i^*)$, $\varphi\alpha\varphi^{-1}\in\mathrm{Aut}_{\mathcal{F}'_i}(P)$, and so $\mathrm{Aut}_{\mathcal{F}'_i}(P)$ is normal in $\mathrm{Aut}_{\mathcal{F}_i}(P)$. When P is fully normalized, $\mathrm{Aut}_{\mathcal{F}'_i}(P)$ contains $\mathrm{Aut}_{S_i}(P)\in\mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}_i}(P))$, and thus $\mathrm{Aut}_{\mathcal{F}'_i}(P)\ge O^{p'}(\mathrm{Aut}_{\mathcal{F}_i}(P))$. Hence $\mathcal{F}'_i$ has index prime to p in $\mathcal{F}_i$ (see Definition 1.21), and so $\mathcal{F}'_i\supseteq O^{p'}(\mathcal{F}_i)$.

Clearly, $\mathcal{F}$ contains $\mathcal{F}'_1\times\cdots\times\mathcal{F}'_m$. By Lemma 3.1 together with Alperin's fusion theorem (Theorem 1.3), each morphism in $\mathcal{F}$ is a composite of restrictions of automorphisms of subgroups of the form $P_1\times\ldots\times P_m$ for $P_i\le S_i$. Since the S_i are strongly closed in $\mathcal{F}$, each such automorphism has the form $(\varphi_1,\ldots,\varphi_m)$ for some $\varphi_i\in\mathrm{Aut}_{\mathcal{F}}(P_i)=\mathrm{Aut}_{\mathcal{F}_i}(P_i)$. Hence for arbitrary $P,Q\le S$, if $P_i,Q_i\le S_i$ denote the images of P and Q under projection, then each $\varphi\in\mathrm{Hom}_{\mathcal{F}}(P,Q)$ extends to some morphism $(\varphi_1,\ldots,\varphi_m)$ where $\varphi_i\in\mathrm{Hom}_{\mathcal{F}}(P_i,Q_i)$. Since $\mathrm{Hom}_{\mathcal{F}}(P_i,Q_i)=\mathrm{Hom}_{\mathcal{F}_i}(P_i,Q_i)$, this shows that $\mathcal{F}\subseteq\mathcal{F}_1\times\cdots\times\mathcal{F}_m$.

Since $\mathcal{F}'_i$ has index prime to p in $\mathcal{F}_i$ for each i, $\mathcal{F}'_1\times\cdots\times\mathcal{F}'_m$ has index prime to p in $\mathcal{F}_1\times\cdots\times\mathcal{F}_m$ by Lemma 3.2(b), and hence has index prime to p in $\mathcal{F}$. So if $O^{p'}(\mathcal{F})=\mathcal{F}$, then $\mathcal{F}=\mathcal{F}'_1\times\cdots\times\mathcal{F}'_m$; and $\mathcal{F}_i=\mathcal{F}'_i$ for each i by definition of $\mathcal{F}_i$. □

Note that if $\mathcal{F}$ is any fusion system (saturated or not) over a finite p-group $S=S_1\times S_2$, and $\mathcal{F}$ factors as a product of fusion systems over S_1 and S_2, then the factors must be the fusion subsystems $\mathcal{F}_i=\mathcal{F}'_i$ as defined in Proposition 3.3. In other words, if there is any such factorization, it must be unique.

We next show that a product of reduced fusion systems is reduced.

Proposition 3.4. *Fix finite p-groups S_1 and S_2 and saturated fusion systems $\mathcal{F}_i$ over S_i. Set $\mathcal{F}=\mathcal{F}_1\times\mathcal{F}_2$. Then*

$$O_p(\mathcal{F})=O_p(\mathcal{F}_1)\times O_p(\mathcal{F}_2),\quad O^p(\mathcal{F})=O^p(\mathcal{F}_1)\times O^p(\mathcal{F}_2),\quad O^{p'}(\mathcal{F})=O^{p'}(\mathcal{F}_1)\times O^{p'}(\mathcal{F}_2)\ .$$

In particular, $\mathcal{F}$ is reduced if and only if $\mathcal{F}_1$ and $\mathcal{F}_2$ are both reduced.

Proof. Set $S=S_1\times S_2$. The decomposition of $O_p(\mathcal{F})$ is clear: if $P\le S$ is normal in $\mathcal{F}$, then so are its projections into S_1 and S_2, and $P_i\trianglelefteq\mathcal{F}_i$ implies $P_1\times P_2\trianglelefteq\mathcal{F}$.

The relation "of index prime to p" among fusion systems is transitive (see Definition 1.21), and hence $O^{p'}(O^{p'}(\mathcal{F}))=O^{p'}(\mathcal{F})$. So by Proposition 3.3, $O^{p'}(\mathcal{F})=\mathcal{F}'_1\times\mathcal{F}'_2$ for some pair of fusion systems $\mathcal{F}'_i$ over S_i. Also, $O^{p'}(\mathcal{F})\subseteq O^{p'}(\mathcal{F}_1)\times O^{p'}(\mathcal{F}_2)$ by Lemma 3.2(b), so $\mathcal{F}'_i\subseteq O^{p'}(\mathcal{F}_i)$, and $\mathcal{F}'_i$ has index prime to p in $\mathcal{F}_i$ since $\mathcal{F}'_1\times\mathcal{F}'_2$ has index prime to p in $\mathcal{F}$. Thus $\mathcal{F}'_i=O^{p'}(\mathcal{F}_i)$.

By definition,

$$\mathfrak{hyp}(\mathcal{F})=\langle s^{-1}\alpha(s)\,|\,s\in P\le S,\ \alpha\in O^p(\mathrm{Aut}_{\mathcal{F}}(P))\rangle=\mathfrak{hyp}(\mathcal{F}_1)\times\mathfrak{hyp}(\mathcal{F}_2)\ .$$

Since $O^p(\mathcal{F})$ is the unique fusion subsystem over $\mathfrak{hyp}(\mathcal{F})$ of p-power index in $\mathcal{F}$ (Theorem 1.22(a)), we have $O^p(\mathcal{F}) = O^p(\mathcal{F}_1) \times O^p(\mathcal{F}_2)$.

The last statement is now immediate. □

By definition, every fusion system $\mathcal{F}$ factors as a product of indecomposable fusion systems. The following lemma is the key step when showing that this factorization is unique (not only up to isomorphism) when $\mathcal{F}$ is reduced.

Lemma 3.5. *Let $\mathcal{F}$ be a reduced fusion system over a finite p-group S. Assume $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 = \mathcal{F}_3 \times \mathcal{F}_4$, where each $\mathcal{F}_i$ is a saturated fusion system over some $S_i \le S$. Set $S_{ij} = S_i \cap S_j$ for $i = 1, 2$ and $j = 3, 4$. Then $\mathcal{F} = \mathcal{F}_{13} \times \mathcal{F}_{14} \times \mathcal{F}_{23} \times \mathcal{F}_{24}$, where $\mathcal{F}_{ij}$ is a reduced fusion system over S_{ij}.*

Proof. By assumption, the subgroups S_i for $i \in \{1, 2, 3, 4\}$ are all strongly closed in $\mathcal{F}$, and $S = S_1 \times S_2 = S_3 \times S_4$. Fix $x, y \in S_1$ which are $\mathcal{F}$-conjugate, and choose $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle x\rangle, \langle y\rangle)$ which sends x to y. Write $x = x_3x_4$ and $y = y_3y_4$, where $x_3, y_3 \in S_3$ and $x_4, y_4 \in S_4$. There are homomorphisms $\varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(\langle x_i\rangle, \langle y_i\rangle)$ for $i = 3, 4$ which send x_i to y_i, and such that φ is the restriction of (φ_3, φ_4). Hence $(\varphi_3, \mathrm{Id}_{S_4})(x) = y_3x_4$, $y_3x_4 \in S_1$ since S_1 is strongly closed, and thus $x_3^{-1}y_3 \in S_{13}$. By a similar argument, $x_4^{-1}y_4 \in S_{14}$, and thus $x^{-1}y \in S_{13} \times S_{14}$. This proves that $\mathfrak{foc}(\mathcal{F}_1) \le S_{13} \times S_{14}$.

By a similar argument, $\mathfrak{foc}(\mathcal{F}_2) \le S_{23} \times S_{24}$. Since $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$, it follows that

$$\mathfrak{foc}(\mathcal{F}) = \mathfrak{foc}(\mathcal{F}_1) \times \mathfrak{foc}(\mathcal{F}_2) \le S_{13} \times S_{14} \times S_{23} \times S_{24} \le S\ .$$

Also, $\mathfrak{foc}(\mathcal{F}) = S$ since $\mathcal{F}$ is reduced (Theorem 1.22(a)), so S is the product of the S_{ij}. Since the intersection of two subgroups which are strongly closed in $\mathcal{F}$ is strongly closed in $\mathcal{F}$, $\mathcal{F}$ splits as a product of reduced fusion systems $\mathcal{F}_{ij}$ over S_{ij} by Propositions 3.3 and 3.4 (recall $O^{p'}(\mathcal{F}) = \mathcal{F}$ since $\mathcal{F}$ is reduced). □

This now implies the uniqueness of any decomposition of a reduced fusion system as a product of indecomposables.

Proposition 3.6. *Each reduced fusion system $\mathcal{F}$ over a finite p-group S has a unique factorization $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_m$ as a product of indecomposable fusion systems $\mathcal{F}_i$ over subgroups $S_i \trianglelefteq S$. Moreover, the $\mathcal{F}_i$ are all reduced, and each fusion preserving automorphism $\alpha \in \mathrm{Aut}(S, \mathcal{F})$ permutes the factors S_i.*

Proof. Let $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_m = \mathcal{F}_1' \times \cdots \times \mathcal{F}_n'$ be two decompositions as products of indecomposable fusion systems. By Lemma 3.5 applied to the decompositions $\mathcal{F} = \mathcal{F}_1 \times \prod_{i\ge2} \mathcal{F}_i = \mathcal{F}_1' \times \prod_{i\ge2} \mathcal{F}_i'$, and since $\mathcal{F}_1$ and $\mathcal{F}_1'$ are indecomposable, either $\mathcal{F}_1 = \mathcal{F}_1'$ and $\prod_{i\ge2} \mathcal{F}_i = \prod_{i\ge2} \mathcal{F}_i'$, or $\mathcal{F}_1$ is a direct factor in $\prod_{i\ge2} \mathcal{F}_i'$. In the latter case, we can assume by induction on $|S|$ that the decomposition of $\prod_{i\ge2} \mathcal{F}_i'$ is unique, and hence that for some j, $\mathcal{F}_1 = \mathcal{F}_j'$ and so $\prod_{i\ne1} \mathcal{F}_i = \prod_{i\ne j} \mathcal{F}_i'$ (Lemma 3.5 again). By the same induction hypothesis, this proves that the two decompositions are equal up to permutation of the factors. The factors $\mathcal{F}_i$ are all reduced by Proposition 3.4.

Fix $\alpha \in \mathrm{Aut}(S, \mathcal{F})$. Since $S = \prod_{i=1}^m \alpha(S_i)$ is a product of subgroups which are strongly closed in $\mathcal{F}$, $\mathcal{F}$ factors as a product of saturated fusion systems over the $\alpha(S_i)$ by Proposition 3.3 (and since $O^{p'}(\mathcal{F}) = \mathcal{F}$). So α permutes the factors S_i by the uniqueness of the decomposition. □

We are now ready to prove that a product of reduced, *indecomposable*, tame fusion systems is tame. Together with Theorem 2.20, this shows that any "minimal" exotic fusion system is indecomposable as well as reduced.

Theorem 3.7. *Fix a reduced fusion system $\mathcal{F}$ over a finite p-group S, and let $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_m$ be its unique factorization as a product of indecomposable fusion systems. If $\mathcal{F}_i$ is tame (strongly tame) for each i, then $\mathcal{F}$ is tame (strongly tame).*

Proof. Let $S = S_1 \times \cdots \times S_m$ be the corresponding decomposition of p-groups; i.e., $\mathcal{F}_i$ is a fusion system over S_i. Assume each $\mathcal{F}_i$ is tame, and let G_i be a finite group which tamely realizes $\mathcal{F}_i$. Assume also that these are chosen so that $G_i \cong G_j$ if $\mathcal{F}_i \cong \mathcal{F}_j$. Set $\mathcal{L}_i = \mathcal{L}^c_{S_i}(G_i)$. Set $G = G_1 \times \cdots \times G_m$, $\mathcal{L} = \mathcal{L}^c_S(G)$, and $\widehat{\mathcal{L}} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_m$. We identify $\widehat{\mathcal{L}}$ with the full subcategory of $\mathcal{L}$ having as objects those $P = P_1 \times \cdots \times P_m$ where $P_i \in \mathrm{Ob}(\mathcal{L}_i)$. Note that $\widehat{\mathcal{L}}$ is not a linking system, since $\mathrm{Ob}(\widehat{\mathcal{L}})$ is not closed under overgroups.

Set $\mathbf{m} = \{1, \ldots, m\}$. Define

$$\mathrm{Aut}^0_{\mathrm{typ}}(\mathcal{L}) = \bigl\{\alpha \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}) \bigm| \alpha_S(\delta_S(S_i)) = \delta_S(S_i) \text{ for each } i \in \mathbf{m}\bigr\} .$$

We first construct a monomorphism

$$\Psi \colon \mathrm{Aut}^0_{\mathrm{typ}}(\mathcal{L}) \longrightarrow \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_1) \times \cdots \times \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}_m)$$

such that for each $\alpha \in \mathrm{Aut}^0_{\mathrm{typ}}(\mathcal{L})$, if $\Psi(\alpha) = (\alpha_1, \ldots, \alpha_m)$, then $\alpha|_{\widehat{\mathcal{L}}} = \prod_{i\in\mathbf{m}} \alpha_i$.

To define Ψ, fix $\alpha \in \mathrm{Aut}^0_{\mathrm{typ}}(\mathcal{L})$, and let $\beta \in \mathrm{Aut}(S, \mathcal{F})$ be the induced automorphism of Lemma 1.15 (i.e., $\delta_S(\beta(g)) = \alpha(\delta_S(g))$ for $g \in S$). Then $\beta(S_i) = S_i$ for each i since δ_S is injective. Also, by Lemma 1.15, $\alpha(P) = \beta(P)$ for each $P \in \mathrm{Ob}(\mathcal{L})$, and $\pi \circ \alpha = c_\beta \circ \pi$, where $c_\beta \in \mathrm{Aut}(\mathcal{F})$ is conjugation by β (and its restrictions).

Fix $i \in \mathbf{m}$, set $S^*_i = \prod_{j\neq i} S_j$ and $\mathcal{L}^*_i = \prod_{j\neq i} \mathcal{L}_j$, and identify $S = S_i \times S^*_i$ and $\widehat{\mathcal{L}} = \mathcal{L}_i \times \mathcal{L}^*_i$. We claim the following:

$$\forall\ \psi \in \mathrm{Mor}(\mathcal{L}_i), \quad \exists\ \alpha_i(\psi) \in \mathrm{Mor}(\mathcal{L}_i) \text{ such that } \alpha(\psi, \mathrm{Id}_{S^*_i}) = (\alpha_i(\psi), \mathrm{Id}_{S^*_i}). \tag{3}$$

For each $\psi \in \mathrm{Mor}(\mathcal{L}_i)$,

$$\pi(\alpha(\psi, \mathrm{Id}_{S^*_i})) = c_\beta(\pi(\psi), \mathrm{Id}_{S^*_i}) = (c_\beta(\pi(\psi)), \mathrm{Id}_{S^*_i}) \in \mathrm{Mor}(\mathcal{F})$$

since $\beta(S_j) = S_j$ for all j. Hence by axiom (A), $\alpha(\psi, \mathrm{Id}_{S^*_i}) = (\alpha_i(\psi), \delta_{S^*_i}(z))$ for some $\alpha_i(\psi) \in \mathrm{Mor}(\mathcal{L}_i)$ and some $z \in Z(S^*_i)$. In particular, (3) holds when ψ is an automorphism of order prime to p. Since $\alpha(\delta_P(x)) = \delta_{\beta(P)}(\beta(x))$ for all $P \in \mathrm{Ob}(\mathcal{L})$ and all $x \in N_S(P)$ (and since $\beta(S_i) = S_i$), (3) also holds when $\psi = \delta_P(x)$ for $P \leq S_i$ and $x \in N_{S_i}(P)$. When $P \in \mathrm{Ob}(\mathcal{L}_i)$ is fully normalized in $\mathcal{F}_i$, $\mathrm{Aut}_{\mathcal{L}_i}(P)$ is generated by elements of order prime to p and by its Sylow p-subgroup $\delta_P(N_{S_i}(P))$ (Proposition 1.11(d)), and hence (3) holds for all $\psi \in \mathrm{Aut}_{\mathcal{L}_i}(P)$. Finally, by Theorem 1.12, all morphisms in $\mathcal{L}_i$ are composites of restrictions of automorphisms of fully normalized subgroups, and hence (3) holds for all $\psi \in \mathrm{Mor}(\mathcal{L}_i)$.

Now let $\alpha_i \in \mathrm{Aut}(\mathcal{L}_i)$ be the automorphism defined by sending $P \in \mathrm{Ob}(\mathcal{L}_i)$ to $\beta(P)$, and $\psi \in \mathrm{Mor}(\mathcal{L}_i)$ to $\alpha_i(\psi)$ as defined in (3). This is clearly a functor, it is isotypical since α is, and it preserves inclusions since α does. Set $\Psi(\alpha) = (\alpha_1, \ldots, \alpha_m)$. Since each morphism in $\widehat{\mathcal{L}}$ is a composite of restrictions of morphisms of the form $(\psi_i, \mathrm{Id}_{S^*_i})$ for $\psi_i \in \mathrm{Mor}(\mathcal{L}_i)$, the restriction of α to $\widehat{\mathcal{L}}$ is $\prod_{i\in\mathbf{m}} \alpha_i$.

By construction, Ψ is a homomorphism. If $\Psi(\alpha) = (\mathrm{Id}_{\mathcal{L}_1}, \ldots, \mathrm{Id}_{\mathcal{L}_m})$, then $\alpha|_{\widehat{\mathcal{L}}} = \mathrm{Id}$ by the above remarks, α is the identity on objects since $\alpha_S = \mathrm{Id}_{\mathrm{Aut}_{\mathcal{L}}(S)}$ (Lemma 1.15), and so $\alpha = \mathrm{Id}_{\mathcal{L}}$ by Theorem 1.12 and since all $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups are objects in $\widehat{\mathcal{L}}$ (Lemma 3.1). Hence Ψ is injective. Finally, since $\mathrm{Aut}_{\mathcal{L}}(S) \cong \prod_{i\in\mathbf{m}} \mathrm{Aut}_{\mathcal{L}_i}(S_i)$, Ψ induces a monomorphism

$$\overline{\Psi} \colon \mathrm{Out}^0_{\mathrm{typ}}(\mathcal{L}) \overset{\mathrm{def}}{=} \mathrm{Aut}^0_{\mathrm{typ}}(\mathcal{L})/\{c_\zeta \,|\, \zeta \in \mathrm{Aut}_{\mathcal{L}}(S)\} \longrightarrow \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_1) \times \cdots \times \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_m) .$$

Next consider the equivalence relation $\sim$ on $\mathbf{m}$, where $i \sim j$ if $G_i \cong G_j$ (equivalently, $\mathcal{F}_i \cong \mathcal{F}_j$). Fix isomorphisms $\tau_{ij} \in \mathrm{Iso}(G_i, G_j)$ for all pairs $i \sim j$ of elements in $\mathbf{m}$, such that $\tau_{ij}(S_i) = S_j$, $\tau_{ii} = \mathrm{Id}_{G_i}$, $\tau_{ji} = \tau_{ij}^{-1}$, and $\tau_{ik} = \tau_{jk} \circ \tau_{ij}$ whenever $i \sim j \sim k$. Let $\widehat{\tau}_{ij} \colon \mathcal{L}_i \xrightarrow{\cong} \mathcal{L}_j$ be the induced isomorphism of linking systems. Then conjugation by $\widehat{\tau}_{ij}$ sends $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_i)$ to $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_j)$. For each i, fix a splitting $s_i \colon \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_i) \longrightarrow \mathrm{Out}(G_i)$ of κ_{G_i}, chosen so that $c_{\tau_{ij}} \circ s_i = s_j \circ c_{\widehat{\tau}_{ij}}$ if $i \sim j$.

Let $\bar{\Sigma} \le \Sigma_m$ be the group of permutations σ of $\mathbf{m}$ such that $\sigma(i) \sim i$ for each i. For each $\sigma \in \bar{\Sigma}$, let $\widehat{\sigma}_G \in \mathrm{Aut}(G)$ be the automorphism which sends G_i to $G_{\sigma(i)}$ via $\tau_{i,\sigma(i)}$, and set $\widehat{\sigma}_{\mathcal{L}} = \widetilde{\kappa}_G(\widehat{\sigma}_G)$. Thus $\widehat{\sigma}_{\mathcal{L}} \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ sends each $\mathcal{L}_i$ to $\mathcal{L}_{\sigma(i)}$ via $\widehat{\tau}_{i,\sigma(i)}$.

Fix $\alpha \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$, and let $\beta \in \mathrm{Aut}(S, \mathcal{F})$ be the restriction of $\alpha_S \in \mathrm{Aut}(\mathrm{Aut}_{\mathcal{L}}(S))$ to $S \cong \delta_S(S)$. By Proposition 3.6, there is $\sigma \in \Sigma_m$ such that $\beta(S_i) = S_{\sigma(i)}$ for each i. Since β is fusion preserving, $\mathcal{F}_i \cong \mathcal{F}_{\sigma(i)}$, and hence $i \sim \sigma(i)$, for each i. Thus $\sigma \in \bar{\Sigma}$, and $\widehat{\sigma}_{\mathcal{L}}^{-1} \circ \alpha \in \mathrm{Aut}^0_{\mathrm{typ}}(\mathcal{L})$. So $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ is generated by $\mathrm{Aut}^0_{\mathrm{typ}}(\mathcal{L})$ and the $\widehat{\sigma}_{\mathcal{L}}$.

Now let $s \colon \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \longrightarrow \mathrm{Out}(G)$ be the composite

$$\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) = \mathrm{Out}^0_{\mathrm{typ}}(\mathcal{L}) \rtimes \{[\widehat{\sigma}_{\mathcal{L}}] \,|\, \sigma \in \bar{\Sigma}\} \xrightarrow[([\widehat{\sigma}_{\mathcal{L}}] \mapsto \sigma)]{\bar{\Psi} \rtimes} \big(\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_1) \times \cdots \times \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_m)\big) \rtimes \bar{\Sigma}$$

$$\xrightarrow[(\sigma \mapsto [\widehat{\sigma}_G])]{(s_1,\ldots,s_m)\rtimes} \big(\mathrm{Out}(G_1) \times \cdots \times \mathrm{Out}(G_m)\big) \rtimes \{[\widehat{\sigma}_G] \,|\, \sigma \in \bar{\Sigma}\} \xrightarrow{\ \mathrm{incl}\ } \mathrm{Out}(G)\ .$$

We must show $\kappa_G \circ s = \mathrm{Id}$. Since $\kappa_G(s([\widehat{\sigma}_{\mathcal{L}}])) = \kappa_G([\widehat{\sigma}_G]) = [\widehat{\sigma}_{\mathcal{L}}]$ for $\sigma \in \bar{\Sigma}$, it will suffice to show $\kappa_G(s([\alpha])) = [\alpha]$ for $\alpha \in \mathrm{Aut}^0_{\mathrm{typ}}(\mathcal{L})$. Let $\mathrm{Out}^0(G) \le \mathrm{Out}(G)$ be the subgroup of classes of automorphisms which leave each G_i invariant, and consider the following composite:

$$\mathrm{Out}^0_{\mathrm{typ}}(\mathcal{L}) \xrightarrow{\ \bar{\Psi}\ } \prod_{i=1}^m \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_i) \xrightarrow{\prod s_i} \prod_{i=1}^m \mathrm{Out}(G_i) \cong \mathrm{Out}^0(G)$$

$$\xrightarrow{\kappa_G|_{\mathrm{Out}^0(G)}} \mathrm{Out}^0_{\mathrm{typ}}(\mathcal{L}) \xrightarrow{\ \bar{\Psi}\ } \prod_{i=1}^m \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_i)\ .$$

Here, $(\prod s_i) \circ \bar{\Psi} = s|_{\mathrm{Out}^0_{\mathrm{typ}}(\mathcal{L})}$, $\bar{\Psi} \circ \kappa_G|_{\mathrm{Out}^0(G)} = \prod \kappa_{G_i}$, and $(\prod \kappa_{G_i}) \circ (\prod s_i) = \mathrm{Id}$. This proves that $\bar{\Psi} \circ \kappa_G|_{\mathrm{Out}^0(G)} \circ s|_{\mathrm{Out}^0_{\mathrm{typ}}(\mathcal{L})} = \bar{\Psi}$. Since $\bar{\Psi}$ is injective, $\kappa_G \circ s = \mathrm{Id}$ on $\mathrm{Out}^0_{\mathrm{typ}}(\mathcal{L})$. Thus s is a splitting for κ_G, and this finishes the proof that $\mathcal{F}$ is tame.

If each $\mathcal{F}_i$ is strongly tame, then we can choose the G_i to all be in the class $\mathfrak{G}(p)$. Hence $G \in \mathfrak{G}(p)$ by Lemma 2.11(b), and $\mathcal{F}$ is strongly tame. □

Theorem 3.7 does *not* say that an arbitrary product of reduced, tame fusion systems is tame: such a product could conceivably have an indecomposable factor which is not tame. However, at least when $p = 2$, a theorem of Goldschmidt implies this is not possible.

Theorem 3.8. *Assume $p = 2$, and let $\mathcal{F}$ be a reduced fusion system over a 2-group S. Assume $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$, where $\mathcal{F}_i$ is a fusion system over S_i and $S = S_1 \times S_2$. Then $\mathcal{F}$ is realizable, tame, or strongly tame if and only if $\mathcal{F}_1$ and $\mathcal{F}_2$ are both realizable, tame, or strongly tame, respectively.*

Proof. Assume $\mathcal{F} = \mathcal{F}_S(G)$, where G is a finite group and $S \in \mathrm{Syl}_2(G)$. If $\mathcal{F}$ is tame, we also assume κ_G is split surjective, and if $\mathcal{F}$ is strongly tame, we also assume $G \in \mathfrak{G}(2)$. By Lemma 2.19, we can assume $O_{2'}(G) = 1$.

Let $G_i \trianglelefteq G$ be the normal closure of S_i in G. Since $\mathcal{F}$ factors as a product $\mathcal{F}_1 \times \mathcal{F}_2$, the subgroups S_1 and S_2 are strongly closed in $\mathcal{F}$, and hence strongly closed in G in the sense of [Gd]. So by Goldschmidt's theorem [Gd, Corollary A1], $G_1 \cap G_2 = 1$. Thus $G_1 \times G_2$ is a normal subgroup of odd index in G. Since $\mathcal{F} = \mathcal{F}_S(G)$ has no proper normal subsystem of odd index (since it is reduced), $\mathcal{F}_S(G) = \mathcal{F}_S(G_1 \times G_2) = \mathcal{F}_{S_1}(G_1) \times \mathcal{F}_{S_2}(G_2)$. Hence $\mathcal{F}_i = \mathcal{F}_{S_i}(G_i)$ for $i = 1, 2$ (there can be at most one way to factor $\mathcal{F}$ as a product of fusion systems over the S_i), and thus each $\mathcal{F}_i$ is realizable.

Set $\mathcal{L} = \mathcal{L}_S^c(G)$ and $\mathcal{L}_i = \mathcal{L}_{S_i}^c(G_i)$. Define $\Phi\colon \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L}_1) \times \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L}_2) \longrightarrow \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L})$ as follows. Fix $\alpha_i \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L}_i)$ $(i = 1, 2)$. Let $\beta_i \in \mathrm{Aut}(S_i, \mathcal{F}_i)$ be the corresponding automorphisms (see Lemma 1.15), and set $\beta = (\beta_1, \beta_2) \in \mathrm{Aut}(S, \mathcal{F})$. Thus $\alpha_i(P_i) = \beta_i(P_i)$ for each $P_i \in \mathrm{Ob}(\mathcal{L}_i)$ and $\pi(\alpha_i(\psi_i)) = \beta_i \pi(\psi_i) \beta_i^{-1}$ for $\psi_i \in \mathrm{Mor}(\mathcal{L}_i)$. Define $\alpha \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L})$ on objects by setting $\alpha(P) = \beta(P)$ for $P \in \mathrm{Ob}(\mathcal{L})$. Fix $\psi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$, let $P_i, Q_i \le S_i$ be the images of P and Q under projection, and set $\widehat{P} = P_1 \times P_2$ and $\widehat{Q} = Q_1 \times Q_2$. Since G and $G_1 \times G_2$ have the same fusion system over S, $\psi = [g]$ for some $g = (g_1, g_2) \in N_G(P, Q)$, where $g_i \in G_i$. Then $g_i \in N_{G_i}(P_i, Q_i)$, and hence ψ extends to $\widehat{\psi} = (\psi_1, \psi_2) \in \mathrm{Mor}_{\mathcal{L}}(\widehat{P}, \widehat{Q})$ where $\psi_i = [g_i] \in \mathrm{Mor}_{\mathcal{L}_i}(P_i, Q_i)$. Also, $\pi(\alpha_1(\psi_1), \alpha_2(\psi_2)) = \beta(\pi(\psi_1), \pi(\psi_2))\beta^{-1}$ sends $\beta(P)$ into $\beta(Q)$, and we define $\alpha(\psi) = (\alpha_1(\psi_1), \alpha_2(\psi_2))|_{\beta(P),\beta(Q)}$. Finally, $\alpha \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L})$ since $\alpha_i \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L}_i)$, and we set $\Phi(\alpha_1, \alpha_2) = \alpha$.

Assume $\mathcal{F}$ is tamely realized by G, and let $s\colon \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \longrightarrow \mathrm{Out}(G)$ be a splitting for κ_G. For each $\alpha_1 \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L}_1)$, $s([\Phi(\alpha_1, \mathrm{Id}_{\mathcal{L}_2})]) = [\gamma]$ for some $\gamma \in \mathrm{Aut}(G, S)$ such that $\gamma|_{S_2} = \mathrm{Id}$. Also, $\gamma(G_2) = G_2$ since G_2 is the normal closure of S_2 in G, and so γ induces $\bar{\gamma} \in \mathrm{Aut}(G/G_2, S_1)$. The class $[\bar{\gamma}] \in \mathrm{Out}(G/G_2)$ is independent of the choice of γ modulo $\mathrm{Inn}(G)$, and hence this gives a well defined homomorphism s_1 from $\mathrm{Out}_{\mathrm{typ}}^I(\mathcal{L}_1)$ to $\mathrm{Out}(G/G_2)$. Also, $\mathcal{F}_{S_1}(G/G_2) \cong \mathcal{F}/S_2 \cong \mathcal{F}_1$, so $\mathcal{L}_{S_1}^c(G/G_2) \cong \mathcal{L}_1$; and s_1 is a splitting for κ_{G/G_2} since s is a splitting for κ_G. Thus $\mathcal{F}_1$ is tame, and $\mathcal{F}_2$ is tame by a similar argument. If $\mathcal{F}$ is strongly tame, then we can choose $G \in \mathfrak{G}(2)$, so $G/G_i \in \mathfrak{G}(2)$ $(i = 1, 2)$ by Lemma 2.11(b), and hence $\mathcal{F}_1$ and $\mathcal{F}_2$ are strongly tame.

This proves the "only if" part of the theorem. Clearly, $\mathcal{F}$ is realizable if both factors are. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are both (strongly) tame, then we have just shown that each of the indecomposable factors of $\mathcal{F}_1$ and $\mathcal{F}_2$ is (strongly) tame, and so $\mathcal{F}$ is (strongly) tame by Theorem 3.7. □

4. Examples

We now give three families of examples, to illustrate some of the techniques which can be used to prove tameness of reduced fusion systems. As an introduction to these techniques, we first list the reduced fusion systems over dihedral and semidihedral groups and prove they are all tame. Next, we prove that certain fusion systems studied in [OV2, § 4–5] are reduced and tame; as a way of explaining how the information about these fusion systems given in [OV2] is just what is needed to prove tameness. As a third example, we prove that the fusion systems of all alternating groups are tame, and that they are reduced with certain obvious exceptions.

In general, tameness is shown by examining, for a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ realized by G, the homomorphisms

$$\mathrm{Out}(G) \xrightarrow{\ \kappa_G\ } \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \xrightarrow{\ \mu_G\ } \mathrm{Out}(S, \mathcal{F})$$

defined in Sections 2.2 and 1.3. By definition, $\mathcal{F}$ is tame if κ_G is split surjective (for some choice of G). However, the group $\mathrm{Out}(S,\mathcal{F})$ is usually much easier to describe than $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$, and the composite $\mu_G\circ\kappa_G$ is induced by restriction to S. So we need some way of describing $\mathrm{Ker}(\mu_G)$.

We first recall some definitions. A proper subgroup $H \lneqq G$ of a finite group G is *strongly p-embedded* if $p\big||H|$, and for each $g\in G\smallsetminus H$, $H\cap gHg^{-1}$ has order prime to p. It is not hard to see that G has a strongly p-embedded subgroup if and only if the poset $\mathcal{S}_p(G)$ of nontrivial p-subgroups is disconnected (cf. [HB3, Theorem X.4.11(b)]), but we will not be using that here.

When $\mathcal{F}$ is a saturated fusion system over a finite p-group S, then a proper subgroup $P\lneqq S$ is *$\mathcal{F}$-essential* if it is $\mathcal{F}$-centric and fully normalized, and $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup. Thus each $\mathcal{F}$-essential subgroup is fully normalized and $\mathcal{F}$-centric by definition, and is $\mathcal{F}$-radical since $O_p(\Gamma)=1$ for any group Γ which has a strongly p-embedded subgroup. See, e.g., [Sz2, Theorem 6.4.3] for a proof of this last statement (it is shown there only for $p=2$, but the same proof works for odd primes). The following proposition is a stronger version of Theorems 1.3 and 1.12, and helps show the importance of essential subgroups when working with fusion systems.

Theorem 4.1. *Let $\mathcal{F}$ be any saturated fusion system over a finite p-group S. Let $\mathcal{E}$ be the set of $\mathcal{F}$-essential subgroups of S, and set $\mathcal{E}_+=\mathcal{E}\cup\{S\}$. Then each morphism in $\mathcal{F}$ is a composite of restrictions of elements of $\mathrm{Aut}_{\mathcal{F}}(P)$ for $P\in\mathcal{E}_+$. If $\mathcal{L}$ is a linking system associated to $\mathcal{F}$, then each morphism in $\mathcal{L}$ is a composite of restrictions of elements of $\mathrm{Aut}_{\mathcal{L}}(P)$ for $P\in\mathcal{E}_+$.*

Proof. The statement about morphisms in $\mathcal{F}$ is shown in [Pg2, § 5], and also in [OV2, Corollary 2.6]. The second statement follows from this together with Proposition 1.11(a) (and since $\mathrm{Ob}(\mathcal{L})$ is closed under overgroups). □

The following proposition will be useful when describing $\mathrm{Ker}(\mu_G)$, and for determining whether or not explicit elements in this group vanish. In fact, it applies to help describe $\mathrm{Ker}(\mu_{\mathcal{L}})$, when $\mathcal{L}$ is an arbitrary linking system (not necessarily induced by a finite group). For any fusion system $\mathcal{F}$ over S and any $P\le S$, we write

$$C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))=\{g\in Z(P)\mid \alpha(g)=g \text{ for all } \alpha\in\mathrm{Aut}_{\mathcal{F}}(P)\}$$

and similarly for $C_{Z(P)}(\mathrm{Aut}_S(P))$ and $C_{Z(P)}(\mathrm{Aut}_{\mathcal{L}}(P))$.

Proposition 4.2. *Let $\mathcal{F}$ be a saturated fusion system over the finite p-group S, and let $\mathcal{L}$ be a linking system associated to $\mathcal{F}$. Let $\mathcal{L}^c\subseteq\mathcal{L}$ be the full subcategory whose objects are the $\mathcal{F}$-centric objects in $\mathcal{L}$. Each element in $\mathrm{Ker}(\mu_{\mathcal{L}})$ is represented by some $\alpha\in\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ such that $\alpha_S=\mathrm{Id}_{\mathrm{Aut}_{\mathcal{L}}(S)}$. For each such α, there are elements $g_P\in C_{Z(P)}(\mathrm{Aut}_S(P))$, defined for each fully normalized subgroup $P\in\mathrm{Ob}(\mathcal{L}^c)$, for which the following hold:*

(a) *$\alpha_P\in\mathrm{Aut}(\mathrm{Aut}_{\mathcal{L}}(P))$ is conjugation by $\delta_P(g_P)$, and g_P is uniquely determined by α modulo $C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$. In particular, $\alpha_P=\mathrm{Id}_{\mathrm{Aut}_{\mathcal{L}}(P)}$ if and only if $g_P\in C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$.*

(b) *Assume $P,Q\in\mathrm{Ob}(\mathcal{L}^c)$ are both fully normalized in $\mathcal{F}$. If $Q=aPa^{-1}$ for some $a\in S$, then we can choose $g_Q=ag_Pa^{-1}$. More generally, if Q is $\mathcal{F}$-conjugate to P, and there is $\zeta\in\mathrm{Iso}_{\mathcal{L}}(P,Q)$ such that $\alpha(\zeta)=\zeta$, then we can choose $g_Q=\pi(\zeta)(g_P)$. In either case, $\alpha_P=\mathrm{Id}_{\mathrm{Aut}_{\mathcal{L}}(P)}$ if and only if $\alpha_Q=\mathrm{Id}_{\mathrm{Aut}_{\mathcal{L}}(Q)}$.*

(c) *If $Q\le P$ are both fully normalized objects in $\mathcal{L}^c$, then $g_P\equiv g_Q$ (mod $C_{Z(Q)}(\mathrm{Aut}_{\mathcal{F}}(P,Q))$), where $\mathrm{Aut}_{\mathcal{F}}(P,Q)$ is the group of those $\varphi\in\mathrm{Aut}_{\mathcal{F}}(P)$ such that $\varphi(Q)=Q$.*

(d) *Let $\mathcal{E}$ be the set of all $\mathcal{F}$-essential subgroups $P \gneqq S$ and let $\mathcal{E}_0 \subseteq \mathcal{E}$ be the subset of those $P \in \mathcal{E}$ such that $C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P)) \gneqq C_{Z(P)}(\mathrm{Aut}_S(P))$. Then $[\alpha] = 1$ in $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ if and only if there is $g \in C_{Z(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$ such that $g_P \in g{\cdot}C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P \in \mathcal{E}_0$.*

(e) *Let $\mathcal{E}_0$ be as in* (d)*, and let $\widehat{\mathcal{E}}_0$ be the set of all $P \in \mathcal{E}_0$ such that $P = C_S(E)$ for some elementary abelian p-subgroup $E \leq S$ which is fully centralized in $\mathcal{F}$. Let $\mathcal{H}$ be a set of subgroups of S such that all subgroups in $\mathcal{H}$ are $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and each $P \in \widehat{\mathcal{E}}_0$ is $\mathcal{F}$-conjugate to some $Q \in \widehat{\mathcal{E}}_0 \cap \mathcal{H}$. Then $[\alpha] = 1$ in $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ if and only if there is $g \in C_{Z(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$ such that $g_P \in g{\cdot}C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P \in \mathcal{H}$.*

Proof. We identify S with $\delta_S(S) \leq \mathrm{Aut}_{\mathcal{L}}(S)$ for short. Fix $\alpha \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ such that $[\alpha] \in \mathrm{Ker}(\mu_{\mathcal{L}})$. Set $\beta = \widetilde{\mu}_{\mathcal{L}}(\alpha)$; thus $\beta \in \mathrm{Aut}_{\mathcal{F}}(S)$. Choose $\zeta \in \mathrm{Aut}_{\mathcal{L}}(S)$ such that $\pi(\zeta) = \beta$. Then $\widetilde{\mu}_{\mathcal{L}}(c_\zeta) = \beta$ by axiom (C) for the linking system $\mathcal{L}$, and so upon replacing α by $\alpha \circ {c_\zeta}^{-1}$, we can arrange that α_S is the identity on $\delta_S(S) \trianglelefteq \mathrm{Aut}_{\mathcal{L}}(S)$. We will show in the proof of (a) how to arrange that $\alpha_S = \mathrm{Id}_{\mathrm{Aut}_{\mathcal{L}}(S)}$.

(a) Fix a fully normalized subgroup $P \in \mathrm{Ob}(\mathcal{L}^c)$. Set $\Gamma = \mathrm{Aut}_{\mathcal{L}}(P)$ for short, and identify P with $\delta_P(P) \trianglelefteq \Gamma$. Set $\mathrm{Out}(\Gamma, P) = \mathrm{Aut}(\Gamma, P)/\mathrm{Inn}(\Gamma)$, where $\mathrm{Aut}(\Gamma, P) \leq \mathrm{Aut}(\Gamma)$ is the subgroup of automorphisms leaving P invariant. By [OV2, Lemma 1.2], there is an exact sequence

$$1 \longrightarrow H^1(\Gamma/P; Z(P)) \xrightarrow{\ \eta\ } \mathrm{Out}(\Gamma, P) \xrightarrow{\ R\ } N_{\mathrm{Out}(P)}(\mathrm{Out}_\Gamma(P))/\mathrm{Out}_\Gamma(P),$$

where R is induced by restriction. Since $\alpha_P \in \mathrm{Aut}(\Gamma)$ and $\alpha_P|_{\delta_P(N_S(P))} = \mathrm{Id}$, $[\alpha_P] \in \mathrm{Ker}(R)$, and $\eta^{-1}([\alpha_P])$ is trivial after restriction to $H^1(N_S(P)/P; Z(P))$. The restriction map from $H^1(\Gamma/P; Z(P))$ to $H^1(N_S(P)/P; Z(P))$ is injective since $\delta_P(N_S(P)) \in \mathrm{Syl}_p(\Gamma)$ (Proposition 1.11(d)), and hence $[\alpha_P] = 1$. Thus $\alpha_P = c_{\delta_P(g_P)}$ for some $g_P \in Z(P)$ which is uniquely determined modulo $C_{Z(P)}(\Gamma) = C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$. Also, $g_P \in C_{Z(P)}(\mathrm{Aut}_S(P))$, since α_P is the identity on $\delta_P(N_S(P))$.

Set $\gamma = \delta_S(g_S) \in \mathrm{Aut}_{\mathcal{L}}(S)$. Upon replacing α by $\alpha \circ c_\gamma^{-1}$, we can arrange that $\alpha_S = \mathrm{Id}$.

(b) Assume $\zeta \in \mathrm{Iso}_{\mathcal{L}}(P, Q)$ and $\alpha(\zeta) = \zeta$. Fix $\psi \in \mathrm{Aut}_{\mathcal{L}}(Q)$, and set $\varphi = \zeta^{-1}\psi\zeta \in \mathrm{Aut}_{\mathcal{L}}(P)$. Set $g = \pi(\zeta)(g_P)$; then $\zeta \circ \delta_P(g_P) \circ \zeta^{-1} = \delta_Q(g)$ by axiom (C) for a linking system. Hence

$$\alpha_Q(\psi) = \alpha_Q(\zeta\varphi\zeta^{-1}) = \zeta\alpha_P(\varphi)\zeta^{-1} = \zeta\delta_P(g_P)\varphi\delta_P(g_P)^{-1}\zeta^{-1} = \delta_Q(g)\psi\delta_Q(g)^{-1} ,$$

and we can choose $g_Q = g$.

If $Q = aPa^{-1}$ and $\zeta = \delta_{P,Q}(a)$, then $\alpha(\zeta) = \zeta$ since $\alpha_S = \mathrm{Id}$ (and since α sends inclusions to inclusions). So again we can choose $g_Q = c_a(g_P)$.

In either case, $g_P \in C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$ if and only if $g_Q \in C_{Z(Q)}(\mathrm{Aut}_{\mathcal{F}}(Q))$, and hence $\alpha_P = \mathrm{Id}$ if and only if $\alpha_Q = \mathrm{Id}$.

(c) Assume $Q \leq P$, and let $\mathrm{Aut}_{\mathcal{L}}(P, Q)$ be the group of elements $\psi \in \mathrm{Aut}_{\mathcal{L}}(P)$ such that $\pi(\psi)(Q) = Q$. Then α commutes with the restriction map

$$\mathrm{Res}^P_Q \colon \mathrm{Aut}_{\mathcal{L}}(P, Q) \longrightarrow \mathrm{Aut}_{\mathcal{L}}(Q)$$

which is injective by Proposition 1.11(f). So if α acts on $\mathrm{Aut}_{\mathcal{L}}(P, Q)$ via conjugation by $\delta_P(g_P)$ and on $\mathrm{Aut}_{\mathcal{L}}(Q)$ via conjugation by $\delta_Q(g_Q)$, they must have the same action on $\mathrm{Aut}_{\mathcal{L}}(P, Q)$. Since g_Q and g_P both lie in $Z(Q) \geq Z(P)$, we conclude $g_Q \equiv g_P$ (mod $C_{Z(Q)}(\mathrm{Aut}_{\mathcal{F}}(P, Q))$).

(d) By Theorem 4.1, all morphisms in $\mathcal{L}$ are composites of restrictions of elements in $\mathrm{Aut}_{\mathcal{L}}(P)$ for P $\mathcal{F}$-essential or $P = S$. Hence if $\alpha \neq \mathrm{Id}_{\mathcal{L}}$, then since $\alpha_S = \mathrm{Id}$ by assumption, $\alpha_P \neq \mathrm{Id}$ for some $P \in \mathcal{E}$. By (a), $g_P \in C_{Z(P)}(\mathrm{Aut}_S(P))$ but $g_P \notin C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$, and so

$P \in \mathcal{E}_0$. The converse is clear: if $\alpha = \mathrm{Id}_{\mathcal{L}}$, then $\alpha_P = \mathrm{Id}$ and hence $g_P \in C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P \in \mathcal{E}_0$.

By Lemma 1.14(a), $[\alpha] = 1$ in $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ if and only if $\alpha = c_\beta$ for some $\beta \in \mathrm{Aut}_{\mathcal{L}}(S)$, and $\beta \in Z(\mathrm{Aut}_{\mathcal{L}}(S))$ since $\alpha_S = \mathrm{Id}$. Since $\beta\delta_S(g) = \delta_S(\pi(\beta)(g))\beta$ for each $g \in S$ by axiom (C) in Definition 1.9, $\pi(\beta) = \mathrm{Id}_S$. Hence $\beta = \delta_S(g)$ for some $g \in Z(S)$ by axiom (A), and $g \in C_{Z(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$ by axiom (C) again. Thus $[\alpha] = 1$ if and only if $\alpha = c_{\delta_S(g)}$ for some $g \in C_{Z(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$, which we just saw is the case exactly when $g^{-1}g_P \in C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P \in \mathcal{E}_0$.

(e) We first prove that

$$\alpha = \mathrm{Id}_{\mathcal{L}} \iff g_P \in C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P)) \text{ for all } P \in \mathcal{H}. \tag{1}$$

The first statement implies the second by (a).

Now assume $\alpha \neq \mathrm{Id}_{\mathcal{L}}$. As was just seen in the proof of (d), there is $P \in \mathcal{E}_0$ such that $g_P \notin C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$. Assume P is such that $|P|$ is maximal among orders of all such subgroups. We will show that $P \in \widehat{\mathcal{E}_0}$ (possibly after replacing P by another subgroup in its $\mathcal{F}$-conjugacy class), and that $g_Q \notin C_{Z(Q)}(\mathrm{Aut}_{\mathcal{F}}(Q))$ for each $Q \in \widehat{\mathcal{E}_0}$ which is $\mathcal{F}$-conjugate to P. In particular, $g_Q \notin C_{Z(Q)}(\mathrm{Aut}_{\mathcal{F}}(Q))$ for some $Q \in \mathcal{H}$, which will prove the remaining implication in (1).

We first check that

$$T \in \mathrm{Ob}(\mathcal{L}) \text{ and } |T| > |P| \quad\Longrightarrow\quad \alpha_T = \mathrm{Id}\ . \tag{2}$$

If $T = S$ or $T \in \mathcal{E}_0$, this follows by assumption. If $T \in \mathcal{E}\smallsetminus\mathcal{E}_0$, then $g_T \in C_{Z(T)}(\mathrm{Aut}_S(T)) = C_{Z(T)}(\mathrm{Aut}_{\mathcal{F}}(T))$, and hence $\alpha_T = \mathrm{Id}$ by definition of g_T. Otherwise, each $\psi \in \mathrm{Aut}_{\mathcal{L}}(T)$ is a composite of restrictions of automorphisms of subgroups in $\mathcal{E} \cup \{S\}$ (Theorem 4.1), each of those automorphisms and its restrictions are sent to themselves by α, and hence $\alpha_T(\psi) = \psi$.

We next claim that

$$\text{for all } Q\ \mathcal{F}\text{-conjugate to } P, \text{ there is } \zeta \in \mathrm{Iso}_{\mathcal{L}}(P,Q) \text{ such that } \alpha_{P,Q}(\zeta) = \zeta. \tag{3}$$

Choose any $\zeta_0 \in \mathrm{Iso}_{\mathcal{L}}(P,Q)$. By Theorem 4.1 again, ζ_0 is the composite of restrictions of automorphisms $\psi_i \in \mathrm{Aut}_{\mathcal{L}}(R_i)$ for subgroups $R_i \leq S$ with $|R_i| \geq |P|$. If we remove from this composite all ψ_i for which $|R_i| = |P|$, we get an isomorphism $\zeta \in \mathrm{Iso}_{\mathcal{L}}(P,Q)$ which is a composite of restrictions of automorphisms of strictly larger subgroups. We just showed that $\alpha_{R_i}(\psi_i) = \psi_i$ whenever $|R_i| > |P|$, and thus $\alpha_{P,Q}(\zeta) = \zeta$.

Set $E = \Omega_1(Z(P))$: the p-torsion subgroup of the center $Z(P)$. If E is not fully normalized in $\mathcal{F}$, then choose $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(E), S)$ such that $\varphi(E)$ is fully normalized (using [BLO2, Proposition A.2(b)]). Then $\varphi(P)$ is fully normalized since $N_S(\varphi(P)) \geq \varphi(N_S(P))$. By (3), there is $\zeta \in \mathrm{Iso}_{\mathcal{L}}(P, \varphi(P))$ such that $\alpha_{P,\varphi(P)}(\zeta) = \zeta$. So $\alpha_{\varphi(P)} \neq \mathrm{Id}$ by (b). Upon replacing P by $\varphi(P)$ and E by $\varphi(E)$, we can now assume E and P are both fully normalized.

Set $P^* = N_{C_S(E)}(P) \geq P$ and $\Gamma = \mathrm{Aut}_{\mathcal{L}}(P)$ for short. To simplify notation, we identify $N_S(P)$ with $\delta_P(N_S(P))$. Then $E \trianglelefteq \Gamma$, so $C_\Gamma(E) \trianglelefteq \Gamma$; and $P^* \in \mathrm{Syl}_p(C_\Gamma(E))$ since $N_S(P) \in \mathrm{Syl}_p(\Gamma)$ (Proposition 1.11(d)). Also, $C_\Gamma(Z(P)) \trianglelefteq \Gamma$, and has p-power index in $C_\Gamma(E)$ since each automorphism of $Z(P)$ which is the identity on its p-torsion subgroup E has p-power order (cf. [G, Theorem 5.2.4]). Hence each Sylow p-subgroup of $C_\Gamma(E)$ is $C_\Gamma(Z(P))$-conjugate to $P^* = C_{N_S(P)}(E)$. By the Frattini argument,

$$\Gamma = N_\Gamma(P^*){\cdot}C_\Gamma(Z(P))\ . \tag{4}$$

Since $\alpha_P \neq \mathrm{Id}_\Gamma$ is conjugation by $g_P \in Z(P)$, α_P is the identity on $C_\Gamma(Z(P))$. Hence by (4), α_P is not the identity on $N_\Gamma(P^*)$. By Proposition 1.11(e), each $\alpha \in N_\Gamma(P^*)$ extends to $\bar{\alpha} \in \mathrm{Aut}_{\mathcal{L}}(P^*)$, and thus $\alpha_{P^*} \neq \mathrm{Id}_{\mathrm{Aut}_{\mathcal{L}}(P^*)}$. If $C_S(E) \gneq P$, then $P^* = N_{C_S(E)}(P) \gneq P$ (cf. [Sz1, Theorem 2.1.6]), which would imply $\alpha_{P^*} = \mathrm{Id}$ by (2). We now conclude that $C_S(E) = P$, and hence that $P \in \widehat{\mathcal{E}}_0$.

Assume $Q \in \widehat{\mathcal{E}}_0$ is $\mathcal{F}$-conjugate to P. By (3), there is $\zeta \in \mathrm{Iso}_{\mathcal{L}}(P, Q)$ such that $\alpha(\zeta) = \zeta$. So by (b), $\alpha_Q \neq \mathrm{Id}$ since $\alpha_P \neq \mathrm{Id}$, and this finishes the proof of (1).

The rest of the proof of (e) is identical to that of (d). □

As one simple application of Proposition 4.2, consider the group $G = A_6 \cong PSL_2(9)$ (cf. [H1, Satz II.8.14]). Set

$$T_1 = \langle (1\,2)(3\,4), (1\,3)(2\,4) \rangle \cong C_2^2 , \qquad T_2 = \langle (1\,2)(3\,4), (3\,4)(5\,6) \rangle \cong C_2^2 ,$$

and $S = \langle T_1, T_2 \rangle \in \mathrm{Syl}_2(G)$, and let $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{L} = \mathcal{L}_S^c(G)$. Then $\mathcal{E} = \mathcal{E}_0 = \widehat{\mathcal{E}}_0 = \{T_1, T_2\}$. Set $g = (5\,6)$, and consider the automorphism $\alpha = \widetilde{\kappa}_G(c_g) \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L})$. Then $\alpha \in \mathrm{Ker}(\widetilde{\mu}_G)$, since $[g, S] = 1$ (and since $\widetilde{\mu}_G \circ \widetilde{\kappa}_G$ sends $\beta \in \mathrm{Aut}(G, S)$ to $\beta|_S$). Since $[g, N_G(T_1)] = 1$, $\alpha_{T_1} = \mathrm{Id}_{\mathrm{Aut}_{\mathcal{L}}(T_1)}$. Since $(1\,2)(3\,4)(5\,6)$ commutes with $N_G(T_2) = \langle T_2, (1\,3)(2\,4), (1\,3\,5)(2\,4\,6) \rangle$, α_{T_2} acts on $\mathrm{Aut}_{\mathcal{L}}(T_2) \cong N_G(T_2)$ as conjugation by $x = (1\,2)(3\,4) \in Z(S)$. So in the notation of Proposition 4.2, $g_{T_1} = 1$ and $g_{T_2} = x$. In both cases, $C_{Z(T_i)}(\mathrm{Aut}_{\mathcal{F}}(T_i)) = 1$, so the g_{T_i} are uniquely determined. Hence by Proposition 4.2(d), $[\alpha] = \kappa_G([c_g])$ represents a nontrivial element in $\mathrm{Ker}(\mu_G)$.

If $[\alpha] \in \mathrm{Ker}(\mu_G)$ is arbitrary, represented by $\alpha \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L})$ such that $\alpha_S = \mathrm{Id}_{\mathrm{Aut}_{\mathcal{L}}(S)}$, then by Proposition 4.2 again, $g_{T_i} \in Z(S)$ for $i = 1, 2$, and $[\alpha] = 1$ if and only if $g_{T_1} = g_{T_2}$. Thus $\mathrm{Ker}(\mu_G) \cong C_2$ is generated by $\kappa_G([c_g])$ as described above. Using this, and the well known description of $\mathrm{Out}(A_6) \cong C_2^2$ (see [Sz1, Theorem 3.2.19(iii)]), it is not hard to see that κ_G is an isomorphism from $\mathrm{Out}(G)$ to $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$.

This example will be generalized in two different ways below: to other groups $PSL_2(q)$ for $q \equiv \pm 1 \pmod 8$ in Proposition 4.3, and to other alternating groups in Proposition 4.8.

4.1. **Dihedral and semidihedral 2-groups.**

As our first examples, we list all reduced fusion systems over dihedral and semidihedral 2-groups, and prove they are all tame. The list of all fusion systems over such groups is well known; it turns out that each of them supports exactly one fusion system which is reduced.

As usual, $v_p(-)$ denotes the p-adic valuation: $v_p(n) = k$ if $p^k | n$ but $p^{k+1} \nmid n$.

Proposition 4.3. *Let S be a dihedral group of order 2^k ($k \geq 3$). Then there is a unique reduced fusion system $\mathcal{F}$ over S, and it is tame. Let q be a prime power such that $v_2(q^2-1) = k+1$, set $G = PSL_2(q)$, and fix $S^* \in \mathrm{Syl}_2(G)$. Then $S \cong S^*$ and $\mathcal{F} \cong \mathcal{F}_{S^*}(G)$; and κ_G is an isomorphism if $q = p^{2^{k-2}}$ for some prime $p \equiv 5 \pmod 8$.*

Proof. Fix $a, b \in S$ such that $\langle a \rangle$ has index two and $S = \langle a, b \rangle$. For each $i \in \mathbb{Z}$, set $T_i = \langle a^{2^{k-2}}, a^i b \rangle \cong C_2^2$. Two subgroups T_i and T_j are S-conjugate if and only if $i \equiv j \pmod 2$. Set $\mathcal{P} = \{T_i \,|\, i \in \mathbb{Z}\}$.

If $P \leq S$ is cyclic of order 2^m, then $\mathrm{Aut}(P) \cong (\mathbb{Z}/2^m)^\times$ is a 2-group. If $P \leq S$ is dihedral of order $2^m \geq 8$, then there is a unique cyclic subgroup of index two in P, and $\mathrm{Aut}(P)$ is a 2-group by Lemma 1.6. Thus the only subgroups $P \leq S$ for which $\mathrm{Aut}(P)$ is not a 2-group are the T_i.

Define $\mathcal{F}$ to be the fusion system over S generated by the automorphisms in $\mathrm{Inn}(S)$, $\mathrm{Aut}(P)$ for $P \in \mathcal{P}$, and their restrictions. Assume $\mathcal{F}$ is saturated (this will be shown later). Then $\mathfrak{foc}(\mathcal{F}) = \langle [S,S], \mathcal{P} \rangle = S$, and hence $O^2(\mathcal{F}) = \mathcal{F}$ (Theorem 1.22(a)). Also, $O^{2'}(\mathcal{F}) = \mathcal{F}$

since any normal subsystem of odd index would have to contain the same automorphism groups, and $O_2(\mathcal{F}) = 1$ by inspection. Thus $\mathcal{F}$ is reduced.

Let $\mathcal{F}^*$ be an arbitrary saturated fusion system over S such that $\mathfrak{foc}(\mathcal{F}^*) = S$. Let $\mathcal{E}$ be the set of all $\mathcal{F}^*$-essential subgroups of S. If $P \in \mathcal{E}$, then $\mathrm{Aut}(P)$ must have elements of odd order, and hence $P \in \mathcal{P}$. For each $T_i \in \mathcal{P}$, $\mathrm{Aut}(T_i) \cong \Sigma_3$ and $\mathrm{Aut}_S(T_i) \cong C_2$. Hence $\mathrm{Aut}_{\mathcal{F}^*}(T_i) = \mathrm{Aut}(T_i)$ if $T_i \in \mathcal{E}$. Since $\mathrm{Aut}(S)$ is a 2-group, Theorem 4.1 implies $\mathcal{F}^*$ is generated by automorphisms in $\mathrm{Aut}_{\mathcal{F}^*}(S) = \mathrm{Inn}(S)$, the $\mathrm{Aut}(P)$ for $P \in \mathcal{E} \subseteq \mathcal{P}$, and their restrictions. In particular, $\mathfrak{foc}(\mathcal{F}^*) \le \langle [S, S], \mathcal{E}\rangle$, and this has index at least two in S if $\mathcal{E} \subsetneqq \mathcal{P}$. Hence $\mathcal{E} = \mathcal{P}$, and so $\mathcal{F}^* = \mathcal{F}$.

Set $G = PSL_2(q)$ for any prime power $q \equiv \pm 1 \pmod 8$, and fix $S^* \in \mathrm{Syl}_2(G)$. As is well known (cf. [G, Lemma 15.1.1(iii)]), S^* is a dihedral group and $|G| = \frac{1}{2}q(q^2 - 1)$, so $S^* \cong D_{2^k}$ where $k = v_2(q^2 - 1) - 1$. So we identify $S^* = S$ for S as above. Since G is simple, $\mathfrak{foc}(\mathcal{F}_S(G)) = S \cap [G, G] = S$ by the focal subgroup theorem (cf. [G, Theorem 7.3.4]), and we have just seen this implies $\mathcal{F}_S(G) = \mathcal{F}$. In particular, $\mathcal{F}$ is saturated, and hence reduced.

Now assume $q = p^{2^{k-2}}$, where $p \equiv 5 \pmod 8$ (and $k \ge 3$). The homomorphism κ_G is an isomorphism in this case by [BLO1, Proposition 7.9], where it is shown more generally for $p \equiv \pm 3 \pmod 8$. But we give a different proof here to illustrate how Proposition 4.2 can be applied.

Set $\widetilde{G} = SL_2(q)$. Fix $u \in \mathbb{F}_q^\times$ of order 2^k. Set $\widetilde{a} = \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right)$ and $\widetilde{b} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and let $a, b \in G$ be their images in the quotient. Then $S \stackrel{\text{def}}{=} \langle a, b\rangle \in \mathrm{Syl}_2(G)$. Let $\delta \in \mathrm{Aut}(G)$ be conjugation by $\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$; then $\delta(a) = a$ and $\delta(b) = ab$. Since u is not a square in $\mathbb{F}_q^\times$, $[\delta]$ generates the subgroup (of order 2) of diagonal automorphisms in $\mathrm{Out}(G)$.

By [St, § 3], $\mathrm{Out}(G) = \langle[\delta]\rangle \times \langle[\psi^p]\rangle \cong C_2 \times C_{2^{k-2}}$, where ψ^p is the field automorphism which acts via $x \mapsto x^p$ on matrix elements. Also, $\psi^p(a) = a^p$ and $\psi^p(b) = b$. Since $p \equiv 5 \pmod 8$, $[\delta|_S]$ and $[\psi^p|_S]$ generate $\mathrm{Out}(S)$. Thus

$$\mu_G \circ \kappa_G \colon \mathrm{Out}(G) \longrightarrow \mathrm{Out}(S, \mathcal{F}) = \mathrm{Out}(S)$$

is surjective with kernel generated by $[\alpha]$, where $\alpha = (\psi^p)^{2^{k-3}}$ is the field automorphism of order 2.

To prove that κ_G is an isomorphism, it remains to show that $\mathrm{Ker}(\mu_G)$ has order 2 and is generated by $\kappa_G([\alpha])$. Set $w = a^{2^{k-2}} \in Z(S)$. We refer to Proposition 4.2. Since α is the identity on $S = N_G(S)$ ($\alpha(a) = a$ since the field automorphism of order two sends u to $-u$), there are elements $g_{T_i} \in C_{Z(T_i)}(\mathrm{Aut}_S(T_i)) = \langle w\rangle$ for each i such that $\widetilde{\kappa}_G(\alpha)$ acts on $\mathrm{Aut}_{\mathcal{L}}(T_i)$ via conjugation by g_{T_i}. These elements are uniquely defined since $C_{Z(T_i)}(\mathrm{Aut}_{\mathcal{F}}(T_i)) = 1$.

When i is even, $T_i \le G_0 \stackrel{\text{def}}{=} PSL_2(\sqrt{q})$ (recall $T_i = \langle a^{2^{k-2}}, a^i b\rangle$), and $N_{G_0}(T_i)$ has index at most two in $N_G(T_i)$. Since $\alpha|_{G_0} = \mathrm{Id}$ and $\alpha|_S = \mathrm{Id}$, α is the identity on $N_G(T_i) \cong \Sigma_4$ in this case, and so $g_{T_i} = 1$.

Now consider T_i for odd i. Let $\widetilde{T}_i \cong Q_8$ be the inverse image in $\widetilde{G}$ of $T_i \le G$, let $\widetilde{w}$ be any lifting of w to $\widetilde{G}$, and set $z = \widetilde{w}^2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in Z(\widetilde{G})$. Since the field automorphism of order two sends u to $-u$, it sends $\widetilde{a}^i\widetilde{b}$ to $z\widetilde{a}^i\widetilde{b}$. If α acted on $N_G(T_i) \cong \Sigma_4$ via the identity, then its action on $N_{\widetilde{G}}(\widetilde{T}_i)$ would be the identity on a subgroup of index two, which necessarily would include $\widetilde{T}_i$. Since this is not the case, we conclude that α acts via conjugation by w, and thus that $g_{T_i} = w$ for i odd.

By Proposition 4.2(d), since $g_{T_0} = 1$ and $g_{T_1} = w$ (and $C_{Z(T_i)}(\mathrm{Aut}_{\mathcal{F}}(T_i)) = 1$), $\kappa_G([\alpha]) \ne 1$ in $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$, and it is the only nontrivial element in $\mathrm{Ker}(\mu_G)$. Thus $\mathrm{Ker}(\mu_G) \cong C_2$, which is what was left to prove. □

We now consider the semidihedral case.

Proposition 4.4. *Let S be a semidihedral group of order 2^k ($k \geq 4$). Then there is a unique reduced fusion system $\mathcal{F}$ over S, and it is tame. Let q be a prime power such that $v_2(q-1) = k-2$, set $G = PSU_3(q)$, and fix $S^* \in \mathrm{Syl}_2(G)$. Then $S \cong S^*$ and $\mathcal{F} \cong \mathcal{F}_{S^*}(G)$, and κ_G is an isomorphism if $3 \nmid (q+1)$ and $q = p^{2^{k-4}}$ for some prime $p \equiv 5$ (mod 8).*

Proof. Fix $a, b \in S$ such that $\langle a \rangle$ has index two, $b^2 = 1$, and $S = \langle a, b \rangle$. Then $|a^i b| = 2$ for i even and $|a^i b| = 4$ for i odd. For each $i \in \mathbb{Z}$, set $T_i = \langle a^{2^{k-2}}, a^{2i}b \rangle \cong C_2^2$, and $R_i = \langle a^{2^{k-3}}, a^{2i+1}b \rangle \cong Q_8$. The T_i are all S-conjugate to each other, and similarly for the R_i. Set $\mathcal{P} = \{T_i, R_i \,|\, i \in \mathbb{Z}\}$.

As shown in the proof of Proposition 4.3, $\mathrm{Aut}(P)$ is a 2-group for each $P \leq S$ which is cyclic, or dihedral of order ≥ 8. The same argument applies when P is quaternion of order ≥ 16, and also to S itself. Thus the only subgroups $P \leq S$ for which $\mathrm{Aut}(P)$ is not a 2-group are those in $\mathcal{P}$.

Define $\mathcal{F}$ to be the fusion system over S generated by the automorphisms in $\mathrm{Inn}(S)$, $\mathrm{Aut}(P)$ for $P \in \mathcal{P}$, and their restrictions. Assume $\mathcal{F}$ is saturated (to be shown later). Then $\mathfrak{foc}(\mathcal{F}) = \langle [S,S], \mathcal{P} \rangle = S$, and hence $O^2(\mathcal{F}) = \mathcal{F}$ (Theorem 1.22(a)). Also, $O^{2'}(\mathcal{F}) = \mathcal{F}$ since any normal subsystem of odd index would have to contain the same automorphism groups, and $O_2(\mathcal{F}) = 1$ by inspection. Thus $\mathcal{F}$ is reduced.

Let $\mathcal{F}^*$ be an arbitrary saturated fusion system over S such that $\mathfrak{foc}(\mathcal{F}^*) = S$. Let $\mathcal{E}$ be the set of all $\mathcal{F}^*$-essential subgroups of S. If $P \in \mathcal{E}$, then $\mathrm{Aut}(P)$ must have elements of odd order, and hence $P \in \mathcal{P}$. For all $P \in \mathcal{P}$, $[\mathrm{Aut}(P){:}\mathrm{Aut}_S(P)] = 3$, and hence $\mathrm{Aut}_{\mathcal{F}^*}(P) = \mathrm{Aut}(P)$ if $P \in \mathcal{E}$. Since $\mathrm{Aut}(S)$ is a 2-group, Theorem 4.1 implies $\mathcal{F}^*$ is generated by automorphisms in $\mathrm{Aut}_{\mathcal{F}^*}(S) = \mathrm{Inn}(S)$, the $\mathrm{Aut}(P)$ for $P \in \mathcal{E}$, and their restrictions. In particular, $\mathfrak{foc}(\mathcal{F}^*) \leq \langle [S,S], \mathcal{E} \rangle$, and this has index at least two in S if $\mathcal{E} \subsetneqq \mathcal{P}$. Hence $\mathcal{E} = \mathcal{P}$, and so $\mathcal{F}^* = \mathcal{F}$.

Fix a prime power $q \equiv 1$ (mod 4), set $G = PSU_3(q)$, and fix $S^* \in \mathrm{Syl}_2(G)$. Then $|G| = \frac{1}{d} q^3(q^2-1)(q^3+1)$ where $d = \gcd(3, q+1)$ [Ta, p. 118], and hence $|S^*| = 2^k$ where $k = v_2(q-1) + 2$. Since $GU_2(q)$ has odd index in $SU_3(q)$, and the Sylow 2-subgroups of $GU_2(q)$ are semidihedral by [CF, p.143], the Sylow 2-subgroups of $SU_3(q)$ and of G are also semidihedral. Thus $S^* \cong SD_{2^k}$, and we identify $S^* = S$ as above. Since G is simple, $\mathfrak{foc}(\mathcal{F}_S(G)) = S$ (cf. [G, Theorem 7.3.4]), and we just saw this implies $\mathcal{F}_S(G) = \mathcal{F}$. In particular, $\mathcal{F}$ is saturated.

Now assume $3 \nmid (q+1)$ and $q = p^{2^{k-4}}$ for some prime $p \equiv 5$ (mod 8). By [St, § 3], $\mathrm{Out}(G)$ is generated by diagonal and field automorphisms; where the group of diagonal automorphisms has order $\gcd(3, q+1) = 1$. Thus $\mathrm{Out}(G) = \langle [\psi^p] \rangle$, generated by the class of the field automorphism $(x \mapsto x^p)$. Since $G = PSU_3(q)$ is defined via matrices over $\mathbb{F}_{q^2}$, ψ^p has order 2^{k-3}.

More explicitly, regard $G = PSU_3(q) = SU_3(q)$ as the group of matrices $M \in SL_3(q^2)$ such that $\psi^q(M^t) = M^{-1}$, where M^t is the transpose $(a_{ij}) \mapsto (a_{4-j,4-i})$. Fix $u \in \mathbb{F}_{q^2}^\times$ of order 2^{k-1} (recall $v_2(q-1) = k-2$), and set $a = \mathrm{diag}(u, -1, u^{-q})$. Since $u^{q-1} = -1$, $a \in SU_3(q)$. Set $b = \left(\begin{smallmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \end{smallmatrix}\right)$. Then $bab^{-1} = a^{-q} = a^{2^{k-2}-1}$, and so $S = \langle a, b \rangle$ is semidihedral. Also, $\psi^p(a) = a^p$, $\psi^p(b) = b$, so $[\psi^p|_S]$ generates $\mathrm{Out}(S)$, and we conclude that

$$\mu_G \circ \kappa_G \colon \mathrm{Out}(G) \xrightarrow{\quad \cong \quad} \mathrm{Out}(S, \mathcal{F}) = \mathrm{Out}(S)$$

is an isomorphism.

It remains to prove that $\mathrm{Ker}(\mu_G) = 1$. Fix $[\alpha] \in \mathrm{Ker}(\mu_G)$, and choose a representative $\alpha \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}^c_S(G))$ for the class $[\alpha]$ such that α_S is the identity on $\mathrm{Aut}_{\mathcal{L}^c_S(G)}(S)$. In the notation of Proposition 4.2, $\mathcal{E}_0$ contains only the subgroups T_i, since $Z(R_i) \cong C_2$ (and hence $C_{Z(R_i)}(\mathrm{Aut}_S(R_i)) = C_{Z(R_i)}(\mathrm{Aut}_{\mathcal{F}}(R_i))$). If α is represented by elements g_P, then $g_{T_i} \in C_{Z(T_i)}(\mathrm{Aut}_S(T_i)) = Z(S)$ for each i, and is uniquely determined since $C_{Z(T_i)}(\mathrm{Aut}_{\mathcal{F}}(T_i)) = 1$. All of the g_{T_i} are equal by point (b) in the proposition, and hence $[\alpha] = 1$ by point (d). □

4.2. **Tameness of some fusion systems studied in** [OV2]**.**

We next consider some fusion systems studied in [OV2, §4–5], and prove they are reduced and tame using the lists of essential subgroups and other information determined there.

Proposition 4.5. *The fusion systems at the prime* 2 *of the group* $PSL_4(5)$*, and of the sporadic simple groups* M_{22}*,* M_{23}*,* McL*,* J_2*, and* J_3*, are all reduced and tame. Moreover, if* G *is any of these groups, then* κ_G *is an isomorphism.*

Proof. By [GL, §1.5], $\mathrm{Out}(G) \cong C_2$ when $G \cong M_{22}$, McL, J_2, or J_3, while $\mathrm{Out}(M_{23}) = 1$. By [St, (3.2)], when $G = PSL_4(5)$, $\mathrm{Out}(G)$ is generated by diagonal automorphisms (induced by conjugation by diagonal matrices in $GL_4(5)$) and a graph automorphism (induced by transpose inverse). Since all multiples of the identity in $GL_4(5)$ have determinant one, the group of diagonal outer automorphisms is isomorphic to $\mathbb{F}_5^\times \cong C_4$. Since the graph automorphism inverts all diagonal matrices, we get $\mathrm{Out}(G) \cong D_8$.

Now let G be any of the above six groups, fix $S \in \mathrm{Syl}_2(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. We prove below in each case that among the homomorphisms

$$\mathrm{Out}(G) \xrightarrow{\ \kappa_G\ } \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}^c_S(G)) \xrightarrow{\ \mu_G\ } \mathrm{Out}(S,\mathcal{F}) \ ,$$

$\mu_G \circ \kappa_G$ is an isomorphism and μ_G is injective. It then follows that κ_G is an isomorphism.

We show that $\mu_G \circ \kappa_G$ is injective for each of these groups, using arguments suggested to us by Richard Lyons. These are based on the following statement, applied to certain subgroups $H \le G$:

$$\left.\begin{array}{l} \alpha \in \mathrm{Aut}(H),\ S \in \mathrm{Syl}_2(H),\ \alpha|_S = \mathrm{Id}_S \\ Q = O_2(H),\ C_H(Q) \le Q, \end{array}\right\} \implies \alpha \in \mathrm{Aut}_{Z(S)}(H). \tag{5}$$

This follows, for example, from [OV2, Lemma 1.2]: $\alpha \in \mathrm{Inn}(H)$ if a certain element in $H^1(H/Q; Z(Q))$ vanishes, and this element does vanish since its restriction to the Sylow subgroup S/Q vanishes. Thus $\alpha \in \mathrm{Inn}(H)$ and is the identity on S, so it must be conjugation by an element of $C_H(S) = Z(S)$.

As in [OV2], we let $S_0 = UT_3(4)$ denote the group of upper triangular 3×3 matrices over $\mathbb{F}_4$ with 1's on the diagonal. For $x \in \mathbb{F}_4$ and $1 \le i < j \le 3$, $e^x_{ij} \in UT_3(4)$ is the matrix with entry x in position (i,j), 1's on the diagonal, and 0's elsewhere. Set

$$E_{ij} = \{e^x_{ij} \,|\, x \in \mathbb{F}_4\} \ , \quad A_1 = \langle E_{12}, E_{13}\rangle \ , \quad \text{and} \quad A_2 = \langle E_{13}, E_{23}\rangle \ .$$

The field automorphism of $\mathbb{F}_4$ is denoted $x \mapsto \bar{x}$, and we write $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$. Also, $\tau, \rho_1^*, \rho_2^*, \gamma_0, \gamma_1, c_\phi \in \mathrm{Aut}(S_0)$ are the automorphisms

$$\tau\left(\left(\begin{smallmatrix}1&a&b\\0&1&c\\0&0&1\end{smallmatrix}\right)\right) = \left(\begin{smallmatrix}1&c&b\\0&1&a\\0&0&1\end{smallmatrix}\right)^{-1}, \quad \rho_1^*\left(\left(\begin{smallmatrix}1&a&b\\0&1&c\\0&0&1\end{smallmatrix}\right)\right) = \left(\begin{smallmatrix}1&a&b+\bar{a}\\0&1&c\\0&0&1\end{smallmatrix}\right), \quad \rho_2^*\left(\left(\begin{smallmatrix}1&a&b\\0&1&c\\0&0&1\end{smallmatrix}\right)\right) = \left(\begin{smallmatrix}1&a&b+\bar{c}\\0&1&c\\0&0&1\end{smallmatrix}\right),$$

$$\gamma_0\left(\left(\begin{smallmatrix}1&a&b\\0&1&c\\0&0&1\end{smallmatrix}\right)\right) = \left(\begin{smallmatrix}1&\omega a&\bar{\omega}b\\0&1&\omega c\\0&0&1\end{smallmatrix}\right), \quad \gamma_1\left(\left(\begin{smallmatrix}1&a&b\\0&1&c\\0&0&1\end{smallmatrix}\right)\right) = \left(\begin{smallmatrix}1&\omega a&b\\0&1&\bar{\omega}c\\0&0&1\end{smallmatrix}\right), \quad c_\phi\left(\left(\begin{smallmatrix}1&a&b\\0&1&c\\0&0&1\end{smallmatrix}\right)\right) = \left(\begin{smallmatrix}1&\bar{a}&\bar{b}\\0&1&\bar{c}\\0&0&1\end{smallmatrix}\right).$$

The group $\mathrm{Out}(S_0) \cong C_2^4 \rtimes (\Sigma_3 \times \Sigma_3)$ is described precisely by [OV2, Lemma 4.5]. In particular, the subgroups $\langle \gamma_0, c_\phi \circ \tau\rangle$ and $\langle \gamma_1, \tau\rangle$ are isomorphic to Σ_3 and commute with each other.

By the focal subgroup theorem (cf. [G, Theorem 7.3.4]), $\mathfrak{foc}(\mathcal{F}) = S \cap [G,G] = S$ in each case, and hence $O^2(\mathcal{F}) = \mathcal{F}$. In each of Cases 1 and 2 below, we prove successively that (i) $\mu_G \circ \kappa_G$ is an isomorphism, (ii) μ_G is injective, (iii) $O_2(\mathcal{F}) = 1$, and (iv) $O^{2'}(\mathcal{F}) = \mathcal{F}$.

Case 1: Assume first that $S = S_\phi = S_0 \rtimes \langle \phi \rangle$: the extension of $UT_3(4)$ by a field automorphism of $\mathbb{F}_4$. Then $S_0 = \langle A_1, A_2 \rangle$ is characteristic in S, since A_1 and A_2 are the unique subgroups of S isomorphic to C_2^4 (cf. [OV2, Lemma 5.1(b)]). Since c_ϕ permutes freely a basis of $Z(S_0) = E_{13}$, [OV2, Corollary 1.3 & Lemma 4.5(a)] imply there is an isomorphism

$$\mathrm{Out}(S) \xrightarrow[\cong]{\text{Res}} C_{\mathrm{Out}(S_0)}(\langle [c_\phi] \rangle)/\langle [c_\phi] \rangle = \langle [\rho_1^*], [\rho_2^*], [\tau] \rangle \cong D_8 \ .$$

Let $\dot\tau, \dot\rho_1^*, \dot\rho_2^* \in \mathrm{Aut}(S)$ be the extensions of $\tau, \rho_1^*, \rho_2^* \in \mathrm{Aut}(S_0)$ which send ϕ to itself.

Set $H_i = \langle A_i, \phi \rangle$, and $N_i = \langle H_i, e_{12}^1 e_{23}^1 \rangle$. By [OV2, Theorem 5.11] and Table 5.2 in its proof, in all cases, S_0 is $\mathcal{F}$-essential, and for $i = 1,2$ either H_i or N_i is $\mathcal{F}$-essential but not both. Also, $\mathrm{Out}_{\mathcal{F}}(S) = 1$ (since $\mathrm{Out}(S)$ is a 2-group), and $\mathrm{Out}_{\mathcal{F}}(S_0) = \langle [\gamma_0], [c_\phi] \rangle$ or $\langle [\gamma_0], [\gamma_1], [c_\phi] \rangle$. By [OV2, Lemma 5.8], there is a unique possibility for $\mathrm{Out}_{\mathcal{F}}(N_i)$ if N_i is essential, and hence this group is normalized by $[\dot\rho_1^*]$ and $[\dot\rho_2^*]$. By [OV2, Lemma 5.7], there are two possibilities for $\mathrm{Out}_{\mathcal{F}}(H_i)$ (if H_i is essential) which are exchanged under conjugation by $[\dot\rho_i^*]$ and invariant under conjugation by $[\dot\rho_{3-i}^*]$.

By inspection, for $i = 1,2$, $[\rho_i^*, \gamma_0] = 1$ but $[\rho_i^*, \gamma_1] \neq 1$. Together with the above observations about the action of $\dot\rho_i^*$ on the possibilities for $\mathrm{Out}_{\mathcal{F}}(H_i)$ and $\mathrm{Out}_{\mathcal{F}}(N_i)$, this shows that $\dot\rho_i^*$ is fusion preserving (contained in $\mathrm{Aut}(S,\mathcal{F})$) exactly when N_i is $\mathcal{F}$-essential and $[\gamma_1] \notin \mathrm{Out}_{\mathcal{F}}(S_0)$ (and $\dot\rho_1^* \dot\rho_2^* \in \mathrm{Aut}(S,\mathcal{F})$ only if N_1 and N_2 are both essential). Also, $\dot\tau$ is fusion preserving if either the N_i are both essential or the H_i are both essential (and the $\mathrm{Out}_{\mathcal{F}}(H_i)$ are chosen appropriately in the latter case), and otherwise $\mathrm{Out}(S,\mathcal{F}) \le \langle [\dot\rho_1^*], [\dot\rho_2^*] \rangle$. Thus $\mathrm{Out}(S,\mathcal{F})$ is as described in Table 4.1, where we refer to [OV2, Table 5.2] for the information about the fusion systems.

G	$\mathcal{F}$-essential	$\mathrm{Out}_{\mathcal{F}}(S_0)$	$\mathrm{Out}(S,\mathcal{F})$	$\mathrm{Out}(G)$
M_{22}	S_0, H_1, N_2	$\langle [\gamma_0], [c_\phi] \rangle$	$\langle [\dot\rho_2^*] \rangle \cong C_2$	C_2
M_{23}	S_0, H_1, N_2	$\langle [\gamma_0], [\gamma_1], [c_\phi] \rangle$	1	1
$PSL_4(5)$	S_0, N_1, N_2	$\langle [\gamma_0], [c_\phi] \rangle$	$\mathrm{Out}(S) \cong D_8$	D_8
McL	S_0, N_1, N_2	$\langle [\gamma_0], [\gamma_1], [c_\phi] \rangle$	$\langle [\dot\tau] \rangle \cong C_2$	C_2

TABLE 4.1

(i) Since $|\mathrm{Out}(G)| = |\mathrm{Out}(S,\mathcal{F})|$, it suffices to prove $\mu_G \circ \kappa_G$ is injective. Fix $\alpha \in \mathrm{Aut}(G,S)$ such that $\mu_G(\kappa_G([\alpha])) = 1$; thus $\alpha|_S = c_g$ for some $g \in N_G(S)$. Upon replacing α by $c_g^{-1} \circ \alpha$, we can assume $\alpha|_S = \mathrm{Id}_S$. When G is one of the three sporadic groups, then by [GL, §1.5], A_i is centric in $N_G(A_i)$ ($i = 1,2$) and $G = \langle N_G(A_1), N_G(A_2) \rangle$. When $G \cong PSL_4(5) \cong P\Omega_6^+(5)$, this is easily checked by identifying $S_0 \le P\Omega_6^+(5)$ as the subgroup generated by classes of diagonal matrices (with respect to an orthonormal basis), together with permutation matrices for the permutations $(1\,2)(3\,4)$ and $(3\,4)(5\,6)$. So by (5), there are elements $z_1, z_2 \in Z(S) = \langle e_{13}^1 \rangle$ such that $\alpha|_{N_G(A_i)} = c_{z_i}$ for $i = 1,2$. Let $g \in N_G(S_0)$ be such that $c_g = \gamma_0 \in \mathrm{Aut}_{\mathcal{F}}(S_0)$. Then $g \in N_G(A_i)$ for $i = 1,2$ since γ_0 leaves the A_i invariant, so $\alpha(g) = c_{z_1}(g) = c_{z_2}(g)$, and hence $z_1 = z_2$ since $[g, Z(S)] \neq 1$. Thus $\alpha \in \mathrm{Aut}_{Z(S)}(G)$.

(ii) Set $\mathcal{L} = \mathcal{L}_S^c(G)$. By Proposition 4.2, each element of $\mathrm{Ker}(\mu_G)$ is represented by some $\alpha \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L})$ which is the identity on objects and on $\mathrm{Aut}_{\mathcal{L}}(S)$, and such that for each fully normalized $P \in \mathrm{Ob}(\mathcal{L})$, $\alpha_P \in \mathrm{Aut}(\mathrm{Aut}_{\mathcal{L}}(P))$ is conjugation by some element $g_P \in C_{Z(P)}(\mathrm{Aut}_S(P))$. Since $Z(N_i) \cong C_2$ (so $C_{Z(N_i)}(\mathrm{Aut}_S(N_i)) = C_{Z(N_i)}(\mathrm{Aut}_{\mathcal{F}}(N_i))$), the only $\mathcal{F}$-essential subgroups which could be in the set $\mathcal{E}_0$ defined in Proposition 4.2(d) are S_0, and H_1 and its S-conjugates if they are essential.

When $P = S_0$,

$$g_P \in C_{Z(P)}(\mathrm{Aut}_S(P)) = \langle e_{13}^1 \rangle = Z(S) = C_{Z(S)}(\mathrm{Aut}_{\mathcal{F}}(S)) \ . \qquad (6)$$

So if H_1 is not $\mathcal{F}$-essential, then $[\alpha] = 1$ in $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ by Proposition 4.2(d).

Assume now that H_1 is $\mathcal{F}$-essential. Then H_1 and A_1 are both $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and (6) holds when P is either of these subgroups. By the description of $\mathrm{Aut}_{\mathcal{F}}(S_0)$ and $\mathrm{Aut}_{\mathcal{F}}(H_1)$ in Table 4.1 and [OV2, Lemma 5.7(a)], A_1 is invariant under all $\mathcal{F}$-automorphisms of S_0 and of H_1, and hence $\mathrm{Aut}_{\mathcal{F}}(S_0, A_1) = \mathrm{Aut}_{\mathcal{F}}(S_0)$ and $\mathrm{Aut}_{\mathcal{F}}(H_1, A_1) = \mathrm{Aut}_{\mathcal{F}}(H_1)$. Also, $C_{A_1}(\mathrm{Aut}_{\mathcal{F}}(S_0)) = C_{A_1}(\mathrm{Aut}_{\mathcal{F}}(H_1)) = 1$. Proposition 4.2(c) now implies $g_{H_1} = g_{A_1} = g_{S_0}$. So $[\alpha] = 1$ by Proposition 4.2(d) again; and thus μ_G is injective.

(iii) By [OV2, Table 5.2], for each $i = 1, 2$, $\mathrm{Out}_{\mathcal{F}}(A_i)$ is isomorphic to one of the groups Σ_5, $(C_3 \times A_5) \rtimes C_2$, A_6, or A_7. Hence A_1 and A_2 are $\mathcal{F}$-radical and $\mathcal{F}$-centric, and $O_2(\mathcal{F}) \le A_1 \cap A_2 = E_{13}$ by Proposition 1.5. Since no proper nontrivial subgroup of E_{13} is invariant under the action of $\gamma_0 \in \mathrm{Aut}_{\mathcal{F}}(S_0)$, and E_{13} itself is not invariant under the action of $\nu_2 \in \mathrm{Aut}_{\mathcal{F}}(N_2)$ (see [OV2, Lemma 5.8]), we conclude that $O_2(\mathcal{F}) = 1$.

(iv) Since $\mathrm{Out}_{\mathcal{F}}(S) = 1$ in each of the four cases, condition (ii) in Definition 1.18 implies that $\mathcal{F}$ cannot contain a proper normal subsystem over S. So $O^{2'}(\mathcal{F}) = \mathcal{F}$.

Case 2: Now assume $S = S_\theta = S_0 \rtimes \langle \theta \rangle$, where $c_\theta = \tau \circ c_\phi \in \mathrm{Aut}(S_0)$. Thus $G = J_2$ or J_3. Again in this case, S_0 is characteristic in S (cf. [OV2, Lemma 4.1(d)]). Since c_θ permutes freely a basis of $Z(S_0) = E_{13}$, [OV2, Corollary 1.3] together with the description of $\mathrm{Out}(S_0)$ in [OV2, Lemma 4.5], imply there is an isomorphism

$$\mathrm{Out}(S) \xrightarrow[\cong]{\mathrm{Res}} C_{\mathrm{Out}(S_0)}(\langle [c_\theta] \rangle)/\langle [c_\theta] \rangle \cong \Sigma_4.$$

Set $Q = \langle E_{13}, e_{12}^1 e_{23}^1, e_{12}^\omega e_{23}^{\bar\omega}, \theta \rangle$, an extraspecial group of type $D_8 \times_{C_2} Q_8$ with $\mathrm{Out}(Q) \cong \Sigma_5$. Let $\dot\gamma_1 \in \mathrm{Aut}(S)$ be the extension of $\gamma_1 \in \mathrm{Aut}(S_0)$ which sends θ to itself. By results in [OV2, §4.2–3], $\mathcal{F} = \mathcal{F}_S(G)$ is isomorphic to the fusion system generated by automorphisms

$$\mathrm{Out}_{\mathcal{F}}(S) = \langle [\dot\gamma_1] \rangle, \qquad \mathrm{Out}_{\mathcal{F}}(S_0) = \langle [\gamma_0], [\gamma_1], [c_\theta] \rangle \cong C_3 \times \Sigma_3, \qquad \mathrm{Out}_{\mathcal{F}}(Q) \cong A_5;$$

and by $\mathrm{Out}_{\mathcal{F}}(A_i) \cong GL_2(4)$ if $G = J_3$. Since $\mathrm{Aut}(S, \mathcal{F})/\mathrm{Inn}(S)$ normalizes $\mathrm{Out}_{\mathcal{F}}(S) \cong C_3$, and the normalizer in Σ_4 of a subgroup of order 3 has order 6, $|\mathrm{Out}(S, \mathcal{F})| \le 2$.

(i) In both cases ($G \cong J_2$ or J_3), $\mathrm{Out}(G) \cong C_2$. So to prove $\mu_G \circ \kappa_G$ is an isomorphism, it suffices to show it is injective. Fix $\alpha \in \mathrm{Aut}(G, S)$ such that $\mu_G(\kappa_G([\alpha])) = 1$; as before, we can assume $\alpha|_S = \mathrm{Id}_S$. By [GL, §1.5], $N_G(Z(S))$ and $N_G(E_{13})$ satisfy the hypotheses of (5), and they generate G since both are maximal proper subgroups. By (5), $\alpha|_{N_G(Z(S))} = \mathrm{Id}$, and $\alpha|_{N_G(E_{13})} = c_z$ for some $z \in Z(S)$. Thus $\alpha \in \mathrm{Aut}_{Z(S)}(G)$.

(ii) Set $\mathcal{L} = \mathcal{L}_S^c(G)$. By Proposition 4.2, each element of $\mathrm{Ker}(\mu_G)$ is represented by some $\alpha \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L})$ which is the identity on objects, and such that for each fully normalized $P \in \mathrm{Ob}(\mathcal{L})$, $\alpha_P \in \mathrm{Aut}(\mathrm{Aut}_{\mathcal{L}}(P))$ is conjugation by some element $g_P \in C_{Z(P)}(\mathrm{Aut}_S(P))$. Since $Z(Q) \cong C_2$ (hence $C_{Z(Q)}(\mathrm{Aut}_S(Q)) = C_{Z(Q)}(\mathrm{Aut}_{\mathcal{F}}(Q))$), $\mathcal{E}_0$ contains at most the subgroups S_0, A_1, and A_2. Note that in both cases, A_1 and A_2 are $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$.

In both cases, $\gamma_0 \in \mathrm{Aut}_{\mathcal{F}}(S_0)$ leaves A_1 and A_2 invariant, and acts on each of the groups $Z(S_0) = E_{13}$, A_1, and A_2 with trivial fixed subgroup. Hence

$$C_{Z(A_1)}(\mathrm{Aut}_{\mathcal{F}}(S_0, A_1)) = C_{Z(A_2)}(\mathrm{Aut}_{\mathcal{F}}(S_0, A_2)) = 1 ,$$

and so $g_{A_1} = g_{S_0} = g_{A_2}$ by Proposition 4.2(c). Also, $g_{S_0} \in C_{Z(S_0)}(\mathrm{Aut}_S(S_0)) = \langle e_{13}^1 \rangle = Z(S) = C_{Z(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$, and Proposition 4.2(d) applies (with $g = g_{S_0}$) to show that $[\alpha] = 1$ in $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$. Thus μ_G is injective.

(iii) Since S_0 and Q are $\mathcal{F}$-centric and $\mathcal{F}$-radical, $O_2(\mathcal{F}) \le S_0 \cap Q$. Also, $\mathrm{Aut}_{\mathcal{F}}(Q)$ acts transitively on the set of elements of order four in Q, and on the set of noncentral elements of order two. Since each of those sets contains elements in S_0 and elements not in S_0, this implies $O_2(\mathcal{F}) \le Z(Q) = \langle e_{13}^1 \rangle$. Since $\gamma_0 \in \mathrm{Aut}_{\mathcal{F}}(S_0)$ and $\gamma_0(e_{13}^1) \ne e_{13}^1$, it follows that $O_2(\mathcal{F}) = 1$.

(iv) Set $\mathcal{F}_0 = O^{2'}(\mathcal{F})$. By Lemma 1.20(e), $O_2(\mathcal{F}_0) = 1$ since $O_2(\mathcal{F}) = 1$. So $\mathcal{F}_0$ is a center-free, nonconstrained fusion system over S, and is included in the list given in [OV2, Theorem 4.8]. Since $\mathrm{Out}_{\mathcal{F}}(Q) \cong A_5$ in all cases, $\mathrm{Out}_{\mathcal{F}_0}(Q)$ contains $O^{2'}(\mathrm{Out}_{\mathcal{F}}(Q)) = \mathrm{Out}_{\mathcal{F}}(Q)$, and so $\mathcal{F}_0$ is the fusion system of J_2 or J_3. Since $\mathrm{Out}_{\mathcal{F}}(S) \cong C_3$ in both cases ($G \cong J_2$ or J_3), neither of these fusion systems can be properly contained as a normal subsystem of the other (see condition (ii) in Definition 1.18). Hence $O^{2'}(\mathcal{F}) = \mathcal{F}$. □

4.3. **Alternating groups.**

We prove here that all fusion systems of alternating groups are tame, and are also (with the obvious exceptions) reduced. Unlike the other examples given in this paper, we prove tameness without first determining the list of essential subgroups.

We first fix some notation when working with alternating and symmetric groups. We always regard $A_n \le \Sigma_n$ as groups of permutations of the set $\mathbf{n} = \{1, \ldots, n\}$. For $\sigma \in \Sigma_n$, we set $\mathrm{supp}(\sigma) = \{i \in \mathbf{n} \,|\, \sigma(i) \ne i\}$ (the *support* of σ). Likewise, for $H \le \Sigma_n$, $\mathrm{supp}(H)$ is defined to be the union of the supports of its elements.

Lemma 4.6. *Fix a prime p and $n \ge p^2$. Assume $n \ge 8$ if $p = 2$. Set $G = A_n$, and fix $S \in \mathrm{Syl}_p(G)$. Set $q = p$ if p is odd, and $q = 4$ if $p = 2$. Then*

$$\mathrm{Out}(S, \mathcal{F}_S(G)) \cong \begin{cases} C_2 & \text{if } n \equiv 0, 1 \pmod{q} \\ 1 & \text{otherwise.} \end{cases}$$

In all cases, $\mu_G \circ \kappa_G$ sends $\mathrm{Out}(G) = \mathrm{Out}_{\Sigma_n}(G) \cong C_2$ onto $\mathrm{Out}(S, \mathcal{F}_S(G))$.

Proof. Set $\mathcal{F} = \mathcal{F}_S(G)$ for short. Set $E_* = \langle (1\,2 \cdots p) \rangle \cong C_p$ if p is odd, and $E_* = \langle (1\,2)(3\,4), (1\,3)(2\,4) \rangle \cong C_2^2$ if $p = 2$. Let $Q \le S$ be the subgroup generated by all subgroups of S which are G-conjugate to E_*. If E_1 and E_2 are G-conjugate to E_*, then either $E_1 = E_2$, or $\mathrm{supp}(E_1) \cap \mathrm{supp}(E_2) = \varnothing$ and $[E_1, E_2] = 1$, or $\langle E_1, E_2 \rangle$ is not a p-group. Since this last case is impossible when $E_1, E_2 \le S$, we conclude that $Q = Q_1 \times \cdots \times Q_k$, where $k = [n/q]$ and the Q_i are pairwise commuting subgroups conjugate to E_*.

Fix $\alpha \in \mathrm{Aut}(S, \mathcal{F})$, and set $R = \alpha(Q)$. We first show that $R = Q$. For $i \ge 1$, let r_i be the number of orbits of length p^i under the action of R on $\mathbf{n}$. Thus

$$\sum_{i \ge 1} p^i r_i = |\mathrm{supp}(R)| \le \begin{cases} q \cdot [n/q] & \text{if } p \text{ is odd or } r_1 = 0 \\ 2 \cdot [n/2] & \text{if } p = 2 \text{ and } r_1 \ge 1 \end{cases} \qquad (7)$$

since $\mathrm{supp}(R)$ has order a multiple of p, and a multiple of 4 when $p = 2$ and $r_1 = 0$. Since $R \cong Q$ is elementary abelian, R is contained in a product $\prod_{i \ge 1} (B_i)^{r_i}$, where $B_i \cong C_p^i$ acts freely on a subset of $\mathbf{n}$ of order p^i, and hence

$$\mathrm{rk}(Q) = \mathrm{rk}(R) \le \begin{cases} \sum_{i\ge1} ir_i & \text{if } p \text{ is odd or } r_1 = 0 \\ \sum_{i\ge1} ir_i - 1 & \text{if } p = 2 \text{ and } r_1 \ge 1\ . \end{cases} \tag{8}$$

In the last case, "-1" appears since R contains only even permutations, and since the only factors B_i which act via odd permutations are those for $i = 1$.

Thus if p is odd or $r_1 = 0$, then by (7) and (8),

$$\sum_{i\ge1} p^i r_i \le q{\cdot}[n/q] = qk = p{\cdot}\mathrm{rk}(Q) \le \sum_{i\ge1} pir_i\ . \tag{9}$$

Also, $p^i \ge pi$, with equality only when $i = 1$ or $p^i = 4$. Hence (9) is possible only when p is odd, $r_1 = k$, and $r_i = 0$ for $i > 1$; or when $p = 2$, $r_2 = k$, and $r_i = 0$ for $i > 2$. In both cases, R is a product of subgroups conjugate to E_*, and thus $R = Q$.

Now assume $p = 2$ and $r_1 \ne 0$. By (7) and (8) again,

$$\sum_{i\ge1} 2^i r_i - 2 \le 2{\cdot}([n/2] - 1) \le 4{\cdot}[n/4] = 4k = 2{\cdot}\mathrm{rk}(Q) \le \sum_{i\ge1} 2ir_i - 2\ ,$$

so $r_i = 0$ for $i \ge 3$, and the inequalities are equalities. In particular, $r_1 + 2r_2 = [n/2] = 2k+1$, so r_1 and $[n/2]$ are both odd. Hence $R \cong (C_2^2)^{r_2} \times C_2^{r_1-1}$ (and $r_1 \ge 3$), where each element in $\mathrm{Aut}_G(R)$ permutes the C_2^2-factors and the C_2-factors. It follows that $\mathrm{Aut}_G(R) \cong (\Sigma_3 \wr \Sigma_{r_2}) \times \Sigma_{r_1}$. Since α is fusion preserving, we have $\mathrm{Aut}_G(R) \cong \mathrm{Aut}_G(Q)$, where $\mathrm{Aut}_G(Q) = \mathrm{Aut}_{\Sigma_n}(Q) \cong \Sigma_3 \wr \Sigma_k$ since $[n/2]$ is odd ($n - 4k \ge 2$ where $4k = |\mathrm{supp}(Q)|$, so there is a transposition which centralizes Q). Thus $\Sigma_3 \wr \Sigma_k \cong (\Sigma_3 \wr \Sigma_{r_2}) \times \Sigma_{r_1}$. Since $(\Sigma_3 \wr \Sigma_\ell)^{\mathrm{ab}} \cong C_2^2$ for all $\ell \ge 2$, we get $r_2 = 1$, $\Sigma_3 \wr \Sigma_k \cong \Sigma_3 \times \Sigma_{r_1}$, and this is clearly impossible.

Thus $\alpha(Q) = Q$. Since α is fusion preserving, it permutes the G-conjugacy classes in Q. For each $1 \le r \le k$, there are $\binom{k}{r}{\cdot}(q-1)^r$ products of r disjoint p-cycles in Q if p is odd, and $\binom{k}{r}{\cdot}(q-1)^r$ products of $2r$ disjoint 2-cycles in Q if $p = 2$. Clearly, $k(q-1) < \binom{k}{r}{\cdot}(q-1)^r$ for $1 < r < k$, and $k(q-1) < (q-1)^k$ since $k > 1$ and $(k, q) \ne (2, 3)$ by assumption. Hence α sends the set of p-cycles in Q (products of two 2-cycles in Q) to itself. Since the p-cycles (products of two 2-cycles) are precisely the nonidentity elements in $\bigcup_{i=1}^k Q_i$, and since $Q_1, \ldots, Q_k$ are the maximal subgroups in this set, α permutes the Q_i.

Thus there is $g \in N_{\Sigma_n}(Q)$ such that $c_g|_Q = \alpha|_Q$, and hence

$$\mathrm{Aut}_{gSg^{-1}}(Q) = (\alpha|_Q)\mathrm{Aut}_S(Q)(\alpha|_Q)^{-1} = \mathrm{Aut}_S(Q)$$

since $\alpha \in \mathrm{Aut}(S)$. Since $Q \trianglelefteq S$ by construction, this implies $gSg^{-1} \le S{\cdot}C_{\Sigma_n}(Q)$, where S normalizes $C_{\Sigma_n}(Q)$ since it normalizes Q. Hence there is $h \in C_{\Sigma_n}(Q)$ such that $hg \in N_{\Sigma_n}(S)$ (and $\alpha|_Q = c_{hg}|_Q$). Upon replacing α by $\alpha \circ c_{hg}^{-1}$, we can now assume $\alpha|_Q = \mathrm{Id}$.

Set $\mathcal{F}_0 = N_{\mathcal{F}}(Q)$. Since Q is fully normalized in $\mathcal{F}$ and $Q \trianglelefteq S$, this is a saturated fusion system over S. Also, $C_S(Q) \le Q$: any permutation which centralizes Q must leave each set $\mathrm{supp}(Q_i)$ invariant, and hence $C_G(Q) \cong (C_{A_q}(E_*))^k \times A_{n-qk} \cong Q \times A_{n-qk}$. Thus Q is normal and centric in $\mathcal{F}_0$, so $\mathcal{F}_0$ is *constrained* in the sense of [BCGLO1, Definition 4.1]. By [BCGLO1, Proposition 4.3], there is a finite group G_0, unique up to isomorphism, such that $O_{p'}(G_0) = 1$, $Q \trianglelefteq G_0$, $C_{G_0}(Q) \le Q$, $S \in \mathrm{Syl}_p(G_0)$, and $\mathcal{F}_0 = \mathcal{F}_S(G_0)$. Thus $G_0/Q \cong \mathrm{Aut}_{\mathcal{F}}(Q)$. The fusion preserving automorphism α induces an automorphism of $\mathcal{F}_0 = N_{\mathcal{F}}(Q)$, and hence by the uniqueness of G_0 (in the strong sense of [AKO, Lemma II.4.3]) induces an automorphism $\beta \in \mathrm{Aut}(G_0)$ such that $\beta|_S = \alpha$. Let $H \trianglelefteq G_0$ be the group of those $g \in G_0$ such that c_g sends each Q_i to itself via an automorphism of order prime to p. Thus $H/Q \le (C_{p-1})^k$ (with index 1 or 2) when p is odd, and $H/Q \cong C_3^k$ when $p = 2$. Since

$\beta|_Q = \mathrm{Id}_Q$ and H/Q has order prime to p, $\beta|_H$ is conjugation by an element $a \in Q$. Upon replacing α and β by $\alpha \circ c_a^{-1}$ and $\beta \circ c_a^{-1}$, we can assume $\beta|_H = \mathrm{Id}_H$. But now, $Z(H) = 1$, so distinct elements of G_0 have distinct conjugation actions on H, and hence $\beta = \mathrm{Id}_{G_0}$. Thus $\alpha = \beta|_S = \mathrm{Id}_S$.

We have now shown that each element of $\mathrm{Aut}(S, \mathcal{F})$ is conjugation by some element of Σ_n. Since $n > 6$, $\mathrm{Out}(G) = \mathrm{Out}_{\Sigma_n}(G)$ by, e.g., [Sz1, Theorem 3.2.17]. Thus $\mu_G \circ \kappa_G$ sends $\mathrm{Out}(G) \cong C_2$ onto $\mathrm{Out}(S, \mathcal{F})$. This last group is trivial exactly when there is $g \in N_{\Sigma_n}(S) \smallsetminus A_n$ such that $c_g|_S \in \mathrm{Aut}_{\mathcal{F}}(S)$; i.e., when $c_g|_S = c_h|_S$ for some $h \in N_G(S)$. Upon replacing g by gh^{-1}, we see that $\mathrm{Out}(S, \mathcal{F}) = 1$ if and only if some odd permutation $g \in \Sigma_n \smallsetminus A_n$ centralizes S.

If $n \not\equiv 0, 1 \pmod{q}$, then there is a transposition $(i\, j)$ which centralizes S: when p is odd because one can choose $i, j \in \mathbf{n} \smallsetminus \mathrm{supp}(S)$, and when $p = 2$ because the S-action on $\mathbf{n}$ has an orbit $\{i, j\}$ of order 2. Thus $\mathrm{Out}(S, \mathcal{F}) = 1$ in this case. If $n \equiv 0, 1 \pmod{q}$, then $|\mathbf{n} \smallsetminus \mathrm{supp}(Q)| \leq 1$, and so

$$C_{\Sigma_n}(S) \leq C_{\Sigma_n}(Q) = Q \leq A_n \ .$$

Thus $\mathrm{Out}(S, \mathcal{F}) = \mathrm{Out}_{\Sigma_n}(S)$ has order two in this case. □

The following well known lemma will be needed when working with elementary abelian subgroups of symmetric groups.

Lemma 4.7. *Fix $n \geq 1$ and an abelian subgroup $G \leq \Sigma_n$. Let $H_1, \dots, H_m \leq G$ be the distinct stabilizer subgroups for the action of G on $\mathbf{n}$, and let $X_i \subseteq \mathbf{n}$ be the set of elements with stabilizer subgroup H_i (so $\mathbf{n}$ is the disjoint union of the X_i). Then each X_i is G-invariant. Let k_i be the number of G-orbits in X_i. Then*

$$C_{\Sigma_n}(G) \cong \prod_{i=1}^{m} (G/H_i) \wr \Sigma_{k_i} \ ,$$

where each factor $(G/H_i) \wr \Sigma_{k_i}$ has support X_i, Σ_{k_i} permutes the k_i G-orbits in X_i, and each factor G/H_i in $(G/H_i)^{k_i}$ has as support one of those G-orbits.

Proof. Let $Y_1, \dots, Y_t$ be the G-orbits in $\mathbf{n}$, and let $C_0 \leq C_{\Sigma_n}(G)$ be the subgroup of elements which leave each of the Y_i invariant. Since G is abelian, y and $g(y)$ have the same stabilizer subgroup for each $g \in G$ and each $y \in \mathbf{n}$. Let K_i be the stabilizer subgroup of the elements in Y_i. Then the homomorphism

$$\chi \colon \prod_{i=1}^{t} (G/K_i) \longrightarrow C_0 \ ,$$

defined by setting $\chi(g_1 K_1, \dots, g_t K_t)(y) = g_i(y)$ for $y \in Y_i$, is an isomorphism.

Since all elements in each orbit have the same stabilizer subgroup, each set X_i is a union of orbits Y_j (i.e., is G-invariant). Also, C_0 is normal in $C_{\Sigma_n}(G)$: it is the kernel of the homomorphism to Σ_t which describes how an element σ permutes the orbits. Each $\sigma \in C_{\Sigma_n}(G)$ sends each orbit in $\mathbf{n}$ to another orbit with the same stabilizer subgroup, and thus leaves each X_i invariant. Since X_i contains k_i orbits, $C_{\Sigma_n}(G)/C_0 \cong \prod_{i=1}^{m} \Sigma_{k_i}$, and $C_{\Sigma_n}(G)$ is isomorphic to the product of the wreath products $(G/H_i) \wr \Sigma_{k_i}$. □

We are now ready to prove that all fusion systems of alternating groups are tame.

Proposition 4.8. *Fix a prime p and $n \geq 2$, set $G = A_n$, and choose $S \in \mathrm{Syl}_p(G)$. Then $\mathcal{F}_S(G)$ is tame. If $p = 2$ and $n \geq 8$; or if p is odd, $n \geq p^2$ and $n \equiv 0, 1$ (mod p); then*

$$\kappa_G \colon \mathrm{Out}(G) \longrightarrow \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G)) \cong C_2$$

is an isomorphism.

Proof. Set $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{L} = \mathcal{L}_S^c(G)$. If $n < p^2$, or if $p = 2$ and $n < 6$, then the Sylow p-subgroups of A_n are abelian, so $\mathcal{F}$ is constrained, $\mathfrak{red}(\mathcal{F}) = 1$ by Proposition 2.4, and so $\mathcal{F}$ is tame by Theorem 2.20. If $p = 2$ and $n = 6, 7$, then since $A_6 \cong PSL_2(9)$ and A_7 has the same fusion system as A_6, $\mathcal{F}$ is tame by Proposition 4.3.

If p is odd and $n \geq p^2$, then $\mu_G\colon \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \xrightarrow{\cong} \mathrm{Out}(S, \mathcal{F})$ is an isomorphism by [BLO1, Theorem E] and [O1, Theorem A]. (Note that while the latter result depends on the classification of finite simple groups, this particular case does not. For example, it follows from [O1, Proposition 3.5], applied with $T_0 = 1$, $T = S$, and $\mathfrak{X} = Q$ the group defined in the proof of Lemma 4.6.) So by Lemma 4.6, either $n \equiv 0, 1 \pmod p$ and κ_G is an isomorphism, or $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) = 1$ and hence κ_G is split surjective. Thus $\mathcal{F}$ is tame in these cases.

It remains to handle the case $p = 2$ and $n \geq 8$. By Lemma 4.6 again, it suffices to prove

$$\mathrm{Ker}(\mu_G) = 1 \text{ if } n \equiv 0, 1 \pmod 4 \quad \text{and} \quad |\mathrm{Ker}(\mu_G)| \leq 2 \text{ if } n \equiv 2, 3 \pmod 4\ , \tag{10}$$

and also

$$n \equiv 2, 3 \pmod 4 \quad \Longrightarrow \quad \text{there is } x \in C_{\Sigma_n}(S) \smallsetminus G \text{ such that } \kappa_G([c_x]) \neq 1\ . \tag{11}$$

Let $Q \leq S$ be as in the proof of Lemma 4.6: the subgroup generated by all subgroups of S G-conjugate to $E_* = \langle (1\,2)(3\,4), (1\,3)(2\,4) \rangle$. We saw in the proof of the lemma that $Q = Q_1 \times \cdots \times Q_k$, where $k = [n/4]$, the Q_i are the only subgroups of S G-conjugate to E_*, and they have pairwise disjoint support. Thus Q is weakly closed: the unique subgroup of S in its G-conjugacy class.

Fix $[\alpha] \in \mathrm{Ker}(\mu_G)$. By Proposition 4.2, we can assume $[\alpha]$ is the class of $\alpha \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L})$ for which $\alpha_S = \mathrm{Id}_{\mathrm{Aut}_{\mathcal{L}}(S)}$. Let $g_P \in C_{Z(P)}(\mathrm{Aut}_S(P))$, for $P \leq S$ $\mathcal{F}$-centric and fully normalized, be the elements defined in Proposition 4.2. Set $g = g_Q \in C_Q(\mathrm{Aut}_S(Q)) = Z(S)$ (the last equality since Q is normal and centric in S). For each fully normalized $P \geq Q$ (including $P = S$), all automorphisms in $\mathrm{Aut}_{\mathcal{F}}(P)$ leave Q invariant since it is weakly closed, so $g_P \equiv g_Q = g \pmod{C_{Z(Q)}(\mathrm{Aut}_{\mathcal{F}}(P)) = C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))}$ by Proposition 4.2(c). So upon replacing α by $\alpha \circ c_{[g]}^{-1}$, we can assume $g = 1$, and $\alpha_P = \mathrm{Id}_{\mathrm{Aut}_{\mathcal{L}}(P)}$ (and $g_P = 1$) for all fully normalized $P \geq Q$.

For each $1 \leq i \leq k$, there is a 3-cycle $h_i \in N_G(Q)$ which permutes transitively the involutions in Q_i and centralizes the other Q_j. Thus $C_Q(\mathrm{Aut}_{\mathcal{F}}(Q)) \leq \bigcap_{i=1}^k C_Q(h_i) = 1$. Recall that $P \in \widehat{\mathcal{E}}_0$ if P is $\mathcal{F}$-essential, $P = C_S(E)$ for some elementary abelian subgroup E fully centralized in $\mathcal{F}$, and $C_{Z(P)}(\mathrm{Aut}_S(P)) \gneqq C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$. Let $\widehat{\mathcal{E}}_0^{\not\geq Q}$ be the set of subgroups $P \in \widehat{\mathcal{E}}_0$ which do not contain Q. Let $\mathcal{X}$ be a set of representatives for $\widehat{\mathcal{E}}_0$ modulo $\mathcal{F}$-conjugacy. By Proposition 4.2(e), applied with $\mathcal{H} = \mathcal{X} \cup \{Q\}$, $[\alpha] = 1$ if and only if $g_P \in C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$ for all P in a set of representatives for $\widehat{\mathcal{E}}_0^{\not\geq Q}$ modulo $\mathcal{F}$-conjugacy.

Fix $P = C_S(E) \in \widehat{\mathcal{E}}_0^{\not\geq Q}$. Since E is fully centralized, $P \in \mathrm{Syl}_2(C_G(E))$. Since P is $\mathcal{F}$-essential, $\mathrm{Out}_{\mathcal{F}}(P)$ has a strongly 2-embedded subgroup, and hence all involutions in any Sylow 2-subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$ are in its center (cf. [OV2, Propositions 3.3(a) & 3.2]). In particular, $\mathrm{Out}_{\mathcal{F}}(P)$ contains no subgroup isomorphic to D_8.

Fix $\overline{P} \in \mathrm{Syl}_2(C_{\Sigma_n}(E))$ which contains P. Thus $P = \overline{P} \cap A_n$. Also, $E \trianglelefteq \overline{P}$, so $\overline{P} \leq \overline{P}{\cdot}C_{\Sigma_n}(\overline{P}) \leq C_{\Sigma_n}(E)$, and hence

$$\overline{P}{\cdot}C_{\Sigma_n}(\overline{P})/\overline{P} \text{ has odd order.} \tag{12}$$

By Lemma 4.7, each union of m E-orbits of order $q = 2^i$ which have the same stabilizer subgroup contributes a factor $E_q \wr \Sigma_m$ to $C_{\Sigma_n}(E)$, where $E_q \cong (C_2)^i$ is acting freely on an orbit of order q in $\mathbf{n}$. Since a Sylow 2-subgroup of Σ_m is a product of wreath products $C_2 \wr \cdots \wr C_2$, $\overline{P} \in \mathrm{Syl}_2(C_{\Sigma_n}(E))$ is a product of subgroups of the form $E_q \wr C_2 \wr \cdots \wr C_2$ (or E_q) with pairwise disjoint support. If $\overline{P}$ contains a factor $E_q \wr C_2 \wr \cdots \wr C_2$ for $q = 2^r \geq 8$, then $\mathrm{Out}_{\mathcal{F}}(P)$ contains $GL_r(2) \geq D_8$, which we just saw is impossible.

Write $\mathbf{n} = X_0 \amalg X_1 \amalg X_2$, where X_0 is the set of points fixed by $\overline{P}$, X_1 is the union of $\overline{P}$-orbits of length 2, and X_2 is the union of $\overline{P}$-orbits of length ≥ 4. By the above description of $\overline{P}$, $\overline{P} = P_1 \times P_2$, where $\mathrm{supp}(P_i) = X_i$ for $i = 1, 2$, $P_1 \cong C_2^m$ where $2m = |X_1|$, and P_2 is a product of subgroups $E_4 \wr C_2 \wr \cdots \wr C_2$ and $C_2 \wr \cdots \wr C_2$ (the latter of order ≥ 8). By (12), $|X_0| \leq 1$, since otherwise there would be a 2-cycle in $C_{\Sigma_n}(\overline{P})$ not in $\overline{P}$.

Each factor E_4 or $C_2 \wr C_2$ (with support of order 4) contains a subgroup conjugate to E_* (thus one of the factors Q_i in Q). Thus $X_2 \subsetneq \mathrm{supp}(Q \cap \overline{P})$. If $n - |X_2| \leq 3$, then $X_2 = \mathrm{supp}(Q)$, so $Q \leq \overline{P} \cap A_n = P$, contradicting the original assumption on P. Thus $|X_0 \cup X_1| > 3$. Since $|X_0| \leq 1$ and $|X_1| = 2m$ is even, we have $m \geq 2$.

If $\{i, j\}$ is any of the m orbits of order 2 in X_1, then $(i\,j) \in C_{\Sigma_n}(P) \smallsetminus A_n$ and $\overline{P} = \langle P, (i\,j) \rangle$. Thus $N_{\Sigma_n}(P) = N_{\Sigma_n}(\overline{P})$, $C_{\Sigma_n}(P) = C_{\Sigma_n}(\overline{P})$, $P{\cdot}C_{\Sigma_n}(P) = \overline{P}{\cdot}C_{\Sigma_n}(\overline{P})$, and so $N_{\Sigma_n}(P)/P{\cdot}C_{\Sigma_n}(P) \cong N_G(P)/P{\cdot}C_G(P)$. This proves that

$$\mathrm{Out}_G(P) = \mathrm{Out}_{\Sigma_n}(P) \cong \mathrm{Out}_{\Sigma_n}(\overline{P}) \cong \Sigma_m \times \mathrm{Out}_{\Sigma_{X_2}}(P_2),$$

where the first isomorphism is induced by restriction. Here, Σ_{X_2} is the group of permutations of X_2.

If $m = 2$, then $O_2(\mathrm{Out}_G(P)) \neq 1$, and if $m \geq 4$, then $\mathrm{Out}_G(P) \geq D_8$. Either of these would contradict the assumption that $\mathrm{Out}_G(P)$ contains a strongly 2-embedded subgroup. Thus $m = 3$, and $X_1 = \mathrm{supp}(P_1)$ has order 6. A group with a strongly 2-embedded subgroup cannot split as a product of two groups of even order, so $|\mathrm{Out}_{\Sigma_{X_2}}(P_2)|$ is odd. Since $P_2{\cdot}C_{\Sigma_{X_2}}(P_2)/P_2$ is isomorphic to a subgroup of $\overline{P}{\cdot}C_{\Sigma_n}(\overline{P})/\overline{P}$, it has odd order by (12), and hence

$$\bigl|N_{\Sigma_{X_2}}(P_2)/P_2\bigr| = \left|\frac{N_{\Sigma_{X_2}}(P_2)}{P_2{\cdot}C_{\Sigma_{X_2}}(P_2)}\right| \cdot \left|\frac{P_2{\cdot}C_{\Sigma_{X_2}}(P_2)}{P_2}\right| = \bigl|\mathrm{Out}_{\Sigma_{X_2}}(P_2)\bigr| {\cdot} \bigl|P_2{\cdot}C_{\Sigma_{X_2}}(P_2)/P_2\bigr|$$

is also odd. If $P_2 \leq T \in \mathrm{Syl}_2(\Sigma_{X_2})$, then $N_T(P_2)/P_2$ has odd order, so $P_2 = T$ (cf. [Sz1, Theorem 2.1.6]), and thus $P_2 \in \mathrm{Syl}_2(\Sigma_{X_2})$.

Since P_2 is a Sylow 2-subgroup of a symmetric group and has no orbits of order 2, it is a product of subgroups $C_2 \wr \cdots \wr C_2$ of order ≥ 8. Since $4\bigl||X_2|$ (a union of orbits of order $2^i \geq 4$) and $|X_0| \leq 1$,

$$n = |X_0| + 6 + |X_2| \equiv 2, 3 \pmod 4 .$$

If R is any other subgroup in $\widehat{\mathcal{E}}_0^{\not\geq Q}$, then $R = \overline{R} \cap G$, $\mathbf{n} = Y_0 \amalg Y_1 \amalg Y_2$ where Y_0 is the set of elements fixed by $\overline{R}$ and Y_1 is the union of R-orbits of order 2, $\overline{R} = R_1 \times R_2$ where $\mathrm{supp}(R_i) = Y_i$, $R_2 \in \mathrm{Syl}_2(\Sigma_{Y_2})$, $|Y_1| = 6 = |X_1|$, and $|Y_2| = |X_2|$ (the largest multiple of 4 which is $\leq n{-}6$). Thus R is Σ_n-conjugate to P, and is A_n-conjugate to P since there are odd permutations which centralize P (the transpositions in P_1).

Now, $Z(\overline{P}) = P_1 \times Z(P_2)$, where $Z(P_2)$ is a product of one copy of C_2 for each factor $C_2 \wr \cdots \wr C_2$ in P_2 (equivalently, for each P_2-orbit in X_2). Also, each of these factors C_2 has support the corresponding P_2-orbit, hence of order a multiple of 4, and hence contained in

A_n. Thus $Z(P_2) \le G = A_n$. Also, $\mathrm{Aut}_{\mathcal{F}}(P)$ acts via the identity on $Z(P_2)$, since all of the factors $C_2 \wr \cdots \wr C_2$ in P_2 have different orders (hence their supports have different orders). Since $\mathrm{Aut}_{A_n}(P_1 \cap A_n) \cong \Sigma_3$ acts on P_1 by permuting the three transpositions, $\mathrm{Aut}_{\mathcal{F}}(P)$ acts on $P_1 \cap A_n \cong C_2^2$ with trivial fixed set. Since $Z(P) = (P_1 \cap A_n) \times Z(P_2)$, it now follows that $C_{Z(P)}(\mathrm{Aut}_S(P))/C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$ has order two.

To summarize, every class in $\mathrm{Ker}(\mu_G)$ is represented by some α such that $\alpha_P = \mathrm{Id}$ when $P \ge Q$, and for such α, $[\alpha] = 1$ if and only if $g_P \in C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$ for some representative in each $\mathcal{F}$-conjugacy class in $\widehat{\mathcal{E}}_0^{\not\ge Q}$. When $n \equiv 0, 1 \pmod 4$, $\widehat{\mathcal{E}}_0^{\not\ge Q} = \varnothing$, so $\mathrm{Ker}(\mu_G) = 1$. When $n \equiv 2, 3 \pmod 4$, all subgroups in $\widehat{\mathcal{E}}_0^{\not\ge Q}$ are $\mathcal{F}$-conjugate to some fixed P, and so $|\mathrm{Ker}(\mu_G)| \le |C_{Z(P)}(\mathrm{Aut}_S(P))/C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))| = 2$. This proves (10).

Assume $n \equiv 2, 3 \pmod 4$, and set $k = [n/4]$ as before. Set $P_1 = \langle (1\,2), (3\,4), (5\,6) \rangle$ and $E = G \cap P_1$. Assume S was chosen so that $\mathrm{supp}(S) = \{1, \ldots, 4k+2\}$, $\mathrm{supp}(Q) = \{3, \ldots, 4k+2\}$, and $P \overset{\text{def}}{=} C_S(E) \in \mathrm{Syl}_2(C_G(E))$. Then $\mathrm{Aut}_{\mathcal{F}}(P) \cong \Sigma_3 \times A$ where A has odd order, and so $P \in \widehat{\mathcal{E}}_0^{\not\ge Q}$.

Set $x = (1\,2)$. Then $\mathrm{Out}(G) = \langle [c_x] \rangle \cong C_2$, $[x, S] = 1$, and c_x is the identity on $N_G(Q)/C'_G(Q) = \mathrm{Aut}_{\mathcal{L}}(Q)$. (Note that if $n = 4k+3$, then $C'_G(Q) = \langle (1\,2\,n) \rangle$ does not commute with x.) Also, $(1\,2)(3\,4)(5\,6)$ centralizes $N_G(P)$, and hence c_x acts on $\mathrm{Aut}_{\mathcal{L}}(P)$ (or on $N_G(P)$) via conjugation by $g_P \overset{\text{def}}{=} (3\,4)(5\,6) \in C_{Z(P)}(\mathrm{Aut}_S(P))$. Since $g_P \notin C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$, $[c_x]$ is sent to a nontrivial element in $\mathrm{Ker}(\mu_G)$. This proves (11), and finishes the proof of the proposition. □

We finish by proving that with the obvious exceptions, most fusion systems of alternating groups are reduced.

Proposition 4.9. *Fix a prime p and $n \ge p^2$ such that $n \equiv 0, 1$ (mod p). Assume $n \ge 8$ if $p = 2$. Set $G = A_n$, and choose $S \in \mathrm{Syl}_p(G)$. Then the fusion system $\mathcal{F}_S(G)$ is reduced.*

Proof. Set $\mathcal{F} = \mathcal{F}_S(G)$. By the focal subgroup theorem (cf. [G, Theorem 7.3.4]), $\mathfrak{foc}(\mathcal{F}) = S \cap [G, G] = S$, so $O^p(\mathcal{F}) = \mathcal{F}$.

Let $Q \le S$ be as in the proof of Lemma 4.6: the subgroup generated by all subgroups of S G-conjugate to E_*, where $E_* = \langle (1\,2\,\cdots\,p) \rangle \cong C_p$ if p is odd, and $E_* = \langle (1\,2)(3\,4), (1\,3)(2\,4) \rangle \cong C_2^2$ if $p = 2$. We saw in the proof of the lemma that $Q = Q_1 \times \cdots \times Q_k$, where $k = [n/p]$ ($p > 2$) or $[n/4]$ ($p = 2$), the Q_i are the only subgroups of S G-conjugate to E_*, and they have pairwise disjoint support. Thus Q is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant. We also saw that $C_S(Q) = Q$, and hence Q is $\mathcal{F}$-centric (since it is the only subgroup in its $\mathcal{F}$-conjugacy class by construction). Finally,

$$\mathrm{Aut}_{\Sigma_n}(Q) \cong \mathrm{Aut}(E_*) \wr \Sigma_k \qquad \text{where} \qquad \mathrm{Aut}(E_*) \cong \begin{cases} C_{p-1} & \text{if } p > 2 \\ \Sigma_3 & \text{if } p = 2\,, \end{cases} \tag{13}$$

and hence $\mathrm{Aut}_{\mathcal{F}}(Q)$ has index at most two in this wreath product. When $p = 2$, since $\Sigma_k \le \mathrm{Aut}_{\Sigma_n}(Q)$ permutes the Q_i with support of order 4, it is contained in $\mathrm{Aut}_{\mathcal{F}}(Q)$.

Set $R = O_p(\mathcal{F})$. Since Q is $\mathcal{F}$-centric, and is $\mathcal{F}$-radical by (13), $R \le Q$ by Proposition 1.5. Assume $R \ne 1$, and fix $g \in R$ of order p. There is $h \in Q$ which is G-conjugate to g (a product of the same number of p-cycles) such that gh is a p-cycle (or a product of two 2-cycles if $p = 2$). Then $h \in R$ since $R \trianglelefteq \mathcal{F}$, and so $gh \in R$. Since each Q_i is generated by elements G-conjugate to gh, this would imply that $R = Q$. But in all cases, there are elements both in Q and in $S \smallsetminus Q$ which are products of p disjoint p-cycles, so Q is not strongly closed in $\mathcal{F}$. We conclude that $R = O_p(\mathcal{F}) = 1$.

Now set $\mathcal{F}_0 = O^{p'}(\mathcal{F})$; we must show $\mathcal{F}_0 = \mathcal{F}$. By [BCGLO2, Theorem 5.4], it suffices to show that $\mathrm{Aut}_{\mathcal{F}_0}(S) = \mathrm{Aut}_{\mathcal{F}}(S)$. Also, by the same theorem,

$$\mathrm{Aut}_{\mathcal{F}_0}(S) = \mathrm{Aut}^0_{\mathcal{F}}(S) \geq \big\langle \alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \,\big|\, \alpha|_P \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)), \text{ some } \mathcal{F}\text{-centric subgroup } P \leq S \text{ with } \alpha(P) = P \big\rangle .$$

For $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$, if $\alpha|_Q \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(Q))$, then $\alpha \in \mathrm{Aut}_{\mathcal{F}_0}(S)$. If $p = 2$, then $O^{2'}(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Aut}_{\mathcal{F}}(Q)$ by the description in (13), so $\mathcal{F}_0 = \mathcal{F}$ in this case.

Assume p is odd. Let p^ℓ be the largest power of p such that $p^\ell \leq n$. Write $S = S_1 \times S_2$, where $\mathrm{supp}(S_1) \cap \mathrm{supp}(S_2) = \varnothing$ and $|\mathrm{supp}(S_1)| = p^\ell$. Fix $T \in \mathrm{Syl}_p(\Sigma_p)$, and identify

$$S_1 = T \wr T \wr \cdots \wr T \leq \Sigma_p \wr \Sigma_p \wr \cdots \wr \Sigma_p \leq \Sigma_{p^\ell} \leq \Sigma_n.$$

Let $\Phi\colon (\Sigma_p)^\ell \longrightarrow \Sigma_p \wr \cdots \wr \Sigma_p \leq \Sigma_{p^\ell}$ be the monomorphism which sends the first factor diagonally to $(\Sigma_p)^{p^{\ell-1}}$, the second factor diagonally to $(1 \wr \Sigma_p)^{p^{\ell-2}}$, etc. Set $P_1 = \Phi(T^\ell)$ and $P = P_1 \times S_2 \leq S$. Fix $u \in \mathbb{F}_p^\times$ of order $p-1$, and choose $h \in N_{\Sigma_p}(T)$ such that $hgh^{-1} = g^u$ for $g \in T$. Let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ be conjugation by $\Phi(h, h^{-1}, 1, \ldots, 1)$. Then $\alpha|_{P_1}$ has matrix $\mathrm{diag}(u, u^{-1}, 1, \ldots, 1) \in \mathrm{SL}_\ell(p)$ with respect to the canonical basis. Since $\mathrm{Aut}_{\mathcal{F}}(P_1)$ has index at most two in $\mathrm{Aut}_{\Sigma_n}(P_1) \cong \mathrm{GL}_\ell(p)$, we get $\alpha|_P \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$, and so $\alpha \in \mathrm{Aut}_{\mathcal{F}_0}(S)$ since P is $\mathcal{F}$-centric. Also, $\alpha|_Q$ represents a generator of $\mathrm{Aut}_{\mathcal{F}}(Q)/O^{p'}(\mathrm{Aut}_{\mathcal{F}}(Q)) \cong \mathbb{F}_p^\times$, so this finishes the proof that $\mathrm{Aut}_{\mathcal{F}_0}(S) = \mathrm{Aut}_{\mathcal{F}}(S)$ and hence that $\mathcal{F}_0 = \mathcal{F}$. Thus $\mathcal{F}$ is reduced. □

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