FUNDAMENTOS DE ÁLGEBRA

HWK 4 (deadline 15/10/2014, in class)

Observation: Exercises numbered from 1 to 5 are worth 20 points total. The bonus exercise A is worth 4 points extra.

1. (Exercise 1.12.4) Let G be a finite nilpotent group and let m be such that $m \mid |G|$. Show that there $H < G$ such that $|H| = m$.

2. (Exercise 1.12.5.) Let G be a finite nilpotent group and $N \lhd G$ s.t. $N \neq \{\mathbf{1}\}$. Show that $N \cap C(G) \neq \{\mathbf{1}\}$.

3. (Exercise 1.12.9.) Let $N \lhd G$. Show that $[N, G] < N$.

4. One of the following exercises:

   (Exercise 1.13.4.) Show that an abelian group has a composition series iff is finite.

   OR

   (Exercise 1.13.5.) Show that any solvable group with a composition series is finite.

5. (Exercise 1.13.7.) Let G be the subgroup of $(\mathbb{H}^\times, \cdot)$ generated by $a = e^{\frac{\pi i}{3}}$ and $b = j$.

   (a) Find the subgroups $C_k(G)$ and $G^{(k)}$, for $k \geq 1$, and decide if G is nilpotent and/or solvable.

   (b) Determine a composition series for G and identify its factors.

   Sugestion: Verify that $|a| = 6$, $|b| = 4$ and $bab^{-1} = a^{-1}$; justify that any element in G can be written in the form $a^r b^s$ with $r, s \geq 0$.

A. Bonus exercise: Show that there isn't any group whose derived subgroup is the dihedral group D_n, $n \geq 3$.
Sugestion: Suppose there is a group G such that $G^{(1)} = D_n$ and let $K < G^{(1)}$ be such that $K \cong \mathbb{Z}_n$. Show that $\varphi \colon G \to \mathrm{Aut}(K)$, given by $\varphi(g) = c_g$, is a group homomorphism and study $\varphi(G^{(1)})$.