1. Consider the trigonometric series  $\sum_{n=-\infty}^{+\infty} c_n e^{int}$, with $\{c_n\}_{n \in \mathbb{Z}}$ a complex sequence. Show that this series is the Fourier series of a positive (finite and regular) Borel measure $\mu \in \mathcal{M}(\mathbb{T})$ if and only if the Cesàro means associated to the trigonometric series 
$$
\sigma_N(t) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N + 1}\right) c_n e^{int},
$$
are positive functions for any $N \in \mathbb{N}$.

2. Let $f \in C(\mathbb{T})$ be such that, for certain $N \in \mathbb{N}$ and $t_0 \in \mathbb{T}$, $\sigma_N(f)(t_0) = \max_{t \in \mathbb{T}} |f(t)|$, where $\sigma_N(f)$ denotes the Cesàro means of the Fourier series of $f$. Show that, in this case, $f$ is necessarily constant.

3. Let $f \in L^\infty(\mathbb{T})$ satisfy $|\hat{f}(n)| \leq K|n|^{-1}$, for some $K > 0$ and all $n \neq 0$.
   a) Show that, for all $t \in \mathbb{T}$ and $N \in \mathbb{N}$, the partial sums of the Fourier series of $f$, $S_N(f)(t) = \sum_{n=-N}^{N} \hat{f}(n)e^{int}$, satisfy the following inequality
      $$|S_N(f)(t)| \leq \|f\|_\infty + 2K.$$
      Hint: Note that $S_N(f)(t) = \sigma_N(f)(t) + \sum_{n=-N}^{N} \frac{|n|}{N+1} \hat{f}(n)e^{int}$.
   b) Using the previous result and the function $f(t) = \frac{t}{2}$, $t \in [-\pi, \pi]$ prove that
      $$\forall N \in \mathbb{N}, t \in \mathbb{T} \quad \left| \sum_{n=1}^{N} \sin(nt) \frac{n}{n}\right| \leq \frac{\pi}{2} + 1.$$

4. We already know that if the Fourier coefficients of $f$ are of the order $O(|n|^{-(k+1+\epsilon)})$ then $f \in C^k(\mathbb{T})$. In this problem we will show how to improve this result slightly, if $L^2$ estimates are used instead. Let $f \in L^1(\mathbb{T})$ be such that $\hat{f}(n) = O(|n|^{-m})$. Show that, if $m > k + \frac{1}{2}$, with $k \geq 1$, then $f$ is $k$ times differentiable a.e. on $\mathbb{T}$, with $\frac{d^k f}{dx^k} \in L^2(\mathbb{T})$.

5. Let $0 < \alpha < 1$. Prove that the function $f_\alpha$ defined by
   $$f_\alpha(t) = \sum_{n=1}^{\infty} \frac{\cos(3^n t)}{3^{n\alpha}},$$
is Hölder continuous with exponent $\alpha$, i.e. $f_\alpha \in \mathcal{A}^\alpha(\mathbb{T})$. This example shows that one cannot improve the decay rate of the Fourier coefficients of Hölder-\(\alpha\) functions.
   Hint: Use the trigonometric formula
   $$\cos x - \cos y = -2 \sin \left(\frac{x + y}{2}\right) \sin \left(\frac{x - y}{2}\right),$$
   and break the series into two sums, estimating each of them adequately.
6. Show that the trigonometric series

\[ \sum_{n=2}^{\infty} \frac{\sin(nt)}{\log(n)}, \]

converges for all \( t \in \mathbb{T} \), but that it is not, however, a Fourier series of any function \( f \in L^1(\mathbb{T}) \).