1. For \( g \in L^p(\mathbb{R}^n) \) fixed, with \( 1 \leq p < \infty \), consider the operator \( T_g : L^1(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \) defined by
\[
T_g f = g * f, \quad f \in L^1(\mathbb{R}^n).
\]
Show that the operator norm of \( T_g \), \( \|T_g\|_{L^1 \to L^p} \), is exactly equal to \( \|g\|_{L^p(\mathbb{R}^n)} \). And what is the operator norm for \( p = \infty \)?

2. Let \( (X, \mu) \) and \( (Y, \nu) \) be \( \sigma \)-finite measure spaces, and \( f : X \times Y \to \mathbb{C} \) a measurable function. The mixed \( L^p \) norm of \( f \) is defined as
\[
\|f\|_{L^p(X,L^q(Y))} = \left\| \|f(x,y)\|_{L^q(Y)} \right\|_{L^p(X)},
\]
and analogously,
\[
\|f\|_{L^q(Y,L^p(X))} = \left\| \|f(x,y)\|_{L^p(X)} \right\|_{L^q(Y)},
\]
for any \( 0 < p, q \leq \infty \). The order in which the partial norms are computed is very important, as they do not commute. Show, however, that for \( 0 < p \leq q \leq \infty \), the following inequality holds
\[
\|f\|_{L^q(Y,L^p(X))} \leq \|f\|_{L^p(X,L^q(Y))}.
\]

3. In the same framework as in the previous problem, consider the integral operator
\[
Kf(x) = \int K(x,y)f(y)dy,
\]
where the kernel \( K(x,y) \) satisfies
\[
\|K\|_{L^q(Y,L^p(X))} \leq C_0 \quad \text{and} \quad \|K\|_{L^p(X,L^q(Y))} \leq C_1,
\]
for any \( 1 \leq p, q \leq \infty \). Prove that, for \( q' \leq s \leq p' \), \( \frac{1}{s} + \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \) and \( f \in L^s(Y) \), then \( Kf(x) \) is well defined a.e. \( x \in X \), \( Kf \in L^r(X) \) and the following inequality holds
\[
\|Kf\|_r \leq C_0^\theta C_1^{1-\theta} \|f\|_s,
\]
where \( 0 \leq \theta \leq 1 \) is such that \( \frac{1}{s} = \theta \frac{1}{q} + (1-\theta) \frac{1}{p} \).

\textit{Obs:} Note that the case \( p = \infty \) was done in class, corresponding to the generalized Schur-Young inequality. The case \( p = q = 2 \) is also very important: in this case \( K \in L^2(X \times Y) \) are called Hilbert-Schmidt kernels and the corresponding integral formulas define linear bounded continuous operators from \( L^2 \) to \( L^2 \), known in the literature as integral Hilbert-Schmidt operators.
4. Consider \( f \in L^1_{loc}(\mathbb{R}^n) \) and \( g \in C^k_c(\mathbb{R}^n) \).

a) Prove rigorously that \( f * g \in C^k(\mathbb{R}^n) \) and that \( \forall |\alpha| \leq k \) \( \partial^\alpha (f * g) = f * \partial^\alpha g \).

b) If also \( f \in C^l(\mathbb{R}^n) \) (maintaining the condition \( g \in C^k_c(\mathbb{R}^n) \)), show then that \( f * g \in C^{k+l}(\mathbb{R}^n) \) and that \( \partial^\alpha (f * g) = \partial^\alpha_1 f * \partial^\alpha_2 g \), with \( \alpha = \alpha_1 + \alpha_2 \), \( |\alpha_1| \leq l \) and \( |\alpha_2| \leq k \).

5. Let \( \varphi \in C^\infty_c(\mathbb{R}^n) \), with \( \int \varphi = 1 \) and \( \text{supp} \ \varphi \subset B_1(0) \). If \( f \in L^1_{loc}(\mathbb{R}^n) \) (which, from the previous problem, we know implies \( \varphi^\epsilon * f \in C^\infty \)), prove that, when \( \epsilon \to 0 \), then \( \varphi^\epsilon * f \to f \) uniformly on any compact subset of an open set where \( f \) is continuous. Show also that, in case \( f \) is \( k \) times continuously differentiable in that open set, then all derivatives \( \partial^\alpha \varphi^\epsilon * f \) converge uniformly to \( \partial^\alpha f \), \( |\alpha| \leq k \), in compact subsets too. Remember that \( \varphi^\epsilon(x) = 1/\epsilon^n \varphi(x/\epsilon) \).

6. This problem shows that, in \( L^\infty(\mathbb{R}^n) \), continuity of the translation holds if and only if \( f \) is uniformly continuous (the “if” part of the equivalence being trivial). Let \( f \in L^\infty(\mathbb{R}^n) \) satisfy \( \lim_{h \to 0} ||f(\cdot - h) - f(\cdot)||_{L^\infty(\mathbb{R}^n)} = 0 \). Prove that, in this case, \( f \) is equal a.e. to a uniformly continuous function. Suggestion: Define the averages of \( f \) over balls of radius \( r \) as

\[
A_r f(x) = \frac{1}{\text{Vol}(B_r(x))} \int_{B_r(x)} f(y) \, dy,
\]

and, trying to recognize here the formula of the convolution of \( f \) by an appropriate approximation of the identity, use it to approximate \( f \) and reach the desired conclusion.