Harmonic Analysis

1st Exam

24 June 2010

Carefully write down all relevant computations and justifications

1. Show that the $L^1(\mathbb{R}^n)$ kernel of a convolution operator in $L^p(\mathbb{R}^n)$ is unique. That is, let $K$ and $\tilde{K}$ be kernels in $L^1(\mathbb{R}^n)$, such that for a certain fixed $1 \leq p \leq \infty$ and all $f \in L^p(\mathbb{R}^n)$, $K * f = \tilde{K} * f$. Show that, necessarily, $K = \tilde{K}$.

2. Let $T$ be a linear and symmetric operator, defined on the space of measurable functions on $\mathbb{R}^n$. Symmetric, here, means that the following holds
\[
\int_{\mathbb{R}^n} T f(x) g(x) \, dx = \int_{\mathbb{R}^n} f(x) T g(x) \, dx,
\]
for any measurable functions $f, g$, for which one of the integrals (and therefore the other) exists.

a) Show that $T$ is then a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, with $1 \leq p, q \leq \infty$, if and only if it is also bounded from $L^{p'}(\mathbb{R}^n)$ to $L^{q'}(\mathbb{R}^n)$ ($p'$ and $q'$ denote, as usual, the conjugate exponents of $p$ and $q$, respectively).

b) Conclude that, if such a symmetric operator $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is bounded, it will also be bounded from $L^r(\mathbb{R}^n)$ to $L^s(\mathbb{R})$, for all exponents $r$ between $p$ and $q'$, with $s$ satisfying $1/p + 1/s = 1/q + 1/r$.

3. Use Plancherel’s theorem to prove that whenever $f, g \in L^1(\mathbb{T})$ satisfy $f * f, g * g \in L^2(\mathbb{T})$, then $f * g \in L^2(\mathbb{T})$. Estimate the norm $\|f * g\|_{L^2(\mathbb{T})}$ in terms of the norms $\|f * f\|_{L^2(\mathbb{T})}$ and $\|g * g\|_{L^2(\mathbb{T})}$.

4. Prove that the Fourier transform map, which takes a finite (and regular) Borel measure $\mu \in \mathcal{M}(\mathbb{T})$, to its Fourier coefficients $\hat{\mu}(n) = \int_{\mathbb{T}} e^{-int} \, d\mu(t)$, is one to one (injective). Equivalently, suppose that $\mu \in \mathcal{M}(\mathbb{T})$ satisfies $\hat{\mu}(n) = 0$ for all $n \in \mathbb{Z}$, and conclude that $\mu$ is the zero measure on $\mathbb{T}$.

5. Show that, if the trigonometric series $\sum_{n=-\infty}^{+\infty} c_n e^{int}$ converges to $f$ in $L^p(\mathbb{T})$, for any fixed $1 \leq p \leq \infty$, i.e. if $\|\sum_{n=-N}^{N} c_n e^{int} - f\|_{L^p(\mathbb{T})} \to 0$ when $N \to \infty$, then the coefficients $c_n$ are necessarily equal to the Fourier coefficients of $f$
\[
c_n = \hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-int} \, dt.
\]

6. Show that $L^2(\mathbb{T}) * L^2(\mathbb{T}) = A(\mathbb{T})$, i.e., that $f \in A(\mathbb{T})$ if and only if there exist two functions $g, h \in L^2(\mathbb{T})$ such that $f = g * h$. 
7. Let \( f : \mathbb{T} \to \mathbb{R}, f \in L^1(\mathbb{T}) \), be given by

\[
    f(x) = \begin{cases} 
        \sin \left( \frac{1}{x} \right), & \text{se } x \in ]-\pi, \pi[ \setminus \{0\} \\ 
        0 & \text{se } x = 0, \pi.
    \end{cases}
\]

Study the pointwise convergence of the partial sums of the Fourier series of this function at every point of \( \mathbb{T} \), justifying your answer carefully and in detail.

8. a) What can you say about the pointwise convergence of Fourier series of arbitrary functions in \( L^1(\mathbb{T}) \)?

b) What if, instead of pointwise convergence, one considers convergence in the norm \( \| \cdot \|_{L^1(\mathbb{T})} \)?

c) Prove that the partial sums of the Fourier series of \( f \in L^1(\mathbb{T}) \) always converge to \( f \) in the sense of distributions, that is, for any \( \phi \in C^\infty(\mathbb{T}) \) show that

\[
    \frac{1}{2\pi} \int_{\mathbb{T}} S_N(f)(t) \phi(t) \, dt \to \frac{1}{2\pi} \int_{\mathbb{T}} f(t) \phi(t) \, dt.
\]

Do only one of the following two problems, noting that they count differently

9. Prove that \( u \in h^1(\mathbb{D}) \) if and only if \( u_r = P_r * \mu \), where \( P_r \) is the Poisson kernel, and \( \mu \) is a unique finite (and regular) Borel measure. (Hint: Notice that this was done in class for the case of the case of Fejér kernel. The proof here is exactly the same).

10. Let \( 1 \leq p, q < \infty \).

a) Define the weak-\( L^p \) space \( L^p_w(X) \), for \( 1 \leq p < \infty \) and state what its quasi-norm \( \| \cdot \|_{L^p_w(X)} \) is.

b) Define what a linear operator of weak type \( (p, q) \) is.

c) Prove that an operator of strong type \( (p, q) \) is necessarily of weak type \( (p, q) \).