LESSON 12 - STUDY GUIDE

Abstract. For the final lesson about L^p spaces, before we start talking at last about Fourier series, I will cover the Riesz-Thorin interpolation theorem. Together with the Marcinkiewicz interpolation theorem, that we will see towards the end of the course, these are the two fundamental theorems about interpolation of operators between L^p and similar spaces.

1. **Riesz-Thorin Interpolation Theorem.**

Study material: In my opinion, by far the best introduction and motivation for interpolation results in real analysis, is Terence Tao's post on his blog **245C, Notes 1: Interpolation of L^p spaces** for the advanced real analysis course that he sometimes teaches at UCLA. From a textbook point of view, the clearest and most perfect presentation is in Stein and Weiss [5], chapter **V - Interpolation of Operators**, in particular Section **5.1 The M. Riesz Convexity Theorem and Interpolation of Operators Defined on L^p Spaces**, pg.177–183. Of course Folland's book [3] also has a very complete and rigorous presentation of both the Riesz-Thorin and the Marcinkiewicz interpolation theorems, in section **6.5 Interpolation of L^p Spaces** but, being very technical results, if one does not do a good job of motivating the results, like Tao does, the presentation ends up being very dry and somewhat painful to read, which I find to be the case with Folland here. Finally, Grafakos [4] also covers both interpolation results in section **1.3 Interpolation**, but except for the fact that he also includes Stein's extension of the Riesz-Thorin interpolation Theorem, for analytic families of operators, it is not as good and complete a presentation as any of the previous ones.

Let us start with a couple of examples. For the first example, we define the linear operator consisting of "multiplication by g", for a fixed $g \in L^q(\mathbb{R}^n)$, $1 \le q \le \infty$. Denoting that operator by T_g, it is defined by $T_g(f) = gf$ and we have, from Hölder's inequality

- if $f \in L^p(\mathbb{R}^n)$, with $p = q'$, then $T_g(f) = gf \in L^1(\mathbb{R}^n)$ and $\|T_g(f)\|_{L^1(\mathbb{R}^n)} \le \|g\|_{L^q(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}$

and, trivially,

- if $f \in L^\infty(\mathbb{R}^n)$ then $T_g(f) = gf \in L^q(\mathbb{R}^n)$ and $\|T_g(f)\|_{L^q(\mathbb{R}^n)} \le \|g\|_{L^q(\mathbb{R}^n)} \|f\|_{L^\infty(\mathbb{R}^n)}$.

We can now ask whether this operator could be interpolated for functions $f \in L^s(\mathbb{R}^n)$, with s between the two previous extremes $q' \le s \le \infty$. The answer comes easily from the generalized Hölder inequality, that tells us that, for $\frac{1}{r} = \frac{1}{q} + \frac{1}{s}$, with $0 < r \le q, s \le \infty$, we have

$$f \in L^s(\mathbb{R}^n) \Rightarrow T_g(f) = gf \in L^r(\mathbb{R}^n) \quad \text{and} \quad \|T_g(f)\|_{L^r(\mathbb{R}^n)} \le \|g\|_{L^q(\mathbb{R}^n)} \|f\|_{L^s(\mathbb{R}^n)}.$$

So, for fixed q and $q' \le s \le \infty$ the relation $\frac{1}{r} = \frac{1}{q} + \frac{1}{s}$ implies $1 \le r \le q$, which are exactly the two extreme exponents of the L^p spaces where T_g is bounded, from the previous two estimates.

In fact, an even more accurate numerology can be obtained if we interpolate the inverses of the extreme L^p exponents. For $q' \le s \le \infty$, let us start by determining the $\theta \in [0,1]$ for which

$$\frac{1}{s} = \frac{\theta}{q'} + \frac{1-\theta}{\infty},$$

Date: May 3, 2020.

which, of course is $\theta = q'/s$. Then, the generalized Hölder relation $\frac{1}{r} = \frac{1}{s} + \frac{1}{q}$ implies

$$\frac{1}{r} = \frac{1}{s} + \frac{1}{q} \Leftrightarrow \frac{1}{r} = \frac{\theta}{q'} + \frac{1}{q} \Leftrightarrow \frac{1}{r} = \frac{\theta}{1} + \frac{1-\theta}{q}.$$

So the interpolated inverses of the extreme exponents of the L^p spaces for T_g correspond precisely to those for f.

Here is another example. At the end of Lesson 10, after studying different L^p estimates for the convolution, we concluded that, for fixed $f \in L^p(\mathbb{R}^n)$, we have, from Young's inequality

- if $f \in L^p(\mathbb{R}^n), g \in L^1(\mathbb{R}^n)$ then $\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}$

and from Hölder's inequality, for $q = p'$,

- if $f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n)$ then $\|f * g\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}$.

Assuming interpolation is possible, let us now see where it would lead us in this case. Interpolating between $g \in L^1(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$,

$$\frac{1}{s} = \frac{\theta}{1} + \frac{1-\theta}{q},$$

then, the interpolated convolution should yield $f * g \in L^r(\mathbb{R}^n)$ for $g \in L^s(\mathbb{R}^n)$, with

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{\infty},$$

which, removing θ from both equations, leads to

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{s}.$$

And this is precisely the numerology of the generalized Young inequality, Theorem 1.9, in Lesson 10.

So, even though both of these examples can be proved by appropriately exploiting Hölder's inequality only, they are indeed interpolation results. The goal of interpolation theorems is, therefore, to obtain abstract and general frameworks to guarantee this type of results.

The first such theorem is due to Marcel Riesz who, in 1926, in the process of proving the convergence of Fourier series in the L^p norm, developed an interpolation method in the framework of bilinear forms in L^p. Later, one of his students, Olof Thorin, extended the result by introducing complex analysis methods. This is what is now called the Riesz-Thorin interpolation theorem, whose current widely presented proof is actually due to a further simplification by Antoni Zygmund. Incidentally, the proof of M. Riesz's theorem on the L^p convergence of Fourier series that is standard today in Fourier analysis courses and books no longer uses the Riesz-Thorin interpolation theorem, but uses the Marcinkiewicz interpolation theorem instead. Marcinkiewicz was a brilliant student of Zygmund, who died at the young age of 30, right at the beginning of World War II, during the German and Soviet invasion of Poland, as a prisoner of war of the USSR (and to whom Zygmund's treatise on trigonometric series is dedicated). Marcinkiewicz used a completely different approach to interpolation of operators, based on real analysis methods only, to prove in 1939 a somewhat more general - but not as sharp - interpolation theorem. The full version that is known today as the Marcinkiewicz interpolation theorem is also due to later work by Zygmund.

These two interpolation theorems, Riesz-Thorin and Marcinkiewicz, have been developed into full theories of interpolation of operators between functions spaces, in advanced functional analysis, with the goal of studying which families of functions spaces and operators between them can be interpolated. But both still represent the quintessential examples of what are currently known as complex versus real interpolation methods. Among the many papers and books written on this subject, [1] and [2] are the best known (although almost unreadable, either of them...).

In order not to delve into interpolation for too long, and delay our introduction to Fourier series even further, I have chosen to cover only the Riesz-Thorin theorem for now. We will definitely need the

Marcinkiewicz theorem too, in particular to prove the convergence of Fourier series in L^p, but it requires setting up a few more concepts in L^p space theory - distribution functions, weak L^p spaces, etc. - and I will leave those for later, once we really need them.

Focusing only on the Riesz-Thorin interpolation theorem now, the first thing that needs to be done is to define an appropriate setting for this result. An issue that should immediately raise some eyebrows is what do we mean by "an operator", defined on two different L^p spaces. After all, in general, for $p_0 \neq p_1$ the spaces L^{p_0} and L^{p_1} are unrelated and none of them is contained in the other. So if we have two bounded operators

$$T_0 : L^{p_0} \to L^{q_0} \qquad \text{and} \qquad T_1 : L^{p_1} \to L^{q_1},$$

from a pure mathematical definition they are different maps, as they have different, unrelated domains. What does it mean then for T_0 to be "the same operator" as T_1? Two answers are possible, and we will now see that they are equivalent.

One possibility is to assume that there is a larger vector space of measurable functions, say X, which contains both L^{p_0} and L^{p_1} and on which a "common" linear operator T is defined a priori, such that its restriction to either L^{p_0} or L^{p_1} coincides with T_0 and T_1, respectively, i.e. $T_{|L^{p_0}} = T_0$ and $T_{|L^{p_1}} = T_1$. But any such larger vector space should then contain the sum $L^{p_0} + L^{p_1} = \{f + g : f \in L^{p_0},\, g \in L^{p_1}\}$. And being itself a vector space, $L^{p_0} + L^{p_1}$ is then the smallest vector space where such a common T should be defined.

Another possibility is to say that $T_0 : L^{p_0} \to L^{q_0}$ and $T_1 : L^{p_1} \to L^{q_1}$ are "the same operator" if they coincide on the intersection $L^{p_0} \cap L^{p_1}$. For $p_0, p_1 \neq \infty$ this intersection is always dense in both L^{p_0} and L^{p_1}, as it contains the space Σ of simple functions which vanish outside sets of finite measure. And it is a simple consequence of sequential continuity that there can only exist one continuous extension of a map from a dense subset of a metric space to the whole space, so T_0 and T_1 are uniquely defined on the whole of L^{p_0} and L^{p_1}, respectively, by their common values on $L^{p_0} \cap L^{p_1}$. Being uniquely defined, it thus makes sense as well to say that T_0 and T_1 are "the same operator" if they agree on $L^{p_0} \cap L^{p_1}$.

We will now see that, in the fact, both of the previous approaches to the meaning of "same operator" are equivalent. In fact, from the inclusion results for L^p spaces we know that if we have, say, $0 < p_0 \leq p \leq p_1 \leq \infty$ then $L^{p_0} \cap L^{p_1} \subset L^p \subset L^{p_0} + L^{p_1}$. So, if an operator is defined on $L^{p_0} + L^{p_1}$ then it is unambiguously defined on all L^p, with intermediate exponents p, as well as on $L^{p_0} \cap L^{p_1}$. On the other hand, if we start with $T_0 : L^{p_0} \to L^{q_0}$ and $T_1 : L^{p_1} \to L^{q_1}$ that coincide on the intersection $L^{p_0} \cap L^{p_1}$ then a linear operator T, defined on $L^{p_0} + L^{p_1}$ in order to agree with T_0 on L^{p_0} and T_1 on L^{p_1}, should necessarily satisfy $T(f + g) = T_0(f) + T_1(g)$ for $f + g \in L^{p_0} + L^{p_1}$ with $f \in L^{p_0}$ and $g \in L^{p_1}$. Of course the problem could be that, $L^{p_0} + L^{p_1}$ not being a direct sum, this definition might not produce the same result for another different pair $\tilde{f} \in L^{p_0}$ and $\tilde{g} \in L^{p_1}$ for which $\tilde{f} + \tilde{g} = f + g$. However, then we would have $\tilde{f} - f = g - \tilde{g}$, and this would mean that these differences would both be in L^{p_0} and L^{p_1}, that is $\tilde{f} - f = g - \tilde{g} \in L^{p_0} \cap L^{p_1}$, where T_0 and T_1 coincide. And we then conclude that $T_0(\tilde{f} - f) = T_1(g - \tilde{g})$ and therefore $T_0(\tilde{f}) + T_1(\tilde{g}) = T_0(f) + T_1(g)$ for whatever pair of functions in L^{p_0} and L^{p_1} that represent elements of $L^{p_0} + L^{p_1}$.

So, if we have two linear bounded operators $T_0 : L^{p_0} \to L^{q_0}$ and $T_1 : L^{p_1} \to L^{q_1}$ we will say that they are *the same operator* if they coincide on the intersection $L^{p_0} \cap L^{p_1}$ in which case, by the previous discussion, a uniquely defined linear operator T on $L^{p_0} + L^{p_1}$ exists that satisfies $T_{|L^{p_0}} = T_0$ and $T_{|L^{p_1}} = T_1$. This operator will also be then automatically defined on $L^p \subset L^{p_0} + L^{p_1}$, for all the intermediate $p_0 \leq p \leq p_1$. We can therefore assume, for the setting of the theorem, that there is a common operator T defined from the start on $L^{p_0} + L^{p_1}$, and thus, on all L^p, $p_0 \leq p \leq p_1$.

Theorem 1.1. (Riesz-Thorin Interpolation Theorem) *Let (X, μ) and (Y, ν) be measure spaces and $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. The measure ν on Y is also required to be semifinite when $q_0 = q_1 = \infty$.*

If $T : (L^{p_0}(X,\mu) + L^{p_1}(X,\mu)) \to (L^{q_0}(Y,\nu) + L^{q_1}(Y,\nu))$ *is a linear operator such that*

$$\|Tf\|_{q_0} \le M_0 \|f\|_{p_0} \qquad \text{and} \qquad \|Tg\|_{q_1} \le M_1 \|g\|_{p_1},$$

for all $f \in L^{p_0}(X,\mu)$ *and* $g \in L^{p_1}(X,\mu)$*, and we consider the interpolated exponents*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

and

$$\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

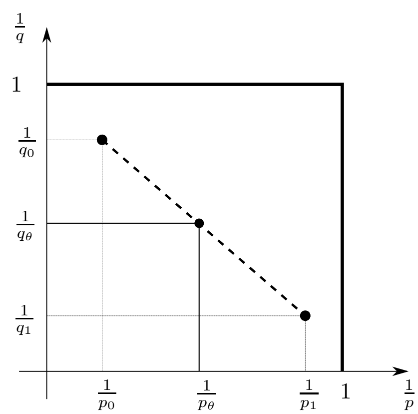
for some $\theta \in [0,1]$*, then* $T : L^{p_\theta}(X,\mu) \to L^{q_\theta}(Y,\nu)$ *is bounded and*

$$\|Tf\|_{q_\theta} \le M_0^{1-\theta} M_1^{\theta} \|f\|_{p_\theta},$$

for all $f \in L^{p_\theta}(X,\mu)$.

Remark 1.2. *Two observations deserve to be made at this point.*

- *The Riesz-Thorin interpolation theorem is often also called the Riesz convexity theorem, as it is done in* [5]*. The reason stems from the fact that, if we denote by* M_θ *the bound for the interpolated operator, the theorem shows that* $M_\theta \le M_0^{1-\theta} M_1^{\theta}$*, and this is equivalent to the fact that* $\log M_\theta$ *is a convex function.*
- *A convenient graphical interpretation of the Riesz-Thorin interpolation numerology can be easily obtained by plotting the pairs* (p_0, q_0) *and* (p_1, q_1) *in a* $\frac{1}{p} \times \frac{1}{q}$ *frame, and observing that the interpolated* (p_θ, q_θ) *pairs correspond to the points on the connecting line segment, as in the following figure*



Yet again, this theorem shows that the appropriate numerology for L^p *spaces always involves the inverses* $1/p$*, and not* p*. Starting right from the beginning with Hölder's inequality, the same happens with many other inequalities, that we will not be able to cover in this course, like the celebrated Sobolev or Strichartz estimates. So it is somewhat regrettable that* L^p *spaces were not historically denoted by* $L^{\frac{1}{p}}$ *instead.*

The proof of the Riesz-Thorin interpolation theorem or, more generally, the interpolation theorems that use the complex method, are usually based on the following simple, but wonderful, result of complex analysis.

Lemma 1.3. (The Three Lines Lemma) *Let f be a bounded continuous function on the complex strip $0 \leq \mathrm{Re}(z) \leq 1$, holomorphic in its interior. If $|f(z)| \leq M_0$ for $\mathrm{Re}(z) = 0$ and $|f(z)| \leq M_1$ for $\mathrm{Re}(z) = 1$ then for $0 \leq \mathrm{Re}(z) = \theta \leq 1$ we have $|f(z)| \leq M_0^{1-\theta} M_1^{\theta}$.*

Proof. The proof requires M_0 and M_1 to be non-zero. So if any of them is zero we can always increase the corresponding bound to make it is strictly positive and then, at the end of the proof, take the limit $M_0 \to 0$ or $M_1 \to 0$ in the final interpolated inequality. we can increase them and substitute them by strictly positive bounds. So assuming that both $M_0 > 0$ and $M_1 > 0$ we can normalize the function f by doing

$$\tilde{f}(z) = \frac{f(z)}{M_0^{1-z} M_1^{z}},$$

to reduce to the case $M_0 = M_1 = 1$. Define now the function $f_\varepsilon(z) = \tilde{f}(z)\, e^{\varepsilon z(z-1)}$. Then f_ε is also continuous, holomorphic and bounded on the strip $0 \leq \mathrm{Re}(z) \leq 1$, but with $|f_\varepsilon(z)| \leq 1$ for $\mathrm{Re}(z) = 0$ and $\mathrm{Re}(z) = 1$. Besides, $|f_\varepsilon(z)| \to 0$ as $|\mathrm{Im}(z)| \to \infty$. So, if we cut the strip into a rectangle sufficiently far off with $-A \leq |\mathrm{Im}(z)| \leq A$, with A large enough, we can ensure that $|f_\varepsilon(z)| \leq 1$ on the boundary of this rectangle. And by the maximum modulus principle this implies that $|f_\varepsilon(z)| \leq 1$ on the whole rectangle. Taking A even larger, if necessary, in order to guarantee that $|f_\varepsilon(z)| \leq 1$ outside the rectangle, the same estimate then holds on the whole strip as well. Finally, for $\mathrm{Re}(z) = \theta$, taking $\varepsilon \to 0$ we conclude

$$\lim_{\varepsilon \to 0} |f_\varepsilon(z)| = |\tilde{f}(z)| = \frac{|f(z)|}{M_0^{1-\theta} M_1^{\theta}} \leq 1,$$

and this implies the desired result. □

The proof of the Riesz-Thorin interpolation theorem then follows by building a function, using duality, that depends holomorphically on z corresponding to a complex parameter such that $\mathrm{Re}(z) = 1/p$ and then using the Three Lines Lemma to obtain the intermediate bounds. Duality again yields the final estimates.

At this point I strongly recommend reading the extraordinary exposition of interpolation results in L^p spaces, and the proof of the Riesz-Thorin's theorem, in Terence Tao's blog post **245C, Notes 1: Interpolation of L^p spaces** as the best I could do at this point would only be to rewrite it here. The textbook presentations in [5] or [3] are also worth taking a look at.

To finish this lesson, it should still be pointed out that E. Stein extended the Riesz-Thorin interpolation theorem to what is now called Stein's theorem on interpolation of analytic families of operators (see [4] or [5]). It is a simple, but very useful and powerful extension where, instead of a fixed common operator T, a family of operators T_z is considered, also depending holomorphically on the complex parameter z, for the same strip $0 \leq \mathrm{Re}(z) \leq 1$. When compared to the classical Riesz-Thorin theorem, the proof of Stein's theorem barely changes because the only difference is that the holomorphic dependence on the complex parameter does not occur on the L^p exponents only, as before, but also on the parameter of the operators themselves. But the general strategy of the proof is pretty much analogous.

References

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[5] Elias Stein & Guido Weiss, *Introduction to Fourier Analysis on Euclidean Spaces,* Princeton University Press, 1971.