RANK-STABLE LIMIT OF COMPLETED MODULI SPACES OF INSTANTONS

JOÃO PAULO SANTOS

Abstract. We show that in the rank-stable limit, the inclusion of the moduli space of instantons over S^4 into the completion introduced in [3] is a homotopy equivalence. We prove also an analogous result for the moduli space of instantons over $\mathbb{P}^2$ at one point.

1. Introduction

Given a principal $SU(r)$ bundle $P \to S^4$, an instanton is a connection ∇ on P whose curvature F_∇ is a minimum of the Yang-Mills functional $\int |F_\nabla|^2$. The moduli space $\mathfrak{M}(P)$ of instantons based at $x_0 \in S^4$ is the quotient of the space of instantons by the action of the gauge group of automorphisms of P which are the identity on x_0. It depends only on r and $c_2(P) = k$ so we represent it by $\mathfrak{M}_k^r(S^4)$. In [3], Nakajima introduced a completion $\overline{\mathfrak{M}}_k^r(S^4)$ of $\mathfrak{M}_k^r(S^4)$ which is a resolution of singularities of the usual Donaldson-Uhlenbeck completion of $\mathfrak{M}_k^r(S^4)$.

this paper we extend this result to instantons on $\mathbb{P}^2$ and show that, in the limit when $r \to \infty$, the inclusions $\mathfrak{M} \to \overline{\mathfrak{M}}$ are homotopy equivalences.

For $r' > r$ there is an inclusion of pairs $(\overline{\mathfrak{M}}_k^r, \mathfrak{M}_k^r) \to (\overline{\mathfrak{M}}_k^{r'}, \mathfrak{M}_k^{r'}$ induced by the inclusion $SU(r) \to SU(r')$. In [2], [4] it was shown that the direct limit $\mathfrak{M}_k^\infty(X) = \lim_r \mathfrak{M}_k^r(X)$ has the homotopy type of $BU(k)$.

for $X = S^4$, and $BU(k) \times BU(k)$ for $X = \mathbb{P}^2$.

2. Instantons on S^4

We summarize the monad description of instantons on S^4 and $\mathbb{P}^2$ and the resolution of singularities of its completion introduced in [3]. Let W be a k-dimensional hermitian vector space. Let $\mathcal{R}_k^r$ be the space of configurations (a_1, a_2, b, c) with $a_i \in \mathrm{End}(W)$, $b \in \mathrm{Hom}(\mathbb{C}^r, W)$, $c \in \mathrm{Hom}(W, \mathbb{C}^r)$, obeying the integrability condition

$$[a_1, a_2] + bc = 0$$

and the perturbed moment map equation

$$[a_1, a_1^*] + [a_2, a_2^*] + bb^* - c^*c = -\zeta$$

for some non-zero real parameter ζ. Let $\tilde{\mathcal{R}}_k^r \subset \mathcal{R}_k^r$ be the subspace of 4-tuples obeying the two non-degeneracy conditions

(1) There is no proper subspace $W' \subset W$ such that

$$\mathrm{Im}\, b \subset W' \text{ and } a_i(W') \subset W' \ (i = 1, 2)$$

(2) There is no nonempty subspace $W' \subset W$ such that

$$W' \subset \mathrm{Ker}\, c \text{ and } a_i(W') \subset W' \ (i = 1, 2)$$

Observe that these conditions are automatically satisfied if both b and c have maximal rank. The group $U(W)$ acts freely on $\mathcal{R}_k^r$ by

$$g \cdot (a_1, a_2, b, c) = (ga_1g^{-1}, ga_2g^{-1}, gb, cg^{-1})$$

The quotient $\tilde{\mathcal{R}}_k^r/U(W)$ is isomorphic to the moduli space $\mathfrak{M}_k^r(S^4)$ and the quotient $\overline{\mathfrak{M}}_k^r(S^4) = \mathcal{R}/U(W)$ is a resolution of singularities of the Donaldson-Uhlenbeck completion of $\mathfrak{M}_k^r(S^4)$.

Now we consider the limit when $r \to \infty$. For $r' > r$, the inclusion $\mathbb{C}^r \to \mathbb{C}^{r'}$ induces a map $i_{r,r'} : \mathcal{R}_k^r \to \mathcal{R}_k^{r'}$. This map preserves the subspace $\tilde{\mathcal{R}}$ and descends to the quotient to give a map of pairs

$$\left(\overline{\mathfrak{M}}_k^r(S^4), \mathfrak{M}_k^r(S^4)\right) \to \left(\overline{\mathfrak{M}}_k^{r'}(S^4), \mathfrak{M}_k^{r'}(S^4)\right)$$

We define $\mathfrak{M}_k^\infty(S^4) = \varinjlim_r \mathfrak{M}_k^r(S^4)$, $\overline{\mathfrak{M}}_k^\infty(S^4) = \varinjlim_r \overline{\mathfrak{M}}_k^r(S^4)$.

Theorem 2.1. *The inclusion $j : \mathfrak{M}_k^\infty(S^4) \to \overline{\mathfrak{M}}_k^\infty(S^4)$ is a homotopy equivalence.*

Proof. We first show that the space $\mathcal{R}_k^\infty$ is contractible. This space is a CW-complex hence it is enough to show that its homotopy groups are trivial, and this will follow if we show that the inclusions $i_{r,r+2k} : \mathcal{R}_k^r \to \mathcal{R}_k^{r+2k}$ are null-homotopic. We identify $\mathbb{C}^{r+2k}$ with $\mathbb{C}^r \oplus W \oplus W$ via a complex hermitian isomorphism. Then we pick $\zeta_b, \zeta_c \in \mathbb{R}$ such that $\zeta_c - \zeta_b = \zeta$ and define a homotopy $H : \mathcal{R}_k^r \to \mathcal{R}_k^{r+2k}$ by

$$H_t(a_1, a_2, b, c) = \left(\sqrt{1-t}\, a_1, \sqrt{1-t}\, a_2, b_t, c_t\right)$$

where

$$b_t = \begin{pmatrix} \sqrt{1-t}\, b & 0 & \sqrt{t\zeta_b}\, 1 \end{pmatrix} \qquad c_t = \begin{pmatrix} \sqrt{1-t}\, c \\ \sqrt{t\zeta_c}\, 1 \\ 0 \end{pmatrix}$$

It is a direct verification that H is a well defined homotopy between $i_{r,r+2k}$ and a constant map. Furthermore, since b_t, c_t both have maximal rank for $t \neq 0$, the restriction of H to $\tilde{\mathcal{R}}_k^r$ defines a homotopy $H : \tilde{\mathcal{R}}_k^r \to \tilde{\mathcal{R}}_k^{r+2k}$. Hence we also conclude that $\tilde{\mathcal{R}}_k^\infty$ is contractible.

We now observe that the principal U-bundle $\tilde{\mathcal{R}}_k^\infty \to \mathfrak{M}_k^\infty(S^4)$ is the pullback $j^*\mathcal{R}_k^\infty$ of the bundle $\mathcal{R}_k^\infty \to \overline{\mathfrak{M}}_k^r$. Applying the five lemma to the long exact sequence of homotopy groups associated with these principal bundles, it follows that j induces isomorphisms on all homotopy groups, hence it is a homotopy equivalence. □

3. Instantons on $\mathbb{P}^2$

We now look at instantons on $\mathbb{P}^2$. We begin by sketching the monad description introduced in [1]. Let W_0, W_1 be k-dimensional hermitian vector spaces. Let $\mathcal{C}_k^r$ be the space of configurations $m = (a_1, a_2, d, b, c)$ where $a_i \in \mathrm{Hom}(W_1, W_0)$, $d \in \mathrm{Hom}(W_0, W_1)$, $b \in \mathrm{Hom}(\mathbb{C}^r, W_0)$, $c \in \mathrm{Hom}(W_1, \mathbb{C}^r)$, such that $a_1(W_1) + a_2(W_1) + b(\mathbb{C}^r) = W_0$, obeying the integrability condition

$$a_1 d a_2 - a_2 d a_1 + bc = 0.$$

Let $\tilde{\mathcal{C}}_k^r \subset \mathcal{C}_k^r$ be the subspace of configurations obeying the non-degeneracy conditions

(1) There are no proper subspaces $W_0' \subset W_0$ and $W_1' \subset W_1$ such that

$$\dim W_0' = \dim W_1',\ \operatorname{Im} b \subset W_0',\ d(W_0') \subset W_1' \text{ and } a_i(W_1') \subset W_0'\ (i = 1, 2)$$

(2) There are no nonempty subspaces $W_0' \subset W_0$ and $W_1' \subset W_1$ such that

$$\dim W_0' = \dim W_1',\ W_1' \subset \operatorname{Ker} c,\ d(W_0') \subset W_1' \text{ and } a_i(W_1') \subset W_0'\ (i = 1, 2)$$

These conditions are automatically satisfied if b and c have maximal rank. The group $Gl(W_0) \times Gl(W_1)$ acts on $\mathcal{C}_k^r$ by

$$(g_0, g_1) \cdot (a_1, a_2, d, b, c) = (g_0 a_1 g_1^{-1}, g_0 a_2 g_1^{-1}, g_1 d g_0^{-1}, g_0 b, c g_1^{-1})$$

The restriction of this action to $\tilde{\mathcal{C}}_k^r$ is free and the quotient is isomorphic to the moduli space $\mathfrak{M}_k^r(\mathbb{P}^2)$.

Now let $\mathcal{R}_k^r \subset \mathcal{C}_k^r$ be the subspace of configurations obeying the moment map equations

$$\begin{cases} a_1 a_1^* + a_2 a_2^* + b b^* = 1 \\ a_1^*(1 + d^* d) a_1 + a_2^*(1 + d^* d) a_2 + c^* c = 1 + d d^* \end{cases}$$

Using the first equation, we can write the second equation in the sometimes more useful form

$$[\, d a_1, (d a_1)^* \,] + [\, d a_1, (d a_1)^* \,] - a_1^* a_1 - a_2^* a_2 + d b (d b)^* - c^* c = -1$$

Let $\tilde{\mathcal{R}}_k^r = \tilde{\mathcal{C}}_k^r \cap \mathcal{R}_k^r$. The $Gl(W_0) \times Gl(W_1)$ action on $\mathcal{C}_k^r$ induces a free $U(W_0) \times U(W_1)$ action on $\tilde{\mathcal{R}}_k^r$ and the quotient is also isomorphic to $\mathfrak{M}_k^r(\mathbb{P}^2)$ (see [1]).

We now perturb the moment map equations by introducing a non-integer positive parameter $\zeta \in \mathbb{R}^+ \setminus \mathbb{Z}$:

$$\begin{cases} a_1 a_1^* + a_2 a_2^* + b b^* = 1 \\ a_1^*(1 + d^* d) a_1 + a_2^*(1 + d^* d) a_2 + c^* c = \zeta + d d^* \end{cases} \tag{1}$$

Again, we can substitute the second equation by

$$[\, d a_1, (d a_1)^* \,] + [\, d a_1, (d a_1)^* \,] - a_1^* a_1 - a_2^* a_2 + d b (d b)^* - c^* c = -\zeta \tag{2}$$

We let $\mathcal{R}_\zeta \subset \mathcal{C}$ denote the sobspace of solutions to these equations.

Theorem 3.1. *The $U(W_0) \times U(W_1)$ action on $\mathcal{R}_\zeta$ is free.*

Proof. Suppose (g_0, g_1) stabilizes (a_1, a_2, d, b, c). Let $W_i(\lambda) \subset W_i$ denote the λ-eigenspace of g_i. Then

$$d(\, W_0(\lambda) \,) \subset W_1(\lambda),\ a_i(\, W_1(\lambda) \,) \subset W_0(\lambda)\ (i = 1, 2)$$

and, for $\lambda \neq 1$,

$$W_0(\lambda) \subset \operatorname{Ker} b^* \text{ and } W_1(\lambda) \subset \operatorname{Ker} c$$

Hence, restricting the perturbed moment map equations (1) to $W_i(\lambda)$. $\lambda \neq 1$, and substituting (2) for the second equation we get

$$\begin{cases} a_1 a_1^* + a_2 a_2^* = 1 \\ [\, d a_1, (d a_1)^* \,] + [\, d a_1, (d a_1)^* \,] - a_1^* a_1 - a_2^* a_2 = -\zeta \end{cases}$$

Taking the trace, since $\operatorname{Tr}(a_1 a_1^* + a_2 a_2^*) = \operatorname{Tr}(a_1^* a_1 + a_2^* a_2)$ we get $\dim W_0(\lambda) = \zeta \dim W_1(\lambda)$. This is impossible unless $W_0(\lambda) = W_1(\lambda) = 0$ for any $\lambda \neq 1$ which implies that $(g_0, g_1) = (1, 1)$. □

Let $\overline{\mathfrak{M}}_k^r(\mathbb{P}^2)$ be the quotient of $\mathcal{R}_{\zeta,k}^r$ by the $U(W_0) \times U(W_1)$ action. We again have maps of pairs

$$\left(\overline{\mathfrak{M}}_k^r(\mathbb{P}^2), \mathfrak{M}_k^r(\mathbb{P}^2)\right) \to \left(\overline{\mathfrak{M}}_k^{r'}(\mathbb{P}^2), \mathfrak{M}_k^{r'}(\mathbb{P}^2)\right)$$

and we define $\mathfrak{M}_k^\infty(S^4) = \varinjlim_r \mathfrak{M}_k^r(S^4)$, $\overline{\mathfrak{M}}_k^\infty(S^4) = \varinjlim_r \overline{\mathfrak{M}}_k^r(S^4)$.

Theorem 3.2. *The inclusion $j : \mathfrak{M}_k^\infty(\mathbb{P}^2) \to \overline{\mathfrak{M}}_k^\infty(\mathbb{P}^2)$ is a homotopy equivalence.*

Proof. The proof follows the same lines as the one for S^4. We will show that the inclusions $i_{r,r+3k} : \mathcal{R}_k^r \to \mathcal{R}_k^{r+2k}$ are null-homotopic. First we identify $\mathbb{C}^{r+3k}$ with $\mathbb{C}^r \oplus W_0 \oplus W_0 \oplus W_1$ via a complex hermitian isomorphism. Then we define a homotopy $H : \mathcal{R}_k^r \to \mathcal{R}_k^{r+2k}$ by

$$H_t(a_1, a_2, d, b, c) = \left(\sqrt{1-t}\, a_1 \,,\, \sqrt{1-t}\, a_2 \,,\, d \,,\, b_t \,,\, c_t \right)$$

where

$$b_t = \begin{pmatrix} \sqrt{1-t}\, b & 0 & \sqrt{t}\, 1 & 0 \end{pmatrix} \qquad c_t = \begin{pmatrix} \sqrt{1-t}\, c \\ \sqrt{t}\, d^* \\ 0 \\ \sqrt{t\zeta}\, 1 \end{pmatrix}$$

It is a direct verification that H is a well defined homotopy between $i_{r,r+2k}$ and the map $f : \mathcal{R}_k^r \to \tilde{\mathcal{R}}_k^{r+3k}$ given by $f(a_1, a_2, d, b, c) = (0, 0, d, b_1, c_1)$. Furthermore, the restriction of H to $\tilde{\mathcal{R}}_k^r$ defines a homotopy $H : \tilde{\mathcal{R}}_k^r \to \tilde{\mathcal{R}}_k^{r+2k}$. To conclude the proof we follow H with another homotopy

$$F_t(a_1, a_2, d, b, c) = (0, 0, (1-t)\, d, b_1, C_t)$$

where

$$C_t = \begin{pmatrix} 0 & (1-t)\, d^* & 0 & \sqrt{\zeta}\, 1 \end{pmatrix}^T$$

F is then a homotopy between f and a constant map, which finishes the proof. $\square$

References

1. A. King, *Instantons and holomorphic bundles on the blown-up plane*, Ph.D. thesis, Worcester College, Oxford, 1989.
2. F. Kirwan, *Geometric invariant theory and the Atiyah-Jones conjecture*, The Sophus Lie Memorial Conference (Oslo, 1992), Scand. Univ. Press, Oslo, 1994, pp. 161–186.
3. H. Nakajima, *Resolutions of moduli spaces of ideal instantons on* $\mathbb{R}^4$, Topology, geometry and field theory, World Sci. Publishing, River Edge, NJ, 1994, pp. 129–136.
4. M. Sanders, *Classifying spaces and Dirac operators coupled to instantons*, Trans. Amer. Math. Soc. **347** (1995), no. 10, 4037–4072.