FRAMED HOLOMORPHIC BUNDLES ON RATIONAL SURFACES

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Abstract. We study the moduli space of framed holomorphic bundles of any rank, over the blow-up of $\mathbb{CP}^2$ at q points. For $c_2 = 1, 2$ we introduce an open cover of the moduli space and describe its nerve. In the limit when $r \to \infty$ we use this result to obtain the homotopy type of the moduli spaces. In particular, we compute the cohomology of the moduli spaces.

1. Introduction

Fix a line $L_\infty \subset \mathbb{CP}^2$ and let X_q denote the blow-up of $\mathbb{CP}^2$ at q points $x_1, \dots, x_q \notin L_\infty$. In this paper we will study the moduli space $\mathfrak{M}^r_k(X_q)$ of equivalence classes of pairs $(\mathcal{E}, \phi)$, where $\mathcal{E}$ is a holomorphic rank r bundle over X_q with $c_1 = 0$ and $c_2 = k$, holomorphically trivial at L_∞, and $\phi : \mathcal{E}|_{L_\infty} \to \mathcal{O}^r_{L_\infty}$ is a holomorphic trivialization. This moduli space is a special case of the moduli of framed sheaves introduced in [15], [11]. See also [16]. The interest on these spaces was motivated by the study of moduli spaces of instantons: in [5], [17] it was shown that $\mathfrak{M}^r_k(X_q)$ is isomorphic as a real analytic space to the moduli space of based charge k $SU(r)$ instantons over a connected sum of q copies of $\overline{\mathbb{CP}}^2$. Monad descriptions for these spaces were introduced in [1], [7], for $X_0 = \mathbb{CP}^2$, [4], [12] for X_1 and [14], [6], for the general case. In this paper we study the moduli spaces using results about bundles on the blow-up of a complex surface (see [19]): analyzing the effect of blowing up on the topology of the moduli space we reduce the study of the moduli spaces over X_q to the moduli spaces over X_0, X_1. Our first result is

Theorem 1.1. *Let $I \subset \{1, \dots, q\}$ and write $|I| = \#I$. Then, for each I with $|I| \le k$ there is an open set $A_I \subset \mathfrak{M}^r_k(X_q)$ such that*
1. *$\{A_I\}_{|I|=k}$ is an open cover of $\mathfrak{M}^r_k(X_q)$;*
2. *$A_I \cap A_J = A_{I\cap J}$;*
3. *There are homeomorphisms $A_I \cong \mathfrak{M}^r_{|I|}(X_q)$.*

For $k = 1$ this describes $\mathfrak{M}^r_1(X_q)$ in terms of the moduli spaces over X_0 and X_1, which are well understood. Hence we can get a cell structure for $\mathfrak{M}^r_1(X_q)$. For $k = 2$ we will prove:

Theorem 1.2. *There are open sets $A_i, N_{ij} \subset \mathfrak{M}^r_2(X_q)$ such that*
1. *$\{A_i\}_i \cup \{N_{ij}\}_{i<j}$ is an open cover of $\mathfrak{M}^r_2(X_q)$;*
2. *There is an open set $A_\emptyset \subset \mathfrak{M}^r_2(X_q)$ such that $A_i \cap A_j = A_\emptyset$ for $i \ne j$.*
3. *For $k \notin \{i, j\}$, $N_{ij} \cap A_k = N_{ij} \cap A_\emptyset$. For different sets $\{i, j\} \ne \{k, l\}$, $N_{ij} \cap N_{kl} = \emptyset$.*

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(4) *There are homotopy equivalences*

$$\begin{array}{ll} A_i \simeq \mathfrak{M}_2^r(X_1) & A_\emptyset \simeq \mathfrak{M}_2^r(X_0) \\ N_{ij} \simeq \mathfrak{M}_1^r(X_1) \times \mathfrak{M}_1^r(X_1) & \\ N_{ij} \cap A_i \simeq \mathfrak{M}_1^r(X_1) \times \mathfrak{M}_1^r(X_0) & N_{ij} \cap A_\emptyset \simeq \mathfrak{M}_1^r(X_0) \times \mathfrak{M}_1^r(X_0) \end{array}$$

We then apply these results to the study of the rank stable moduli space, which is defined as follows: when $r_2 > r_1$, there is a map $\mathfrak{M}_k^{r_1}(X_q) \to \mathfrak{M}_k^{r_2}(X_q)$ induced by taking direct sum with a trivial rank $r_2 - r_1$ bundle. We define the rank stable moduli space as the direct limit $\mathfrak{M}_k^\infty(X_q) \stackrel{\text{def}}{=} \varinjlim_r \mathfrak{M}_k^r(X_q)$. Using the monad descriptions, it was shown in [13], [18] and [2] that

$$\mathfrak{M}_k^\infty(X_0) \simeq BU(k) \ , \ \mathfrak{M}_k^\infty(X_1) \simeq BU(k) \times BU(k) \tag{1}$$

We will prove as a corollary of theorem 1.1 that

Theorem 1.3. *There is a homotopy equivalence*

$$\mathfrak{M}_1^\infty(X_q) \simeq BU(1) \times \left(\bigvee_{i=1}^q BU(1) \right)$$

Together with the results of [5] and [17], this shows that for a large class of metrics conjecture 1.1 in [2] is false.

Using theorem 1.2 we will compute the cohomology of $\mathfrak{M}_2^\infty(X_q)$:

Theorem 1.4. *Let $K_C \subset \mathbb{Z}[x_1, x_2, x_3, x_4] \cong H^*\left(BU(1)^{\times 4} \right)$ be the ideal generated by the product $x_1 x_2$ and let $K_A \subset \mathbb{Z}[a_1, k_1, a_2, k_2] \simeq H^*\left(BU(2)^{\times 2} \right)$ be the ideal generated by k_1, k_2. Then, as graded modules over $\mathbb{Z}$, we have an isomorphism*

$$H^*\left(\mathfrak{M}_2^\infty(X_q) \right) \cong \mathbb{Z}[a_1, a_2] \oplus K_A^{\oplus q} \oplus K_C^{\oplus \frac{q(q-1)}{2}}$$

The plan of this paper is as follows: In section 2 we prove theorem 1.1 and in section 3 we use it to prove theorem 1.3. In section 4 we prove theorem 1.2 and in the next two sections we apply it to prove theorem 1.4: in section 5 we study the open cover from theorem 1.2 in the limit when $r \to \infty$; in section 6 we use the spectral sequence associated with this open cover (see [21]) to prove theorem 1.4. In the appendix we gather some results about the monad constructions of the moduli spaces for $q = 0, 1$.

This paper is based on results in the author's thesis [20].

2. An open cover of $\mathfrak{M}_k^r(X_q)$

Fix a subset $I \subset \{1, \dots, q\}$ and let $\pi_I : X_q \to X_{|I|}$ ($|I| = \#I$) be the blow up at points x_j, $j \notin I$. π_I induces a map

$$\pi_I^* : \mathfrak{M}_k^r(X_{|I|}) \to \mathfrak{M}_k^r(X_q) \tag{2}$$

Let $q \geq k$. The objective of this section is to prove

Theorem 2.1. $\left\{ \pi_I^* \mathfrak{M}_k^r(X_{|I|}) \right\}_{|I|=k}$ *is an open cover of* $\mathfrak{M}_k^r(X_q)$. *Furthermore*

$$\pi_I^* \mathfrak{M}_k^r(X_{|I|}) \cap \pi_J^* \mathfrak{M}_k^r(X_{|J|}) = \pi_{I \cap J}^* \mathfrak{M}_k^r(X_{|I \cap J|})$$

and we have isomorphisms

$$\mathfrak{M}^r_k(X_{|I|}) \underset{\pi_{I*}}{\overset{\pi_I^*}{\underset{\longleftarrow}{\overset{\cong}{\longrightarrow}}}} \pi_I^*\mathfrak{M}^r_k(X_{|I|})$$

From this open cover we can build a spectral sequence converging to $H^*(\mathfrak{M}^r_k(X_q))$. The case $k = 2$ will be treated in section 7. For the general case see [20], section 4.3. We turn now to the proof of theorem 2.1. We begin by proving the last statement:

Proposition 2.2. *We have isomorphisms*

$$\mathfrak{M}^r_k(X_{|I|}) \underset{\pi_{I*}}{\overset{\pi_I^*}{\underset{\longleftarrow}{\overset{\cong}{\longrightarrow}}}} \pi_I^*\mathfrak{M}^r_k(X_{|I|})$$

where π_I^* and π_{I*} are inverses of each other. We also have

$$\pi_I^*\mathfrak{M}^r_k(X_{|I|}) = \{\mathcal{E} \in \mathfrak{M}^r_k(X_q) \,|\, \mathcal{E}|_{L_i} \textit{ is trivial for } i \notin I\}$$

Proof. From theorem 3.2 in [9] it follows that, if a bundle is trivial on the exceptional divisor then it is also trivial on a neighborhood of the exceptional divisor. Hence, a bundle $\mathcal{E} \to \tilde{X}$ on a blow up $\pi : \tilde{X} \to X$ is trivial on the exceptional divisor if and only if $\tilde{\mathcal{E}} = \pi^*\pi_*\tilde{\mathcal{E}}$. The statement of the proposition follows. □

Proof of theorem 2.1. From proposition 2.2 it follows that

$$\pi_I^*\mathfrak{M}^r_k(X_{|I|}) \cap \pi_J^*\mathfrak{M}^r_k(X_{|J|}) = \pi_{I\cap J}^*\mathfrak{M}^r_k(X_{|I\cap J|})$$

To show that $\mathfrak{M}^r_k(X_q) \subset \bigcup_{|I|=k} \pi_I^*\mathfrak{M}^r_k(X_k)$ we only need to show that a bundle $\mathcal{E} \in \mathfrak{M}^r_k(X_q)$ is trivial in at least $q-k$ exceptional lines ($q > k$). We prove this result by induction in q. Assume $\mathcal{E}$ is not trivial in L_1. Let $p : X_q \to X_{q-1}$ be the blow up at x_1 and let $\mathcal{E}' = (\pi_*\mathcal{E})^{\vee\vee}$. Then $c_2(\mathcal{E}') < k$ so we can apply induction. The proof is completed by noting that we cannot have bundles with negative c_2 by Bogomolov inequality for framed bundles (see [15]).

Finally we have to show that $\pi_I^*\mathfrak{M}^r_k(X_{|I|})$ is open. Let H be an ample divisor. Choose N such that $H^i(\mathcal{E}(NH)) = 0$ for all $\mathcal{E} \in \mathfrak{M}^r_k(\tilde{X})$. Then choose M such that $\pi_*\mathcal{E}(NH + ML)$ is locally free. Consider the function $h^1 : \mathfrak{M}^r_k(\tilde{X}) \to \mathbb{Z}$ defined by $h^1 = \dim H^1(\mathcal{E}(NH + ML))$. Then, from the exact sequence

$$0 \to \mathcal{E}(NH) \to \mathcal{E}(NH + ML) \to \mathcal{T} \to 0$$

($\mathcal{T}$ has support contained in L) we get $H^2(\mathcal{E}(NH + ML)) = H^2(\mathcal{T}) = 0$. Now notice that

$$H^0(\mathcal{E}(NH + ML)) \cong H^0(\pi_*\mathcal{E}(NH + ML))$$

and, since by assumption $\pi_*\mathcal{E}(NH + ML)$ is locally free and π_*H is ample, for N large enough we get

$$H^i(\pi_*\mathcal{E}(NH + ML)) = 0 \text{ for } i > 0$$

Hence, we get that

$$h^1 = \chi(\pi_*\mathcal{E}(NH + ML)) - \chi(\mathcal{E}(NH + ML))$$

From Riemann-Roch theorem it follows that

$$h^1 = c_2(\mathcal{E}) - c_2(\pi_*\mathcal{E}^{\vee\vee}) + f(N, M, c_1(X))$$

where f does not depend on $\mathcal{E}$. The result then follows from the upper-semicontinuity of h^1 (see [10], chapter III, theorem 12.8). □

3. The charge one moduli space

The objective of this section is to prove theorem 1.3:

Theorem 3.1. *There is a homotopy equivalence*

$$\mathfrak{M}_1^\infty(X_q) \simeq BU(1) \times \left(\bigvee_{i=1}^q BU(1) \right)$$

From theorem 2.1 we know that

$$\mathfrak{M}_1^r(X_q) = \bigcup_{l=1}^q \pi_l^* \mathfrak{M}_1^r(X_1)\ ,\ \pi_i^*\mathfrak{M}_1^r(X_1) \cap \pi_j^*\mathfrak{M}_1^r(X_1) = \pi_\emptyset^*\mathfrak{M}_1^r(X_0)\ (i \neq j)$$

We begin by studying the maps $\pi_\emptyset^*\mathfrak{M}_1^r(X_0) \to \pi_i^*\mathfrak{M}_1^r(X_1)$.

Lemma 3.2. *Let $\iota_1 : \mathbb{CP}^r \to \mathbb{CP}^r \times \mathbb{CP}^r$ be the inclusion into the first factor: $\iota_1([u]) = ([u], *)$, where $*$ denotes the base point. Then there are homotopy equivalences $h_0 : \mathbb{CP}^\infty \to \mathfrak{M}_1^\infty(X_0)$ and $h_1 : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathfrak{M}_1^\infty(X_1)$ such that the following diagram*

$$\begin{tikzcd}
\pi_\emptyset^*\mathfrak{M}_1^\infty(X_0) \arrow[d] & \mathfrak{M}_1^\infty(X_0) \arrow[l, "\pi_\emptyset^*", "\cong"'] \arrow[d, "\pi^*"] & \mathbb{CP}^\infty \arrow[l, "h_0", "\cong"'] \arrow[d, "\iota_1"] \\
\pi_i^*\mathfrak{M}_1^\infty(X_1) & \mathfrak{M}_1^\infty(X_1) \arrow[l, "\pi_i^*", "\cong"'] & \mathbb{CP}^\infty \times \mathbb{CP}^\infty \arrow[l, "h_1", "\cong"']
\end{tikzcd}$$

is homotopy commutative.

Proof. We will use the monad description of $\mathfrak{M}_1^r(X_1), \mathfrak{M}_1^r(X_0)$ (see appendix A). We define the following maps:

$$\begin{array}{ll}
p_0 : \mathfrak{M}_1^r(X_0) \to \mathbb{CP}^r & p_0 : [a_1, a_2, b, c] \to [b] \\
p_1 : \mathfrak{M}_1^r(X_1) \to \mathbb{CP}^r \times \mathbb{CP}^r & p_1 : [a_1, a_2, d, b, c] \to \left([b], \left[\dfrac{\bar{c}^t}{\|c\|^2} \right] \right) \\
\Delta : \mathbb{CP}^r \to \mathbb{CP}^r \times \mathbb{CP}^r & \Delta : [u] \to ([u], [u]) \\
f : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty \times \mathbb{CP}^\infty & f : (x, y) \mapsto (x, xy^{-1})
\end{array}$$

where to define f we observe that $\mathbb{CP}^\infty = BU(1)$ is homotopic to the free abelian group on $U(1)$. Now observe that the diagram

$$\begin{tikzcd}
\pi_\emptyset^*\mathfrak{M}_1^\infty(X_0) \arrow[d] & \mathfrak{M}_1^\infty(X_0) \arrow[l, "\pi_\emptyset^*", "\cong"'] \arrow[r, "p_0"] \arrow[d, "\pi^*"] & \mathbb{CP}^\infty \arrow[d, "\Delta"] \arrow[dr, "\iota_1"] & \\
\pi_i^*\mathfrak{M}_1^\infty(X_1) & \mathfrak{M}_1^\infty(X_1) \arrow[l, "\pi_i^*", "\cong"'] \arrow[r, "p_1"] & \mathbb{CP}^\infty \times \mathbb{CP}^\infty \arrow[r, "f"] & \mathbb{CP}^\infty \times \mathbb{CP}^\infty
\end{tikzcd}$$

is homotopy commutative and the maps $p_0, p_1, f, \pi_\emptyset^*, \pi_i^*$ are homotopy equivalences. The statement of the lemma then follows by writing $h_0 = p_0^{-1}$ and $h_1 = p_1^{-1} f^{-1}$, where $p_0^{-1}, p_1^{-1}, f^{-1}$ are the homotopy inverses. $\square$

We are ready to prove theorem 3.1.

Proof of theorem 3.1. Let C be the cone on q points $v_1, \dots, v_q$. Let

$$M = \frac{\left(\coprod_{i=1}^{q} BU(1) \times BU(1) \times \{v_i\}\right) \coprod (\, BU(1) \times C\,)}{(\,[u]\,,\, v_i\,) \sim (\,\iota_1([u])\,,\, v_i)}$$

We first show that M is homotopically equivalent to $\mathfrak{M}_1^\infty(X_q)$.

Denote the points in C by

$$[t, v_i] \in C = \frac{[0,1] \times \coprod_i \{v_i\}}{(0, v_i) \sim (0, v_j) \sim *}$$

Define a map

$$\zeta : \left(\coprod_{i=1}^{q} BU(1) \times BU(1) \times \{v_i\}\right) \coprod (\, BU(1) \times C\,) \to \mathfrak{M}_1^\infty(X_q)$$

as follows: denote the points in C by $[t, v_i] \in C = \frac{[0,1]\times\coprod_i\{v_i\}}{(0,v_i)\sim(0,v_j)\sim *}$. Then define ζ by

$$BU(1) \times BU(1) \times \{v_i\} \ni ([u_1], [u_2], v_i) \mapsto \pi_i^* h_1([u_1], [u_2])$$
$$BU(1) \times C \ni ([u], [t, v_i]) \mapsto \pi_\emptyset^* h_0([u]) \text{ for } t < \frac{1}{3}$$
$$BU(1) \times C \ni ([u], [t, v_i]) \mapsto \pi_i^* h_1 \iota_1([u]) \text{ for } t > \frac{2}{3}$$

For $\frac{1}{3} \le t \le \frac{2}{3}$ use the homotopy between $\pi_\emptyset^* h_0$ and $p_i^* h_1 \iota_1$ from lemma 3.2. ζ descends to the quotient to give a map $\zeta : M \to \mathfrak{M}_1^\infty(X_q)$. We want to apply Whitehead theorem to show ζ is a homotopy equivalence. The van Kampen theorem implies both M and $\mathfrak{M}_1^\infty(X_q)$ are simply connected hence we only have to show ζ is an isomorphism in homology groups. We prove it by induction in $q' = 1, \dots, q$. We apply the five lemma to the Meyer-Vietoris long exact sequence corresponding to open neighborhoods of the sets

$$\pi_{q'+1}^* \mathfrak{M}_1^\infty(X_1),\ \pi_{(1,\dots,q')}^* \mathfrak{M}_1^\infty(X_{q'}) \subset \mathfrak{M}_1^\infty(X_q)$$
$$BU(1) \times BU(1) \times \{v_i\},\ \bigcup_{l=1}^{q'} BU(1) \times BU(1) \times \{v_l\} \subset M$$

using the fact that the restrictions

$$\zeta : BU(1) \times BU(1) \times \{v_i\} \to \pi_i^* \mathfrak{M}_1^\infty(X_1)$$
$$\zeta : BU(1) \times C \to \pi_\emptyset^* \mathfrak{M}_1^\infty(X_0)$$

are homotopy equivalences. It follows that ζ induces isomorphisms in all homology groups.

To conclude the proof we only have to show that M is homotopically equivalent to

$$BU(1) \times \left(\bigvee_{i=1}^{q} BU(1)\right) = \frac{\coprod_{l=1}^{q} BU(1) \times BU(1) \times \{v_l\}}{(x, *, v_i) \sim (x, *, v_j)}$$

where $* \in BU(1)$ is the base point. Define an open cover of $BU(1) \times (\bigvee_i BU(1))$ by $U_i = BU(1) \times BU(1) \times \{v_i\}$. Then the claim is a special case of proposition 4.1 in [21]. $\square$

4. An open cover of $\mathfrak{M}_2^r(X_2)$

The objective of this section is to prove theorem 1.2. We begin by studying the case $q = 2$. We will adopt, in this section and the next, the following notation: Denote the blow up points by $x_L, x_R \in X_0$. Let $\pi : X_2 \to X_0$ be the blow up map at x_L, x_R. By abuse of notation we will denote by π_L the maps $X_2 \to X_1$ and $X_1 \to X_0$ corresponding to the blow up at x_L and in the same way π_R will denote the blow up at x_R. We have the diagram

$$\begin{tikzcd} X_2 \arrow[r, "\pi_L"] \arrow[d, "\pi_R"'] & X_{1R} \arrow[d, "\pi_R"] \\ X_{1L} \arrow[r, "\pi_L"] & X_0 \end{tikzcd}$$

of blow up maps where $X_{1L} \cong X_{1R} \cong X_1$. Denote by L_L and L_R the exceptional divisors above x_L and x_R respectively. Again, by abuse of notation we identify $L_L \subset X_2$ with $L_L \subset X_{1L}$ and the same for L_R. Write $x_L = [x_{1L}, x_{2L}, 1]$, $x_R = [x_{1R}, x_{2R}, 1]$, $x_L, x_R \in X_0 = \mathbb{CP}^2$. Since $x_L \neq x_R$ we may assume without loss of generality that $x_{1L} \neq x_{1R}$. Let $z_i = x_{iR} - x_{iL}$. z_1, z_2 determine a point $([z_1, z_2, 1], [z_1, z_2]) \in X_1 \setminus L_\infty = \widetilde{\mathbb{CP}^2} \setminus L_\infty \subset \mathbb{CP}^2 \times \mathbb{CP}^1$. We are ready to state the first theorem of this section:

Theorem 4.1. *Let*

$$A_L = \pi_R^* \mathfrak{M}_2^r(X_{1L}) = \{\mathcal{E} \in \mathfrak{M}_2^r(X_2) \,:\, \mathcal{E}|_{L_R} \textit{ is trivial}\}$$
$$A_R = \pi_L^* \mathfrak{M}_2^r(X_{1R}) = \{\mathcal{E} \in \mathfrak{M}_2^r(X_2) \,:\, \mathcal{E}|_{L_L} \textit{ is trivial}\}$$

and let $C = \mathfrak{M}_2^r(X_2) \setminus (A_L \cup A_R)$. Let $N_L \subset \mathfrak{M}_2^r(X_{1L})$ be the set of non-degenerate configurations $m = (a_1, a_2, d, b, c)$ such that the eigenvalues of da_1 (equal to the eigenvalues of $a_1 d$) are in a δ neighborhood of $0, z_1$. In a similar way define $N_R \subset \mathfrak{M}_2^r(X_{1R})$. Let $N_2 = \pi_R^* N_L \cup \pi_L^* N_R \cup C$. Then $\{A_L, A_R, N_2\}$ is an open cover of $\mathfrak{M}_2^r(X_2)$. There are homotopy equivalences

1. $A_L \simeq A_R \simeq \mathfrak{M}_2^r(X_1)$
2. $C \simeq \mathfrak{M}_1^r(X_1) \times \mathfrak{M}_1^r(X_1)$
3. $A_L \cap A_R \simeq \mathfrak{M}_2^r(X_0)$
4. $A_L \cap N_2 \simeq N_L \simeq A_R \cap N_2 \simeq N_R \simeq \mathfrak{M}_1^r(X_1) \times \mathfrak{M}_1^r(X_0)$
5. $A_L \cap A_R \cap N_2 \simeq \mathfrak{M}_1^r(X_0) \times \mathfrak{M}_1^r(X_0)$
6. $N_2 \simeq C$

From this open cover we get, in a standard way (see [21]), a spectral sequence:

Corollary 4.2. *There is a spectral sequence converging to the cohomology of $\mathfrak{M}_2^r(X_2)$ with E_1 term*

$$E_1^{0,n} = H^n(A_L) \oplus H^n(A_R) \oplus H^n(N_2)$$
$$E_1^{1,n} = H^n(A_L \cap A_R) \oplus H^n(A_L \cap N_2) \oplus H^n(A_R \cap N_2)$$
$$E_1^{2,n} = H^n(A_L \cap A_R \cap N_2)$$

In the next section we will study the d_1 differential of this spectral sequence.

We turn now to the proof of theorem 4.1. We will delay the proof that N_2 is open and begin by proving the homotopy equivalences (1), (2) and (3):

Proposition 4.3. *A_L, A_R are open sets,*

$$C = \{[\mathcal{E}, \phi] \in \mathfrak{M}_2^r(X_2) \,:\, c_2\left(\,(\pi_{i*}\mathcal{E})^{\vee\vee}\,\right) = 1\,,\, i = L, R\}$$

and the following maps are isomorphisms (where $\pi_{i*}^{\vee\vee}(\mathcal{E}) \overset{def}{=} (\pi_{i*}\mathcal{E})^{\vee\vee}$):

$$\begin{aligned}
&\pi_R^* : \mathfrak{M}_2^r(X_{1L}) \to A_L \subset \mathfrak{M}_2^r(X_2)\\
&\pi_L^* : \mathfrak{M}_2^r(X_{1R}) \to A_R \subset \mathfrak{M}_2^r(X_2)\\
&\pi_{R*}^{\vee\vee} \times \pi_{L*}^{\vee\vee} : C \to S_0\mathfrak{M}_1^r(X_{1L}) \times S_0\mathfrak{M}_1^r(X_{1R})\\
&\pi^* : \mathfrak{M}_2^r(X_0) \to A_L \cap A_R \subset \mathfrak{M}_2^r(X_2)
\end{aligned}$$

where $S_0\mathfrak{M}_1^r(X_1) \subset \mathfrak{M}_1^r(X_1)$ is the subspace of bundles $\mathcal{E}$ verifying $(\pi_*\mathcal{E})^{\vee\vee} = \mathcal{O}_{X_0}^r$.

Proof. The isomorphisms for $A_L, A_R, A_L \cap A_R$ follows from theorem 2.1. That theorem also implies A_L, A_R are open. It remains to look at the map $\pi_{R*}^{\vee\vee} \times \pi_{L*}^{\vee\vee} : C \to S_0\mathfrak{M}_1^r(X_{1L}) \times S_0\mathfrak{M}_1^r(X_{1R})$. The continuity of this map was proved in proposition 3.1 in [19]. We will construct an inverse for $\pi_{R*}^{\vee\vee} \times \pi_{L*}^{\vee\vee}$. Let $(\mathcal{E}_L, \phi_L) \in S_0\mathfrak{M}_1^r(X_{1L})$, $(\mathcal{E}_R, \phi_R) \in S_0\mathfrak{M}_1^r(X_{1R})$. Hartogs' theorem implies there are unique extensions of ϕ_L, ϕ_R to maps

$$\phi_L : \mathcal{E}_L|_{X_0\setminus\{x_L\}} \to \mathcal{O}^r_{X_0\setminus\{x_L\}}\ ,\ \phi_R : \mathcal{E}_R|_{X_0\setminus\{x_R\}} \to \mathcal{O}^r_{X_0\setminus\{x_R\}}$$

These maps induce an isomorphism $\mathcal{E}_L \cong \mathcal{E}_R$ over $X_0 \setminus \{x_L, x_R\}$ which we use to glue $\mathcal{E}_L, \mathcal{E}_R$ and obtain a bundle $\mathcal{E} \to X_2$. The continuity of this map was proved in proposition 3.3 in [19]. This concludes the proof. □

Before we continue we need a lemma. Let $\overline{\mathfrak{M}_2^r(X_1)}$ be the Donaldson-Uhlenbeck completion of the moduli space $\mathfrak{M}_2^r(X_1)$ (see the appendix). A blow-up $\pi : X_2 \to X_1$ induces a map $\pi_* : \mathfrak{M}_2^r(X_2) \to \overline{\mathfrak{M}_2^r(X_1)}$ given by $\mathcal{E} \mapsto ((\pi_*\mathcal{E})^{\vee\vee}, \ell((\pi_*\mathcal{E})^{\vee\vee}/\pi_*\mathcal{E}))$

Lemma 4.4. *Let $\mathcal{E}_m \in \overline{\mathfrak{M}_2^r(X_1)}$ and let $m = (a_1, a_2, d, b, c)$ be the configuration associated to $\mathcal{E}_m$. The following are equivalent:*

1. *$\mathcal{E}_m$ is in the image of $\pi_{R*} : C \to \overline{\mathfrak{M}_2^r(X_1)}$;*
2. *$cdb = 0$ and the eigenvalues of da_i (equal to the ones of a_id) are 0 and z_i;*
3. *After a change of basis we can write*

$$a_1 = \begin{bmatrix} a_1' & 0 \\ 0 & z_1 \end{bmatrix}\ ,\ a_2 = \begin{bmatrix} a_2' & \frac{b'c''}{z_1} \\ -\frac{b''c'}{z_1} & z_2 \end{bmatrix}\ ,\ d = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\ ,\ b = \begin{bmatrix} b' \\ b'' \end{bmatrix}\ ,\ c = \begin{bmatrix} c' & c'' \end{bmatrix}$$

with $c''b'' = 0$.

Proof. We will show that $1 \Rightarrow 2$, $2 \Rightarrow 3$ and $3 \Rightarrow 1$.

- $1 \Rightarrow 2$: Suppose $\mathcal{E}_m = \pi_{R*}\tilde{\mathcal{E}}$ for some $\tilde{\mathcal{E}} \in C$. Then, by proposition 4.3, $\mathcal{E}_m^{\vee\vee} \in S_0\mathfrak{M}_1^r(X_{1L})$ and $\mathcal{E}_m$ is not locally free at the blow up point x_R. So, from proposition A.7, $\mathcal{E}_m^{\vee\vee}$ corresponds to a configuration of the form $[a_1', a_2', 0, b', c']$. Since m is degenerate, by proposition A.5 after a change of basis it can be written in one of two forms, corresponding to the two types of special pairs. If m is b-special then

$$a_i = \begin{bmatrix} a_i' & * \\ 0 & a_i'' \end{bmatrix}\ ,\ d = \begin{bmatrix} d' & * \\ 0 & d'' \end{bmatrix}\ ,\ b = \begin{bmatrix} b' \\ 0 \end{bmatrix}\ ,\ c = \begin{bmatrix} c' & c'' \end{bmatrix}$$

in which case the configuration is equivalent to the completely reducible configuration (see proposition A.5)

$$(a_1', a_2', d', b', c') \oplus (a_1'', a_2'', d'', 0, c'')$$

corresponding to an ideal bundle with singularity at $(a_1''d'', a_2''d'')$ and charge one bundle given by (a_1', a_2', d', b', c'). So we should have $d' = 0$ and $a_i''d'' = z_i$. Hence the eigenvalues of da_i are $0, z_i$ and $cdb = 0$. A similar argument applies if m is c-special.

- $2 \Rightarrow 3$: Now assume the configuration (a_1, a_2, d, b, c) satisfies 2. Fix a basis of eigenvectors $v_0, v_1 \in V$ of $a_1 d$ and $w_0, w_1 \in W$ of da_1 with v_0, w_0 corresponding to the eigenvalue 0. Normalize v_1, w_1 so that $dv_1 = w_1$. Then

$$a_1 = \begin{bmatrix} a_1' & 0 \\ 0 & a_1'' \end{bmatrix}, \; a_2 = \begin{bmatrix} a_2' & \frac{b'c''}{a_1'' - d'a_1'} \\ \frac{b''c'}{d'a_1' - a_1''} & a_2'' \end{bmatrix} \tag{3}$$

$$d = \begin{bmatrix} d' & 0 \\ 0 & 1 \end{bmatrix}, \; b = \begin{bmatrix} b' \\ b'' \end{bmatrix}, \; c = \begin{bmatrix} c' & c'' \end{bmatrix} \tag{4}$$

From $cdb = 0$ we get $(b'c'')(b''c') = 0$. If $b'c'' = 0$ then a_2 is lower triangular. If $b''c' = 0$ then a_2 is upper triangular. In both cases the diagonal entries of $a_2 d$ are its eigenvalues. Hence, the condition about the eigenvalues of $a_1 d$ and $a_2 d$ yields the equations $a_1'd' = a_2'd' = 0$, $a_1'' = z_1$ and $a_2'' = z_2$. Since $a_1(W) + a_2(W) + b(\mathbb{C}^r) = V$ we must have $d' = 0$.

- $3 \Rightarrow 1$: Let $m = [a_1, a_2, d, b, c]$ be a configuration satisfying 3. $c''b'' = 0$ implies either $c'' = 0$ or $b'' = 0$. It follows that the pair $(\mathrm{Span}\{(0,1)\}, \mathrm{Span}\{(0,1)\})$ is a special pair hence the configuration is degenerate. Now, from proposition A.5 it follows that m is equivalent to the completely reducible configuration

$$m' \oplus m'' = (a_1', a_2', 0, b', c') \oplus (z_1, z_2, 0, 0)$$

Notice that $(a_1', a_2', 0, b', c') \in S_0\mathfrak{M}_1^r(X_{1L})$. Then, from proposition 4.3, there is $\tilde{m} \in C$ such that $\pi_{R*}\tilde{m}^{\vee\vee} = m'$. Then, from the characterization of points in the completion it follows that $\pi_{R*}\tilde{m} = m$. □

Now we turn to the proof of homotopy equivalences (4) and (5) in theorem 4.1.

Definition 4.1. Let

$$N_z = \{\, (a_{1z}, a_{2z}, b_z, c_z) \in \mathfrak{M}_1^r(X_0) \,|\, |a_{1z} - z| < \delta \,\}$$
$$N' = \{\, (a_1', a_2', d', b', c') \in \mathfrak{M}_1^r(X_1) \,|\, |d'a_1'| < \delta \,\}$$

Let $N_0 \subset \mathfrak{M}_2^r(X_0)$ be the subset of points (a_1, a_2, b, c) with the eigenvalues of a_1 lying in δ neighborhoods of x_L and x_R. Then define the map $\boxplus_0 : N_{x_{1L}} \times N_{x_{1R}} \to N_0$ by $[a_{1L}, a_{2L}, b_L, c_L] \boxplus_0 [a_{1R}, a_{2R}, b_R, c_R] = [a_1, a_2, b, c]$ with

$$a_1 = \begin{bmatrix} a_{1L} & 0 \\ 0 & a_{1R} \end{bmatrix}, \; a_2 = \begin{bmatrix} a_{2L} & \frac{b_L c_R}{a_{1R} - a_{1L}} \\ \frac{b_R c_L}{a_{1L} - a_{1R}} & a_{2R} \end{bmatrix}, \; b = \begin{bmatrix} b_L \\ b_R \end{bmatrix}, \; c = \begin{bmatrix} c_L & c_R \end{bmatrix}$$

and $\boxplus_L : N' \times N_{z_1} \to N_L$ by $[a'_1, a'_2, d', b', c'] \boxplus_L [a''_1, a''_2, b'', c''] = [a_1, a_2, d, b, c]$ with

$$a_1 = \begin{bmatrix} a'_1 & 0 \\ 0 & a''_1 \end{bmatrix}, \; a_2 = \begin{bmatrix} a'_2 & \frac{b'c''}{a''_1 - d'a'_1} \\ \frac{b''c'}{d'a'_1 - a''_1} & a''_2 \end{bmatrix} \tag{5}$$
$$d = \begin{bmatrix} d' & 0 \\ 0 & 1 \end{bmatrix}, \; b = \begin{bmatrix} b' \\ b'' \end{bmatrix}, \; c = \begin{bmatrix} c' & c'' \end{bmatrix}$$

Proposition 4.5.

(1) *The maps $\boxplus_0, \boxplus_L$ are homeomorphisms;*
(2) *The inclusions $N_z \to \mathfrak{M}^r_1(X_0)$, $N' \to \mathfrak{M}^r_1(X_1)$ are homotopy equivalences;*
(3) $\pi^*_R N_L \cap \pi^*_L N_R = \pi^*_\emptyset N_0$.

Proof. Statement (2) is clear from the definition. To prove statement (3) we observe that

$$\pi^*_R N_L \cap \pi^*_L N_R = \pi^*_R N_L \cap \pi^*_\emptyset \mathfrak{M}^r_2(X_0) \cong N_L \cap \pi^*_L \mathfrak{M}^r_2(X_0)$$

The result now follows easily from proposition A.8. We turn to the proof of statement (1). It is an easy consequence of proposition A.5 that $\boxplus_0$ and $\boxplus_L$ preserve the nondegeneracy of the configurations so the maps are well defined.

Now we look at $\boxplus_L$. For δ small enough the eigenvalues of $a_1 d$ are distinct. Hence we can choose, up to the action of $(\mathbb{C}^*)^{\times 4}$, eigenvector basis $\{v_0, v_1\} \subset V$ of $a_1 d$ and $\{w_0, w_1\} \subset W$ of da_1, where v_0, w_0 correspond to the eigenvalues near 0. Normalize v_1, w_1 so that $dv_1 = w_1$. Then the action of $(\mathbb{C}^*)^{\times 4}$ is reduced to an action of $(\mathbb{C}^*)^{\times 3}$. We can thus write (see also equation (3))

$$a_1 = \begin{bmatrix} a'_1 & 0 \\ 0 & a''_1 \end{bmatrix}, \; a_2 = \begin{bmatrix} a'_2 & \frac{b'c''}{a''_1 - d'a'_1} \\ \frac{b''c'}{d'a'_1 - a''_1} & a''_2 \end{bmatrix} \tag{6}$$
$$d = \begin{bmatrix} d' & 0 \\ 0 & 1 \end{bmatrix}, \; b = \begin{bmatrix} b' \\ b'' \end{bmatrix}, \; c = \begin{bmatrix} c' & c'' \end{bmatrix}$$

The group $(\mathbb{C}^*)^{\times 3}$ acts transitively on equivalence classes of such configurations written in the above canonical form. This shows the existence of an inverse, hence $\boxplus_L$ is a homeomorphism. The proof for $\boxplus_0$ is similar. □

We will need the following identity:

Proposition 4.6. *Let $\tau : \mathfrak{M}^r_k(X_0) \to \mathfrak{M}^r_k(X_0)$ be defined by*

$$\tau(a_1, a_2, b, c) = (a_1 - x_{1L}\mathbb{1}, a_2 - x_{2L}\mathbb{1}, b, c)$$

Let $m_1, m_2 \in \mathfrak{M}^r_1(X_0)$. Then $\pi^*_L(m_1 \boxplus_0 m_2) = \pi^*_L m_1 \boxplus_L \tau(m_2)$.

Proof. It follows easily from proposition A.8. □

The maps $\boxplus_0, \boxplus_L$ extend to the closure $\bar{N}', \bar{N}_z$ of N', N_z. The following proposition is a direct consequence of proposition A.5:

Proposition 4.7.

- *Let $m_L = [a_{1L}, a_{2L}, b_L, c_L] \in \bar{N}_{x_{1L}}$, $m_R = [a_{1R}, a_{2R}, b_R, c_R] \in \bar{N}_{x_{1R}}$. Then the following are equivalent:*
  (1) *$m_L \boxplus_0 m_R$ is degenerate;*
  (2) *Either m_L or m_R is degenerate.*
  (3) *At least one of the 4 vectors b_L, b_R, c_L, c_R is zero.*

- *Let* $m' = [a_1', a_2', d', b', c'] \in \bar{N}'$, $m'' = [a_1'', a_2'', b'', c''] \in \bar{N}_{z_1}$. *The following are equivalent:*
  (1) $m' \boxplus_L m''$ *is degenerate;*
  (2) *Either* m' *or* m'' *is degenerate;*
  (3) *One of the 4 vectors* b', b'', c', c'' *is zero.*

We are ready to prove

Proposition 4.8. *N_2 is an open neighborhood of C.*

Proof. From lemma 4.4 it follows immediately that $\pi_{R*}C \subset \bar{N}_L$.
Suppose there is a sequence $y_n \in \mathfrak{M}_2^r(X_{1L})$ such that $y_n \to y \in \pi_{R*}C$. Write $y_n = [a_{1n}, a_{2n}, d_n, b_n, c_n]$. Then, by property 2 in lemma 4.4 the eigenvalues of $d_n a_{in}$ converge to $0, z_i$. Hence, for n large enough $y_n \in N_L$. Hence $N_L \cup \pi_{R*}C$ is an open neighborhood of $\pi_{R*}C$.
Suppose there is a sequence $x_n \to x \in C$ such that $x_n \notin N_2$. Hence $x_n \notin C$ so, by passing to a subsequence we may assume without loss of generality that $x_n \in \pi_R^*\mathfrak{M}_2^r(X_{1L})$. Let $y_n = \pi_{R*}x_n \in \mathfrak{M}_2^r(X_{1L})$ and write $y_n = [a_{1n}, a_{2n}, d_n, b_n, c_n]$. Then $y_n \to y = \pi_{R*}x$ by continuity of π_{R*}, and $y_n \notin N_L$. But by property 2 in lemma 4.4 the eigenvalues of $d_n a_{in}$ converge to $0, z_i$ which implies, for n large enough, that $y_n \in N_L$. □

Finaly we prove the homotopy equivalence (6):

Proposition 4.9. *The inclusion $C \to N_2$ is a strong deformation retract.*

Proof. We will construct a homotopy $H_2 : N_2 \times [0,1] \to N_2$ betwen the identity and a retraction $N_2 \to C$. Let $H_{x_1,x_2} : \bar{N}_z \times [0,1] \to \bar{N}_z$ be defined by

$$H_{x_1,x_2}(a_1, a_2, b, c, t) = \big(t^2 a_1 + (1-t^2)x_1, t^2 a_2 + (1-t^2)x_2, tb, tc\big)$$

and let $H_1 : \bar{N}' \times [0,1] \to \bar{N}'$ be defined by

$$H_1(a_1', a_2', d', b', c', t) = (a_1', a_2', t^2 d', b', c')$$

Then we defined $H_L : \bar{N}_L \times [0,1] \to \bar{N}_L$ by

$$H_L(m' \boxplus_L m'', t) \stackrel{\text{def}}{=} H_1(m', t) \boxplus_L H_{z_1,z_2}(m'', t)$$

We define H_2 as the unique solution of the system of equations

$$\begin{aligned} \pi_{R*}H_2(x,t) &= H_L(\pi_{R*}x, t) \\ \pi_{L*}H_2(x,t) &= H_R(\pi_{L*}x, t) \end{aligned} \tag{7}$$

We have to show existence and uniqueness of solution. Then we will show that H_2 defines a homotopy between the identity on N_2 and a retraction $N_2 \to C$. We define the auxiliary map $H_0 : \bar{N}_0 \times [0,1] \to \bar{N}_0$ by

$$H_0(m_L \boxplus_0 m_R, t) \stackrel{\text{def}}{=} H_{x_{1L},x_{2L}}(m_L, t) \boxplus_0 H_{x_{1R},x_{2R}}(m_R, t)$$

To prove existence and uniqueness of solution of the system (7) we consider two cases:

(1) Assume that either $t = 0$ or $x \in C$. Then we claim that $H_L(\pi_{R*}x, t) \in \pi_{R*}C$, $H_R(\pi_{L*}x, t) \in \pi_{L*}C$. If $t = 0$ this follows directly from lemma 4.4. If $x \in C$ then, from lemma 4.4 we can write

$$\pi_{R*}x = x' \boxplus_L x'' = (a_1', a_2', 0, b', c') \boxplus_L (a_1'', a_2'', b'', c'')$$

with $c''b'' = 0$. It then follows from the definition of H_L that $H_L(\pi_{R*}x, t) = \pi_{R*}x$ for all t. In the same way we see that $H_R(\pi_{L*}x, t) = \pi_{L*}x$. This proves the claim. Then, existence and uniqueness follows from proposition 4.3.

(2) Assume $t \neq 1$ and $x \notin C$. Then we may assume $\pi_{R*}x \in N_L$. Then, since $H_L(\pi_{R*}x, t) \in N_L$, we get from (7)

$$\pi_{R*}H_2(x, t) = H_L(\pi_{R*}x, t) \Rightarrow H_2(x, t) = \pi_R^* H_L(\pi_{R*}x, t)$$

This proves uniqueness. To prove existence we need to show that

$$\pi_{L*}H_2(x, t) = \pi_{L*}\pi_R^* H_L(\pi_{R*}x, t) = H_R(\pi_{L*}x, t)$$

It is enough to show this for the case where $x = \pi_L^* \pi_R^* y$ for some $y \in N_0$ since the set of points of this form is dense and $H_L, H_R, \pi_{L*}, \pi_R^*, \pi_{R*}$ are continuous. It is an easy computation to show that $H_L(\pi_L^* y, t) = \pi_L^* H_0(y, t)$, $H_R(\pi_R^* y, t) = \pi_R^* H_0(y, t)$. It follows that

$$\pi_{L*}\pi_R^* H_L(\pi_{R*}x, t) = \pi_{L*}\pi_R^*\pi_L^* H_0(y, t) = \pi_R^* H_0(y, t) = H_R(\pi_{L*}x, t)$$

Now we need to show that H_2 is the desired homotopy. Direct inspection shows $H_2(x, 1) = x$. We saw in (1) above that, for $x \in C$, $H_2(x, t) = x$ and $H_2(x, 0) \in C$. The continuity of H_2 follows from the continuity of $\pi_{L*}, \pi_{R*}, H_L, H_R$: let $(x_n, t_n) \to (x, t) \in N_2$. Then, after passing to a subsequence, we get $H_2(x_{n_k}, t_{n_k}) \to (\tilde{x}, \tilde{t}\,) \in \bar{N}_2$. Then equations 7 imply that $(\tilde{x}, \tilde{t}\,) \in N_2$ and unicity of solution implies $(\tilde{x}, \tilde{t}\,) = H_2(x, t)$. Applying this reasoning to every sublimit of $H_2(x_n, t_n)$ we conclude that $H_2(x_n, t_n) \to H_2(x, t)$. Hence H_2 is continuous. □

We now prove the general case (theorem 1.2). What remains to be proven is:

Theorem 4.10. *Let $A_i, A_\emptyset$ be as in theorem 2.1 and let $N_{ij} = \pi_{ij}^* N_2$. Then*

(1) *$\{A_i\} \cup \{N_{ij}\}$ is an open cover of $\mathfrak{M}_2^r(X_q)$;*
(2) *There are homotopy equivalences*

$$N_{ij} \cap A_i \simeq \mathfrak{M}_1^r(X_1) \times \mathfrak{M}_1^r(X_0) \qquad\qquad N_{ij} \cap A_\emptyset \simeq \mathfrak{M}_1^r(X_0) \times \mathfrak{M}_1^r(X_0)$$

(3) *For $k \notin \{i, j\}$, $N_{ij} \cap A_k = N_{ij} \cap A_\emptyset$;*
(4) *For different sets $\{i, j\} \neq \{k, l\}$, $N_{ij} \cap N_{kl} = \emptyset$.*

Proof. Statement (1) follows from theorems 2.1 and 4.1. Statement (2) follows from proposition 2.2. To prove statement (3) observe that $N_{ij} \cap A_k = N_{ij} \cap A_{ij} \cap A_k$. So we turn to statement (4). First we look at $N_{ij} \cap N_{jk}$. Let $x_1, \ldots, x_q \subset \mathbb{CP}^2$ be the blow-up points and write $x_i = [x_{1i}, x_{2i}, 1]$. We may assume without loss of generality that the balls $B_\delta(x_{1i})$ are disjoint. Now observe that

$$N_{ij} \cap N_{jk} = N_{ij} \cap A_{ij} \cap A_{jk} \cap N_{jk} = N_{ij} \cap A_j \cap N_{jk}$$

Let $z_i = x_{1i} - x_{1j}$ and let $z_k = x_{1k} - x_{1j}$. Then, theorem 4.1 states that $N_{ij} \cap A_j$ is the set of non-degenerate configurations (a_1, a_2, d, b, c) such that the eigenvalues of da_1 are in a δ neighborhood of 0 and z_i. Hence $N_{ij} \cap A_j$ and $N_{jk} \cap A_j$ are disjoint. This shows $N_{ij} \cap N_{jk} = \emptyset$. The case $N_{ij} \cap N_{kl}$ for $\{i, j\} \cap \{k, l\} = \emptyset$ is treated similarly. □

5. The differential d_1

The objective of this section is to obtain, in the limit when $r \to \infty$, the homotopy type of the inclusion maps (see theorem 4.1)

$$
\begin{tikzcd}
A_L & A_0 \arrow[l] \arrow[r] & A_R \\
N_L \arrow[u] \arrow[dr] & N_0 \arrow[u] \arrow[l] \arrow[r] & N_R \arrow[u] \arrow[dl] \\
& N_2 &
\end{tikzcd}
$$

where $A_0 = A_L \cap A_R = \pi_\emptyset^* \mathfrak{M}_2^r(X_0)$. Since these spaces are classifying spaces it is enough to study the pullback under these maps of the tautological bundles. This will allow us to compute the d_1 differential in the spectral sequence introduced in corolary 4.2, from where we will compute the homology of $\mathfrak{M}_2^\infty(X_2)$.

Definition 5.1. Let

$$\tilde{F}_0(k,r) = \frac{\left\{(u,v) \in \mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^r) \times \mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^r) \,|\, u,v \text{ are injective}\right\}}{(u,v) \sim (u(\bar{g}^t)^{-1}, vg)\,,\; g \in Gl(k,\mathbb{C})}$$

$$\tilde{F}_1(k,r) = \frac{\left\{(u,v) \in \mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^r) \times \mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^r) \,|\, u,v \text{ are injective}\right\}}{(u,v) \sim (u(\bar{g}_u^t)^{-1}, vg_v)\,,\; g_u, g_v \in Gl(k,\mathbb{C})}$$

and define $F_0(k,r) \subset \tilde{F}_0(k,r)$ and $F_1(k,r) \subset \tilde{F}_1(k,r)$ by

$$F_0(k,r) = \left\{ \left[\bar{b}^t, c\right] \in \tilde{F}_0(k,r)\,,\; bc = 0 \right\}$$

$$F_1(k,r) = \left\{ \left[\bar{b}^t, c\right] \in \tilde{F}_1(k,r)\,,\; bc = 0 \right\}$$

Lemma 5.1. *Let $j_0 : F_0(k,r) \to \mathfrak{M}_k^r(X_0)$, $j_1 : F_1(k,r) \to \mathfrak{M}_k^r(X_1)$ be the inclusion maps given by $\left[\bar{b}^t, c\right] \mapsto [b,c]$. Then we have the homotopy commutative diagram*

$$
\begin{tikzcd}
\mathfrak{M}_k^r(X_0) \arrow[d, "\pi^*"] & F_0(k,r) \arrow[l, "j_0"'] \arrow[r, "\imath_0"'] \arrow[d, "pr"] \arrow[rr, bend left, "p_0"] & \tilde{F}_0(k,r) \arrow[r, "\tilde{p}_0"'] \arrow[d, "\widetilde{pr}"] & Gr(k,\mathbb{C}^r) \\
\mathfrak{M}_k^r(X_1) & F_1(k,r) \arrow[l, "j_1"'] \arrow[r, "\imath_1"] & \tilde{F}_1(k,r) &
\end{tikzcd}
\tag{8}
$$

where p_0 is the projection $[b,c] \mapsto [c]$. Moreover, in the rank stable limit, the maps $\imath_0, \imath_1, p_0, \tilde{p}_0, j_0, j_1$ are homotopy equivalences.

Proof. We divide the proof into three steps:

(1) $p_0, \tilde{p}_0$ are fibrations with fibers $M(k, r-k)$ and $M(k,r)$ respectively where $M(k,r) = \frac{U(r)}{U(k)}$ is the space of injective maps from $\mathbb{C}^k$ to $\mathbb{C}^r$ which is contractible in the stable range. That proves $p_0, \tilde{p}_0$ are homotopy equivalences. It imediatelly follows that $\imath_0$ is a homotopy equivalence.

(2) Now we look at $\imath_1$. Consider the projection $p_1 : F_1(k,r) \to Gr(k,\mathbb{C}^r)$ given by $[b,c] \mapsto [c]$. When $r \to \infty$, the spectral sequence associated with the fibration $p_1 : F_1(k,r) \to Gr(k,\mathbb{C}^r)$ (whose fiber is $Gr(k,\mathbb{C}^{r-k})$) collapses since all homology is in even dimensions. It easilly follows that $\imath_1$ is an isomorphism in all homology groups, hence an homotopy equivalence.

(3) Finally we need to prove the statements about j_0, j_1. First we look at j_0. Let $\mathcal{R}$ be the space of configurations (a_1, a_2, b, c) and let $\mathcal{R}^F \subset \mathcal{R}$ be the subspace configurations of the form $(0, 0, b, c)$. Then we have the fibration map

$$\begin{tikzcd} \mathcal{R}^F \arrow[r] \arrow[d] & F_0(k,r) \arrow[d, "j_0"] \\ \mathcal{R} \arrow[r] & \mathfrak{M}_k^r(X_0) \end{tikzcd}$$

In the rank stable limit the spaces $\mathcal{R}^F$ and $\mathcal{R}$ are contractible (see [18], [2]) so, by the five lemma j_0 is an isomorphism in homotopy groups hence an homotopy equivalence. A similar proof works for j_1. □

Now we turn to the main theorem of this section.

Definition 5.2. Let E, L be the tautological bundles over $Gr(2, \infty)$ and $Gr(1, \infty)$ respectivelly. Consider the compositions

$$A_0 \xrightarrow{\pi_{\emptyset *}} \mathfrak{M}_2^\infty(X_0) \xrightarrow{j_0^{-1}} F_0(2, \infty) \xrightarrow{p_0} Gr(2, \mathbb{C}^\infty) \tag{9}$$

$$N_0 \xrightarrow{\boxplus_0^{-1}} N_{x_{1L}} \times N_{x_{1R}} \xrightarrow{p_L} N_{x_{1L}} \longrightarrow \mathfrak{M}_1^\infty(X_0) \xrightarrow{p_0 j_0^{-1}} Gr(1, \mathbb{C}^\infty) \tag{10}$$

$$N_0 \xrightarrow{\boxplus_0^{-1}} N_{x_{1L}} \times N_{x_{1R}} \xrightarrow{p_R} N_{x_{1R}} \longrightarrow \mathfrak{M}_1^\infty(X_0) \xrightarrow{p_0 j_0^{-1}} Gr(1, \mathbb{C}^\infty) \tag{11}$$

$$N_L \xrightarrow{\boxplus_L^{-1}} N' \times N_{z_1} \xrightarrow{p''} N_{z_1} \longrightarrow \mathfrak{M}_1^\infty(X_0) \xrightarrow{p_0 j_0^{-1}} Gr(1, \mathbb{C}^\infty) \tag{12}$$

Then we define the following bundles:

- $E_0 \to A_0$ is the pullback of E under the composition 9.
- $L_{0L,0} \to N_0$ is the pullback of L under 10
- $L_{0R,0} \to N_0$ is the pullback of L under 11
- $L_{0R,L} \to N_L$ is the pullback of L under 12

Now let $\tilde{E}_u, \tilde{E}_v \to \tilde{F}_1(2, r)$ be the tautological bundles corresponding to u, v and let $\tilde{L}_u, \tilde{L}_v$ be the tautological line bundles over $\tilde{F}_1(1, \infty)$. Consider the compositions

$$A_L \xrightarrow{\pi_{R*}} \mathfrak{M}_2^\infty(X_{1L}) \xrightarrow{j_1^{-1}} F_1(2, \infty) \xrightarrow{\imath_1} \tilde{F}_1(2, \infty) \tag{13}$$

$$N_L \xrightarrow{\boxplus_L^{-1}} N' \times N_{z_1} \xrightarrow{p'} N' \longrightarrow \mathfrak{M}_1^\infty(X_1) \xrightarrow{\imath_1 j_1^{-1}} \tilde{F}_1(1, \infty) \tag{14}$$

$$N_2 \xrightarrow{\simeq} C \xrightarrow{\pi_{L*}^{\vee\vee}} S_0\mathfrak{M}_1^\infty(X_{1L}) \rightarrowtail \mathfrak{M}_1^\infty(X_{1L}) \xrightarrow{\imath_1 j_1^{-1}} \tilde{F}_1(1, \infty) \tag{15}$$

$$N_2 \xrightarrow{\simeq} C \xrightarrow{\pi_{R*}^{\vee\vee}} S_0\mathfrak{M}_1^\infty(X_{1R}) \rightarrowtail \mathfrak{M}_1^\infty(X_{1R}) \xrightarrow{\imath_1 j_1^{-1}} \tilde{F}_1(1, \infty) \tag{16}$$

We define the bundles

- $E_{bL}, E_{cL} \to A_L$ are the pullback of $\tilde{E}_u, \tilde{E}_v$ under 13.
- $L_{bL,L}, L_{cL,L} \to N_L$ are the pullback of $\tilde{L}_u, \tilde{L}_v$ under 14
- $L_{bL,2}, L_{cL,2} \to N_2$ are the pullback of $\tilde{L}_u, \tilde{L}_v$ under 15
- $L_{bR,2}, L_{cR,2} \to N_2$ are the pullback of $\tilde{L}_u, \tilde{L}_v$ under 16

Theorem 5.2. *We have the following bundle isomorphisms:*

(1) $E_{bL}|_{A_0} = E_0$, $E_{cL}|_{A_0} = E_0$
(2) $L_{bL,L}|_{N_0} \cong L_{0L,0}$, $L_{cL,L}|_{N_0} \cong L_{0L,0}$, $L_{0R,L}|_{N_0} \cong L_{0R,0}$
(3) $E_{bL}|_{N_L} \cong L_{bL} \oplus L_{0R}$, $E_{cL}|_{N_L} \cong L_{cL} \oplus L_{0R}$
(4) $E_0|_{N_0} \cong L_{0L,0} \oplus L_{0R,0}$.
(5) $L_{bR,2}|_{N_L} \cong L_{cR,2}|_{N_L} \cong L_{0R,L}$
(6) $L_{bL,2}|_{N_L} \cong L_{bL,L}$, $L_{cL,2}|_{N_L} \cong L_{cL,L}$

Similar statements hold for the spaces A_R, N_R and the maps $N_R \to A_R$, $N_R \to N_2$ and $N_0 \to N_R$.

Proof.

(1) First we show that $E_{bL}|_{A_0} \cong E_{cL}|_{A_0} \cong E_0$. Consider diagram (8). We will start by defining a homotopy inverse $q : Gr(k, \mathbb{C}^r) \to \tilde{F}_0(k, r)$ to the map $\tilde{p}_0 : \tilde{F}_0 \to Gr(k, \mathbb{C}^\infty)$ as follows: choose a map $c : \mathbb{C}^k \to \mathbb{C}^r$ representing an element $[c] \in Gr(k, \mathbb{C}^r)$. Choose $h \in Gl(k, \mathbb{C})$ such that ch is orthogonal. Then define $q([c]) = [ch, ch]$. This map is well defined and independent of the choice of h. Also $p_0 q = \mathbb{1}$ hence $p_0 = q^{-1}$. Now observe that the composition

$$\tilde{pr} \circ q : Gr(k, \mathbb{C}^r) \to \tilde{F}_1(k, r) = Gr(k, \mathbb{C}^r) \times Gr(k, \mathbb{C}^r)$$

is the diagonal map. It follows that, if E is the tautological bundle over $Gr(k, \mathbb{C}^\infty)$, then

$$q^* \tilde{pr}^* \tilde{E}_u \cong q^* \tilde{pr}^* \tilde{E}_v \cong E$$

To show that $E_{bL}|_{A_0} \cong E_{cL}|_{A_0} \cong E_0$ it suffices to show that $pr^* \imath_1^* \tilde{E}_u \cong pr^* \imath_1^* \tilde{E}_v \cong p_0^* E$. We have

$$pr^* \imath_1^* \tilde{E}_u = \imath_0^* \tilde{pr}^* \tilde{E}_u \cong p_0^* q^* \tilde{pr}^* \tilde{E}_u = p_0^* E$$

and a similar statement is true for $\tilde{E}_v$. This concludes the proof.

(2) We want to show that $L_{bL,L}|_{N_0} \cong L_{0L,0}$, $L_{cL,L}|_{N_0} \cong L_{0L,0}$ and $L_{0R,L}|_{N_0} \cong L_{0R,0}$. We have the commutative diagram (see proposition 4.6)

$$\begin{tikzcd}
N_0 \arrow[d, "\pi^*"] & N_{x_{1L}} \times N_{x_{1R}} \arrow[l, "\boxplus_0"'] \arrow[r, "p_R"] \arrow[d, "\pi^* \times \tau"] & N_{x_{1R}} \arrow[r] \arrow[d, "\tau"] & \mathfrak{M}_1^\infty(X_0) \arrow[d, "\tau"] & F_0 \arrow[l, "j_0"'] \arrow[r, "p_0"] \arrow[d] & Gr \arrow[d] \\
N_L & N' \times N_z \arrow[l, "\boxplus_L"'] \arrow[r, "p''"] & N_z \arrow[r] & \mathfrak{M}_1^\infty(X_0) & F_0 \arrow[l, "j_0"'] \arrow[r, "p_0"] & Gr
\end{tikzcd}$$

from which it follows that $L_{0R,L}|_{N_0} \cong L_{0R,0}$. We also have the commutative diagram

$$\begin{tikzcd}
N_0 \arrow[d, "\pi^*"] & N_{x_{1L}} \times N_{x_{1R}} \arrow[l, "\boxplus_0"'] \arrow[r, "p_L"] \arrow[d, "\pi^* \times \tau"] & N_{x_{1L}} \arrow[r] \arrow[d, "\tau"] & \mathfrak{M}_1^\infty(X_0) \arrow[d, "\tau"] & F_0 \arrow[l, "j_0"'] \arrow[r, "\imath_0"'] \arrow[d, "pr"] \arrow[rr, bend left, "p_0"] & \tilde{F}_0 \arrow[r, "\tilde{p}_0"'] \arrow[d, "\tilde{pr}"] & Gr \\
N_L & N' \times N_z \arrow[l, "\boxplus_L"'] \arrow[r, "p'"] & N' \arrow[r] & \mathfrak{M}_1^\infty(X_1) & F_1 \arrow[l, "j_1"'] \arrow[r, "\imath_1"] & \tilde{F}_1 &
\end{tikzcd}$$

from which it follows, as in step (1), that $L_{bL,L}|_{N_0} \cong L_{cL,L}|_{N_0} \cong L_{0L,0}$.

(3) We want to show that $E_{bL}|_{N_L} \cong L_{bL,L} \oplus L_{0R,L}$ and $E_{cL}|_{N_L} \cong L_{cL,L} \oplus L_{0R,L}$. Consider the following diagram:

(17)
$$\begin{tikzcd}
A_L & & F_1 \arrow[ll, "j_1", "\simeq"'] \arrow[rr, "\imath_1", "\simeq"'] & & \tilde{F}_1 \\
N_L \arrow[u] & N' \times N_{z_1} \arrow[l, "\boxplus_L", "\cong"'] \arrow[r, "\simeq"'] & \mathfrak{M}_1^\infty(X_1) \times \mathfrak{M}_1^\infty(X_0) & F_1 \times F_0 \arrow[l, "j_1 \times j_0", "\simeq"'] \arrow[r, "\imath_1 \times \imath_0", "\simeq"'] & \tilde{F}_1 \times \tilde{F}_0 \arrow[u, dashed, "\tilde{w}"]
\end{tikzcd}$$

Since $\tilde{pr}^* \tilde{L}_u \cong \tilde{pr}^* \tilde{L}_v \cong \tilde{p}_0^* L$, the proof will be complete if we show there is a map $\tilde{w} : \tilde{F}_1(1,\infty) \times \tilde{F}_0(1,\mathbb{C}^\infty) \to \tilde{F}_1(2,\infty)$ making the diagram homotopy commutative, such that

$$\tilde{w}^* \tilde{E}_u = \tilde{L}_u \oplus \tilde{pr}^* \tilde{L}_u \ , \ \tilde{w}^* \tilde{E}_v = \tilde{L}_v \oplus \tilde{pr}^* \tilde{L}_v \tag{18}$$

We begin by building $\tilde{w}$. Define maps $s_L, s_R : Gr(1,\mathbb{C}^\infty) \to Gr(1,\mathbb{C}^\infty)$ as follows: let $v : \mathbb{C} \to \mathbb{C}^\infty$ and write $v = (v^1, v^2, \ldots)$. Then

$$s_L([v]) \stackrel{\text{def}}{=} [(v^1, 0, v^2, 0, \ldots)] \ , \ s_R([v]) \stackrel{\text{def}}{=} [(0, v^1, 0, v^2, \ldots)]$$

We observe that s_L, s_R are homotopic to the identity. It follows that, if we define

$$\tilde{w} : ([b_L, c_L], [b_R, c_R]) \mapsto [s_L(b_L) \oplus s_R(b_R), s_L(c_L) \oplus s_R(c_R)]$$

then

$$(\tilde{w})^* \tilde{E}_u = \tilde{L}_u \oplus \tilde{pr}^* \tilde{L}_u \ , \ (\tilde{w})^* \tilde{E}_v = \tilde{L}_v \oplus \tilde{pr}^* \tilde{L}_v$$

It remain to show diagram 17 is commutative. Let $j_z : F_0(1,\infty) \to N_z$ be defined by $j_z : [b,c] \mapsto [z,0,b,c]$. Then the diagram

$$\begin{tikzcd}
N' \times N_{z_1} \arrow[dr] & & F_1 \times F_0 \arrow[ll, "j_1 \times j_{z_1}"'] \arrow[dl] \\
& \mathfrak{M}_1^\infty(X_1) \times \mathfrak{M}_1^\infty(X_0) &
\end{tikzcd}$$

is homotopy commutative. We are left with the diagram

$$\begin{tikzcd}
A_L & & F_1 \arrow[ll, "j_1", "\simeq"'] \arrow[rr, "\imath_1", "\simeq"'] & & \tilde{F}_1 \\
N_L \arrow[u] & N' \times N_{z_1} \arrow[l, "\boxplus_L", "\cong"'] & & F_1 \times F_0 \arrow[ll, "j_1 \times j_{z_1}", "\simeq"'] \arrow[r, "\imath_1 \times \imath_0", "\simeq"'] & \tilde{F}_1 \times \tilde{F}_0 \arrow[u, "\tilde{w}"]
\end{tikzcd}$$

Now define the map $w : F_1(1,\infty) \times F_0(1,\infty) \to F_1(2,\infty)$ by

$$w : ([b_L, c_L], [b_R, c_R]) \mapsto [s_L(b_L) \oplus s_R(b_R), s_L(c_L) \oplus s_R(c_R)]$$

Clearly we have the commutative diagram

$$\begin{tikzcd}
F_1(2,\infty) \arrow[r, "\imath_1"] & \tilde{F}_1(2,\infty) \\
F_1(1,\infty) \times F_0(1,\infty) \arrow[u, "w"] \arrow[r, "\imath_1 \times \imath_0"] & \tilde{F}_1(1,\infty) \times \tilde{F}_0(1,\infty) \arrow[u, "\tilde{w}"']
\end{tikzcd}$$

We are thus left with the diagram

$$\begin{tikzcd} A_L & & F_1 \arrow[ll, "j_1"', "\simeq"] \\ N_L \arrow[u] & N' \times N_{z_1} \arrow[l, "\boxplus_L"', "\cong"] & F_1 \times F_0 \arrow[l, "j_1 \times j_{z_1}"', "\simeq"] \arrow[u, "w"] \end{tikzcd}$$

Next we introduce maps

$$S_L([b,c]) = [s_L(b), s_L(c)] \ , \ S_R([b,c]) = [s_R(b), s_R(c)]$$

These maps are homotopic to the identity hence we only have to show the diagram

$$\begin{tikzcd} A_L & & & F_1 \arrow[lll, "j_1"'] \\ N_L \arrow[u] & N' \times N_{z_1} \arrow[l, "\boxplus_L"'] & F_1 \times F_0 \arrow[l, "j_1 \times j_{z_1}"'] & F_1 \times F_0 \arrow[l, "S_L \times S_R"'] \arrow[u, "w"] \end{tikzcd}$$

is homotopy commutative. This is an easy direct verification.

(4) We want to show that $E_0|_{N_0} \cong L_{0L} \oplus L_{0R}$. Consider the diagram

$$\begin{tikzcd} N_0 \arrow[r, "i_1"] \arrow[d, "i_2"] & A_0 \arrow[d, "i_3"] \\ N_L \arrow[r, "i_4"] & A_L \end{tikzcd}$$

Then $E_0 = i_3^* E_{bL}$ so

$$E_0|_{N_0} = i_1^* E_0 = i_1^* i_3^* E_{bL} = i_2^* i_4^* E_{bL} = L_{0L} \oplus L_{0R}$$

(5) We want to show that $L_{bR}|_{N_L} \cong L_{cR}|_{N_L} \cong L_{0R}$. The result will follow if we show that the following diagram is homotopy commutative:

$$\begin{tikzcd} N_L \arrow[d] & N' \times N_{z_1} \arrow[l, "\boxplus_L"', "\cong"] \arrow[r] & N_{z_1} \arrow[r] & \mathfrak{M}_1^\infty(X_0) \arrow[d, "\pi^*"] & F_0 \arrow[l] \arrow[r] & \tilde{F}_0 \arrow[d, "\tilde{pr}"] \\ N_2 & C \arrow[l, "\simeq"'] \arrow[r, "\pi_{L*}^{\vee\vee}"] & S_0\mathfrak{M}_1^\infty(X_1) \arrow[r, "\simeq"] & \mathfrak{M}_1^\infty(X_1) & F_1 \arrow[l] \arrow[r] & \tilde{F}_1 \end{tikzcd} \tag{19}$$

Let

$$\begin{aligned} S_{1L}N_2 &= \{(\mathcal{E}, \phi) \in N_2 \mid c_2\left((\pi_{L*}\mathcal{E})^{\vee\vee}\right) = 1\} \\ S_1 N_L &= \{(\mathcal{E}, \phi) \in N_L \mid c_2\left((\pi_{L*}\mathcal{E})^{\vee\vee}\right) = 1\} \\ S_0 N' &= \{(\mathcal{E}, \phi) \in N' \mid c_2\left((\pi_{L*}\mathcal{E})^{\vee\vee}\right) = 0\} \end{aligned}$$

Then the commutativity of (19) follows from the commutativity of

$$
\begin{tikzcd}
N_2 & & N_L \arrow[ll, "\pi_R^*"'] & & N' \times N_{z_1} \arrow[ll, "\boxplus_L"'] \arrow[dd] \\
C \arrow[u, hook] \arrow[r, hook] \arrow[d, "\pi_{L*}^{\vee\vee}"'] & S_{1L}N_2 \arrow[ul] \arrow[dl, "\pi_{L*}^{\vee\vee}"'] & S_1N_L \arrow[l, "\pi_R^*"] \arrow[u, hook] \arrow[d, "\pi_{L*}^{\vee\vee}"] & S_0N' \times N_{z_1} \arrow[l, "\boxplus_L"'] \arrow[ur] \arrow[dr] & \\
\mathfrak{M}_1^\infty(X_{1R}) & & \mathfrak{M}_1^\infty(X_0) \arrow[ll, "\pi_R^*"'] & & N_{z_1} \arrow[ll, hook']
\end{tikzcd} \tag{20}
$$

We need to check the image of $\boxplus_L : S_0N' \times N_{z_1} \to N_L$ is contained in S_1N_L. Then, analyzing the commutativity of diagram (20) boils down do analyzing the diagram

$$
\begin{tikzcd}
S_1N_L \arrow[d, "\pi_{L*}^{\vee\vee}"] & S_0N' \times N_{z_1} \arrow[l, "\boxplus_L"'] \arrow[d] \\
\mathfrak{M}_1^\infty(X_0) & N_{z_1} \arrow[l]
\end{tikzcd} \tag{21}
$$

Let $m' \in S_0N' \subset S_0\mathfrak{M}_1^\infty(X_1)$, $m' = [a_1', a_2', 0, b', c']$. Let $m'' \in N_z$. Then a direct computation shows that $(\pi_{L*}(m' \boxplus_L m''))^{\vee\vee} = m''$. This shows that the image of $S_0N' \times N_{z_1}$ under $\boxplus_L$ is contained in S_1N_L and that diagram (21) is commutative.

(6) We want to show that $L_{bL,2}|_{N_L} \cong L_{bL,L}$, $L_{cL,2}|_{N_L} \cong L_{cL,L}$. This will follow from the commutativity of the diagram

$$
\begin{tikzcd}
N_L \arrow[d, "\pi_R^*"] & N' \times N_{z_1} \arrow[l, "\boxplus_L"', "\cong"] \arrow[r] & N' \arrow[r] & \mathfrak{M}_1^\infty(X_1) \arrow[d, equal] \\
N_2 & C \arrow[l, "\simeq"'] \arrow[r, "\pi_{R*}^{\vee\vee}"] & S_0\mathfrak{M}_1^\infty(X_1) \arrow[r, "\simeq"] & \mathfrak{M}_1^\infty(X_1)
\end{tikzcd} \tag{22}
$$

We showed in proposition 4.9 that the map $H_2(\cdot, 0)$ is the homotopy inverse of the inclusion $C \to N_2$. Let $(m', m'') \in N' \times N_z$. Then, by definition of H_2,

$$\pi_{R*}H_2(\pi_R^*(m' \boxplus_L m''), 0) = H_L(m' \boxplus_L m'', 0) = H_1(m', 0) \boxplus H_z(m'', 0)$$

Hence the diagram

$$
\begin{tikzcd}
N_L \arrow[d, "\pi_R^*"] & N' \times N_{z_1} \arrow[l, "\boxplus_L"', "\cong"] \arrow[r] & N' \arrow[d, "H_1(\cdot,0)"] \\
N_2 \arrow[r, "H_2(\cdot,0)"] & C \arrow[r, "\pi_{R*}^{\vee\vee}"] & S_0\mathfrak{M}_1^\infty(X_1)
\end{tikzcd}
$$

is commutative. From here it follows easily that diagram (22) is commutative. □

6. The cohomology of $\mathfrak{M}_2^\infty(X_q)$

The objective of this section is to prove theorem 1.4. We begin by proving it for the special case $q = 2$:

Theorem 6.1. *There is an exact sequence*

$$0 \to H^*(\mathfrak{M}_2^\infty(X_2)) \to H^*(A_L) \oplus H^*(A_R) \oplus H^*(N_2) \to \\ \to H^*(A_0) \oplus H^*(N_L) \oplus H^*(N_R) \to H^*(N_0) \to 0$$

which splits to give an isomorphism

$$H^*(\mathfrak{M}_2^\infty(X_2)) \cong \operatorname{Ker}\left(H^*(A_L) \oplus H^*(A_R) \to H^*(A_0) \right) \oplus \\ \oplus \operatorname{Ker}\left(H^*(N_2) \to H^*(N_L) \oplus H^*(N_R) \right)$$

Proof. Recall corolary 4.2. We will use this spectral sequence to compute $H_*(\mathfrak{M}_2^\infty(X_2))$. Clearly the map $d_1 : E_{1,n} \to E_{2,n}$ is surjective hence $E_2^{2,n} = 0$. Also we notice that $E_1^{p,2n+1} = 0$ for any p. It follows that the spectral sequence collapses at the term E_2. We get then

$$H^{2n}(\mathfrak{M}_2^\infty(X_2)) = E_\infty^{0,2n} = \operatorname{Ker}\left(d_1 : E_1^{0,2n} \to E_1^{1,2n}\right) \tag{23}$$

$$H^{2n+1}(\mathfrak{M}_2^\infty(X_2)) = E_\infty^{1,2n} = \frac{\operatorname{Ker}\left(d_1 : E_1^{1,2n} \to E_1^{2,2n}\right)}{\operatorname{Im}\left(d_1 : E_1^{0,2n} \to E_1^{1,2n}\right)} \tag{24}$$

When performing calculations we will use the following sign conventions:

$$\begin{tikzcd}
A_L & A_0 \arrow[l, "+"'] \arrow[r, "+"] & A_R \\
N_L \arrow[u, "-"] \arrow[dr, "+"'] & N_0 \arrow[l, "+"'] \arrow[r, "+"] \arrow[u, "+"] & N_R \arrow[u, "-"'] \arrow[dl, "-"] \\
 & N_2 &
\end{tikzcd} \tag{25}$$

We begin by defining the following generators of the cohomology of $E_1^{0,2n}$:

$$\begin{aligned}
a^i_{\Delta L} &= c_i(E_{cL}) - c_i(E_{bL}) & a^i_{\Delta R} &= c_i(E_{cR}) - c_i(E_{bR}) \\
a^i_{bL} &= c_i(E_{bL}) & a^i_{bR} &= c_i(E_{bR}) \\
c_{\Delta L} &= c_1(L_{cL}) - c_1(L_{bL}) & c_{\Delta R} &= c_1(L_{cR}) - c_1(L_{bR}) \\
c_{bL} &= c_1(L_{bL}) & c_{bR} &= c_1(L_{bR})
\end{aligned}$$

We do the same for $E_1^{1,2n}$:

$$\begin{aligned}
n_{\Delta L} &= c_1(L_{cL}) - c_1(L_{bL}) & n_{\Delta R} &= c_1(L_{cR}) - c_1(L_{bR}) \\
n_{bL} &= c_1(L_{bL}) & n_{bR} &= c_1(L_{bR}) \\
n_{0R} &= c_1(L_{0R}) & n_{0L} &= c_1(L_{0L}) \\
a^i &= c_i(E_0) & &
\end{aligned}$$

and for $E_1^{2,n}$:

$$n_{0R} = c_1(L_{0R}) \qquad\qquad n_{0L} = c_1(L_{0L})$$

Then, from theorem 5.2 it follows that the map $d_1 : E_1^{0,2n} \to E_1^{1,2n}$ may be represented by the following diagram, whose entries correspond to those in diagram

(25):

$$\begin{tikzcd}
\left(a^1_{\Delta L}, a^1_{bL}, a^2_{\Delta L}, a^2_{bL}\right) \arrow[r, mapsto] \arrow[d, mapsto] & \left(0, a^1, 0, a^2\right) & \left(a^1_{\Delta R}, a^1_{bR}, a^2_{\Delta R}, a^2_{bR}\right) \arrow[l, mapsto] \arrow[d, mapsto] \\
\begin{matrix}(-n_{\Delta L}, -n_{bL} - n_{0R}, -n_{\Delta L}n_{0R}, -n_{bL}n_{0R}) \\ (n_{\Delta L}, n_{bL}, 0, n_{0R})\end{matrix} & & \begin{matrix}(-n_{\Delta R}, -n_{bR} - n_{0L}, -n_{\Delta R}n_{0L}, -n_{bR}n_{0L}) \\ (0, -n_{0L}, -n_{\Delta R}, -n_{bR})\end{matrix} \\
& (c_{\Delta L}, c_{bL}, c_{\Delta R}, c_{bR}) \arrow[ul, mapsto] \arrow[ur, mapsto] &
\end{tikzcd}$$

Also the map $d_1 : E_1^{1,2n} \to E_1^{2,2n}$ is given by

$$\begin{aligned}
(a^1, a^2) &\mapsto (n_{0L} + n_{0R}, n_{0L}n_{0R}) \\
(n_{\Delta L}, n_{bL}, n_{0R}) &\mapsto (0, n_{0L}, n_{0R}) \\
(n_{0L}, n_{\Delta R}, n_{bR}) &\mapsto (n_{0L}, 0, n_{0R})
\end{aligned}$$

Now let

$$\begin{aligned}
K_{AL} &= \mathrm{Ker}(H^*(A_L) \to H^*(A_0)) & \qquad K_{AR} &= \mathrm{Ker}(H^*(A_R) \to H^*(A_0)) \\
K_{NL} &= \mathrm{Ker}(H^*(N_L) \to H^*(N_0)) & \qquad K_{NR} &= \mathrm{Ker}(H^*(N_R) \to H^*(N_0))
\end{aligned}$$

Then

$$\begin{aligned}
&H^*(A_L) \cong \mathbb{Z}[a^1, a^2] \oplus K_{AL}\ ,\ H^*(A_R) \cong \mathbb{Z}[a^1, a^2] \oplus K_{AR} \\
&H^*(C) \cong \mathbb{Z}[n_L, n_R] \oplus K_{NL} \oplus K_{NR} \oplus K_C \\
&H^*(N_L) \cong \mathbb{Z}[n_L, n_R] \oplus K_{NL}\ ,\ H^*(N_R) \cong \mathbb{Z}[n_L, n_R] \oplus K_{NR}
\end{aligned}$$

Notice that $K_C \subset H^*(C)$ is the ideal generated by $c_{\Delta L}c_{\Delta R}$. The restriction of the map $H^*(A_L) \to H^*(N_L)$ to K_{AL} induces a map $s_L : K_{AL} \to K_{NL}$. Similarly we have a map $s_R : K_{AR} \to K_{NR}$. Let also $s : \mathbb{Z}[a^1, a^2] \to \mathbb{Z}[n_L, n_R]$ be the map induced by the direct sum map $BU(1) \times BU(1) \to BU(2)$. Then the map $d_1 : E_1^{0,2n} \to E_1^{1,2n}$ is given by

$$\begin{aligned}
&d_1\left(a_L + k_{AL}, a_R + k_{AR}, x + k_{NL} + k_{NR} + k_C\right) = \\
&\qquad = \left(-s(a_L) - s_L(k_{AL}) + x + k_{NL}, -s(a_R) - s_R(k_{AR}) - x - k_{NR}, a_L + a_R\right)
\end{aligned}$$

and the map $d_1 : E_1^{1,2n} \to E_1^{2,2n}$ is given by

$$d_1\left(x_L + k_{NL}, x_R + k_{NR}, a\right) = \left(x_L + x_R + s(a)\right)$$

Now we can finish the proof:

(1) We prove first that $H^{2n+1}(\mathfrak{M}_2^\infty(X_2)) = 0$. We need to show $\mathrm{Ker}\,(d_1 : E_1^{1,2n} \to E_1^{2,2n}) \subset \mathrm{Im}\,(d_1 : E_1^{0,2n} \to E_1^{1,2n})$. Let $(x_L + k_{NL}, x_R + k_{NR}, a) \in \mathrm{Ker}\, d_1$. Then $x_L + x_R + s(a) = 0$. It follows that

$$d_1(a, 0, -x_R + k_L - k_R) = (x_L + k_L, x_R + k_R, a)$$

(2) Now we will show that $H^{2n} \cong \mathbb{Z}[a^1, a^2] \oplus K_{AL} \oplus K_{AR} \oplus K_C$ which conpletes the proof. We first define a map $\mathbb{Z}[a^1, a^2] \oplus K_{AL} \oplus K_{AR} \oplus K_C \to E_1^{0,2n}$ by

$$(a, k_{AL}, k_{AR}, k_C) \mapsto (a + k_{AL}, -a + k_{AR}, s(a) + s_L(k_{AL}) - s_R(k_{AR}) + k_C)$$

We want to show this map is injective onto the kernel of d_1. Injectivity is clear and a direct verification shows the image is contained in the kernel of d_1. To show surjectivity let

$$(a_L + k_{AL}, a_R + k_{AR}, x + k_{NL} + k_{NR} + k_C) \in \operatorname{Ker} d_1.$$

Then $a_L = -a_R$, $k_{NL} = s_L(k_{AL})$, $k_{NR} = -s_R(k_{AR})$ and $x = s(a_L) = -s(a_R)$. The result follows. □

We are ready to prove the general case:

Theorem 6.2. *With notations as in theorem 2.1 let*

$$K_i = \operatorname{Ker}\left(H^*(\pi_i^*\mathfrak{M}_2^\infty(X_1)) \to H^*(\pi_\emptyset^*\mathfrak{M}_2^\infty(X_0)) \right)$$
$$K_{ij} = \operatorname{Ker}\left(H^*(\pi_{ij}^*\mathfrak{M}_2^\infty(X_2)) \to H^*(\pi_i^*\mathfrak{M}_2^\infty(X_1)) \oplus H^*(\pi_j^*\mathfrak{M}_2^\infty(X_1)) \right)$$

Then, as modules over $\mathbb{Z}$, we have an isomorphism

$$H^*\left(\mathfrak{M}_2^\infty(X_q)\right) \cong H^*\left(\mathfrak{M}_2^\infty(X_0)\right) \oplus \bigoplus_i K_i \oplus \bigoplus_{i<j} K_{ij} \tag{26}$$

Proof. We divide the proof into two steps:

(1) We will use theorem 2.1 to build a spectral sequence converging to the cohomology of $H^*(\mathfrak{M}_2^\infty(X_q)$. Let Δ be the $q-1$ simplex. Label its vertices by v_i, $i = 1, \ldots, q$, and the e_{ij} be the middle point of the edge joining v_i and v_j. We define a filtration $\Delta_0 \subset \Delta_1 \subset \Delta$ of Δ where $\Delta_0 = \bigcup_{i<j} e_{ij}$ and Δ_1 is the 1-skeleton of Δ. Write $\Delta_1 = \bigcup_i \Delta_{1i}$ where Δ_{1i} is the closure of the connected component of $\Delta_1 \setminus \Delta_0$ containing v_i. Then we define

$$M = \frac{\bigcup_{i,j}\left(e_{ij} \times \pi_{ij}^*\mathfrak{M}_2^\infty(X_2)\right) \cup \bigcup_i \left(\Delta_{1i} \times \pi_i^*\mathfrak{M}_2^\infty(X_1)\right) \cup \left(\Delta \times \pi_\emptyset^*\mathfrak{M}_2^\infty(X_0)\right)}{\sim}$$

where $\sim$ is induced by the inclusions $e_{ij} \subset \Delta_{1i} \subset \Delta$ and $\pi_\emptyset^*\mathfrak{M}_2^\infty(X_0) \subset \pi_i^*\mathfrak{M}_2^\infty(X_1) \subset \pi_{ij}^*\mathfrak{M}_2^\infty(X_2)$. Then, the arguments in [21] can be applied to show that M is homotopically equivalent to $\mathfrak{M}_2^\infty(X_q)$. The filtration of Δ by Δ_0, Δ_1 induces a filtration $F_0 \subset F_1 \subset F_2 = M$ of M which leads to a spectral sequence with

$$E_1^{0,n} = H^n(F_0) \cong \bigoplus_{i<j}\left(H^0(e_{ij}) \otimes H^n(\pi_{ij}^*\mathfrak{M}_2^\infty(X_2))\right)$$
$$E_1^{1,n} = H^n(F_1, F_0) \cong \bigoplus_i \left(H^1(\Delta_{1i}, \partial\Delta_{1i}) \otimes H^n(\pi_i^*\mathfrak{M}_2^\infty(X_1))\right)$$
$$E_1^{2,n} = H^n(F_2, F_1) \cong H^1(F_1) \otimes H^n(\mathfrak{M}_2^\infty(X_0))$$

(2) The d_1 differential is induced by the inclusions $\pi_\emptyset^*\mathfrak{M}_2^\infty(X_0) \to \pi_i^*\mathfrak{M}_2^\infty(X_1) \to \pi_{ij}^*\mathfrak{M}_2^\infty(X_2)$. We will use the sign conventions $(i < j)$:

$$\begin{tikzcd} H^*\left(\pi_{ij}^*\mathfrak{M}_2^\infty(X_2)\right) \arrow[r, "+"] \arrow[d, "-"] & H^*\left(\pi_i^*\mathfrak{M}_2^\infty(X_1)\right) \arrow[d, "+"] \\ H^*\left(\pi_j^*\mathfrak{M}_2^\infty(X_1)\right) \arrow[r, "+"] & H^*\left(\pi_\emptyset^*\mathfrak{M}_2^\infty(X_0)\right) \end{tikzcd} \tag{27}$$

Let

$K_i = \mathrm{Ker}\left(H^*(\pi_i^*\mathfrak{M}_2^\infty(X_1)) \to H^*(\pi_\emptyset^*\mathfrak{M}_2^\infty(X_0))\right)$

$K_{ij} = \mathrm{Ker}\left(H^*(\pi_{ij}^*\mathfrak{M}_2^\infty(X_2)) \to H^*(\pi_i^*\mathfrak{M}_2^\infty(X_1)) \oplus H^*(\pi_j^*\mathfrak{M}_2^\infty(X_1))\right)$

Then, from theorem 6.1 we have

$$H^*(\pi_i^*\mathfrak{M}_2^\infty(X_1)) \cong H^*(\pi_\emptyset^*\mathfrak{M}_2^\infty(X_0)) \oplus K_i$$
$$H^*(\pi_{ij}^*\mathfrak{M}_2^\infty(X_2)) \cong H^*(\pi_\emptyset^*\mathfrak{M}_2^\infty(X_0)) \oplus K_i \oplus K_j \oplus K_{ij}$$

Then the sequence of maps $E_1^{0,n} \xrightarrow{d_1} E_1^{1,n} \xrightarrow{d_1} E_1^{2,n}$ splits into three sequences

(28)
$$\begin{tikzcd} \bigoplus_{i<j} H^0(e_{ij}) \otimes K_{ij} \arrow[r] & 0 \arrow[r] & 0 \\ \bigoplus_{i<j} H^0(e_{ij}) \otimes (K_i \oplus K_j) \arrow[r] & \bigoplus_l H^1(\Delta_{1l}, \partial\Delta_{1l}) \otimes K_l \arrow[r] & 0 \\ H^0(\Delta_0) \otimes K^n \arrow[r] & H^1(\Delta_1, \Delta_0) \otimes K^n \arrow[r] & H^1(\Delta_1) \otimes K^n \end{tikzcd}$$

where K^n stands for $H^n\left(\mathfrak{M}_2^\infty(X_0)\right)$. The bottom maps are easily analyzed using the exact sequence

$$0 \to H^0(\Delta_1) \to H^0(\Delta_0) \to H^1(\Delta_1, \Delta_0) \to H^1(\Delta_1) \to 0$$

It follows that the map $d_1 : E_1^{1,n} \to E_1^{2,n}$ is surjective. Since $E_1^{r,n} = 0$ for $r > 2$ and n even, this implies the spectral sequence collapses and

$$H^n(\mathfrak{M}_2^\infty(X_q)) = \frac{\mathrm{Ker}(d_1 : E_1^{1,n} \to E_1^{2,n})}{\mathrm{Im}(d_1 : E_1^{0,n} \to E_1^{1,n})}$$
$$H^n(\mathfrak{M}_2^\infty(X_q)) = \mathrm{Ker}(d_1 : E_1^{0,n} \to E_1^{1,n})$$

Lets look more closely at the map

(29)
$$\bigoplus_{i<j} H^0(e_{ij})(K_i \oplus K_j) \to \bigoplus_i H^1\left(\Delta_{1i}, \partial\Delta_{1i}\right) \otimes K_i$$

Observe that

$$\bigoplus_{i<j} H^0(e_{ij}) \otimes (\, K_i \oplus K_j\,) = \bigoplus_i H^0(\partial\Delta_{1i}) \otimes K_i$$

It follows that the map (29) can be easily analysed using the exact sequence

$$0 \to H^0(\Delta_{1i}) \to H^0(\partial\Delta_{1i}) \to H^1(\Delta_{1i}, \partial\Delta_{1i}) \to 0$$

We gather together our conclusions:

(a) The top sequence in (28) contributes a term
$$\bigoplus_{i<j} H^0(e_{ij}) \otimes K_{ij}$$
to $H^{2n}(\mathfrak{M}_2^\infty(X_q))$.

(b) The bottom sequence in (28) does not contribute to $H^{2n+1}(\mathfrak{M}_2^\infty(X_q))$ since it is exact in the middle.

(c) The bottom sequence in (28) contributes a term
$$H^0(\Delta_1) \otimes H^*\left(\mathfrak{M}_2^\infty(X_0)\right)$$
to $H^{2n}(\mathfrak{M}_2^\infty(X_q))$.

(d) The map (29) is surjective hence it does not contribute to $H^{2n+1}(\mathfrak{M}_2^\infty(X_q))$.
(e) The map (29) contributes a term

$$\bigoplus_i H^0(\Delta_{1i}) \otimes K_i$$

to $H^{2n}(\mathfrak{M}_2^\infty(X_q))$.

From (b) and (d) it follows that $H^{2n+1}(\mathfrak{M}_2^\infty(X_q)) = 0$ and from (a), (c) and (e) equation (26) follows. □

Appendix A. Monads

In this appendix we will sketch the monad description of the spaces $\mathfrak{M}_k^r(\mathbb{CP}^2)$ and $\mathfrak{M}_k^r(\tilde{\mathbb{CP}}^2)$. We follow [12]. See also [3].

Let $L_\infty \subset \mathbb{CP}^2$ be a rational curve and let $L \subset \tilde{\mathbb{CP}}^2$ be the exceptional divisor. Choose sections x_1, x_2, x_3 spanning $H^0(\mathcal{O}(L_\infty))$ and y_1, y_2 spanning $H^0(\mathcal{O}(L_\infty - L))$ so that x_3 vanishes on L_∞ and $x_1y_1 + x_2y_2$ spans the kernel of

$$H^0(\mathcal{O}(L_\infty)) \otimes H^0(\mathcal{O}(L_\infty - L)) \longrightarrow H^0(\mathcal{O}(2L_\infty - L))$$

A.1. **The moduli space over $\mathbb{CP}^2$.** Let W be a k-dimensional vector space. Let $\mathcal{R}$ be the space of 4-tuples $m = (a_1, a_2, b, c)$ with $a_i \in \mathrm{End}(W)$, $b \in \mathrm{Hom}(\mathbb{C}^r, W)$, $c \in \mathrm{Hom}(W, \mathbb{C}^r)$, obeying the integrability condition $[a_1, a_2] + bc = 0$. For each $m = (a_1, a_2, b, c) \in \mathcal{R}$ we define maps A_m, B_m

$$W(-L_\infty) \xrightarrow{A_m} W^{\oplus 2} \oplus \mathbb{C}^n \xrightarrow{B_m} W(L_\infty)$$

by

$$A_m = \begin{bmatrix} x_1 - a_1x_3 \\ x_2 - a_2x_3 \\ cx_3 \end{bmatrix}, \quad B_m = \begin{bmatrix} -x_2 + a_2x_3 & x_1 - a_1x_3 & bx_3 \end{bmatrix}$$

Then $B_mA_m = 0$. The assignement $m \mapsto \mathcal{E}_m = \mathrm{Ker}\, B_m / \mathrm{Im}\, A_m$ induces a map $f : \mathcal{R} \to \overline{\mathfrak{M}_k^r(\mathbb{CP}^2)}$. m is called non degenerate if A_m, B_m have maximal rank at every point in $\mathbb{CP}^2$.

Theorem A.1. *f induces an isomorphism betwen the quotient of the space of non degenerate points in $\mathcal{R}$ by the action of $Gl(W)$:*

$$g \cdot (a_1, a_2, b, c) = (g^{-1}a_1g, g^{-1}a_2g, g^{-1}b, cg)$$

and the moduli space $\mathfrak{M}_k^r(\mathbb{CP}^2)$.

For a proof see [7], proposition 1.

Theorem A.2. *The algebraic quotient $\mathcal{R}/Gl(W)$ is isomorphic to the Donaldson-Uhlenbeck completion of the moduli space of instantons over S^4.*

For a proof see [8], sections 3.3, 3.4, 3.4.4.

We sketch here how the map from $\mathcal{R}/Gl(W)$ to the Donaldson-Uhlenbeck completion of the moduli space of instantons is constructed (see [12] for details): Let $m = (a_1, a_2, b, c) \in \mathcal{R}$. A subspace $W' \subset W$ is called b-special with respect to m if

$$a_i(W') \subset W' \ (i = 1, 2) \text{ and } \mathrm{Im}\, b \subset W'. \tag{30}$$

A subspace $W' \subset W$ is called c-special with respect to m if

$$a_i(W') \subset W' \ (i = 1, 2) \text{ and } W' \subset \mathrm{Ker}\, c. \tag{31}$$

m is called completely reducible if for every $W' \subset W$ which is b-special or c-special, there is a complement $W'' \subset W$ which is c-special or b-special respectively.

Proposition A.3. *Let $m = (a_1, a_2, b, c) \in \mathcal{R}$.*

(1) *m is non degenerate if and only if the only b-special subspace is W and the only c-special subspace is 0;*
(2) *For every m, the orbit of m under $Gl(W)$ contains in its closure a canonical completely reducible orbit and completely reducible orbits have disjoint closures;*
(3) *If m is completely reducible then, after acting with some $g \in Gl(W)$ we can write*
$$a_i = \begin{bmatrix} a_i^{red} & 0 \\ 0 & a_i^{\Delta} \end{bmatrix}\ ,\ b = \begin{bmatrix} b^{red} \\ 0 \end{bmatrix}\ ,\ c = \begin{bmatrix} c^{red} & 0 \end{bmatrix}$$
where $(a_1^{red}, a_2^{red}, b^{red}, c^{red})$ is non-degenerate and the matrices $a_1^{\Delta}, a_2^{\Delta}$ can be simultaneously diagonalized. Such a configuration is equivalent to the following data:
   - *An irreducible integrable configuration $(a_1^{red}, a_2^{red}, b^{red}, c^{red})$ corresponding to a bundle with $c_2 = l \leq k$;*
   - *$k - l$ points in $\mathbb{C}^2 = \mathbb{CP}^2 \setminus L_\infty$ given by the eigenvalue pairs of $a_1^{\Delta}, a_2^{\Delta}$*

   This is precisely the Donaldson-Uhlenbeck completion.

A.2. **The moduli space over $\tilde{\mathbb{CP}}^2$.** Let $\tilde{\mathcal{R}}$ be the space of 5-tuples $\tilde{m} = (a_1, a_2, d, b, c)$ where $a_i \in \mathrm{Hom}(W, V)$, $d \in \mathrm{Hom}(V, W)$, $b \in \mathrm{Hom}(\mathbb{C}^r, V)$, $c \in \mathrm{Hom}(W, \mathbb{C}^r)$, such that $a_1(W) + a_2(W) + b(\mathbb{C}^r) = V$, obeying the integrability condition $a_1 d a_2 - a_2 d a_1 + bc = 0$. For each $\tilde{m} = (a_1, a_2, d, b, c) \in \tilde{\mathcal{R}}$ we define maps $A_{\tilde{m}}, B_{\tilde{m}}$
$$W(-L_\infty) \oplus V(L - L_\infty) \xrightarrow{A_{\tilde{m}}} (V \oplus W)^{\oplus 2} \oplus \mathbb{C}^n \xrightarrow{B_{\tilde{m}}} V(L_\infty) \oplus W(L_\infty - L)$$
by
$$A_{\tilde{m}} = \begin{bmatrix} a_1 x_3 & -y_2 \\ x_1 - d a_1 x_3 & 0 \\ a_2 x_3 & y_1 \\ x_2 - d a_2 x_3 & 0 \\ c x_3 & 0 \end{bmatrix}\ ,\ B_{\tilde{m}} = \begin{bmatrix} x_2 & a_2 x_3 & -x_1 & -a_1 x_3 & b x_3 \\ d y_1 & y_1 & d y_2 & y_2 & 0 \end{bmatrix}$$
Then $B_{\tilde{m}} A_{\tilde{m}} = 0$. The assignement $\tilde{m} \mapsto \mathcal{E}_{\tilde{m}} = \mathrm{Ker}\, B_{\tilde{m}} / \mathrm{Im}\, A_{\tilde{m}}$ induces a map $\tilde{f} : \tilde{\mathcal{R}} \to \overline{\mathfrak{M}_k^r(\tilde{\mathbb{CP}}^2)}$. A point $\tilde{m} \in \tilde{\mathcal{R}}$ is called non-degenerate if $A_{\tilde{m}}$ and $B_{\tilde{m}}$ have maximal rank at every point in $\tilde{\mathbb{CP}}^2$.

Theorem A.4. *The map $\tilde{f}$ induces an isomorphism betwen the quotient of the space of non degenerate points in $\tilde{\mathcal{R}}$ by the action of $Gl(V) \times Gl(W)$:*
$$(g_0, g_1) \cdot (a_1, a_2, b, c, d) = (g_0^{-1} a_1 g_1, g_0^{-1} a_2 g_1, g_0^{-1} b, c g_1, g_1^{-1} d g_0)$$
and the moduli space $\mathfrak{M}_k^r(\tilde{\mathbb{CP}}^2)$.

See [12] for a proof.

Consider the algebraic quotient $\tilde{\mathcal{R}}/Gl(V) \times Gl(W)$. This space is a completion of the moduli space $\mathfrak{M}_k^r(\tilde{\mathbb{CP}}^2)$. We proceed to give an interpretation of the points in this completion in terms of the Donaldson-Uhlenbeck completion. See [12] for

details. Let $\tilde{m} = (a_1, a_2, d, b, c)$. Let $V' \subset V$ and $W' \subset W$ and assume $\dim V' = \dim W'$. The pair (V', W') is called b-special with respect to $\tilde{m}$ if

$$a_i(W') \subset V' \ (i = 1, 2),\ d(V') \subset W' \text{ and } \operatorname{Im} b \subset V' \tag{32}$$

The pair (V', W') is called c-special with respect to $\tilde{m}$ if

$$a_i(W') \subset V' \ (i = 1, 2),\ d(V') \subset W' \text{ and } W' \subset \operatorname{Ker} c \tag{33}$$

$\tilde{m}$ is called completely reducible if for every pair (V', W') which is either b-special or c-special, there are complements V'', W'' to V' and W' such that the pair (V'', W'') is c-special or b-special respectively.

Proposition A.5. *Let $\tilde{m} = (a_1, a_2, d, b, c) \in \tilde{R}$.*

1. *$\tilde{m}$ is non-degenerate if and only if the only b-special pair is (V, W) and the only c-special pair is $(0, 0)$;*
2. *For every $\tilde{m}$, the orbit of $\tilde{m}$ under $Gl(V) \times Gl(W)$ contains in its closure a canonical completely reducible orbit and completely reducible orbits have disjoint closures;*
3. *If $\tilde{m}$ is completely reducible then, after acting with some $(g_0, g_1) \in Gl(V) \times Gl(W)$, we can write*

$$a_i = \begin{bmatrix} a_i^{red} & 0 \\ 0 & a_i^{\Delta} \end{bmatrix}\ ,\ d = \begin{bmatrix} d^{red} & 0 \\ 0 & d^{\Delta} \end{bmatrix} b = \begin{bmatrix} b^{red} \\ 0 \end{bmatrix}\ ,\ c = \begin{bmatrix} c^{red} & 0 \end{bmatrix}$$

where $(a_1^{red}, a_2^{red}, d^{red}, b^{red}, c^{red})$ is non-degenerate effective and integrable and the matrices $a_1^{\Delta}, a_2^{\Delta}, d^{\Delta}$ can be simultaneously diagonalized. Such a configuration is equivalent to the following data:

- *An irreducible configuration $(a_1^{red}, a_2^{red}, d, b^{red}, c^{red})$ associated to a bundle with $c_2 = l \leq k$;*
- *$k - l$ points in the blow up $\widetilde{\mathbb{C}^2}$ of $\mathbb{C}^2$ at the origin. This points are determined as follows: $a_1^{\Delta}, a_2^{\Delta}, d^{\Delta}$ determine $k - l$ unique points $(\lambda_1^r, \lambda_2^r), [\mu_1^r, \mu_2^r] \in \widetilde{\mathbb{C}^2}$ corresponding to vectors $v_1, \ldots, v_{k-l}$ such that $da_i v^r = \lambda_i^r v^r$ (λ_1, λ_2 are the eigenvalue pairs of da_1, da_2) and $(\mu_1^r a_1 + \mu_2^r a_2) v^r = 0$.*

A.3. **Direct image.** In this section we gather some results concerning the direct image map π_* induced by the blowup map $\pi : \tilde{\mathbb{CP}}^2 \to \mathbb{CP}^2$.

Proposition A.6. *Let $\pi_{\#} : \tilde{\mathcal{R}} \to \mathcal{R}$ be given by $\pi_{\#}(a_1, a_2, d, b, c) = (da_1, da_2, db, c)$. Let $\tilde{m} \in \tilde{\mathcal{R}}$, $m = \pi_{\#}\tilde{m}$. Then $\mathcal{E}_{\tilde{m}}|_{\tilde{\mathbb{CP}}^2 \setminus L}$ is isomorphic to $\mathcal{E}_m|_{\mathbb{CP}^2 \setminus [0,0,1]}$.*

For the proof see [19], proposition 5.6.

Proposition A.7. *Let $S_0\mathfrak{M}_1^r(\tilde{\mathbb{CP}}^2) = \{(\mathcal{E}, \phi) \in \mathfrak{M}_1^r(\mathbb{CP}^2) : (\pi_*\mathcal{E})^{\vee\vee} = \mathcal{O}_{\mathbb{CP}^2}^r\}$. Then*

1. *$m \in S_0\mathfrak{M}_1^r(\mathbb{CP}^2)$ if and only if m is of the form $(a_1, a_2, 0, b, c)$.*
2. *The inclusion $S_0\mathfrak{M}_1^r \to \mathfrak{M}_1^r$ is a homotopy equivalence.*

Proof. First we observe that $\mathfrak{M}_1^r(\tilde{\mathbb{CP}}^2) = S_0\mathfrak{M}_1^r(\tilde{\mathbb{CP}}^2) \cup \pi_{\emptyset}^*\mathfrak{M}_1^r(\mathbb{CP}^2)$. Now $m \in \pi_{\emptyset}^*\mathfrak{M}_1^r(\mathbb{CP}^2)$ if and only if d is an isomorphism (see [12]). The first statement follows. The second statement follows easily from the first: just consider the homotopy $(a_1, a_2, d, b, c) \mapsto (a_1, a_2, td, b, c)$. □

Proposition A.8. *Let $x = [x_1, x_2, 1] \in \mathbb{CP}^2$ and let $\pi_x : \tilde{\mathbb{CP}}^2 \to \mathbb{CP}^2$ be the blow up at x. Then the map $\pi_x^* : \mathfrak{M}_k^r(\mathbb{CP}^2) \to \mathfrak{M}_k^r(\tilde{\mathbb{CP}}^2)$ is given by*

$$[a_1, a_2, b, c] \mapsto [a_1 - x_1\mathbb{1}, a_2 - x_2\mathbb{1}, \mathbb{1}, b, c]$$

Proof. For $x = [0, 0, 1]$ see [3]. For the general case consider the translation $[w_1, w_2, w_3] \mapsto [w_1 - x_1 w_3, w_2 - x_2 w_3, w_3]$. This induces a map $\tau : \mathfrak{M}_k^r(X_0) \to \mathfrak{M}_k^r(X_0)$ given by

$$[a_1, a_2, b, c] \mapsto [a_1 - x_1\mathbb{1}, a_2 - x_2\mathbb{1}, b, c]$$

The result follows. □

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