A STRATIFICATION OF THE MODULI OF HOLOMORPHIC VECTOR BUNDLES ON THE BLOWUP OF A SURFACE

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Abstract. We study a stratification of the moduli space of framed holomorphic bundles over the blowup $\tilde{X}$ of a complex surface X, induced by the behavior of the bundles on a neighborhood of the exceptional divisor. We show that each strata is a trivial fibration over a moduli space over X.

1. Introduction

Let X be a smooth algebraic surface and let $C \subset X$ be a curve with positive self-intersection. An important example is $X = \mathbb{P}^2$ and $C \subset \mathbb{P}^2$ a rational curve. Fix a point $x_0 \in X \setminus C$ and let $\pi : \tilde{X} \to X$ denote the blowup of X at x_0. We let $\tilde{C} = \pi^{-1}(C)$. In this paper we will study the moduli space $\mathfrak{M}_k^r(\tilde{X})$ of pairs $(\mathcal{E}, \phi)$ where $\mathcal{E}$ is a rank r holomorphic bundle over $\tilde{X}$ with Chern classes $c_1 = 0$ and $c_2 = k$, and ϕ is a trivialization of $\mathcal{E}$ on $\tilde{C}$. We describe a stratification of this space whose strata are trivial fiber bundles over the corresponding moduli space over X. This approach is inspired by [2].

The motivation for this paper came from the study of the moduli space of based instantons over a connected sum of q copies of $\mathbb{P}^2$: in [1], [9], it was shown that this space is isomorphic as a real analytic space to the moduli space of holomorphic bundles over a blow up of $\mathbb{P}^2$ at q points, framed at a rational curve $C_\infty \subset \mathbb{P}^2$. This space can be analyzed by understanding the effect of blowing up since the moduli space over $\mathbb{P}^2$, which is isomorphic to the moduli space of based instantons in S^4, has been studied extensively. In an upcoming paper we will use this approach to investigate the homotopy type of the moduli space in the limit when $r \to \infty$.

1.1. Results.

Given a holomorphic bundle $\mathcal{E}$ over $\tilde{X}$ let $\mathcal{E}_X = (\pi_*\mathcal{E})^{\vee\vee}$ be the double-dual of the direct image sheaf $\pi_*\mathcal{E}$. Since $\mathcal{E}_X$ is reflexive and X has complex dimension 2, $\mathcal{E}_X$ is locally free. Hence $\mathcal{E}_X$ is a holomorphic bundle over X. We define $S_i\mathfrak{M}_k^r(\tilde{X}) \subset \mathfrak{M}_k^r(\tilde{X})$ as the subspace of bundles $\mathcal{E}$ such that $c_2(\mathcal{E}_X) = i$. We define a map $\pi_i : S_i\mathfrak{M}_k^r(\tilde{X}) \to \mathfrak{M}_i^r(X)$ as follows: given a holomorphic bundle $\mathcal{E}$ over $\tilde{X}$ trivialized on $\tilde{C}$ by $\phi : \mathcal{E}|_{\tilde{C}} \to \tilde{C} \times \mathbb{C}^r$, there is a canonical isomorphism of holomorphic bundles $f : \mathcal{E}|_{\tilde{X}\setminus L} \to \mathcal{E}_X|_{X\setminus\{x_0\}}$ which induces a trivialization $\phi \circ f^{-1}$ of $\mathcal{E}_X$ on C. We set $\pi_i(\mathcal{E}, \phi) = (\mathcal{E}_X, \phi \circ f^{-1})$.

Fix a rational curve $C_\infty \subset \mathbb{P}^2$ and let $\tilde{\mathbb{P}}^2$ be the blowup of $\mathbb{P}^2$ at a point $y_0 \notin C_\infty$. Then we have subspaces $S_i\mathfrak{M}_k^r(\tilde{\mathbb{P}}^2) \subset \mathfrak{M}_k^r(\tilde{\mathbb{P}}^2)$. Our first result is

Theorem 1.1. *π_i is a topologically trivial fiber bundle with fiber $S_0\mathfrak{M}_{k-i}^r(\tilde{\mathbb{P}}^2)$.*

We then provide a description of the fiber $S_0\mathfrak{M}_{k-i}^r(\tilde{\mathbb{P}}^2)$ in terms of polynomial equations involving finite dimensional matrices (a monad description). Let W_0, W_1

be rank k complex vector spaces and let $\tilde{\mathcal{R}}^{reg}$ be the space of 5-tuples (a_1, a_2, d, b, c) where $a_i \in \mathrm{Hom}(W_1, W_0)$, $d \in \mathrm{Hom}(W_0, W_1)$, $b \in \mathrm{Hom}(\mathbb{C}^r, W_0)$, $c \in \mathrm{Hom}(W_1, \mathbb{C}^r)$, obeying the integrability condition $a_1 da_2 - a_2 da_1 + bc = 0$, and some non-degeneracy condition (see section 4.2). Then (see [7]) the quotient of the space of $\tilde{\mathcal{R}}^{reg}$ by the action of $Gl(W_0) \times Gl(W_1)$

$$(g_0, g_1) \cdot (a_1, a_2, b, c, d) = (g_0^{-1} a_1 g_1, g_0^{-1} a_2 g_1, g_0^{-1} b, c g_1, g_1^{-1} d g_0)$$

is isomorphic to $\mathfrak{M}_k^r(\tilde{\mathbb{P}}^2)$.

Theorem 1.2. *$S_0\mathfrak{M}_k^r(\tilde{\mathbb{P}}^2) \subset \mathfrak{M}_k^r(\tilde{\mathbb{P}}^2)$ is the subspace of equivalence classes of 5-tuples (a_1, a_2, d, b, c) such that da_1, da_2 are nilpotent and, for any sequence $i_1, \dots, i_n \in \{1, 2\}$, we have*

$$c \left(\prod_{j=1}^{n} da_{i_j} \right) db = 0$$

This last result appeared in the author's thesis, [10]. Another description of this space was provided in [3], using a different monad construction. In an upcoming paper, we will show that, in the limit when $r \to \infty$, the inclusion $S_0\mathfrak{M}_k^r(\tilde{\mathbb{P}}^2) \to \mathfrak{M}_k^r(\tilde{\mathbb{P}}^2)$ is a homotopy equivalence.

The paper is organized as follows: In section §2 we fix some notation, gather some lemmas derived from Hartog's theorem and prove the continuity of the map π_i. In section §3 we prove theorem 1.1 using a gluing construction. In section §4 we prove theorem 1.2 by constructing, for each $(\mathcal{E}, \phi) \in S_0\mathfrak{M}$, the trivialization of $\mathcal{E}$ outside of the exceptional divisor which extends ϕ.

2. The moduli space and the map π_i

We will define the moduli space of holomorphic bundles on $\tilde{X}$ as the space of holomorphic structures on a fixed smooth vector bundle (see [8]). Let $E \to \tilde{X}$ be a smooth bundle with $c_1(E) = 0$. A holomorphic structure on E is a semi-connection of type $(0, 1)$ $\bar{\partial} : \Omega^0(E) \to \Omega^{0,1}(E)$ which is integrable, i.e. $\bar{\partial}^2 = 0$. Let $\mathcal{C}(\tilde{X}, E)$ be the space of pairs $(\bar{\partial}, \phi)$ where $\bar{\partial}$ is a holomorphic structure on E holomorphically trivial on $\tilde{C}$, and $\phi : E|_{\tilde{C}} \to \tilde{C} \times \mathbb{C}^r$ is a holomorphic trivialization. Then $\mathfrak{M}(\tilde{X}, E)$ is the quotient of $\mathcal{C}(\tilde{X}, E)$ by the action of the group $Aut(E)$ of smooth bundle automorphisms of E given by $g \cdot (\bar{\partial}, \phi) = (g \circ \bar{\partial} \circ g^{-1}, \phi \circ g^{-1})$. By theorem 1.1 and lemma 2.6 in [8], this action is free and $\mathfrak{M}(\tilde{X}, E)$ has the structure of a finite dimensional Hausdorff complex analytic space. In a similar way we can define the moduli space $\mathfrak{M}(X, E_X)$ of holomorphic structures on E_X trivialized on C. $\mathfrak{M}(\tilde{X}, E), \mathfrak{M}(X, E_X)$ depend only on the rank r and second Chern class k of E, E_X so we will write $\mathfrak{M}_k^r(\tilde{X}), \mathfrak{M}_k^r(X)$ when the specific smooth bundle is not important. We will often write an element in $\mathfrak{M}_k^r(\tilde{X})$ as a pair $(\mathcal{E}, \phi)$ where $\mathcal{E}$ is a holomorphic bundle over $\tilde{X}$ and ϕ is a tivialization of $\mathcal{E}$ on $\tilde{C}$.

We now prove the continuity of the map π_i. First we need some lemmas which will also be used in section 3.

Lemma 2.1. *Let $B \subset \mathbb{C}^2$ be a 4 dimensional ball centered at 0. Then any holomorphic map $g : B \setminus \{0\} \to Gl(r, \mathbb{C})$ can be extended to a map $\tilde{g} : B \to Gl(r, \mathbb{C})$.*

Proof. $Gl(r, \mathbb{C}) \subset \mathbb{C}^{r^2}$ so, by Hartog's theorem g extends to a map $\tilde{g} : B \to \mathbb{C}^{r^2}$. Composing with the determinant we get a map $det \circ \tilde{g} : B \to \mathbb{C}$ which can only vanish at $0 \in B$, hence it never vanishes. So the image of $\tilde{g}$ lies in $Gl(r, \mathbb{C})$. □

Lemma 2.2. *Let $\mathcal{E}_1, \mathcal{E}_2$ be holomorphic vector bundles over X. Let $\phi : \mathcal{E}_1|_{X \setminus \{x_0\}} \to \mathcal{E}_2|_{X \setminus \{x_0\}}$ be a bundle isomorphism. Then ϕ extends to an isomorphism $\tilde{\phi} : \mathcal{E}_1 \to \mathcal{E}_2$.*

Proof. Let U be a neighborhood of x_0. Fix trivializations of $\mathcal{E}_1, \mathcal{E}_2$ on a neighborhood U of x_0. The restriction of ϕ to U is equivalent to a map $g : U \setminus \{x_0\} \to Gl(r, \mathbb{C})$. By lemma 2.1, g can be extended to U so ϕ can be extended to X. □

Lemma 2.3. *Let $\mathcal{F} \to T \times \tilde{X}$ be a holomorphic bundle. Then there is a holomorphic bundle $\mathcal{F}_X$ over $T \times X$ such that*

(1) *There is an isomorphism of holomorphic bundles*
$$F : \mathcal{F}|_{T \times (\tilde{X} \setminus L)} \to \mathcal{F}_X|_{T \times (X \setminus x_0)}$$
(2) *If $\mathcal{F}_t, \mathcal{F}_{Xt}$ are the restrictions of $\mathcal{F}, \mathcal{F}_X$ to $t \times \tilde{X}$ and $t \times X$ respectively, then $\mathcal{F}_{Xt} \cong (\pi_* \mathcal{F}_t)^{\vee\vee}$.*

Proof. Let $p : T \times \tilde{X} \to \tilde{X}$ be the projection. For each $N \in \mathbb{Z}$ define the sheaf $\pi_*^N \mathcal{F} \to T \times X$ by
$$\pi_*^N \mathcal{F} = (\mathbf{1} \times \pi)_* \left(\mathcal{F} \otimes p^* \mathcal{O}(NL) \right)$$
We then have a natural isomorphism of holomorphic bundles
$$F : \mathcal{F}|_{T \times (\tilde{X} \setminus L)} \to \pi_*^N \mathcal{F}|_{T \times (X \setminus x_0)}$$
Now fix $t \in T$ and let $\imath_t$ denote the inclusions $t \times \tilde{X} \to T \times \tilde{X}$ and $t \times X \to T \times X$. Then
$$(\pi_*^N \mathcal{F})_t = \imath_t^* \pi_*^N \mathcal{F} = \pi_* \imath_t^* \left(\mathcal{F} \otimes p^* \mathcal{O}(NL) \right) = \pi_* \left(\mathcal{F}_t(NL) \right)$$
This sheaf is locally free for $N \geq 2k$ (see [2]). Since $(\pi_*^N \mathcal{F})_t$ and $(\pi_* \mathcal{F}_t)^{\vee\vee}$ are naturally isomorphic on $X \setminus x_0$, lemma 2.2 implies that $(\pi_*^N \mathcal{F})_t \cong (\pi_* \mathcal{F}_t)^{\vee\vee}$ on the whole of X. It follows that $\pi_*^N \mathcal{F}$ is flat over T since the Hilbert polynomial of $(\pi_*^N \mathcal{F})_t$ is constant with $t \in T$ (see [6], proposition 2.1.2). So we just have to take $\mathcal{F}_X = \pi_*^N \mathcal{F}$. □

A family of framed holomorphic bundles on $\tilde{X}$ parameterized by T is a holomorphic bundle $\mathcal{F} \to T \times \tilde{X}$ together with a holomorphic trivialization Φ o $\mathcal{F}|_{T \times \tilde{C}}$. Such a family induces a continuous map $T \to \mathfrak{M}_k^r(\tilde{X})$.

Proposition 2.4. *The map π_i is continuous.*

Proof. let T be a small neighborhood of $(\mathcal{E}, \phi) \in S_i \mathfrak{M}_k^r(\tilde{X})$ We just have to show that the restriction of π_i to T is continuous. Let $\mathcal{F} \to T \times \tilde{X}$ be the universal bundle over T, with a holomorphic trivialization Φ on $T \times \tilde{C}$. We use the isomorphism F from lemma 2.3 to define a trivialization Φ_X of $\mathcal{F}_X$ on $T \times C$. Then the pair $(\mathcal{F}_X, \Phi_X)$ defines a continuous map $T \to \mathfrak{M}_i^r(X)$ which equals the restriction of π_i to T. □

3. Gluing holomorphic bundles

The objective of this section is to prove theorem 1.1. This proof is based on a gluing construction: Given spaces A, B, bundles $E_A \to A$, $E_B \to B$ and a bundle isomorphism $\psi : E_A|_{A\cap B} \to E_B|_{A\cap B}$ we can get a bundle $E_A \bigcup_\psi E_B$ over $A \cup B$ by gluing E_A, E_B along $A \cap B$. We will apply this construction to the following pairs (A, B):

(1) We let $A = U$ be a neighborhood of $x_0 \in X$ biholomorphic to a ball in $\mathbb{C}^2$, with $U \cap C \neq \emptyset$ and with the property that any holomorphic map on $U \cap C$ can be extended to U. Let $B = X \setminus x_0$. Then $A \cup B = X$ and $A \cap B = U \setminus x_0$.

(2) Let $\tilde{U} = \pi^{-1}(U) \subset \tilde{X}$ and let L denote the exceptional divisor over $x_0 \in X$. Let $A = \tilde{U}$ and $B = \tilde{X}\setminus L$. Then $\tilde{X} = A\cup B$ with intersection $A\cap B = \tilde{U}\setminus L$. Observe that we have canonical isomorphisms $\pi : \tilde{X} \setminus L \to X \setminus \{x_0\}$ and $\pi : \tilde{U} \setminus L \to U \setminus \{x_0\}$.

(3) An important special case of the construction above is when $X = \mathbb{P}^2$, with a rational curve $C_\infty \subset \mathbb{P}^2$. We let $\tilde{\mathbb{P}}^2$ be the blowup of $\mathbb{P}^2$ at a point $y_0 \in \mathbb{P}^2 \setminus C_\infty$. Then we fix a neighborhood V of y_0 like U in (1) and we let $\tilde{V} = \pi^{-1}(V) \subset \tilde{\mathbb{P}}^2$. We fix once and for all a biholomorphic correspondence between U and V taking x_0 to y_0 and restricting to a biholomorphic map $U \cap C \to V \cap C_\infty$, using it to identify U with V, and $\tilde{U}$ with $\tilde{V}$.

To perform the gluing we need to have the bundle isomorphisms on the intersections. So we introduce enlarged moduli spaces:

Definition 3.1. Let Y be a surface, $C \subset Y$ a curve with positive self-intersection and $A \subset Y$ an open set. Given a smooth vector bundle E over Y, let $\tilde{\mathcal{C}}^A(X, E_X)$ be the space of triples $(\bar\partial, \phi, \psi)$ where $\bar\partial$ is a $(0, 1)$ connection on E and ϕ, ψ are C^∞ trivializations of E on C and A respectively. An element $(\bar\partial_0, \phi_0, \psi_0) \in \tilde{\mathcal{C}}^A(Y, E)$, induces an isomorphism

$$\tilde{\mathcal{C}}^A(Y, E) \cong \Omega^{0,1}(Y, E) \times C^\infty(C, Gl(r, \mathbb{C})) \times C^\infty(A, Gl(r, \mathbb{C})) \tag{1}$$

which we use to topologize $\tilde{\mathcal{C}}^A(Y, E)$. We let $\mathcal{C}^A(Y, E) \subset \tilde{\mathcal{C}}^A(Y, E)$ be the subspace of triples $(\bar\partial, \phi, \psi)$ such that $(\bar\partial, \phi) \in \mathcal{C}(Y, E)$, the holomorphic structure $\bar\partial$ is holomorphically trivial on A and ψ is a holomorphic trivialization of E on A that agrees with ϕ on $A \cap C$. Then we define $\mathfrak{M}^A(Y, E)$ as the quotient of $\mathcal{C}^A(Y, E)$ by the group $Aut(E)$.

We fix smooth complex vector bundles $E \to \tilde{X}$, $E_X \to X$ and $E_P \to \tilde{\mathbb{P}}^2$, with $c_2(E) = k$, $c_2(E_X) = i$, $c_2(E_P) = k - i$ and $c_1(E) = c_1(E_X) = c_1(E_P) = 0$. Let $S_i\mathfrak{M}^{\tilde{U}\setminus L}(\tilde{X}, E_{\tilde{X}}) \subset \mathfrak{M}^{\tilde{U}\setminus L}(\tilde{X}, E)$ be the subspace of triples $[(\bar\partial, \phi, \psi)]$ such that $[(\bar\partial, \phi)] \in S_i\mathfrak{M}(\tilde{X}, E)$ and similarly for $S_0\mathfrak{M}^{\tilde{\mathbb{P}}^2\setminus L}(\tilde{\mathbb{P}}^2, E_P)$. We have projection maps

$$pr_{\tilde{X}} : S_i\mathfrak{M}^{\tilde{U}\setminus L}(\tilde{X}, E) \to S_i\mathfrak{M}(\tilde{X}, E)$$
$$pr_X : \mathfrak{M}^U(X, E_X) \to \mathfrak{M}(X, E_X)$$
$$pr_P : S_0\mathfrak{M}^{\tilde{\mathbb{P}}^2\setminus L}(\tilde{\mathbb{P}}^2, E_P) \to S_0\mathfrak{M}(\tilde{\mathbb{P}}^2, E_P)$$

Proposition 3.1. *pr_P is a homeomorphism and $pr_{\tilde{X}}, pr_X$ are principal bundle maps with contractible fiber.*

Proof. First we observe that these maps are surjective. Given any $(\mathcal{E},\phi)\in\mathfrak{M}(X,E_X)$, $\mathcal{E}|_U$ is trivial, so pr_X is surjective. By definition of the stratification, given any $(\mathcal{E},\phi)\in S_0\mathfrak{M}_k^{\tilde{\mathbb{P}}^2\setminus L}(\tilde{\mathbb{P}}^2)$, $(\pi_*\mathcal{E})^{\vee\vee}$ is trivial so $\mathcal{E}|_{\tilde{\mathbb{P}}^2\setminus L}\cong(\pi_*\mathcal{E})^{\vee\vee}|_{\mathbb{P}^2\setminus y_0}$ is trivial. Hence pr_P is surjective. Finally the same argument shows that $pr_{\tilde{X}}$ is surjective: given $(\mathcal{E},\phi)\in\mathfrak{M}(\tilde{X},E)$, $(\pi_*\mathcal{E})^{\vee\vee}|_U$ is trivial so $\mathcal{E}|_{\tilde{U}\setminus L}\cong(\pi_*\mathcal{E})^{\vee\vee}|_{U\setminus x_0}$ is trivial. Now let

$$\begin{aligned}
G &= \left\{\eta\in\mathrm{Hol}(\tilde{U}\setminus L, Gl(r,\mathbb{C}))\,:\,\eta|_{\tilde{U}\cap\tilde{C}}=\mathbf{1}\right\}\\
&\cong\{\eta\in\mathrm{Hol}(U\setminus x_0, Gl(r,\mathbb{C}))\,:\,\eta|_{U\cap C}=\mathbf{1}\}\\
G_X &= \{\eta\in\mathrm{Hol}(U, Gl(r,\mathbb{C}))\,:\,\eta|_{A\cap C}=\mathbf{1}\}\\
G_P &= \left\{\eta\in\mathrm{Hol}(\tilde{\mathbb{P}}^2\setminus L, Gl(r,\mathbb{C}))\,:\,\eta|_{\tilde{C}_\infty}=\mathbf{1}\right\}\\
&\cong\left\{\eta\in\mathrm{Hol}(\mathbb{P}^2\setminus y_0, Gl(r,\mathbb{C}))\,:\,\eta|_{C_\infty}=\mathbf{1}\right\}
\end{aligned}$$

Then G acts freely on $S^i\mathfrak{M}^{\tilde{U}\setminus L}(\tilde{X},E)$ by $\eta[\mathcal{E},\phi,\psi]=[\mathcal{E},\phi,\eta\psi]$ and its orbits are the fibers of $pr_{\tilde{X}}$. Similarly, G_X, G_P act freely on $\mathfrak{M}^U(X,E_X)$, $S_0\mathfrak{M}^{\tilde{\mathbb{P}}^2\setminus L}(\tilde{\mathbb{P}}^2,E_P)$ with orbits the fibers of pr_X, pr_P respectively. G_X is clearly contractible and by lemma 2.1, $G=G_X$ and $G_P=\{1\}$. So we only need to show the existence of local sections.

In section 4 we will show that pr_P is a homeomorphism. We turn our attention to pr_X. Pick a small open set $T\subset\mathfrak{M}(X,E_X)$ such that the restriction of the universal bundle $\mathcal{F}\to T\times X$ to $T\times U$ is trivial. Let $\Phi:\mathcal{F}|_{T\times C}\to T\times C\times\mathbb{C}^r$ be the trivialization inducing the trivializations of the bundles $\mathcal{F}_t$ on $t\times C$. We can find a trivialization Ψ of $\mathcal{F}|_{T\times U}$ which extends the trivialization $\Phi|_{T\times(U\cap C)}$. Then Ψ induces a continuous choice of trivializations of the bundles on U, which is the same as a section of pr_X.

Now let $T\subset\mathfrak{M}(\tilde{X},E)$ and let $\mathcal{F}$ be the universal bundle over T trivialized on $T\times\tilde{C}$ by Φ. Using lemma 2.3 we get a bundle $\mathcal{F}_X$ over $T\times X$ and an isomorphism $F:\mathcal{F}|_{T\times(\tilde{X}\setminus L)}\to\mathcal{F}_X|_{T\times(X\setminus x_0)}$ which, together with Φ, induces a trivialization Φ_X of $\mathcal{F}_X$ on $T\times C$. Pick a trivialization Ψ_X of $\mathcal{F}_X$ on $T\times U$ which agrees with Φ_X on $T\times(U\cap C)$. Then F,Ψ_X induce a trivialization Ψ of $\mathcal{F}$ on $T\times(\tilde{U}\setminus L)$ which agrees with Φ on $\tilde{U}\cap\tilde{C}$. From Ψ we get the desired section of $pr_{\tilde{X}}$. □

We are now going to show that $S_i\mathfrak{M}^{\tilde{U}\setminus L}(\tilde{X},E)$ is homeomorphic to the product $\mathfrak{M}^U(X,E_X)\times S_0\mathfrak{M}^{\tilde{\mathbb{P}}^2\setminus L}(\tilde{\mathbb{P}}^2,E_P)$. The proof is based on a gluing construction. We will define three maps

$$\begin{aligned}
&g:\mathfrak{M}^U(X,E_X)\times S_0\mathfrak{M}^{\tilde{\mathbb{P}}^2\setminus L}(\tilde{\mathbb{P}}^2,E_P)\to\mathfrak{M}^{\tilde{U}\setminus L}(\tilde{X},E)\\
&g_X:S_i\mathfrak{M}^{\tilde{U}\setminus L}(\tilde{X},E)\to\mathfrak{M}^U(X,E_X)\\
&g_P:S_i\mathfrak{M}^{\tilde{U}\setminus L}(\tilde{X},E)\to S_0\mathfrak{M}^{\tilde{\mathbb{P}}^2\setminus L}(\tilde{\mathbb{P}}^2,E_P)
\end{aligned}$$

First we define g. Let $(\bar{\partial}_X,\phi_X,\psi_X)\in\mathcal{C}^U(X,E_X)$, $(\bar{\partial}_P,\phi_P,\psi_P)\in S_0\mathcal{C}^{\tilde{\mathbb{P}}^2\setminus L}(\tilde{\mathbb{P}}^2,E_P)$. Let

$$E_{XP}=E_X|_{X\setminus\{x_0\}}\bigcup_{\psi_P^{-1}\psi_X}E_P|_{\tilde{U}}$$

Then $c_2(E_{XP})=k$ hence E_{XP} is isomorphic to E as smooth vector bundles. $\bar{\partial}_X$ and $\bar{\partial}_P$ induce a holomorphic structure $\bar{\partial}_{XP}$ on E_{XP}, and ϕ, $\psi|_{U\setminus x_0}$ induce

holomorphic trivializations of E_{XP} on $\tilde{C}$ and $\tilde{U} \setminus L$ respectively. Hence we have $[\bar{\partial}_{XP}, \phi_X, \psi_X|_{U\setminus x_0}] \in \mathfrak{M}^{\tilde{U}\setminus L}(\tilde{X}, E_{XP})$. Choose an isomorphism $f : E_{XP} \to E$. Then f induces an isomorphism $f_{\#} : \mathfrak{M}^{\tilde{U}\setminus L}(\tilde{X}, E_{XP}) \to \mathfrak{M}^{\tilde{U}\setminus L}(\tilde{X}, E)$. We define

$$g([\bar{\partial}_X, \phi_X, \psi_X], [\bar{\partial}_P, \phi_P, \psi_P]) = f_{\#}([\bar{\partial}_{XP}, \phi_X, \psi_X|_{\tilde{U}\setminus L}])$$

This map does not depend on the choice of isomorphism f since given another isomorphism $\hat{f}$, $\hat{f} \circ f^{-1} \in Aut(E)$.

In a similar way we define the maps g_X, g_P: given $[\bar{\partial}, \phi, \psi] \in S_i\mathfrak{M}^{\tilde{U}\setminus L}(\tilde{X}, E)$. define the bundles

$$\hat{E}_X = E|_{\tilde{X}\setminus L} \bigcup_{\psi} U \times \mathbb{C}^r \qquad\qquad \hat{E}_P = E|_{\tilde{U}} \bigcup_{\psi} (\tilde{\mathbb{P}}^2 \setminus L) \times \mathbb{C}^r$$

Then $\bar{\partial}$ induces holomorphic structures $\bar{\partial}_X$ on $\hat{E}_X$ and $\bar{\partial}_P$ on $\hat{E}_P$. Also, $\hat{E}_X, \hat{E}_P$ have canonical trivializations ψ_X and ψ_P on U and $\tilde{\mathbb{P}}^2 \setminus L$ respectively. Proceeding as above we choose isomorphisms $f_X : \hat{E}_X \to E_X$ and $f_P : \hat{E}_P \to E_P$. We define

$$g_X([\bar{\partial}, \phi, \psi]) = (f_X)_{\#}([\bar{\partial}_X, \phi, \psi_X])$$
$$g_P([\bar{\partial}, \phi, \psi]) = (f_P)_{\#}([\bar{\partial}_P, \psi_P|_{C_\infty}, \psi_P])$$

As above, g_X, g_P are independent of the choice of isomorphisms f_X, f_P.

Proposition 3.2. $pr_X \circ g_X = \pi_i \circ pr_{\tilde{X}}$

Proof. It is enough to observe that the bundles $pr_X(g_X(\mathcal{E}, \phi))$ and $\pi_i \circ pr_{\tilde{X}}(\mathcal{E}, \phi)$ are isomorphic on $X \setminus x_0$ and apply lemma 2.2. □

Proposition 3.3. *The map* $g : \mathfrak{M}^U(X, E_X) \times S_0\mathfrak{M}^{\tilde{\mathbb{P}}^2\setminus L}(\tilde{\mathbb{P}}^2, E_P) \to \mathfrak{M}^{\tilde{U}\setminus L}(\tilde{X}, E)$ *is a homeomorphism with inverse* (g_X, g_P).

Proof. It is straightforward to check that g and (g_X, g_P) are inverses. We need to show the continuity of g, g_X, g_P. We start with g. Fix $(\bar{\partial}_{0X}, \phi_{0X}, \psi_{0X}) \in \mathcal{C}^U(X, E_X)$, $(\bar{\partial}_{0P}, \phi_{0P}, \psi_{0P}) \in \mathcal{C}^{\tilde{\mathbb{P}}^2\setminus L}(\tilde{\mathbb{P}}^2, E_P)$. Then, these elements induce isomorphisms

$$\tilde{\mathcal{C}}^U(X, E_X) \cong \Omega^{0,1}(X, E_X) \times C^\infty(C, Gl(r, \mathbb{C})) \times C^\infty(U, Gl(r, \mathbb{C}))$$
$$\tilde{\mathcal{C}}^{\tilde{\mathbb{P}}^2\setminus L}(\tilde{\mathbb{P}}^2, E_P) \cong \Omega^{0,1}(\tilde{\mathbb{P}}^2, E_P) \times C^\infty(L_\infty, Gl(r, \mathbb{C})) \times C^\infty(\tilde{\mathbb{P}}^2 \setminus L, Gl(r, \mathbb{C}))$$

(see definition 3.1). Fix balls $B_r(x_0) \subset B_R(x_0) \subset \overline{B_R(x_0)} \subset U$ and let $K = \overline{B_R(x_0)} \setminus B_r(x_0)$. Choose $\varepsilon > 0$ such that $W = B_\varepsilon(\mathbf{1}) \subset Gl(r, \mathbb{C})$. K, W define an open neighborhood $S(K, W) \subset C^\infty(U, Gl(r, \mathbb{C}))$ of $\mathbf{1}$ in the compact-open topology. Identifying $U \setminus \{x_0\}$ with $\tilde{U} \setminus L$, we can also think of $S(K, W)$ as an open set in $C^\infty(\tilde{\mathbb{P}}^2 \setminus L, Gl(r, \mathbb{C}))$. We will build a continuous map

$$\tilde{g} : \Omega^{0,1}(E_X) \times C^\infty(C, Gl) \times S(K, W) \times \Omega^{0,1}(E_P) \times C^\infty(\mathbb{P}^1, Gl) \times S(K, W) \to \tilde{\mathcal{C}}(\tilde{X}, E)$$

whose restriction to $\mathcal{C}^U(X, E_X) \times \mathcal{C}^{\tilde{\mathbb{P}}^2\setminus L}(\tilde{\mathbb{P}}^2, E_P)$ has image inside $\mathcal{C}^{\tilde{U}\setminus L}(\tilde{X}, E_{\tilde{X}})$ and projects down to g. The continuity of g will follow. Let $q = (A_X^{0,1}, g_X, h_X, A_P^{0,1}, g_P, h_P)$ with

$$(A_X^{0,1}, g_X, h_X) \in \Omega^{0,1}(E_X) \times C^\infty(C, Gl) \times S(K, W)$$
$$(A_P^{0,1}, g_P, h_P) \in \Omega^{0,1}(E_P) \times C^\infty(C, Gl) \times S(K, W)$$

For each such q let

$$E_q = E_X|_{X\setminus\{x_0\}} \bigcup_{\psi_{0P}^{-1} h_P^{-1} h_X \psi_{0X}} E_P|_{\tilde{U}}$$

Fix a bundle isomorphism $f : E_{q_0} \to E$. To complete the proof all we need is a continuous family of bundle isomorphisms $f_q : E_q \to E_{q_0}$ since then we can define $\tilde{g}$ by

$$\tilde{g}(q) = (f \circ f_q)_\#(\bar{\partial}_{XP}, g_X\phi_{0X}, h_X\psi_{0X}|_{\tilde{U}\setminus L})$$

Let $\tilde{U}_r = \pi^{-1}(B_r(x_0))$. We will build f_q by giving three maps

$$\begin{aligned} &f_X : E_X|_{X\setminus\overline{B_R(x_0)}} \to E_X|_{X\setminus\overline{B_R(x_0)}} \\ &f_U : (U \setminus \{x_0\}) \times \mathbb{C}^r \to (U \setminus \{x_0\}) \times \mathbb{C}^r \\ &f_P : E_P|_{\tilde{U}_r} \to E_P|_{\tilde{U}_r} \end{aligned}$$

such that the following diagrams commute:

$$\begin{tikzcd} E_X|_{U\setminus\overline{B_R(x_0)}} \arrow[r, "f_X"] \arrow[d, "h_X\psi_{0X}"] & E_X|_{U\setminus\overline{B_R(x_0)}} \arrow[d, "\psi_{0X}"] \\ (U\setminus\overline{B_R(x_0)})\times\mathbb{C}^r \arrow[r, "f_U"] & (U\setminus\overline{B_R(x_0)})\times\mathbb{C}^r \end{tikzcd} \qquad \begin{tikzcd} (\tilde{U}_r\setminus L)\times\mathbb{C}^r \arrow[r, "f_U"] & (\tilde{U}_r\setminus L)\times\mathbb{C}^r \\ E_P|_{\tilde{U}_r\setminus L} \arrow[r, "f_P"] \arrow[u, "h_P\psi_{0P}"] & E_P|_{\tilde{U}_r\setminus L} \arrow[u, "\psi_{0P}"] \end{tikzcd}$$

Fix a monotonous C^∞ function $\eta : [0, +\infty[\to [0, 1]$ such that $\eta(\rho) = 0$ for $\rho < r$ and $\eta(\rho) = 1$ for $\rho > R$. η induces a map $\tilde{\eta} : U \to \mathbb{R}$ given by $\tilde{\eta}(x) = \eta(\|x\|)$ for $x \in K$ and $\tilde{\eta} = 1$ otherwise. We define

$$\begin{aligned} &f_X = \mathbf{1} && x \in X \setminus \overline{B_R(x_0)} \\ &f_U = \tilde{\eta}\psi_{X0}\psi_X^{-1} + (1-\eta)\psi_{P0}\psi_P^{-1} && x \in U \setminus \{x_0\} \\ &f_P = \mathbf{1} && x \in \tilde{U}_r \end{aligned}$$

This completes the proof of continuity of g. The proof for g_X, g_P is completely analogous. □

An immediate corollary is

Corollary 3.4. *The spaces $S_i\mathfrak{M}(\tilde{X}, E)$ and $\mathfrak{M}(X, E_X) \times S_0\mathfrak{M}(\tilde{\mathbb{P}}^2, E_P)$ are homotopically equivalent.*

We are ready to prove theorem 1.1:

Theorem 3.5. *The spaces $\mathfrak{M}_i(X, E_X) \times S_0\mathfrak{M}_{k-i}(\tilde{\mathbb{P}}^2, E_P)$ and $S_i\mathfrak{M}_k(\tilde{X}, E)$ are homeomorphic.*

Proof. Fix a global section $s : \mathfrak{M}(X, E_X) \to \mathfrak{M}^U(X, E_X)$ of pr_X. Define $G_s : \mathfrak{M}(X, E_X) \times S_0\mathfrak{M}(\tilde{\mathbb{P}}^2, E_P) \to S_i\mathfrak{M}(\tilde{X}, E)$ by $G_s = pr \circ g \circ (s \times \mathbf{1})$. It is easy to check that G_s is a bijection. We need to show that its inverse is continuous. First we describe the inverse. Let $(\mathcal{E}, \phi) \in \mathfrak{M}(\tilde{X}, E)$ and let $(\mathcal{E}_X, \phi_X, \psi_X) = s \circ \pi_i(\mathcal{E}, \phi)$. Then there is a unique isomorphism $f : \mathcal{E}|_{\tilde{X}\setminus L} \to \mathcal{E}_X|_{X\setminus x_0}$ that sends ϕ to ϕ_X. Indeed, if $\hat{f}$ is another such isomorphism, $\hat{f} \circ f^{-1}$ is an automorphism of $\mathcal{E}_X|_{X\setminus x_0}$ which, by lemma 2.2, extends to an automorphism of $\mathcal{E}_X$ which is the identity on $\mathcal{E}_X|_C$. But then $\hat{f} \circ f^{-1} = \mathbf{1}$. f and $\psi_X|_{U\setminus x_0}$ induce a trivialization ψ of $\mathcal{E}$ on $\tilde{U} \setminus L$ and $(\mathcal{E}, \phi, \psi) \in \mathfrak{M}^{\tilde{U}\setminus L}(\tilde{X}, E)$. Then

$$G_s^{-1}(\mathcal{E}, \phi) = (\ \pi_i(\mathcal{E}, \phi)\,,\ pr_P \circ g_P(\mathcal{E}, \phi, \psi)\ )$$

To prove that this map is continuous all we need is to show that the isomorphism f depends continuously on $(\mathcal{E}, \phi)$. We apply the construction from lemma 2.3. Let $T \subset \mathfrak{M}(\tilde{X}, E)$ and let $\mathcal{F}$ be the universal bundle over $T \times \tilde{X}$ trivialized on $T \times \tilde{C}$ by Φ. Then, we have a bundle $\mathcal{F}_X$ over $T \times X$ with trivialization Φ_X on $T \times C$ such that $\pi_i(\mathcal{F}_t, \Phi_t) = (\mathcal{F}_{Xt}, \Phi_{Xt})$ and the isomorphism $f_t : \mathcal{F}_t|_{\tilde{X}\setminus L} \to \mathcal{F}_{Xt}|_{X \setminus x_0}$ is precisely the restriction of the isomorphism F from 2.3 to $\mathcal{F}_t$. Hence f_t depends continuously on t. □

4. The fiber $S_0\overline{\mathfrak{M}}_k^r(\tilde{\mathbb{P}}^2)$

The objective of this section is twofold: to give a characterization of points in $S_0\mathfrak{M}_k^r(\tilde{\mathbb{P}}^2)$ in terms of a monad description, and to show that the projection map $pr_P : \mathfrak{M}_k^{\tilde{\mathbb{P}}^2\setminus L}(\tilde{\mathbb{P}}^2) \to \mathfrak{M}_k(\tilde{\mathbb{P}}^2)$ is a homeomorphism. This will be done by giving an explicit description of the inverse $pr_P^{-1} : \mathfrak{M}(\tilde{\mathbb{P}}^2) \to S_0\mathfrak{M}^{\tilde{\mathbb{P}}^2\setminus L}(\tilde{\mathbb{P}}^2)$. We begin by sketching the monad description of the spaces $\mathfrak{M}_k^r(\mathbb{P}^2)$ and $\mathfrak{M}_k^r(\tilde{\mathbb{P}}^2)$. We follow [7].

4.1. **The moduli space over $\mathbb{P}^2$.** Let $L_\infty \subset \mathbb{P}^2$ be a rational curve. Let W be a k dimensional complex vector space. Choose sections x_1, x_2, x_3 spanning $H^0(\mathcal{O}(L_\infty))$. so that x_3 vanishes on L_∞. Let $\mathcal{R}$ be the space of 4-tuples $m = (a_1, a_2, b, c)$ with $a_i \in \mathrm{End}(W)$, $b \in \mathrm{Hom}(\mathbb{C}^r, W)$, $c \in \mathrm{Hom}(W, \mathbb{C}^r)$, obeying the integrability condition $[a_1, a_2] + bc = 0$. For each $m = (a_1, a_2, b, c) \in \mathcal{R}$ we define maps A_m, B_m

$$W(-L_\infty) \xrightarrow{A_m} W^{\oplus 2} \oplus \mathbb{C}^n \xrightarrow{B_m} W(L_\infty)$$

by

$$A_m = \begin{bmatrix} x_1 - a_1x_3 \\ x_2 - a_2x_3 \\ cx_3 \end{bmatrix}, \quad B_m = \begin{bmatrix} -x_2 + a_2x_3 & x_1 - a_1x_3 & bx_3 \end{bmatrix}$$

Then $B_mA_m = 0$. A point $m \in \mathcal{R}$ is called non-degenerate if A_m and B_m have maximal rank at every point in $\mathbb{P}^2$. Then $\mathcal{E}_m$ is locally free. The assignment $m \mapsto \mathcal{E}_m = \mathrm{Ker}\, B_m/\mathrm{Im}\, A_m$ induces a map f from the space $\mathcal{R}^{reg}$ of non-degenerate configurations to the moduli space $\mathfrak{M}_k^r(\mathbb{P}^2)$. We have (see [5])

Theorem 4.1 (Donaldson)**.** *Let $\mathcal{M}_k^r(\mathbb{P}^2)$ denote the quotient of the space of non-degenerate points in $\mathcal{R}$ by the action $g \cdot (a_1, a_2, b, c) = (g^{-1}a_1g, g^{-1}a_2g, g^{-1}b, cg)$ of $Gl(W)$. Then the map f induces an isomorphism of moduli spaces $\mathcal{M}_k^r(\mathbb{P}^2) \to \mathfrak{M}_k^r(\mathbb{P}^2)$.*

Let $\overline{\mathcal{M}}_k^r(\mathbb{P}^2)$ be the algebraic quotient $\mathcal{R}/Gl(W)$. This space is isomorphic to the Donaldson-Uhlenbeck completion of the moduli space of instantons over S^4.

For a proof see [5], [4]. We sketch here how the map from $\mathcal{R}/Gl(W)$ to the Donaldson-Uhlenbeck completion of the moduli space of instantons is constructed (see [7] for details): Let $m = (a_1, a_2, b, c)$. A subspace $V \subset W$ is called b-special with respect to m if

$$a_i(V) \subset V \;\; (i = 1, 2) \text{ and } \mathrm{Im}\, b \subset V \tag{2}$$

A subspace $V \subset W$ is called c-special with respect to m if

$$a_i(V) \subset V \;\; (i = 1, 2) \text{ and } V \subset \mathrm{Ker}\, c \tag{3}$$

m is called completely reducible if for every $V \subset W$ which is b-special or c-special, there is a complement $V' \subset W$ which is c-special or b-special respectively.

Proposition 4.2. *Let $m = (a_1, a_2, b, c) \in \mathcal{R}$.*

(1) *m is non degenerate if and only if the only b-special subspace is W and the only c-special subspace is 0;*
(2) *For every m, the orbit of m under $Gl(W)$ contains in its closure a canonical completely reducible orbit and completely reducible orbits have disjoint closures;*
(3) *If m is completely reducible then, after acting with some $g \in Gl(W)$ we can write*

$$a_i = \begin{bmatrix} a_i^{red} & 0 \\ 0 & a_i^{\Delta} \end{bmatrix} \; , \; b = \begin{bmatrix} b^{red} \\ 0 \end{bmatrix} \; , \; c = \begin{bmatrix} c^{red} & 0 \end{bmatrix}$$

where $(a_1^{red}, a_2^{red}, b^{red}, c^{red})$ is non-degenerate and the matrices $a_1^{\Delta}, a_2^{\Delta}$ can be simultaneously diagonalized. Such a configuration is equivalent to the following data:

- *An irreducible integrable configuration $(a_1^{red}, a_2^{red}, b^{red}, c^{red})$ corresponding to a bundle with $c_2 = l \leq k$;*
- *$k - l$ points in $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$ given by the eigenvalue pairs of $a_1^{\Delta}, a_2^{\Delta}$*

This is precisely the Donaldson-Uhlenbeck completion.

4.2. **The moduli space over $\tilde{\mathbb{P}}^2$.** Let $L_\infty \subset \mathbb{P}^2$ be a rational curve and let L be the exceptional divisor. Let W_0, W_1 be k dimensional complex vector spaces. Choose sections x_1, x_2, x_3 spanning $H^0(\mathcal{O}(L_\infty))$ and y_1, y_2 spanning $H^0(\mathcal{O}(L_\infty - L))$ so that x_3 vanishes on L_∞ and $x_1 y_1 + x_2 y_2$ spans the kernel of the map

$$H^0(\mathcal{O}(L_\infty)) \otimes H^0(\mathcal{O}(L_\infty - L)) \longrightarrow H^0(\mathcal{O}(2L_\infty - L))$$

Let $\tilde{\mathcal{R}}$ be the space of 5-tuples (a_1, a_2, d, b, c) where $a_i \in \operatorname{Hom}(W_1, W_0)$, $d \in \operatorname{Hom}(W_0, W_1)$, $b \in \operatorname{Hom}(\mathbb{C}^r, W_0)$, $c \in \operatorname{Hom}(W_1, \mathbb{C}^r)$, such that we have $a_1(W_1) + a_2(W_1) + b(\mathbb{C}^r) = W_0$, obeying the integrability condition $a_1 d a_2 - a_2 d a_1 + bc = 0$. For each $\tilde{m} = (a_1, a_2, d, b, c) \in \tilde{\mathcal{R}}$ we define maps $A_{\tilde{m}}, B_{\tilde{m}}$

$$\begin{aligned} W_1(-L_\infty) \oplus W_0(L - L_\infty) \xrightarrow{A_{\tilde{m}}} (W_0 \oplus W_1)^{\oplus 2} \oplus \mathbb{C}^n \xrightarrow{B_{\tilde{m}}} \\ \to W_0(L_\infty) \oplus W_1(L_\infty - L) \end{aligned}$$

by

$$A_{\tilde{m}} = \begin{bmatrix} a_1 x_3 & -y_2 \\ x_1 - d a_1 x_3 & 0 \\ a_2 x_3 & y_1 \\ x_2 - d a_2 x_3 & 0 \\ c x_3 & 0 \end{bmatrix} \; , \; B_{\tilde{m}} = \begin{bmatrix} x_2 & a_2 x_3 & -x_1 & -a_1 x_3 & b x_3 \\ d y_1 & y_1 & d y_2 & y_2 & 0 \end{bmatrix}$$

Then $B_{\tilde{m}} A_{\tilde{m}} = 0$. A point $\tilde{m} \in \tilde{\mathcal{R}}$ is called non-degenerate if $A_{\tilde{m}}$ and $B_{\tilde{m}}$ have maximal rank at every point in $\tilde{\mathbb{P}}^2$. The assignment $\tilde{m} \mapsto \mathcal{E}_{\tilde{m}} = \operatorname{Ker} B_{\tilde{m}} / \operatorname{Im} A_{\tilde{m}}$ induces a map $\tilde{f} : \tilde{\mathcal{R}}^{reg} \to \mathfrak{M}_k^r(\tilde{\mathbb{P}}^2)$.

Theorem 4.3 (King, [7])**.** *Let $\mathcal{M}_k^r(\tilde{\mathbb{P}}^2)$ denote the quotient of the space of non degenerate points in $\tilde{\mathcal{R}}$ by the action of $Gl(W_0) \times Gl(W_1)$:*

$$(g_0, g_1) \cdot (a_1, a_2, b, c, d) = (g_0^{-1} a_1 g_1, g_0^{-1} a_2 g_1, g_0^{-1} b, c g_1, g_1^{-1} d g_0)$$

Then the map $\tilde{f}$ induces an isomorphism $\mathcal{M}_k^r(\tilde{\mathbb{P}}^2) \to \mathfrak{M}_k^r(\tilde{\mathbb{P}}^2)$.

4.3. **The space $S_0\mathfrak{M}^r_k(\tilde{\mathbb{P}}^2)$.** The objective of this section is to prove the theorem

Theorem 1.2. *Let $\tilde{m} = (a_1, a_2, d, b, c) \in \overline{\mathfrak{M}}^r_k(\tilde{\mathbb{P}}^2)$. Then $\mathcal{E}_{\tilde{m}} \in S_0\overline{\mathfrak{M}}^r_k(\tilde{\mathbb{P}}^2)$ if and only if da_1, da_2 are nilpotent and, for any sequence $i_1, \ldots, i_n \in \{1, 2\}$, we have*

$$c\left(\prod_{j=1}^{n} da_{i_j}\right) db = 0$$

In the special case $k = 1, 2$ we will show that

Corollary 4.4. *Let $k = 1, 2$ and let $\tilde{m} = (a_1, a_2, d, b, c) \in \mathfrak{M}^r_k(\tilde{\mathbb{P}}^2)$. Then $\mathcal{E}_{\tilde{m}} \in S_0\mathfrak{M}^r_k(\tilde{\mathbb{P}}^2)$ if and only if $d = 0$.*

We will divide the proof of theorem 1.2 into several propositions:

Proposition 4.5. *Define $\pi_\# : \tilde{\mathcal{R}} \to \mathcal{R}$ by $\pi_\#(a_i, d, b, c) = (da_i, db, c)$. Let $m = \pi_\#\tilde{m}$. Then $\mathcal{E}_{\tilde{m}}|_{\tilde{X}\setminus L}$ is isomorphic to $\mathcal{E}_m|_{X\setminus\{x_0\}}$.*

Proof. Let $m = \pi_\#\tilde{m}$. Fix an isomorphism $W \cong W_1$. Let p be the projection $p : W^{\oplus 2} \oplus \mathbb{C}^r \to (W_0 \oplus W_1)^{\oplus 2} \oplus \mathbb{C}^r$ with kernel $W_0^{\oplus 2}$. After restricting to $\tilde{X} \setminus L$ we can rescale the sections so that $y_2 = -x_1$, $y_1 = x_2$. Then a direct verification shows that, for any $\tilde{m}$, p induces maps $\operatorname{Ker} B_{\tilde{m}} \to \operatorname{Ker} B_m$ and $\operatorname{Im} A_{\tilde{m}} \to \operatorname{Im} A_m$. Hence we get a map $\mathcal{E}_{\tilde{m}} \to \mathcal{E}_m$. It is a direct computation to check that this map is an isomorphism. □

It follows from this proposition and theorem 4.1 that, for $\tilde{m} \in \tilde{\mathcal{R}}$, $\mathcal{E}_{\tilde{m}} \in S_0\mathfrak{M}^r_k(\tilde{\mathbb{P}}^2)$ if and only if $\mathcal{E}_{\pi_\#\tilde{m}}$ is a sheaf whose whole charge is concentrated at $[0, 0, 1] \in \mathbb{P}^2$, that is, $\ell\left(\mathcal{E}^{\vee\vee}_{\pi_\#\tilde{m}}/\mathcal{E}_{\pi_\#\tilde{m}}\right) = k$. We proceed to study this situation:

Lemma 4.6. *Let $m = (a_1, a_2, b, c)$ be such that $\mathcal{E}_m$ has its whose whole charge concentrated at $[0, 0, 1]$. Then, after a change of basis we can write*

$$a_i = \begin{bmatrix} J & * \\ 0 & J \end{bmatrix}, \ b = \begin{bmatrix} * \\ 0 \end{bmatrix}, \ c = \begin{bmatrix} 0 & * \end{bmatrix}$$

where J represents any nilpotent matrix in the Jordan canonical form.

Before we begin the proof we observe that this lemma implies one direction of theorem 1.2.

Proof. We begin by proving by induction on the charge that, after acting with an element $g \in Gl(W)$, we can write.

$$a_i = \begin{bmatrix} a_{iu} & * \\ 0 & a_{id} \end{bmatrix}, \ b = \begin{bmatrix} * \\ 0 \end{bmatrix}, \ c = \begin{bmatrix} 0 & * \end{bmatrix}$$

Clearly the configuration (a_1, a_2, b, c) cannot be non-degenerate hence there is a subspace V which is either b-special or c-special (proposition 4.2, 1). We consider both cases:

(1) If m is b special, after a change of basis we can write

$$a_i = \begin{bmatrix} a'_i & * \\ 0 & f_i \end{bmatrix}, \ b = \begin{bmatrix} b' \\ 0 \end{bmatrix}, \ c = \begin{bmatrix} c' & * \end{bmatrix}$$

So the point $\left(\left[\begin{smallmatrix} a'_i & 0 \\ 0 & f_i \end{smallmatrix}\right], \left[\begin{smallmatrix} b' \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} c' & 0 \end{smallmatrix}\right]\right)$ is in the closure of the orbit of m. It follows then from proposition 4.2 that $m' = (a'_1, a'_2, b', c')$ corresponds to

an ideal bundle with charge concentrated at $[0,0,1]$. Hence we can apply the induction hypothesis to m'. We get

$$a_i = \begin{bmatrix} a'_{iu} & * & * \\ 0 & a'_{id} & * \\ 0 & 0 & f_i \end{bmatrix}, \; b = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}, \; c = \begin{bmatrix} 0 & * & * \end{bmatrix}$$

which is in the desired form.

(2) If m is c-special, after a change of basis we can write

$$a_i = \begin{bmatrix} f_i & * \\ 0 & a'_i \end{bmatrix}, \; b = \begin{bmatrix} * \\ b' \end{bmatrix}, \; c = \begin{bmatrix} 0 & c' \end{bmatrix}$$

Applying induction hypothesis to (a'_1, a'_2, b', c') as in the previous case, we can write

$$a_i = \begin{bmatrix} f_i & * & * \\ 0 & a'_{iu} & * \\ 0 & 0 & a'_{id} \end{bmatrix}, \; b = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}, \; c = \begin{bmatrix} 0 & 0 & * \end{bmatrix}$$

This is in the desired form.

Now the condition $[a_1, a_2] + bc = 0$ implies $[a_{1u}, a_{2u}] = [a_{1d}, a_{2d}] = 0$. So, after a change of basis we can put all these matrices in the Jordan canonical form. Since all charge is concentrated at $[0,0,1]$, proposition 4.2 implies the eigenvalues of these matrices are all 0. □

The other direction of theorem 1.2 follows directly from the proposition

Proposition 4.7. *Let $m \in \mathcal{R}$ be such that a_1, a_2 are nilpotent and, for any n_1, n_2 we have*

$$ca_1^{n_1}a_2^{n_2}b = 0 \tag{4}$$

Then $\mathcal{E}_m|_{\mathbb{P}^2\setminus[0,0,1]}$ is free.

Proof. We will build an explicit trivialization. First we need to introduce some notation. Let $\mathbb{P}^2 = \{[x_1, x_2, x_3]\}$ and define an open cover of $\mathbb{P}^2 \setminus [0,0,1]$ by $U_1 = \{x_1 \neq 0\}$, $U_2 = \{x_2 \neq 0\}$. Let $e_1, \ldots, e_r$ be the canonical basis of $\mathbb{C}^r$. Choose coordinates $(\alpha_2, \alpha_3) \mapsto [1, \alpha_2, \alpha_3]$ in U_1 and $(\beta_1, \beta_3) \mapsto [\beta_1, 1, \beta_3]$ in U_2. Then define functions $s_i^j : U_j \to \operatorname{Ker} B_m$ by

$$s_i^1 = \big(0, -\alpha_3(\mathbf{1} - \alpha_3 a_1)^{-1} be_i, e_i\big)$$
$$s_i^2 = \big(\beta_3(\mathbf{1} - \beta_3 a_2)^{-1} be_i, 0, e_i\big)$$

(since a_1, a_2 are nilpotent, $\mathbf{1} - \lambda a_i$ is invertible for any λ). It is a direct verification that indeed the image of s_i^1, s_i^2 lie in $\operatorname{Ker} B_m$. We want to show that s_i^1, s_i^2 induce a trivialization of $\mathcal{E}_m$. We will have to show that $S_i^1 - s_i^2 \in \operatorname{Im} A_m$ in $U_1 \cap U_2$. Then s_i^1, s_i^2 induce a section of $\mathcal{E}_m$. We will show these sections are linearly independent.

(1) We begin by looking at the space $\operatorname{Im} A + \operatorname{Im} s_1^1 + \ldots + \operatorname{Im} s_r^1$. We can represent the image of s_i^1 in terms of column vectors in matrix form. Then, joining this matrix with A_m we have

$$\mathfrak{A}_1 = \begin{bmatrix} \mathbf{1} - \alpha_3 a_1 & 0 \\ \alpha_2 - \alpha_3 a_2 & -\alpha_3(\mathbf{1} - \alpha_3 a_1)^{-1} b \\ \alpha_3 c & \mathbf{1} \end{bmatrix}$$

This matrix clearly has maximum rank since $\mathbf{1} - \alpha_3 a_1$ is non-singular. Hence, for dimensional reasons its columns form a basis for $\operatorname{Ker} B$. In particular s_i^1 are linearly independent.

(2) Now we repeat the argument for s_i^2. In U_2 we have a similar matrix:

$$\mathfrak{A}_2 = \begin{bmatrix} \beta_1 - \beta_3 a_1 & \beta_3(\mathbf{1} - \beta_3 a_2)^{-1} b \\ \mathbf{1} - \beta_3 a_2 & 0 \\ \beta_3 c & \mathbf{1} \end{bmatrix}$$

which has also clearly maximum rank. Its columns form a basis for $\operatorname{Ker} B$. In particular s_i^2 are linearly independent.

(3) Now we will show $s_i^1 - s_i^2 \in \operatorname{Im} A$ in $U_1 \cap U_2$. We have, with $\beta_1 = \alpha_2^{-1}$ and $\beta_3 = \alpha_2^{-1}\alpha_3$,

$$s^2(\alpha_2^{-1}, \alpha_2^{-1}\alpha_3) = \begin{bmatrix} \alpha_3(\alpha_2 - \alpha_3 a_2)^{-1} b \\ 0 \\ \mathbf{1} \end{bmatrix}$$

Since s^2 is in the kernel of B, it must be in the image of the surjective matrix $\mathfrak{A}_1$. So we can solve

$$\begin{bmatrix} \mathbf{1} - \alpha_3 a_1 & 0 \\ \alpha_2 - \alpha_3 a_2 & -\alpha_3(\mathbf{1} - \alpha_3 a_1)^{-1} b \\ \alpha_3 c & \mathbf{1} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \alpha_3(\alpha_2 - \alpha_3 a_2)^{-1} b \\ 0 \\ \mathbf{1} \end{bmatrix}$$

We obtain immediately $\xi_1 = \alpha_3(\mathbf{1} - \alpha_3 a_1)^{-1}(\alpha_2 - \alpha_3 a_2)^{-1} b$. Now equation 4 implies $c\xi_1 = 0$. From here it follows immediately that $\xi_2 = \mathbf{1}$ hence $s_i^2 - s_i^1 = A_m \xi_1$.

This completes the proof. □

Notice that this proof gives an explicit description of the map $S_0\mathfrak{M}(\tilde{\mathbb{P}}^2) \to S_0\mathfrak{M}^{\tilde{\mathbb{P}}^2\setminus L}(\tilde{\mathbb{P}}^2)$. We now prove corollary 4.4:

Proof. $\mathcal{E}_{\tilde{m}} \in S_k\mathfrak{M}_k^r(\tilde{\mathbb{P}}^2)$ if and only if $\det d \neq 0$. This proves the case $k = 1$. For $k = 2$ assume $\operatorname{rank} d = 1$. Then, after a change of basis we can write $d = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Suppose $\mathcal{E}_{\tilde{m}} \in S_0\mathfrak{M}_k^r(\tilde{\mathbb{P}}^2)$. Then, from lemma 4.6 we can write

$$a_i = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}, \quad b = \begin{pmatrix} b' \\ * \end{pmatrix}, \quad c = \begin{pmatrix} c' & * \end{pmatrix},$$

But this shows (a_1, a_2, d, b, c) is degenerate. This completes the proof. □

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