# HOLOMORPHIC BUNDLES ON THE BLOWN-UP PLANE AND THE BAR CONSTRUCTION

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Abstract. Let $\mathfrak{M}_0 = \coprod BU(k)$, $\mathfrak{M}_1 = \coprod (BU(k) \times BU(k))$. We construct a map from $Bar\left(\mathfrak{M}_0, \mathfrak{M}_0^n, \mathfrak{M}_1^n\right)$ to the rank-stable moduli space of holomorphic bundles on the blowup of $\mathbb{P}^2$ at $n$ points, framed on a rational curve. We show that this map is a homotopy equivalence in the degree 1 and 2 components and in the limit when $k \to \infty$.

## 1. Introduction

In this paper we will study the moduli space of holomorphic bundles over a rational surface with vanishing first Chern class, trivialized on a rational curve, in the limit when the rank of the bundles goes to infinity. This space is isomorphic to the moduli space of based instantons over a positive definite simply connected closed four-manifold (see [2], [9]), and has been studied in [7], [1], [12].

Write $\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{P}^1$ and, given a finite subset $I \subset \mathbb{C}^2$, let $\mathbb{P}^2_I$ be the blowup of $\mathbb{P}^2$ along $I$. Let $\mathfrak{M}^r_I$ be the moduli space of rank $r$ holomorphic bundles on $\mathbb{P}^2_I$ with first Chern class $c_1(E) = 0$, trivialized at $\mathbb{P}^1$. Direct sum with a trivial rank $r' - r$ bundle induces a map $\mathfrak{M}^r_I \to \mathfrak{M}^{r'}_I$. Let $\mathfrak{M}^\infty_I$ be the direct limit when $r \to \infty$. In [7], [11], [1] it was shown that $\mathfrak{M}^\infty_\emptyset \simeq \coprod BU(k)$ and, for each $x \in \mathbb{C}^2$, $\mathfrak{M}^\infty_x \simeq \coprod BU(k) \times BU(k)$. Then, Whitney sum induces a map

$$(\mathfrak{M}^\infty_\emptyset)^n \times \mathfrak{M}^\infty_\emptyset \to \mathfrak{M}^\infty_\emptyset$$

For $J \subset I$, pullback of bundles induces a map $\pi^* : \mathfrak{M}^\infty_J \to \mathfrak{M}^\infty_I$. Combining with Whitney sum we get maps $\mathfrak{M}^\infty_\emptyset \times \mathfrak{M}^\infty_x \to \mathfrak{M}^\infty_x$. Let

$$\mathcal{B}_I = Bar\left( \mathfrak{M}^\infty_\emptyset \,,\, \prod_{x\in I} \mathfrak{M}^\infty_\emptyset \,,\, \prod_{x\in I} \mathfrak{M}^\infty_x \right) \tag{1}$$

Pullback and Whitney sum give maps

$$\mathfrak{M}^\infty_\emptyset \to \mathfrak{M}^\infty_I \qquad \prod_{x\in I} \mathfrak{M}^\infty_x \to \mathfrak{M}^\infty_I$$

which induce a map

$$h_I : \mathcal{B}_I \to \mathfrak{M}^\infty_I \tag{2}$$

The second Chern class of the bundles gives a grading of the spaces $\mathfrak{M}^\infty_I, \mathcal{B}_I$ and we write $\mathfrak{M}^\infty_{I,k}, \mathcal{B}_{I,k}, h_{I,k}$ for the degree $k$ components and map. Our first result is

**Theorem 1.** *If, for every $J \subset I$ with $\#J \leq k$, $h_{J,k}$ is a homotopy equivalence, then $h_{I,k}$ is a homotopy equivalence.*

Using Theorem 1 and results from [12] we then show that

**Theorem 2.** *For $k = 1, 2$, $h_{I,k}$ is a homotopy equivalence.*

In [14], Taubes described, for $k' > k$, gluing maps $\mathfrak{M}^r_{I,k} \to \mathfrak{M}^r_{I,k'}$. He showed that, in the limit when $k \to \infty$, we get $\mathfrak{M}^\infty_{I,\infty} \simeq BU \times \prod_{x \in I} BU$. We will show that

**Theorem 3.** *The map*

$$h_{I,\infty} : \mathcal{B}_{I,\infty} \to \mathfrak{M}^\infty_{I,\infty}$$

*is a homotopy equivalence.*

**Conjecture.** *The map $h_I$ is a homotopy equivalence.*

Let $\mathscr{L}$ denote the linear isometries operad and let $\mathfrak{C}$ denote the category whose objects are the finite subsets of $\mathbb{C}^2$ and whose morphisms are the inclusions. The paper is organized as follows: In §2 we describe the moduli space $\mathfrak{M}_I$ as an $\mathscr{L}$-space. In §3 we describe the bar construction, the space $\mathcal{B}_I$ and the map $h_I$. $\mathfrak{M}_I$ and $\mathcal{B}_I$ are functors from $\mathfrak{C}$ to $Top$, the morphisms being induced by the pullback of holomorphic bundles. In §4 we prove theorem 1. This theorem is a consequence of the following fact (see [12]): $\mathfrak{M}_{I,k}$ is the colimit $\varinjlim \mathfrak{M}_{J,k}$ taken over the subsets $J \subset I$ with $\#J \leq k$. In §5 we use theorem 1 and results from [12] to prove theorem 2. In §6 we prove theorem 3. We make use of some classifying maps $\mathfrak{M}_{I,\infty} \to BU$ described in [13]. In the appendix we gather some results needed in section §5. The proofs use the monad descriptions of holomorphic bundles introduced in [4], [6].

## 2. $\mathscr{L}$-SPACES

The objective of this section is to define a functor from the category $\mathfrak{C}$ of finite subsets of $\mathbb{C}^2$ to the category of $\mathscr{L}$-spaces. Following [10], we first define a functor from $\mathfrak{C}$ to the category of $\mathcal{L}_*$-functors. We briefly review the notion of $\mathcal{L}_*$ functor. Let $\mathcal{L}$ be the category whose objects are the complex hermitian vector spaces with finite or countably infinite dimension, topologized as the limits of their finite dimensional subspaces, and the space of morphisms from $V$ to $W$ is the space linear isometries, with the compact open topology. Let $\mathcal{L}_n$ be the full subcategory whose objects have dimension $n$ and let $\mathcal{L}_*$ be the graded subcategory given by the union. An $\mathcal{L}_*$ functor is a pair $(\mathfrak{M}, \omega)$ where $\mathfrak{M} : \mathcal{L}_* \to (Top)$ is a continuous functor and $\omega : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M} \circ \oplus$ is a commutative, associative and continuous natural transformation satisfying

(1) Let 0 be the 0-dimensional vector space and let $* \in \mathfrak{M}0$ be the basepoint. Then $\omega(x, *) = x$.
(2) Suppose $V = V' \oplus V''$. Let $* \in \mathfrak{M}V''$ be the basepoint. Then the map $\mathfrak{M}V' \to \mathfrak{M}V$ given by $x \mapsto \omega(x, *)$ is a closed embedding.

A morphism of $\mathcal{L}_*$ functors is a continuous natural transformation that commutes with $\omega$.

Now let $\mathbb{H} = \mathbb{C}^\infty$. Let $\mathscr{L}$ denote the linear isometries operad defined by $\mathscr{L}(j) = \mathcal{L}(\mathbb{H}^{\oplus j}, \mathbb{H})$. $\mathscr{L}$ is an $E_\infty$ operad. There is a functor from the category of $\mathcal{L}_*$-functors to the category of $\mathscr{L}$-spaces: given an $\mathcal{L}_*$ functor $(\mathfrak{M}, \omega)$, we let $\mathfrak{M}\mathbb{H} = \varinjlim \mathfrak{M}V$ where $V$ runs over the finite dimensional hermitian subspaces of $\mathbb{H}$. Then $\mathfrak{M}\mathbb{H}$ in an $\mathscr{L}$-space. We proceed to define, for each rational surface $X$, an $\mathcal{L}_*$-functor $(\mathfrak{M}_X, \omega)$.

2.1. **Objects.** Let $X$ be a rational surface. We want to associate to each complex hermitian vector space $V$ a topological space $\mathfrak{M}_X(V)$. Fix a rational curve $L_\infty \subset X$ with positive self-intersection. Let $E \to X$ be a complex smooth vector bundle with first Chern class $c_1(E) = 0$ and rank $rk(E) = dim(V)$. A holomorphic structure on $E$ is a semi-connection $\bar{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E)$ satisfying the integrability condition $\bar{\partial}_E^2 = 0$. Let $\mathcal{C}(X, E, V)$ be the space of pairs $(\bar{\partial}_E, \phi)$ where $\bar{\partial}_E$ is a holomorphic structure on $E$ holomorphically trivial on $L_\infty$ and $\phi : E|_{L_\infty} \to V \otimes \mathcal{O}_{L_\infty} = V \times L_\infty$ is an isomorphism of holomorphic bundles. Given an isomorphism of smooth vector bundles $\psi : E_1 \to E_2$ we define the map $\psi_* : \mathcal{C}(X, E_1, V) \to \mathcal{C}(X, E_2, V)$ by

$$\psi_*(\bar{\partial}, \phi) = (\psi \circ \bar{\partial} \circ \psi^{-1}, \phi \circ \psi^{-1})$$

For $E_1 = E_2 = E$ we get an action of the group $Aut(E)$ of smooth bundle automorphisms of $E$ on $\mathcal{C}(X, E, V)$. Now, applying theorem 1.1 and lemma 2.6 from [8], since, for $k > 0$, $H^0(\text{End}\,(V \otimes \mathcal{O}_{L_\infty}(-k)) = 0$, the action of $Aut(E)$ on $\mathcal{C}(X, E, V)$ is free and the quotient has the structure of a finite dimensional Hausdorff complex analytic space. We define $\mathfrak{M}(X, E, V)$ as the quotient $\mathcal{C}(X, E, V)/Aut(E)$.

**Proposition 2.1.** *Let $E_1, E_2$ be two isomorphic complex vector bundles. Then there is a canonical isomorphism $\mathfrak{M}(X, E_1, V) \cong \mathfrak{M}(X, E_2, V)$.*

*Proof.* Let $\psi, \tilde{\psi} : E_1 \to E_2$ be bundle automorphisms and let $g \in \text{Aut}(E_1)$. Then $\psi \circ g \circ \psi^{-1}, \tilde{\psi} \circ \psi^{-1} \in \text{Aut}(E_2)$ and

$$\psi_* \circ g_* = (\psi \circ g \circ \psi^{-1})_* \circ \psi_*$$
$$\tilde{\psi}_* = (\tilde{\psi} \circ \psi^{-1})_* \circ \psi_*$$

The first equation tells us that $\psi_*$ descends to the quotient to give a map $\psi_* : \mathfrak{M}(X, E_1, V) \to \mathfrak{M}(X, E_2, V)$. The second equation tells us that $\psi_* = \tilde{\psi}_*$ hence the map is independent of the choice of isomorphism $\psi$. □

The smooth isomorphism class of $E$ is completely determined by its rank and its second Chern class $c_2(E) = k$. Hence we will write $\mathfrak{M}_{X,k}(V)$ instead of $\mathfrak{M}(X, E, V)$. We define

$$\mathfrak{M}_X(V) = \coprod_{k=0}^{\infty} \mathfrak{M}_{X,k} V$$

2.2. **Morphisms.** We now show how to associate to a linear isometry $\alpha : V \to W$ a continuous map $\mathfrak{M}_X \alpha : \mathfrak{M}_X(V) \to \mathfrak{M}_X(W)$. For each $k$ pick a smooth complex vector bundle $E$ with $c_1(E) = 0$, $c_2(E) = k$ and $rk(E) = dim(V)$. Let $V_\alpha = \alpha(V)^\perp$. Then $\alpha$ extends canonically to as isomorphism $\tilde{\alpha} : V \oplus V_\alpha \to W$. Let $\epsilon_\alpha$ be the trivial bundle over $X$ with fiber $V_\alpha$. We will define a map

$$\mathcal{C}_E \alpha : \mathcal{C}(X, E, V) \to \mathcal{C}(X, E \oplus \epsilon_\alpha, W)$$

Given a trivialization $\phi : E|_{L_\infty} \to L_\infty \times V$ we define $\phi_\alpha : E \oplus \epsilon_\alpha \to W \times L_\infty$ as the composition

$$(E \oplus \epsilon_\alpha)|_{L_\infty} \xrightarrow{\phi \oplus \mathbf{1}} (V \oplus V_\alpha) \times L_\infty \xrightarrow{\tilde{\alpha} \times \mathbf{1}} W \times L_\infty$$

Let $\bar{\partial}$ be the canonical holomorphic structure on $\epsilon_\alpha$. Then, given $(\bar{\partial}_E, \phi) \in \mathcal{C}(X, E, V)$ we define $\mathcal{C}_E \alpha(\bar{\partial}_E, \phi) = (\bar{\partial}_E \oplus \bar{\partial}, \phi_\alpha)$.

**Lemma 2.2.** *$\mathcal{C}_E \alpha$ descends to the quotient to give a map $\mathfrak{M}_{X,k} \alpha : \mathfrak{M}_{X,k} V \to \mathfrak{M}_{X,k} W$.*

*Proof.* An automorphism $\psi \in \mathrm{Aut}(E)$ induces an automorphism $\psi_\alpha = \psi \oplus \mathbf{1} \in \mathrm{Aut}(E \oplus \epsilon_\alpha)$ and it's a direct verification that $\mathcal{C}_E\alpha \circ \psi_* = (\psi_\alpha)_* \circ \mathcal{C}_E\alpha$. The result follows. □

$\mathfrak{M}_X\alpha : \mathfrak{M}_X V \to \mathfrak{M}_X W$ is the map whose degree $k$ components are $\mathfrak{M}_{X,k}$.

**Proposition 2.3.** *$\mathfrak{M}_X : \mathcal{L}_* \to (Top)$ is a functor.*

*Proof.* Given isometries $\alpha : U \to V$ and $\beta : V \to W$, we want to show that $\mathfrak{M}_X\beta \circ \mathfrak{M}_X\alpha = \mathfrak{M}_X(\beta \circ \alpha)$. Let $E$ be a bundle with rank equal to $\dim U$ and $c_2(E) = k$. Let $\epsilon_\alpha = \alpha(U)^\perp \times X$, $\epsilon_\beta = \beta(V)^\perp \times X$, $\epsilon_{\beta\alpha} = (\beta \circ \alpha)(U)^\perp \times X$. Then the map $\beta \oplus \mathbf{1} : \alpha(U)^\perp \oplus \beta(V)^\perp \to (\beta \circ \alpha)(U)^\perp$ is an isomorphism of hermitian vector spaces inducing an isomorphism of bundles $\psi : E \oplus \epsilon_\alpha \oplus \epsilon_\beta \to E \oplus \epsilon_{\beta\alpha}$ and we have the commutative diagram

$$\begin{tikzcd}
\mathcal{C}(X,E,U) \arrow[r, "\mathcal{C}_E\alpha"] \arrow[d, "\mathcal{C}_E(\beta\circ\alpha)"] & \mathcal{C}(X, E\oplus\epsilon_\alpha, V) \arrow[d, "\mathcal{C}_E\beta"] \\
\mathcal{C}(X, E\oplus\epsilon_{\beta\alpha}, W) & \mathcal{C}(X, E\oplus\epsilon_\alpha\oplus\epsilon_\beta, W) \arrow[l, "\psi_*"']
\end{tikzcd}$$

These maps descend to the quotient and we get $\mathfrak{M}_X\beta \circ \mathfrak{M}_X\alpha = \mathfrak{M}_X(\beta \circ \alpha)$. □

**Proposition 2.4.** *The assignement $\beta \mapsto \mathfrak{M}_X\beta$ induces a continuous map betwen the space of isometries $\mathcal{L}(V, W)$ and the space of maps $\mathrm{Map}(\mathfrak{M}_X V, \mathfrak{M}_X W)$.*

*Proof.* When $\dim V = \dim W$, an isometry $\alpha : V \to W$ induces a map $\alpha \times \mathbf{1} : V \times L_\infty \to W \times L_\infty$ and $\mathcal{C}_E : \mathcal{C}(X, E, V) \to \mathcal{C}(X, E, W)$ is just the map $\mathcal{C}_E(\partial_E, \phi) = (\partial_E, (\alpha \times \mathbf{1}) \circ \phi)$. The assignement $\alpha \mapsto \mathcal{C}_E\alpha$ is clearly continuous, and hence the assignement $\alpha \mapsto \mathfrak{M}_X\alpha$ is continuous. Consider now the general case. Let $r, R$ be the dimensions of $V, W$ respectivelly, and let $V_0 = \mathbb{C}^{R-r}$. Consider the restriction map $\rho : \mathcal{L}(V \oplus V_0, W) \to \mathcal{L}(V, W)$. $\rho$ is a principal $U(R - r)$ bundle. We fix a section $\theta$ of $\rho$. Given an isometry $\alpha : V \to W$, let $V_\alpha = \alpha(V)^\perp$, $\epsilon_0 = V_0 \times X$ and $\epsilon_\alpha = V_\alpha \times X$. Then, restricting $\theta(\alpha)$ to $V_0$ we get an isomorphism $\theta(\alpha)|_{V_0} : V_0 \to V_\alpha$ which induces an isomorphism of holomorphic bundles $\psi : E \oplus \epsilon_0 \to E \oplus \epsilon_\alpha$. Let $i : V \to V \oplus V_0$ be the canonical inclusion. We have the commutative diagram

$$\begin{tikzcd}
\mathcal{C}(X,E,V) \arrow[r, "\mathcal{C}_E i"] \arrow[d, "\mathcal{C}_E\alpha"] & \mathcal{C}(X, E\oplus\epsilon_0, V\oplus V_0) \arrow[d, "\mathcal{C}_{E\oplus\epsilon_0}\theta(\alpha)"] \\
\mathcal{C}(X, E\oplus\epsilon_\alpha, W) & \mathcal{C}(X, E\oplus\epsilon_0, W) \arrow[l, "\psi_*", "\cong"']
\end{tikzcd}$$

This shows that the following diagram is commutative

$$\begin{tikzcd}
\mathcal{L}(V,W) \arrow[r, "\mathfrak{M}_X"] \arrow[d, "\theta"] & Map(\mathfrak{M}_X V, \mathfrak{M}_X W) \\
\mathcal{L}(V\oplus V_0, W) \arrow[r, "\mathfrak{M}_X"] & Map(\mathfrak{M}_X(V\oplus V_0), \mathfrak{M}_X W) \arrow[u, "(\mathfrak{M}_X i)^*"']
\end{tikzcd}$$

where $(\mathfrak{M}_X i)^*$ is the adjoint of the map $\mathfrak{M}_X i : \mathfrak{M} V \to \mathfrak{M}(V \oplus V_0)$. The bottom map is continuous since $\dim(V \oplus V_0) = \dim W$. We conclude that $\mathfrak{M}_X : \mathcal{L}(V, W) \to Map(\mathfrak{M}_X V, \mathfrak{M}_X W)$ is continuous. □

2.3. **The natural transformation.** We now define a natural transformation $\omega : \mathfrak{M}_X \times \mathfrak{M}_X \to \mathfrak{M}_X \circ \oplus$. We must define, for each pair $V_1, V_2$, maps $\omega_{V_1,V_2} : \mathfrak{M}_X V_1 \times \mathfrak{M}_X V_2 \to \mathfrak{M}_X(V_1 \oplus V_2)$. Given smooth vector bundles $E_1, E_2$ with ranks equal to the dimensions of $V_1$ and $V_2$ respectively, Whitney sum induces a map

$$\mathcal{C}(X, E_1, V_1) \times \mathcal{C}(X, E_2, V_2) \to \mathcal{C}(X, E_1 \oplus E_2, V_1 \oplus V_2)$$

This map descends to the quotient giving a map

$$\omega : \mathfrak{M}_X(V_1) \times \mathfrak{M}_X(X_2) \to \mathfrak{M}_X(V_1 \oplus V_2)$$

We can easily check that

**Proposition 2.5.** *Let $\alpha : V \to W$ be an isometry, and let $U = \alpha(V) \subset W$. Let $* \in \mathfrak{M}_X 0$ be the base-point. Then we have the commutative diagram*

$$\begin{tikzcd}
\mathfrak{M}_X V \arrow[r, "\mathfrak{M}\alpha"] \arrow[d, shift left, "\mathbf{1}\times *"] & \mathfrak{M}_X W \\
\mathfrak{M}_X V \times \mathfrak{M}_X 0 \arrow[r, "\mathfrak{M}\alpha"] \arrow[u, shift left, "\omega"] & \mathfrak{M}_X U \times \mathfrak{M}_X(U^\perp) \arrow[u, "\omega"']
\end{tikzcd}$$

**Proposition 2.6.** *$\omega$ is a commutative and associative natural transformation. Furthermore*

(a) *Let $x \in \mathfrak{M}_X V$, and let $* \in \mathfrak{M}_X 0$ be the base point. Then $\omega(x, *) = x$.*
(b) *The map $\mathfrak{M}_X V \to \mathfrak{M}_X(V \oplus W)$ given by $x \mapsto \omega(x, *)$ is a homeomorphism onto a closed subset.*

*Proof.* Property (a) is clear. We show property (b). Using for example the monad description of $\mathfrak{M}_X V$ in [3] we see that the map $\mathfrak{M}_X V \to \mathfrak{M}_X(V \oplus W)$ embeds $\mathfrak{M}_X V$ as a close submanifold. Property (b) follows. Now we check commutativity. Fix bundles $E_1, E_2$ with the right ranks and let $\psi : E_1 \oplus E_2 \to E_2 \oplus E_1$ and $\alpha : V_1 \oplus V_2 \to V_2 \oplus V_1$ be the canonical isomorphisms. Let $\tau$ be the canonical map $A \times B \leftrightarrow B \times A$. We claim that the diagram

$$\begin{tikzcd}
\mathcal{C}(X, E_1, V_1) \times \mathcal{C}(X, E_2, V_2) \arrow[r, leftrightarrow, "\tau"] \arrow[d, "\omega_{V_1,V_2}"] & \mathcal{C}(X, E_2, V_2) \times \mathcal{C}(X, E_1, V_1) \arrow[dd, "\omega_{V_2,V_1}"] \\
\mathcal{C}(X, E_1 \oplus E_2, V_1 \oplus V_2) \arrow[d, "\psi_*"] & \\
\mathcal{C}(X, E_2 \oplus E_1, V_1 \oplus V_2) \arrow[r, "\mathfrak{M}\alpha"] & \mathcal{C}(X, E_2 \oplus E_1, V_2 \oplus V_1)
\end{tikzcd}$$

is commutative. This follows because, given $\phi_i : E_i|_{L_\infty} \to L_\infty \times V_i$ we have $\alpha \circ (\phi_1 \oplus \phi_2) \circ \psi^{-1} = \phi_2 \oplus \phi_1$. Commutativity of $\omega$ then follows. Associativity follows from a similar argument.

We turn to the proof that $\omega$ is a natural transformation. Consider isomorphisms $\alpha : V \to W$ and $\beta : V' \to W'$. Fix bundles $E, E' \to X$ with convenient rank. Then the result follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{C}(X, E, V) \times \mathcal{C}(X, E', V') \arrow[r, "\omega"] \arrow[d, "\mathcal{C}_E\alpha \times \mathcal{C}_{E'}\beta"] & \mathcal{C}(X, E \oplus E', V \oplus V') \arrow[d, "\mathcal{C}_{E\oplus E'}(\alpha\oplus\beta)"] \\
\mathcal{C}(X, E, W) \times \mathcal{C}(X, E', W') \arrow[r, "\omega"] & \mathcal{C}(X, E \oplus E', W \oplus W')
\end{tikzcd}$$

For general isometries $\alpha_0 : V_0 \to W_0$ and $\alpha_1 : V_1 \to W_1$ let $U_i = \alpha_i(V_i)$ and let $U_i^\perp, (U_0 \oplus U_1)^\perp$ be the orthogonal complements inside $W_i, W_0 \oplus W_1$ respectivelly. Then $U_0^\perp \oplus U_1^\perp = (U_0 \oplus U_1)^\perp$. The result then follows from the diagram

$$\begin{tikzcd}
\mathfrak{M}V_0 \times \mathfrak{M}V_1 \arrow[r, "\mathfrak{M}\alpha_0 \times \mathfrak{M}\alpha_1"] \arrow[dddd, bend right=60, "\omega"'] & \mathfrak{M}W_0 \times \mathfrak{M}W_1 \arrow[dddd, bend left=60, "\omega"] \\
\mathfrak{M}V_0 \times \mathfrak{M}0 \times \mathfrak{M}V_1 \times \mathfrak{M}0 \arrow[r, "\mathfrak{M}\alpha_0 \times * \times \mathfrak{M}\alpha_1 \times *"] \arrow[u, "\omega\times\omega"] \arrow[d, leftrightarrow, "\tau"] & \mathfrak{M}U_0 \times \mathfrak{M}U_0^\perp \times \mathfrak{M}U_1 \times \mathfrak{M}U_1^\perp \arrow[u, "\omega\times\omega"'] \arrow[d, leftrightarrow, "\tau"'] \\
\mathfrak{M}V_0 \times \mathfrak{M}V_1 \times \mathfrak{M}0 \times \mathfrak{M}0 \arrow[r, "\mathfrak{M}\alpha_0 \times \mathfrak{M}\alpha_1 \times * \times *"] \arrow[d, "\omega\times\omega"] & \mathfrak{M}U_0 \times \mathfrak{M}U_1 \times \mathfrak{M}U_0^\perp \times \mathfrak{M}U_1^\perp \arrow[d, "\omega\times\omega"] \\
\mathfrak{M}(V_0 \oplus V_1) \times \mathfrak{M}0 \arrow[r, "\mathfrak{M}(\alpha_0\oplus\alpha_1)\times *"] \arrow[d, "\omega"] & \mathfrak{M}(U_0 \oplus U_1) \times \mathfrak{M}(U_0^\perp \oplus U_1^\perp) \arrow[d, "\omega"] \\
\mathfrak{M}(V_0 \oplus V_1) \arrow[r, "\mathfrak{M}(\alpha_0\oplus\alpha_1)"] & \mathfrak{M}(W_0 \oplus W_1)
\end{tikzcd}$$

□

2.4. **Pullback.** We are now going to define a functor from $\mathfrak{C}$ to the category of $\mathcal{L}_*$ functors. Let $I \in \mathfrak{C}$. Recall that $\mathbb{P}^2_I$ is the blowup of $\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{P}^1$ along $I$. Then we have a canonical rational curve $L_\infty = \mathbb{P}^1 \subset \mathbb{P}^2$. We associate to each $I \in \mathfrak{C}$ the $\mathcal{L}_*$ functor $(\mathfrak{M}_{\mathbb{P}^2_I}, \omega)$, which we write simply as $(\mathfrak{M}_I, \omega)$. Now let $J \subset I$. Fix a bundle $E \to \mathbb{P}^2_J$ with $\operatorname{rk} E = \dim V$ and $c_2(E) = k$. We have a map $\pi_{JI} : \mathbb{P}^2_I \to \mathbb{P}^2_J$ corresponding to blowing up $\mathbb{P}^2_J$ along $I \setminus J$, inducing a map $\tilde{\pi}_{JI} : \pi^*_{JI}E \to E$. We define a pullback map $\pi^*_{JI} : \mathcal{C}(\mathbb{P}^2_J, E, V) \to \mathcal{C}(\mathbb{P}^2_I, \pi^*_{JI}E, V)$ by

$$\pi^*_{JI}(\bar{\partial}, \phi) = (\pi^*_{JI}\bar{\partial}, \phi \circ \tilde{\pi}_{JI})$$

This map descends to the quotient to give a map $\pi^*_{JI} : \mathfrak{M}_{J,k}V \to \mathfrak{M}_{I,k}V$.

**Proposition 2.7.** *The assignments $I \to \mathfrak{M}_I$ and $(J \subset I) \to \pi^*_{JI}$ define a functor from $\mathfrak{C}$ to the category of $\mathcal{L}_*$-functors. Furthermore, this functor preserves cartesian squares and the maps $\pi^*_{JI}$ are open embeddings.*

*Proof.* We have to show that $\pi^*_{JI}$ are morphisms in the category of $\mathcal{L}_*$-functors. Let $\alpha : V \to W$ be an isometry. Let $\epsilon_{\alpha J} = \alpha(V)^\perp \times \mathbb{P}^2_J$, $\epsilon_{\alpha I} = \alpha(V)^\perp \times \mathbb{P}^2_I$ be the trivial bundles. Fix a bundle $E \to \mathbb{P}^2_J$ with the right rank. Then we have the commutative diagram

$$\begin{tikzcd}
\mathcal{C}(\mathbb{P}^2_J, E, V) \arrow[r, "\pi^*_{JI}(V)"] \arrow[d, "\mathcal{C}_E\alpha"] & \mathcal{C}(\mathbb{P}^2_I, \pi^*_{JI}E, V) \arrow[d, "\mathcal{C}_{\pi^*_{JI}E}\alpha"] \\
\mathcal{C}(\mathbb{P}^2_J, E \oplus \epsilon_{\alpha J}, W) \arrow[r, "\pi^*_{JI}(W)"] & \mathcal{C}(\mathbb{P}^2_I, \pi^*_{JI}E \oplus \epsilon_{\alpha I}, W)
\end{tikzcd}$$

which shows that $\pi^*_{JI}$ is a natural transformation. Now we show that it commutes with $\omega$. Given vector spaces $V, W$, we pick bundles $E_V, E_W \to \mathbb{P}^2_J$ with the right

rank. Then we have a canonical isomorphism $\psi : \pi_{JI}^* E_V \oplus \pi_{JI}^* E_W \to \pi_{JI}^*(E_V \oplus E_W)$ and the commutative diagram

$$\begin{tikzcd}
\mathcal{C}(\mathbb{P}^2_J, E_V, V) \times \mathcal{C}(\mathbb{P}^2_J, E_W, W) \arrow[r, "\pi_{JI}^*"] \arrow[dd, "\omega"] & \mathcal{C}(\mathbb{P}^2_I, \pi_{JI}^* E_V, V) \times \mathcal{C}(\mathbb{P}^2_I, \pi_{JI}^* E_W, W) \arrow[d, "\omega"] \\
 & \mathcal{C}(\mathbb{P}^2_I, \pi_{JI}^* E_V \oplus \pi_{JI}^* E_W, V \oplus W) \arrow[d, "\psi_*"] \\
\mathcal{C}(\mathbb{P}^2_J, E_V \oplus E_W, V \oplus W) \arrow[r, "\pi_{JI}^*"] & \mathcal{C}(\mathbb{P}^2_I, \pi_{JI}^*(E_V \oplus E_W), V \oplus W)
\end{tikzcd}$$

which shows $\pi_{JI}^*$ commutes with $\omega$. Finally we show that, given $K \subset J \subset I$ we have $\pi_{JI}^* \circ \pi_{KJ}^* = \pi_{KI}^*$. Given a vector space $V$ we fix a bundle $E \to \mathbb{P}^2_K$ with the right rank. Then we have a canonical isomorphism of bundles $\psi : \pi_{KI}^* E \to \pi_{JI}^* \pi_{KJ}^* E$ and a commutative diagram

$$\begin{tikzcd}
\mathcal{C}(\mathbb{P}^2_K, E, V) \arrow[r, "\pi_{KJ}^*"] \arrow[d, "\pi_{KI}^*"] & \mathcal{C}(\mathbb{P}^2_J, \pi_{KJ}^* E, V) \arrow[d, "\pi_{JI}^*"] \\
\mathcal{C}(\mathbb{P}^2_I, \pi_{KI}^* E, V) \arrow[r, "\psi_*"] & \mathcal{C}(\mathbb{P}^2_I, \pi_{JI}^* \pi_{KJ}^* E, V)
\end{tikzcd}$$

This completes the proof. For the last 2 statements see [12]. □

We finish this section, we define a functor $\mathfrak{M}\mathbb{H}$ from $\mathfrak{C}$ to the category of $\mathscr{L}$-spaces by setting $\mathfrak{M}_I\mathbb{H} = \lim_{\to} \mathfrak{M}_I V$.

## 3. Modules of topological spaces

We are now going to introduce the space $\mathcal{B}_I$ and the map $h_I : \mathcal{B}_I \to \mathfrak{M}_I\mathbb{H}$ (see equations (1) and (2)). We follow [5]. A topological category $\mathscr{C}$ is a small category with topologized morphism sets such that composition is continuous and the inclusion $ob\mathscr{C} \subset mor\mathscr{C}$ is a closed cofibration. An equivalence of topological categories is a continuous functor which is the identity on objects and a homotopy equivalence on morphism spaces. Let $\mathfrak{R}$ be the category whose objects are the pairs $(\mathscr{C}, X)$ where $\mathscr{C}$ is a topological category and $X$ is a continuous contravariant functor from $\mathscr{C}$ to the category $(Top)$ of topological spaces, and a morphism from $(\mathscr{C}, X)$ to $(\mathscr{D}, Y)$ is a pair $(a, b)$ where $a : \mathscr{D} \to \mathscr{C}$ is an equivalence of topological categories and $b : X \circ a \to Y$ is a natural transformation.

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "X"] \arrow[dr, phantom, "\searrow b"] & Top \arrow[d, equal] \\
\mathscr{D} \arrow[u, "a"] \arrow[r, "Y"] & Top
\end{tikzcd}$$

Given two morphisms $(a_1, b_1) : (\mathscr{B}, Y) \to (\mathscr{C}, Z)$ and $(a_2, b_2) : (\mathscr{A}, X) \to (\mathscr{B}, Y)$, composition is given by $(a_1, b_1) \circ (a_2, b_2) = (a_2 \circ a_1, b_1 \circ (a_1^* b_2))$. We now define a "geometrical realization" functor from $\mathfrak{R}$ to the homotopy category $hTop$. Given $(\mathscr{C}, X) \in \mathfrak{R}$, we define $|(\mathscr{C}, X)|$ as the homotopy colimit $B(X, \mathscr{C}, *)$. Given a

morphism $(a,b) : (\mathscr{C}_1, X_1) \to (\mathscr{C}_2, X_2)$ we have induced maps

$$B(X_1, \mathscr{C}_1, *) \underset{\simeq}{\xleftarrow{a}} B(X_1 \circ a, \mathscr{C}_2, *) \xrightarrow{b} B(X_2, \mathscr{C}_2, *)$$

Let $a^{-1}$ denote the homotopy inverse of $a$. We define $|(a,b)| = b \circ a^{-1}$.

We will now define two functors $\mathscr{S}, \mathscr{T}$ from $\mathfrak{C}$ to $\mathfrak{R}$, and a natural transformation $\mathfrak{H} : \mathscr{S} \to \mathscr{T}$. The geometrical realization of $\mathscr{S}$ will induce the functor $\mathcal{B} : \mathfrak{C} \to hTop$ from equation (1) and we will show that there is a weak equivalence $|\mathscr{T}| \simeq \mathfrak{M}\mathbb{H}$. Then $\mathfrak{H}$ induces the natural transformation $h : \mathcal{B} \to |\mathscr{T}| \simeq \mathfrak{M}\mathbb{H}$ from equation (2). Given an object $I \in \mathfrak{C}$ we will define a category $\Delta_I$, functors $S_I, T_I : \Delta_I \to Top$ and a natural transformation $\eta_I : S_I \to T_I$, and set

$$\mathscr{S}(I) = (\Delta_I, S_I) \qquad \mathscr{T}(I) = (\Delta_I, T_I) \qquad \mathfrak{H}(I) = (\mathbf{1}, \eta_I)$$

Given a morphism $i : J \to I$, we will define $\mathscr{S}(I) = (a_i, s_i)$, $\mathscr{T}(I) = (a_i, t_i)$ (see the diagram bellow).

$$\begin{tikzcd} \Delta_J \arrow[r, "S_J"] & Top \arrow[d, equal] \\ \Delta_I \arrow[u, "a_i"] \arrow[r, "S_I"'] \arrow[ur, phantom, "\Searrow s_i"] & Top \end{tikzcd} \qquad \begin{tikzcd} \Delta_J \arrow[r, "T_J"] & Top \arrow[d, equal] \\ \Delta_I \arrow[u, "a_i"] \arrow[r, "T_I"'] \arrow[ur, phantom, "\Searrow t_i"] & Top \end{tikzcd}$$

3.1. **Objects: the category $\Delta_I$.** Let $\Delta$ denote the topological category whose objects are the sets $[n] = \{0, 1, \dots, n\} \subset \mathbb{Z}$ and the morphisms are the order preserving maps $\mu : [m] \to [n]$. We now define the category $\Delta_I$. It has the same objects as $\Delta$. To define the morphisms we introduce some notation. Recall that $\mathscr{L}(j) = \mathcal{L}(\mathbb{H}^j, \mathbb{H})$. Given $I \in \mathfrak{C}$ we let

$$\mathbb{H}^{1+jI} = \mathbb{H} \oplus \left( \bigoplus_{x \in I} \mathbb{H} \right)^{\oplus j} \qquad \mathscr{L}(1 + jI) = \mathcal{L}(\mathbb{H}^{1+jI}, \mathbb{H})$$

Given a topological space $X$ we will write

$$X^{jI} = \left( \prod_{x \in I} X \right)^{\times j}$$

We proceed to defining the morphisms. Let $[m], [n] \in \mathrm{Ob}(\Delta_I)$. Then we set

$$\Delta_I([m], [n]) = \coprod_{\mu \in \Delta([m],[n])} \Delta_{I,\mu}([m], [n])$$

where

$$\begin{aligned} &\Delta_{I,\mu}([m], [n]) \\ &\qquad = \mathscr{L}(1 + \mu(0)I) \times \left( \prod_{\alpha=1}^{m} \mathscr{L}(\mu(\alpha) - \mu(\alpha - 1))^I \right) \times \mathscr{L}(n - \mu(m) + 1)^I \end{aligned}$$

Composition of morphisms is defined as follows: for each $\mu \in \Delta([m], [n])$ and each $\nu \in \Delta([n], [p])$ we define a map

$$\Delta_{I,\nu}([n], [p]) \times \Delta_{I,\mu}([m], [n]) \to \Delta_{I,\nu \circ \mu}([m], [p])$$

by using the operad maps

$$\left(\mathscr{L}(\nu(0)I+1)\times\prod_{\beta=1}^{\mu(0)}\mathscr{L}(\nu(\beta)-\nu(\beta-1))^I\right)\times\mathscr{L}(\mu(0)I+1)$$
$$\longrightarrow\mathcal{L}\left(\mathbb{H}^{\nu(0)I+1}\oplus\bigoplus_{\beta=1}^{\mu(0)}\mathbb{H}^{\nu(\beta)I-\nu(\beta-1)I},\mathbb{H}\right)=\mathscr{L}(\nu(\mu(0))I+1)$$

$$\left(\prod_{\beta=\mu(\alpha-1)+1}^{\mu(\alpha)}\mathscr{L}(\nu(\beta)-\nu(\beta-1))\right)\times\mathscr{L}(\mu(\alpha)-\mu(\alpha-1))$$
$$\longrightarrow\mathcal{L}\left(\bigoplus_{\beta=\mu(\alpha-1)+1}^{\mu(\alpha)}\mathbb{H}^{\nu(\beta)-\nu(\beta-1)},\mathbb{H}\right)=\mathscr{L}(\nu(\mu(\alpha))-\nu(\mu(\alpha-1)))$$

$$\left(\mathscr{L}(p-\nu(n)+1)\times\prod_{\beta=\mu(m)+1}^{n}\mathscr{L}(\nu(\beta)-\nu(\beta-1))\right)\times\mathscr{L}(n-\mu(m)+1)$$
$$\longrightarrow\mathcal{L}\left(\mathbb{H}^{p-\nu(n)+1}\oplus\bigoplus_{\beta=\mu(m)+1}^{n}\mathbb{H}^{\nu(\beta)-\nu(\beta-1)},\mathbb{H}\right)=\mathscr{L}(p-\nu(\mu(m))+1)$$

Associativity then follows from the properties of the operad. The connected components $\Delta_{I,\mu}([m],[n])$ of $\Delta_I([m],[n])$ are contractible and in one to one correspondence with the morphisms $\Delta([m],[n])$. Hence the categories $\Delta_I,\Delta$ are equivalent.

3.2. **Objects: The functors $S_I$ and $T_I$.** To each $[m]\in Ob(\Delta_I)$ we associate the $\mathscr{L}$-spaces

$$S_I([m])=\mathfrak{M}_\emptyset\mathbb{H}\times(\mathfrak{M}_\emptyset\mathbb{H})^{mI}\times\prod_{x\in I}\mathfrak{M}_x\mathbb{H}$$
$$T_I([m])=\mathfrak{M}_I(\mathbb{H})\times\mathfrak{M}_I(\mathbb{H})^{mI}\times\mathfrak{M}_I(\mathbb{H})^I$$

Now let $\mu\in\Delta([m],[n])$ and observe that

$$S_I([n])=\left(\mathfrak{M}_\emptyset\times\mathfrak{M}_\emptyset^{\mu(0)I}\right)\times\left(\prod_\alpha\mathfrak{M}_\emptyset^{(\mu(\alpha)-\mu(\alpha-1))I}\right)\times\left(\prod_{x\in I}\mathfrak{M}_\emptyset^{n-\mu(m)}\times\mathfrak{M}_x\right)$$
$$T_I([n])=\left(\mathfrak{M}_I\times\mathfrak{M}_I^{\mu(0)I}\right)\times\left(\prod_\alpha\mathfrak{M}_I^{(\mu(\alpha)-\mu(\alpha-1))I}\right)\times\left(\mathfrak{M}_I^{(n-\mu(m))I}\times\mathfrak{M}_I^I\right)$$

(where we dropped the $\mathbb{H}$ to shorten the notation). Given $f\in\Delta_{I,\mu}([m],[n])$ we write $f=\left(f_0\,,\,(f_{\alpha x})_{\substack{x\in I\\ \alpha=1,\dots,m}}\,,\,(f_x)_{x\in I}\right)$ with

$$f_0\in\mathscr{L}(\mu(0)I+1)$$
$$(f_{\alpha x})_{x\in I}\in\mathscr{L}(\mu(\alpha)-\mu(\alpha-1))^I\quad(\alpha=1,\dots,m)$$
$$(f_x)_{x\in I}\in\mathscr{L}(n-\mu(m)+1)^I$$

Using the action of the operad $\mathscr{L}$ on $\mathfrak{M}\mathbb{H}$ we get maps

$$f_0 : \begin{array}{l} \mathfrak{M}_\emptyset\mathbb{H} \times (\mathfrak{M}_\emptyset\mathbb{H})^{\mu(0)I} \to \mathfrak{M}_\emptyset\mathbb{H} \\ \mathfrak{M}_I\mathbb{H} \times (\mathfrak{M}_I\mathbb{H})^{\mu(0)I} \to \mathfrak{M}_I\mathbb{H} \end{array}$$

$$\prod_{x\in I} f_{\alpha x} : \begin{array}{l} (\mathfrak{M}_\emptyset\mathbb{H})^{(\mu(\alpha)-\mu(\alpha-1))I} \to (\mathfrak{M}_\emptyset\mathbb{H})^I \\ (\mathfrak{M}_I\mathbb{H})^{(\mu(\alpha)-\mu(\alpha-1))I} \to (\mathfrak{M}_I\mathbb{H})^I \end{array} \qquad (\alpha = 1, \dots, m)$$

$$\prod_{x\in I} f_x \circ \pi^*_{\emptyset,x} : \prod_{x\in I} (\mathfrak{M}_\emptyset\mathbb{H})^{n-\mu(m)} \times \mathfrak{M}_x\mathbb{H} \to \prod_{x\in I} (\mathfrak{M}_x\mathbb{H})^{n-\mu(m)+1} \to \prod_{x\in I} \mathfrak{M}_x\mathbb{H}$$

$$\prod_{x\in I} f_x : (\mathfrak{M}_I\mathbb{H})^{(n-\mu(m))I} \times (\mathfrak{M}_I\mathbb{H})^I \to (\mathfrak{M}_I\mathbb{H})^I$$

These maps define the desired morphisms $S_I(f) : S_I([n]) \to S_I([m])$ and $T_I(f) : T_I([n]) \to T_I([m])$.

3.3. **Morphisms.** Now consider a morphism $i : J \to I$ in $\mathfrak{C}$. We will define two morphisms

$$\mathscr{S}(i) = (a_i, s_i) : (\Delta_J, S_J) \to (\Delta_I, S_I)$$
$$\mathscr{T}(i) = (a_i, t_i) : (\Delta_J, T_J) \to (\Delta_I, T_I)$$

where the functor $a_i : \Delta_I \to \Delta_J$. is the identity on objects and on morphisms is induced by the maps

$$\mathscr{L}(\mu(0)I + 1) \to \mathscr{L}(\mu(0)J + 1)$$
$$\mathscr{L}(\mu(\alpha) - \mu(\alpha - 1))^I \to \mathscr{L}(\mu(\alpha) - \mu(\alpha - 1))^J$$
$$\mathscr{L}(n + 1 - \mu(m))^I \to \mathscr{L}(n + 1 - \mu(m))^J$$

where the first map is the adjoint to the inclusion $\mathbb{H}^{\mu(0)J+1} \to \mathbb{H}^{\mu(0)I+1}$ and the second and third maps are the projections. The verification that this is indeed a functor is straightforward.

It remains to define the natural transformations $s_i : S_J \circ a_i \to S_I$ and $t_i : T_J \circ a_i \to T_I$. Let $\imath : (\mathfrak{M}\mathbb{H})^J \to (\mathfrak{M}\mathbb{H})^I$ be the canonical inclusion. Then for each $[m] \in \Delta_I$, we let $s_i(m), t_i(m)$ be the maps

$$s_i(m) : S_J \circ a_i(m) = \mathfrak{M}_\emptyset \times \mathfrak{M}_\emptyset^{mJ} \times \prod_{x\in J} \mathfrak{M}_x \xrightarrow{\imath} \mathfrak{M}_\emptyset \times \mathfrak{M}_\emptyset^{mI} \times \prod_{x\in I} \mathfrak{M}_x = S_I(m)$$

$$\begin{aligned} t_i(m) : T_J \circ a_i(m) = \ & \mathfrak{M}_J \times \mathfrak{M}_J^{mJ} \times \mathfrak{M}_J^J \xrightarrow{\pi^*_{J,I}} \mathfrak{M}_I \times \mathfrak{M}_I^{mJ} \times \mathfrak{M}_I^J \\ & \xrightarrow{\imath} \mathfrak{M}_I \times \mathfrak{M}_I^{mI} \times \mathfrak{M}_I^I = T_I(m) \end{aligned}$$

We now check that $s_i, t_i$ are natural transformations. Let $\mu \in \Delta([m], [n])$, $f \in \Delta_{I,\mu}([m], [n])$. Write $f = \left( f_0 \,,\, (f_{\alpha x})_{\substack{x\in I \\ \alpha=1,\dots,m}} \,,\, (f_x)_{x\in I} \right)$. Then each one of the following diagrams

$$\begin{tikzcd} S_J \circ a_i(n) \arrow[r, "s_i(n)"] \arrow[d, "S_J\circ a_i(f)"] & S_I(n) \arrow[d, "S_I(f)"] \\ S_J \circ a_i(m) \arrow[r, "s_i(m)"] & S_I(m) \end{tikzcd} \qquad \begin{tikzcd} T_J \circ a_i(n) \arrow[r, "t_i(n)"] \arrow[d, "S_J\circ a_i(f)"] & T_I(n) \arrow[d, "S_I(f)"] \\ T_J \circ a_i(m) \arrow[r, "t_i(m)"] & T_I(m) \end{tikzcd}$$

splits into three diagrams. For $s_i$ we have the diagrams

$$\begin{tikzcd}
\mathfrak{M}_\emptyset \times \mathfrak{M}_\emptyset^{\mu(0)J} \arrow[r, "\imath"] \arrow[d, "a_i(f_0)"] & \mathfrak{M}_\emptyset \times \mathfrak{M}_\emptyset^{\mu(0)I} \arrow[d, "f_0"] \\
\mathfrak{M}_\emptyset \arrow[r, equal] & \mathfrak{M}_\emptyset
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{M}_\emptyset^{(\mu(\alpha)-\mu(\alpha-1))J} \arrow[r, "\imath"] \arrow[d, "\prod f_{\alpha x}"] & \mathfrak{M}_\emptyset^{(\mu(\alpha)-\mu(\alpha-1))I} \arrow[d, "\prod f_{\alpha x}"] \\
\mathfrak{M}_\emptyset^J \arrow[r, "\imath"] & \mathfrak{M}_\emptyset^I
\end{tikzcd}$$

$$\begin{tikzcd}
\prod_{x\in J}\left(\mathfrak{M}_\emptyset^{n-\mu(m)} \times \mathfrak{M}_x\right) \arrow[r, "\imath"] \arrow[d, "\prod f_x\circ\pi^*_{\emptyset,x}"] & \prod_{x\in I}\left(\mathfrak{M}_\emptyset^{n-\mu(m)} \times \mathfrak{M}_x\right) \arrow[d, "\prod f_x\circ\pi^*_{\emptyset,x}"] \\
\prod_{x\in J}\mathfrak{M}_x \arrow[r, "\imath"] & \prod_{x\in I}\mathfrak{M}_x
\end{tikzcd}$$

and for $\mathcal{T}$:

$$\begin{tikzcd}
\mathfrak{M}_J \times \mathfrak{M}_J^{\mu(0)J} \arrow[r, "\imath\circ\pi^*_{J,I}"] \arrow[d, "a_i(f_0)"] & \mathfrak{M}_I \times \mathfrak{M}_I^{\mu(0)I} \arrow[d, "f_0"] \\
\mathfrak{M}_J \arrow[r, "\pi^*_{J,I}"] & \mathfrak{M}_I
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{M}_J^{(\mu(\alpha)-\mu(\alpha-1))J} \arrow[r, "\imath\circ\pi^*_{J,I}"] \arrow[d, "\prod f_{\alpha x}"] & \mathfrak{M}_I^{(\mu(\alpha)-\mu(\alpha-1))I} \arrow[d, "\prod f_{\alpha x}"] \\
\mathfrak{M}_J^J \arrow[r, "\imath\circ\pi^*_{J,I}"] & \mathfrak{M}_I^I
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{M}_J^{(n-\mu(m))J} \times \mathfrak{M}_J^J \arrow[r, "\imath\circ\pi^*_{J,I}"] \arrow[d, "\prod f_x"] & \mathfrak{M}_I^{(n-\mu(m))J} \times \mathfrak{M}_I^I \arrow[d, "\prod f_x"] \\
\mathfrak{M}_J^J \arrow[r, "\imath\circ\pi^*_{J,I}"] & \mathfrak{M}_I^I
\end{tikzcd}$$

It is straightforward to check that these diagrams are commutative.

3.4. **The natural transformation.** We now want to define a natural transformation $\mathfrak{H} : \mathscr{S} \to \mathscr{T}$. Let $I \in \mathfrak{C}$. Then we set

$$\mathfrak{H}(I) = (\mathbf{1}, \eta_I) : (\Delta_I, S_I) \to (\Delta_I, T_I)$$

where the natural transformation $\eta_I : S_I \to T_I$ is defined as follows: for each $[n] \in ob(\Delta_I)$ we let $\eta_I : S_I([n]) \to T_I([n])$ be the map

$$\eta_I = \pi^*_{\emptyset,I} \times (\pi^*_{\emptyset,I})^{mI} \times \prod_x \pi^*_{x,I} : \mathfrak{M}_\emptyset \times \mathfrak{M}_\emptyset^{mI} \times \prod_x \mathfrak{M}_x \to \mathfrak{M}_I \times \mathfrak{M}_I^{mI} \times \mathfrak{M}_I^I$$

**Lemma 3.1.** *$\eta_I$ is a natural transformation.*

*Proof.* Let $[m], [n] \in \Delta$, $\mu \in \Delta([m],[n])$ and let $f \in \Delta_{I,\mu}([m],[n])$. Write, as usual, $f = \left( f_0\, ,\, (f_{\alpha x})_{\substack{x\in I\\ \alpha=1,\dots,m}}\, ,\, (f_x)_{x\in I}\right)$. Then the diagram

$$\begin{tikzcd}
S_I([n]) \arrow[r, "\eta_I"] \arrow[d, "S_I(f)"] & T_I([n]) \arrow[d, "T_I(f)"] \\
S_I([m]) \arrow[r, "\eta_I"] & T_I([m])
\end{tikzcd}$$

splits into the three diagrams

$$\begin{tikzcd}
\mathfrak{M}_\emptyset \times \mathfrak{M}_\emptyset^{\mu(0)I} \arrow[r, "\pi^*_{\emptyset,I}"] \arrow[d, "f_0"] & \mathfrak{M}_I \times \mathfrak{M}_I^{\mu(0)I} \arrow[d, "f_0"] \\
\mathfrak{M}_\emptyset \arrow[r, "\pi^*_{\emptyset,I}"] & \mathfrak{M}_I
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{M}_\emptyset^{(\mu(\alpha)-\mu(\alpha-1))I} \arrow[r, "\pi^*_{\emptyset,I}"] \arrow[d, "\prod_x f_{\alpha x}"] & \mathfrak{M}_I^{(\mu(\alpha)-\mu(\alpha-1))I} \arrow[d, "\prod_x f_{\alpha x}"] \\
\mathfrak{M}_\emptyset^I \arrow[r, "\pi^*_{\emptyset,I}"] & \mathfrak{M}_I^I
\end{tikzcd}$$

$$\begin{tikzcd}
\prod_x \mathfrak{M}_\emptyset^{n-\mu(m)} \times \mathfrak{M}_x \arrow[r, "\prod_x \pi^*_{\emptyset,I}\times\pi^*_{x,I}"] \arrow[d, "\prod_x f_x\circ(\pi^*_{\emptyset,x}\times\mathbf{1})"] & \mathfrak{M}_I^{(n-\mu(m))I} \times \mathfrak{M}_I \arrow[d, "\prod_x f_x"] \\
\prod_x \mathfrak{M}_x \arrow[r, "\prod_x \pi^*_{x,I}"] & \mathfrak{M}_I^I
\end{tikzcd}$$

These diagrams commute since $\pi^*_{\emptyset,x}$, $\pi^*_{\emptyset,I}$ and $\pi^*_{x,I}$ are morphisms of $\mathscr{L}$-spaces with $\pi^*_{x,I} \circ \pi^*_{\emptyset,x} = \pi^*_{\emptyset,I}$. □

**Proposition 3.2.** *$\mathfrak{H}$ is a natural transformation between $\mathscr{S}$ and $\mathscr{T}$.*

*Proof.* Given a morphism $i : J \to I$ in $\mathfrak{C}$, we need to check that the diagram

$$\begin{tikzcd}
(\Delta_J, S_J) \arrow[r, "(\mathbf{1},\eta_J)"] \arrow[d, "(a_i,s_i)"] & (\Delta_J, T_J) \arrow[d, "(a_i,t_i)"] \\
(\Delta_I, S_I) \arrow[r, "(\mathbf{1},\eta_I)"] & (\Delta_I, T_I)
\end{tikzcd}$$

commutes. That is, we need to check the equality of the natural transformations $t_i \circ a_i^*\eta_J$ and $\eta_I \circ s_i$

$$\begin{tikzcd}
\Delta_I \arrow[r, bend left, "S_J\circ a_i"] \arrow[r, bend right, "T_I"'] \arrow[r, phantom, "t_i\circ a_i^*\eta_J \Downarrow\ \Downarrow \eta_I\circ s_i"] & (Top)
\end{tikzcd}$$

This boils down to checking, for each $[n] \in \Delta$, the commutativity of the diagram

$$\begin{tikzcd}
S_J([n]) \arrow[r, "\eta_J"] \arrow[d, "s_i"] & T_J([n]) \arrow[d, "t_i"] \\
S_I([n]) \arrow[r, "\eta_I"] & T_I([n])
\end{tikzcd}$$

which commutes due to the functoriality of $\pi^*$. □

$\mathfrak{H}$ induces a natural transformation betwen the geometrical realizations:

**Conjecture.** *The natural transformation $h : \mathcal{B} = |\mathscr{S}| \to |\mathscr{T}|$ induced by $\mathfrak{H}$ is a weak equivalence.*

3.5. **Geometrical realization.** The objective of this section is to prove two results: $h_x : B(S_x, \Delta_x, *) \to B(T_x, \Delta_x, *)$ is a homotopy equivalence and $|\mathscr{T}|, \mathfrak{M}\mathbb{H} : \mathfrak{C} \to hTop$ are weakly equivalent, that is, for any $I \in \mathfrak{C}$, $B(T_I, \Delta_I, *) \simeq \mathfrak{M}_I$.

Before we begin the proof we need some lemmas. Let $\bar{\Delta}$ be the category whose objects are the objects of $\Delta$ plus an extra object which we will denote by $[-1]$, and such that $\bar{\Delta}([m], [n]) \subset \Delta([m+1], [n+1])$ as the morphisms which send $m+1$ to

$n+1$. Clearly restriction gives an isomorphism $\bar{\Delta}([m],[n]) \cong \Delta([m],[n+1])$ hence we can see $\Delta$ as a subcategory of $\bar{\Delta}$. It's easily seen that $[-1] \in \bar{\Delta}$ is an initial (and final) object. Furthermore we have

**Lemma 3.3.** *The inclusion functor $F : \Delta^{op} \to \bar{\Delta}^{op}$ is cofinal.*

*Proof.* To prove this, it is enough to show that, for any $[n] \in \bar{\Delta}$, the undercategory $[n] \downarrow F$ has a final element. This is the element $[n] \xleftarrow{I} [n+1]$ with $I \in \bar{\Delta}^{op}([n],[n+1])$ the morphism that sends $n+2$ to $n+1$ and is the identity on the remaining elements. Indeed, given an element $[n] \xleftarrow{\bar{\mu}} [m]$ with $\bar{\mu} \in \bar{\Delta}^{op}([n],[m])$, there is a unique morphism $\mu \in \Delta^{op}([n+1],[m])$, namely $\mu$ corresponds to $\bar{\mu}$ under the isomorphism $\bar{\Delta}([m],[n]) \cong \Delta([m],[n+1])$. □

Now we define a category $\bar{\Delta}_I$ equivalent to $\bar{\Delta}$. Therefore the category $\bar{\Delta}_I$ has the same objects as $\bar{\Delta}$. Given $[m],[n] \in \mathrm{ob}(\bar{\Delta}_I)$, the morphisms are given by

$$\bar{\Delta}_I([m],[n]) = \coprod_{\mu \in \bar{\Delta}([m],[n])} \bar{\Delta}_{I,\mu}([m],[n])$$

where

$$\begin{aligned}\bar{\Delta}_{I,\mu}&([m],[n]) \\ &= \mathscr{L}(1+\mu(0)I) \times \left(\prod_{\alpha=1}^{m} \mathscr{L}(\mu(\alpha)-\mu(\alpha-1))^I\right) \times \mathscr{L}(n-\mu(m)+1)^I\end{aligned}$$

**Lemma 3.4.** *The inclusion functor $\imath_I : \Delta_I^{op} \to \bar{\Delta}_I^{op}$ is cofinal.*

*Proof.* We have to show that $B(\imath_I^* \bar{\Delta}_I, \Delta_I, *) \simeq *$. Consider the functors

$$\begin{array}{ccc} \Delta_I & \xrightarrow{F} & \Delta \\ \downarrow{\scriptstyle \imath_I} & & \downarrow{\scriptstyle \imath} \\ \bar{\Delta}_I & \xrightarrow{\bar{F}} & \bar{\Delta} \end{array}$$

Let $\bar{F}_{h*}\bar{\Delta}_I = B(\bar{\Delta}_I, \bar{\Delta}_I, \bar{\Delta})$ denote Segal's pushdown. Since $\bar{F}$ is an equivalence of categories,

$$B(\imath_I^*\bar{\Delta}_I, \Delta_I, *) \simeq B(\imath^* \bar{F}^* \bar{F}_{h*} \bar{D}elta_I, \Delta_I, *) = B(F^* \imath^* \bar{F}_{h*} \bar{D}elta_I, \Delta_I, *)$$

Since $F$ is an equivalence of categories, $\imath$ is cofinal and $\bar{F}_{h*}\bar{\Delta}_I \simeq \bar{\Delta}$, we get

$$B(F^* \imath^* \bar{F}_{h*} \bar{D}elta_I, \Delta_I, *) \simeq B(\bar{F}_{h*}\bar{D}elta_I, \bar{\Delta}, *) \simeq B(\bar{\Delta}, \bar{\Delta}, *) \simeq *$$

□

**Lemma 3.5.** *The functor $T_I : \Delta_I \to Top$ can be extended to a functor $\bar{T}_I : \bar{\Delta}_I \to Top$.*

*Proof.* We define $\bar{T}_I([-1]) = \mathfrak{M}_I \mathbb{H}$. We now define the functor on morphisms. Let $\mu \in \bar{\Delta}([m],[n])$ and observe that

$$\bar{T}_I([n]) = \left(\mathfrak{M}_I \times \mathfrak{M}_I^{\mu(0)I}\right) \times \left(\prod_{\alpha} \mathfrak{M}_I^{(\mu(\alpha)-\mu(\alpha-1))I}\right) \times \left(\mathfrak{M}_I^{(n-\mu(m)+1)I}\right)$$

Hence,given $f \in \Delta_{I,\mu}([m],[n])$, the linear isometry operad action on $\mathfrak{M}_I\mathbb{H}$ induces, as before, a map $\bar{T}_I(f) : \bar{T}_I([n]) \to \bar{T}_I([m])$. □

We can now prove

**Proposition 3.6.** *The maps*

$$|\mathscr{T}| = B(T_I, \Delta_I, *) \xrightarrow{\imath_I} B(\bar{T}_I, \bar{\Delta}_I, *) \xleftarrow{[-1]} \bar{T}_I(-1) = \mathfrak{M}_I$$

*are homotopy equivalences.*

*Proof.* $\imath_I : \Delta_I \to \bar{\Delta}_I$ is cofinal and $T_I = \bar{T}_I \circ \imath_I$ hence the first map is a homotopy equivalence. To prove that the second map is a homotopy equivalence we consider the composition

$$\begin{tikzcd}
B(\bar{T}_I, \bar{\Delta}_I, *) & B(\mathbf{1}^*\mathbf{1}_{h*}\bar{T}_I, \bar{\Delta}_I, *) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & B(\bar{F}_{h*}\bar{T}_I, \bar{\Delta}, *) \\
\bar{T}_I(-1) \arrow[u] & \mathbf{1}^*\mathbf{1}_{h*}\bar{T}_I(-1) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \bar{F}_{h*}\bar{T}_I(-1) \arrow[u]
\end{tikzcd}$$

Since $[-1]$ is a final object in $\bar{\Delta}$, the right vertical map is a homotopy equivalence. Hence the left vertical map is a homotopy equivalence. □

**Theorem 4.** $h_x : B(S_x, \Delta_x, *) \to B(T_x, \Delta_x, *)$ *is a homotopy equivalence.*

This shows that conjecture 3.4 holds for $\#I = 1$.

*Proof.* The proof follows the same lines. This time we define a category $\hat{\Delta}$ with the same objects as $\bar{\Delta}$ but whose morphisms $\hat{\Delta}([m], [n])$ are the order preserving maps $\mu : \{-1, \dots, m\} \to \{-1, \dots, n\}$ which fix $-1$. We define a category $\hat{\Delta}_x$ by setting

$$\hat{\Delta}_x([m], [n]) = \coprod_{\mu \in \hat{\Delta}([m],[n])} \hat{\Delta}_{x,\mu}([m], [n])$$

with

$$\begin{aligned}
&\hat{\Delta}_{x,\mu}([m], [n]) \\
&\qquad = \mathscr{L}(1 + \mu(0)) \times \left( \prod_{\alpha=1}^{m} \mathscr{L}(\mu(\alpha) - \mu(\alpha - 1)) \right) \times \mathscr{L}(n - \mu(m) + 1)
\end{aligned}$$

Then $\Delta_x \to \hat{\Delta}_x$ is cofinal, both $S_x$ and $T_x$ can be extend to functors $\hat{S}_x, \hat{T}_x : \hat{\Delta}_x \to Top$ by setting $\hat{S}_x([-1]) = \hat{T}_x([-1]) = \mathfrak{M}_x$ and the natural transformation $h_x$ induces a natural transformation $\hat{h}_x : \hat{S}_x \to \hat{T}_x$. Then we have the commutative diagram

$$\begin{tikzcd}
B(S_x, \Delta_x, *) \arrow[r, "\simeq"] \arrow[d, "h_x"] & B(\hat{S}_x, \hat{\Delta}_x, *) \arrow[d, "\hat{h}_x"] & \hat{S}_x(-1) \arrow[l, "\simeq"'] \arrow[d, equal] \\
B(T_x, \Delta_x, *) \arrow[r, "\simeq"] & B(\hat{T}_x, \hat{\Delta}_x, *) & \hat{T}_x(-1) \arrow[l, "\simeq"']
\end{tikzcd}$$

which shows that $h_x$ is a homotopy equivalence. □

## 4. Grading

The objective of this section is to prove theorem 1. For any $[n] \in \Delta_I$, the topological spaces $S_I(n)$ and $T_I(n)$ are naturally graded as products of graded spaces and given $f \in \Delta_I([m],[n])$, the maps $S_I(f)$ and $T_I(f)$ preserve this grading. Hence we can define functors $S_{I,k}, T_{I,k} : \Delta_I \to Top$ such that $S_{I,k}(n), T_{I,k}(n)$ are the $k$th components of $S_I(n), T_I(n)$. The natural transformation $\eta$ also preserve the grading so we get natural transformations $\eta_{I,k} : S_{I,k} \to T_{I,k}$ inducing maps $h_{I,k} : B(S_{I,k}, \Delta_I, *) \to \mathfrak{M}_{I,k}\mathbb{H}$.

We now fix $I \in \mathfrak{C}$. Let $\mathfrak{C}_I \subset \mathfrak{C}$ be the subcategory whose objects are the subsets of $I$. Then $\mathscr{S}$ induces a functor $\mathscr{S}_I : \mathfrak{C}_I \times \Delta_I^{op} \to Top$ defined as follows: On objects $\mathscr{S}_I(J,[n]) = S_J(n)$. Given morphisms $i : J \to J'$ and $f : [m] \to [m']$, we let $j : J \to I$ and $j' : J' \to I$ be the unique morphisms. Then we have the diagram

$$\begin{tikzcd} S_J(m) \arrow[r, "(S_J\circ a_j)(f)"] \arrow[d, "s_i(m)"] & S_J(m') \arrow[d, "s_i(m')"] \\ S_{J'}(m) \arrow[r, "(S_{J'}\circ a_{j'})(f)"] & S_{J'}(m') \end{tikzcd}$$

which commutes since $s_i$ is a natural transformation and $a_j = a_i \circ a_{j'}$. We define $\mathscr{S}_I(i,f) = s_i(m') \circ (S_J \circ a_j)(f) = (S_{J'} \circ a_{j'})(f) \circ s_i(m)$. In a similar way, the functor $\mathscr{T} : \mathfrak{C} \to \mathfrak{R}$ induces a functor $\mathscr{T}_I : \mathfrak{C}_I \times \Delta_I^{op} \to Top$ and the natural transformation $\mathfrak{H}$ induces a natural transformation $\mathfrak{H}_I : \mathscr{S}_I \to \mathscr{T}_I$. Consider the functor $B(\mathscr{S}_I, \Delta_I, *) : \mathfrak{C}_I \to Top$.

**Proposition 4.1.** *The functors*

$$\begin{tikzcd} \mathfrak{C}_I \arrow[r, "B(\mathscr{S}_I,\Delta_I,*)"] & Top \arrow[r] & hTop \end{tikzcd}$$

$$\begin{tikzcd} \mathfrak{C}_I \arrow[r] & \mathfrak{C} \arrow[r, "\mathscr{S}"] & \mathfrak{R} \arrow[r, "|\cdot|"] & hTop \end{tikzcd}$$

*are weakly equivalent.*

*Proof.* The assignement $B(S_J, \Delta_I, *) \xrightarrow{\simeq} B(S_J, \Delta_J, *)$ $(J \in \mathfrak{C}_I)$ provides the natural transformation. □

We define functors $\mathscr{S}_{I,k}, \mathscr{T}_{I,k} : \mathfrak{C}_I \times \Delta_I^{op} \to Top$ by setting $\mathscr{S}_{I,k}(J,n) = S_{J,k}(n)$, $\mathscr{T}_{I,k}(J,n) = T_{J,k}(n)$ and a natural transformation $\mathfrak{H}_{I,k} = (\mathbf{1}, \eta_{I,k}) : \mathscr{S}_{I,k} \to \mathscr{T}_{I,k}$.

Let $J \in \mathfrak{C}_I$ and let $j : J \to I$ be the unique morphism. Then we get a map $s_j : \mathscr{S}_I(J,n) = S_J(n) \to S_I(n)$ and toghether these maps define a map $s : B(*, \mathfrak{C}_I, \mathscr{S}_I) \to S_I$ of right $\Delta_I$-modules. Let $\mathfrak{C}_{I,k}$ be the subcategory of $\mathfrak{C}_I$ whose objects are the subsets $J \subset I$ with cardinality $\#J \leq k$.

**Proposition 4.2.** *The map $s : B(*, \mathfrak{C}_{I,k}, \mathscr{S}_{I,k}) \to S_{I,k}$ is a weak equivalence of $\Delta_I$ modules.*

*Proof.* The basic observation is that given a morphism $i : J \to J'$, the maps $s_i(n) : S_J(n) \to S_{J'}(n)$ restricted to each component of $S_J(n)$ are homeomorphisms onto a component of $S_{J'}(n)$. Fix $[n] \in \Delta_I$ and let $\mathbb{Z}_{I,k} \subset \mathbb{Z} \times \mathbb{Z}^{nI} \times \mathbb{Z}^I$ be the subset of the tuples on non-negative integers whose sum is $k$. We write an element $\vec{k} \in \mathbb{Z}_{I,k}$ as $\vec{k} = \left( k_0\,,\, (k_{\alpha x})_{\substack{x\in I \\ \alpha=1,\ldots,n}}\,,\, (k_x)_{x\in I} \right)$. To each $\vec{k} \in \mathbb{Z}_{I,k}$ we associate an

object $J_{\vec{k}} \in \mathfrak{C}_{I,k}$ by setting

$$J_{\vec{k}} = \{x \in I \,|\, k_x \neq 0 \text{ or } \exists \alpha : k_{\alpha x} \neq 0\}$$

We define a functor $S_{\vec{k},n} : \mathfrak{C} \to Top$ as follows: on objects

$$S_{\vec{k},n}(J) = \begin{cases} \emptyset & \text{if } J_{\vec{k}} \not\subset J \\ \mathfrak{M}_{\emptyset,k_0} \times \Big( \prod_{\substack{\alpha=1,\ldots,n \\ x\in J}} \mathfrak{M}_{\emptyset,k_{\alpha x}} \Big) \times \Big( \prod_{x\in J} \mathfrak{M}_{x,k_x} \Big) & \text{if } J_{\vec{k}} \subset J \end{cases}$$

Then $S_{\vec{k},n}(J) \subset S_{J,k}(n)$ and, given a morphism $i : J \to J'$ in $\mathfrak{C}$, we define $S_{\vec{k},n}(i)$ as the restriction of $s_i(n) : S_J(n) \to S_{J'}(n)$ to $S_{\vec{k},n}(J)$, which is a homeomorphism onto $S_{\vec{k},n}(J')$ whenever $J_{\vec{k}} \subset J$. Then, for any $J \subset I$,

$$\mathscr{S}_{I,k}(J,n) = S_{J,k}(n) = \coprod_{\vec{k}\in\mathbb{Z}_{I,k}} S_{\vec{k},n}(J)$$

and, given morphisms $i : J \to J'$, $\mathbf{1} : [n] \to [n]$,

$$\mathscr{S}_{I,k}(i,\mathbf{1}) = s_i(n) = \coprod_{\vec{k}\in\mathbb{Z}_{I,k}} S_{\vec{k},n}(i)$$

Hence,

$$B(*,\mathfrak{C}_{I,k},\mathscr{S}_{I,k})([n]) \cong \coprod_{\vec{k}\in\mathbb{Z}_{I,k}} B(*,\mathfrak{C}_{I,k},S_{\vec{k},n})$$

Now consider the subcategory $\mathfrak{C}_{I,\vec{k}}$ of $\mathfrak{C}_{I,k}$ whose objects are the sets $J$ such that $J_{\vec{k}} \subset J \subset I$ and $\#J \leq k$. Then $B(*,\mathfrak{C}_{I,k},S_{\vec{k},n}) \cong B(*,\mathfrak{C}_{I,\vec{k}},S_{\vec{k},n})$. The maps $s_j : S_J \to S_I$ induce homeomorphisms $B(*,\mathfrak{C}_{I,\vec{k}},S_{I,\vec{k}}) \xrightarrow{\cong} B(*,\mathfrak{C}_{I,\vec{k}},S_{I,\vec{k}}(I))$ and since $\mathfrak{C}_{I,\vec{k}}$ has an initial object ($J_{\vec{k}}$), we have

$$B(*,\mathfrak{C}_{I,\vec{k}},S_{I,\vec{k}}(I)) \cong B(*,\mathfrak{C}_{I,\vec{k}},*) \otimes_* S_{I,\vec{k}}(I) \simeq S_{I,\vec{k}}(I)$$

This shows that the map $s : B(*,\mathfrak{C}_{I,k},\mathcal{S}_{I,k}) \to S_I$ is a weak equivalence. □

As a corollary we can now prove theorem 1:

**Theorem 1.** *Let $I \subset \mathbb{C}^2$ and suppose that, for any $J \in \mathfrak{C}_{I,k}$, $h_{J,k}$ is a homotopy equivalence. Then $h_{I,k}$ is a homotopy equivalence.*

*Proof.* Let $M_{I,k} : \mathfrak{C}_{I,k} \to Top$ denote the functor given by $M_{I,k}(J) = \mathfrak{M}_{J,k}\mathbb{H}$. The pullback maps $\pi^*_{JI}\mathfrak{M}_{J,k}\mathbb{H} \to \mathfrak{M}_{I,k}\mathbb{H}$ induce a map $\pi : B(*,\mathfrak{C}_{I,k},M_{I,k}) \to \mathfrak{M}_{I,k}\mathbb{H}$. We have a commutative diagram

$$\begin{tikzcd} B(S_{I,k},\Delta_I,*) \arrow[r, "h_{I,k}"] & \mathfrak{M}_{I,k}\mathbb{H} & B(*,\mathfrak{C}_{I,k},M_{I,k}) \arrow[l, "\pi"'] \\ B(B(*,\mathfrak{C}_{I,k},\mathscr{S}_{I,k}),\Delta_I,*) \arrow[u, "s", "\simeq"'] \arrow[rr, leftrightarrow, "\cong"] & & B(*,\mathfrak{C}_{I,k},B(\mathscr{S}_{I,k},\Delta_I,*)) \arrow[u, "h"'] \end{tikzcd}$$

where $h$ is induced by the maps $h_J$, for $J \in \mathfrak{C}_{I,k}$. Hence $h$ is a homotopy equivalence. So we only need to show that $\pi$ is a homotopy equivalence. It was shown in [12] that $M_{I,k}$ and the maps $\pi^*_{JI}$ provide the nerve of a cover of $\mathfrak{M}_I$ by Zariski open sets. The result follows. □

From theorem 4 we get as an immediate corollary theorem 2 for $k = 1$:

**Corollary 4.3.** *$h_{I,1}$ is a homotopy equivalence.*

## 5. The case $k = 2$

We now prove theorem 2 for $k = 2$. By theorem 1, it is enough to consider the case when $\#I = 2$. So in this section we fix $I = \{x, y\} \in \mathfrak{C}$. We will show in appendix A that there is an open cover $\{A_x, A_y, N_2\}$ of $\mathfrak{M}_{I,2}V$, together with spaces $N'_x, N'_y \subset \mathfrak{M}_{I,1}V$ and a homeomorphism $\boxplus : N'_x \times N'_y \to N_2$ such that, if we let $A_\emptyset = A_x \cap A_y$ and $N'_\emptyset = N'_x \cap N'_y$, there are homeomorphisms

$$\begin{array}{lll} A_x \cong \mathfrak{M}_{x,2}V & N'_x \cong \mathfrak{M}_{x,1}V & \boxplus : N'_x \times N'_\emptyset \cong A_x \cap N_2 \\ A_y \cong \mathfrak{M}_{y,2}V & N'_y \cong \mathfrak{M}_{y,1}V & \boxplus : N'_y \times N'_\emptyset \cong A_y \cap N_2 \\ A_\emptyset \cong \mathfrak{M}_{\emptyset,2}V & N'_\emptyset \cong \mathfrak{M}_{\emptyset,1}V & \boxplus : N'_\emptyset \times N'_\emptyset \cong A_\emptyset \cap N_2 \end{array}$$

Hence we have maps

$$\begin{array}{ll} (\mathfrak{M}_\emptyset V \times \mathfrak{M}_\emptyset V)_k \to \mathfrak{M}_{\emptyset,k}V & (\mathfrak{M}_\emptyset V \times \mathfrak{M}_x V)_k \to \mathfrak{M}_{x,k}V \\ (\mathfrak{M}_\emptyset V \times \mathfrak{M}_y V)_k \to \mathfrak{M}_{y,k}V & (\mathfrak{M}_x V \times \mathfrak{M}_y V)_k \to \mathfrak{M}_{I,k}V \end{array} \tag{3}$$

where $(\cdots)_k$ denotes the degree k component, $k = 0, 1, 2$. We define the graded simplicial spaces $M, N : \Delta^{op} \to Top$ by setting

$$\begin{array}{ll} MV(n)_k = (\,\mathfrak{M}_x V \times (\mathfrak{M}_\emptyset V)^n \times \mathfrak{M}_\emptyset V\,)_k & \\ NV(n)_k = (\,\mathfrak{M}_x V \times (\mathfrak{M}_\emptyset V)^n \times \mathfrak{M}_y V\,)_k & \qquad (k = 0, 1, 2) \end{array}$$

and defining morphisms in the usual way. Let $\nabla : \Delta \to Top$ be the functor which assigns to $[n]$ the affine $n$-simplex. Write $|MV| = MV \otimes_\Delta \nabla$, $|NV| = NV \otimes_\Delta \nabla$ for the geometrical realizations. $NV$ is just the nerve of the cover $\{A_x, A_y, N_2\}$. Then the natural maps $|MV| \to \mathfrak{M}_{x,2}V$ and $|NV| \to \mathfrak{M}_{I,2}V$ are homotopy equivalences. Now consider the bi-simplicial space $FV : \Delta^{op} \times \Delta^{op} \to Top$ given by

$$FV(n, m) = (\,\mathfrak{M}_x V \times (\mathfrak{M}_\emptyset V)^n \times \mathfrak{M}_\emptyset V \times (\mathfrak{M}_\emptyset V)^m \times \mathfrak{M}_y V\,)_2$$

The morphisms are defined as follows: given morphisms $i : n_1 \to n_2$ and $j : m_1 \to m_2$, $FV(i, j) : FV(n_1, m_1) \to FV(n_2, m_2)$ is induced by the maps

$$\begin{array}{l} MV(i) : (\mathfrak{M}_x \times (\mathfrak{M}_\emptyset)^{n_1} \times \mathfrak{M}_\emptyset)_k \to (\mathfrak{M}_x \times (\mathfrak{M}_\emptyset)^{n_2} \times \mathfrak{M}_\emptyset)_k \\ MV(j) : (\mathfrak{M}_\emptyset \times (\mathfrak{M}_\emptyset)^{m_1} \times \mathfrak{M}_y)_k \to (\mathfrak{M}_\emptyset \times (\mathfrak{M}_\emptyset)^{m_2} \times \mathfrak{M}_y)_k \end{array}$$

The maps (3) induce a map $\tilde{h}_V : FV \otimes_{\Delta\times\Delta} \nabla \times \nabla \to \mathfrak{M}_{I,2}V$. Taking the direct limit over all subspaces $V \subset \mathbb{H}$, we define functors $M\mathbb{H}$, $N\mathbb{H}$, $F\mathbb{H}$ and a map $h_\mathbb{H}$.

**Proposition 5.1.** *$\tilde{h}_\mathbb{H}$ is a homotopy equivalence.*

*Proof.* Let $F\mathbb{H}_m$ be the simplicial space obtained from $F\mathbb{H}$ by fixing $m$. Then

$$F\mathbb{H} \otimes_{\Delta\times\Delta} \nabla \times \nabla \cong (\,F\mathbb{H}_m \otimes_\Delta \nabla\,) \otimes_\Delta \nabla$$

The simplicial space $F\mathbb{H}_m \otimes_\Delta \nabla$ is isomorphic to the simplicial space

$$[m] \longrightarrow (\,M\mathbb{H} \otimes_\Delta \nabla\,) \times (\mathfrak{M}_\emptyset\mathbb{H})^m \times \mathfrak{M}_y\mathbb{H}$$

We have then a weak equivalence $\tilde{\eta} : F\mathbb{H}_m \otimes_\Delta \nabla \to N\mathbb{H}$. $\tilde{\eta}$ induces a map betwen the spectral sequences associated with these simplicial spaces. These spectral sequences collapse at the $E_2$ term (see [12]) and it easily follows that the induced map on geometrical realizations $|\tilde{\eta}| : |F\mathbb{H}_m \otimes \Delta\nabla| \to |N\mathbb{H}|$ is an isomorphism in homology. Since the spaces are simply-connected, $|\tilde{\eta}|$ is a homotopy equivalence. Since $\tilde{h}_\mathbb{H}$ is

the composition $|F\mathbb{H}_m \otimes \Delta\nabla| \to |N\mathbb{H}| \to \mathfrak{M}_{I,2}\mathbb{H}$ we conclude that $\tilde{h}_{\mathbb{H}}$ is a homotopy equivalence. □

Now let $d : \Delta \to \Delta \times \Delta$ be the diagonal inclusion. Then

$$(FV \circ d) \otimes_\Delta \nabla \cong FV \otimes_{\Delta\times\Delta} \nabla \times \nabla$$

We can represent the main morphisms of $FV \circ d$ with the diagram

$$\begin{tikzcd}
\mathfrak{M}_{x,2} & \mathfrak{M}_{\emptyset,2} \arrow[l, "\pi^*"'] \arrow[r, "="] & \mathfrak{M}_{\emptyset,2} & \mathfrak{M}_{\emptyset,2} \arrow[l, "="'] \arrow[r, "\pi^*"] & \mathfrak{M}_{y,2} \\
\mathfrak{M}_{x,1} \times \mathfrak{M}_{\emptyset,1} \arrow[u, "\boxplus"] \arrow[dddd, "="] & \mathfrak{M}_{\emptyset,1} \times \mathfrak{M}_{\emptyset,1} \arrow[l, "\pi^*"'] \arrow[u, "\boxplus"] \arrow[d, "="] & & \mathfrak{M}_{\emptyset,1} \times \mathfrak{M}_{\emptyset,1} \arrow[u, "\boxplus"] \arrow[d, "="] \arrow[r, "\pi^*"] & \mathfrak{M}_{\emptyset,1} \times \mathfrak{M}_{y,1} \arrow[u, "\boxplus"] \arrow[dddd, "="] \\
& \mathfrak{M}_{\emptyset,1} \times \mathfrak{M}_{\emptyset,1} \arrow[uur, "\boxplus"] \arrow[ddl, "\pi^*"'] & & \mathfrak{M}_{\emptyset,1} \times \mathfrak{M}_{\emptyset,1} \arrow[uul, "\boxplus"'] \arrow[ddr, "\pi^*"] & \\
& \mathfrak{M}_{\emptyset,1} \times \mathfrak{M}_{\emptyset,1} \arrow[u, "="] \arrow[r, "="] \arrow[d, "\pi^*"] & \mathfrak{M}_{\emptyset,1} \times \mathfrak{M}_{\emptyset,1} \arrow[uuu, "\boxplus"] \arrow[d, "\pi^*"] & \mathfrak{M}_{\emptyset,1} \times \mathfrak{M}_{\emptyset,1} \arrow[u, "="'] \arrow[l, "="'] \arrow[d, "\pi^*"] & \\
\mathfrak{M}_{x,1} \times \mathfrak{M}_{\emptyset,1} & \mathfrak{M}_{x,1} \times \mathfrak{M}_{\emptyset,1} \arrow[l, "="'] \arrow[r, "\pi^*"] & \mathfrak{M}_{x,1} \times \mathfrak{M}_{y,1} & \mathfrak{M}_{\emptyset,1} \times \mathfrak{M}_{y,1} \arrow[l, "\pi^*"'] \arrow[r, "="] & \mathfrak{M}_{\emptyset,1} \times \mathfrak{M}_{y,1}
\end{tikzcd}$$

Direct inspection of the open cover $\{A_x, A_y, N_2\}$ shows that the spaces $|FV \circ d|$ and $B(FV \circ d, \Delta, *)$ are homotopically equivalent (or we can show that we have an isomorphism in homology in the limit over $V \subset \mathbb{H}$). Hence, theorem 2 will follow from

**Proposition 5.2.** *There is a homotopy equivalence*

$$B(S_{I,\infty}, \Delta_I, *) \xrightarrow{\simeq} B(F\mathbb{H} \circ d, \Delta, *)$$

*such that the diagram*

$$\begin{tikzcd}
B(S_{I,2}, \Delta_I, *) \arrow[r, "h_{I,2}"] \arrow[d, "\simeq"] & B(T_{I,2}, \Delta_I, *) \arrow[r, "\simeq"] & B(\bar{T}_{I,2}, \bar{\Delta}_I, *) \\
B(F\mathbb{H} \circ d, \Delta, *) \arrow[rr, "\tilde{h}_{\mathbb{H}}"] & & \mathfrak{M}_{I,2} \arrow[u, "{[-1]}"']
\end{tikzcd} \tag{4}$$

*is homotopy commutative.*

*Proof.* The functors $S_{I,2}$ and $F\mathbb{H} \circ d$ are equal on objects. Given $f \in \mathscr{L}(1)$, we define $\boxplus_f : (\mathfrak{M}_I\mathbb{H} \times \mathfrak{M}_I\mathbb{H})_2 \to \mathfrak{M}_{I,2}\mathbb{H}$ by

$$m \boxplus_f m' = \mathfrak{M}f(m) \boxplus \mathfrak{M}f(m') = \mathfrak{M}f(m \boxplus m')$$

Let $\Delta_{\mathscr{L}}$ denote the category with same objects as $\Delta$ and whose morphisms are $\Delta_{\mathscr{L}}(m,n) = \Delta(m,n) \times \mathscr{L}(1)$. Let $F\mathbb{H}_{\mathscr{L}} : \Delta_{\mathscr{L}} \to \mathit{Top}$ be the functor which coincides with $F\mathbb{H} \circ d$ and $S_{I,2}$ on objects, and on morphisms is induced by $\boxplus_f$. Notice that, since the maximum degree is 2, morphisms involve at most one non-trivial product $\boxplus_f$. The canonical inclusion $\Delta \to \Delta_{\mathscr{L}}$ is an equivalence of categories

so it induces a homotopy equivalence $B(F\mathbb{H}\circ d,\Delta,*)\to B(F\mathbb{H}_{\mathscr{L}},\Delta_{\mathscr{L}},*)$. Now consider the isometry $\alpha_t\in\mathcal{L}(\mathbb{H},\mathbb{H}\oplus\mathbb{H})$ defined by

$$\alpha_t(v)=(\sin t\,v,\cos t\,v)$$

Fix $f\in\mathscr{L}(2)$ and consider the composition

$$\mathfrak{M}\mathbb{H}\times\mathfrak{M}\mathbb{H}\xrightarrow{\alpha_0\times\alpha_t}\mathfrak{M}(\mathbb{H}\oplus\mathbb{H})\times\mathfrak{M}(\mathbb{H}\oplus\mathbb{H})\xrightarrow{\boxplus}\mathfrak{M}(\mathbb{H}\oplus\mathbb{H})\xrightarrow{\mathfrak{M}f}\mathfrak{M}\mathbb{H}$$

For $t=0$ this map equals $\boxplus_{f\circ\alpha_0}$, and for $t=1$ it equals the operad action of $f$ on $\mathfrak{M}\mathbb{H}$ (see appendix A). So we have a homotopy equivalence $B(S_{I,2},\Delta_I,*)\simeq B(F\mathbb{H}_{\mathscr{L}},\Delta_{\mathscr{L}},*)$. It is then straightforward to check that diagram (4) is homotopy commutative. □

## 6. The limit when $k\to\infty$

Let $n=\#I$ and let $r=\dim V$. In [2], [9] it was shown that $\mathfrak{M}_{I,k}V$ is isomorphic to the moduli space of charge $k$ $SU(V)$ instantons over a connected sum $X_n$ of $n$ copies of $\mathbb{P}^2$, based at a point $x_\infty\in X_n$. We represent this moduli space by $\mathcal{M}_k^r(X_n)$. In [14] Taubes described, for $k'>k$ gluing maps $\mathcal{M}_k^r(X_n)\to\mathcal{M}_{k'}^r(X_n)$ and showed that in the direct limit when $k\to\infty$, we have a homotopy equivalence

$$\mathcal{M}_\infty^r(X_n)\simeq Map_0(X_n,BSU(V))$$

Let $K$ be the homotopy fiber of the map $(\mathfrak{M}_{\emptyset,\infty}V)^I\xrightarrow{\omega}\mathfrak{M}_{\emptyset,\infty}V$.

**Theorem 5.** *There is a commutative diagram*

$$\begin{tikzcd}
K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
(\mathfrak{M}_{\emptyset,\infty}V)^I \arrow[r, "\pi^*"] \arrow[d, "\omega"] & \prod_{x\in I}\mathfrak{M}_{x,\infty}V \arrow[r] \arrow[d, "\omega\circ\pi^*"] & (\Omega^2BSU(V))^I \arrow[d, equal] \\
\mathfrak{M}_{\emptyset,\infty}V \arrow[r, "\pi^*"] & \mathfrak{M}_{I,\infty}V \arrow[r] & (\Omega^2BSU(V))^I
\end{tikzcd}$$

*where the horizontal and vertical lines are fiber sequences.*

*Proof.* $X_n$ is a connected sum of $n$ copies of $X_1=\mathbb{P}^2$ so we have a cofibration $\bigvee_{n-1}S^3\to X_n$ with quotient $\bigvee X_1$. On the other hand the inclusion of the 2-skeleton gives a cofiber sequence $\bigvee S^2\to X_n\to S^4$. We have the diagram

$$\begin{tikzcd}
 & \bigvee S^3 \arrow[r, equal] \arrow[d] & \bigvee S^3 \arrow[d] \\
\bigvee S^2 \arrow[r] \arrow[d, equal] & X_n \arrow[r] \arrow[d] & S^4 \arrow[d] \\
\bigvee S^2 \arrow[r] & \bigvee X_1 \arrow[r] & \bigvee S^4
\end{tikzcd}$$

Applying the functor $Map(\cdot,BSU(V))$ we get the desired diagram of fiber sequences. □

Taking the direct limit over all subspaces $V \subset \mathbb{H}$, and applying Bott periodicity, we get $\mathfrak{M}_{I,\infty}\mathbb{H} = \varinjlim \mathfrak{M}_{I,\infty}V \simeq BU \times BU^I$.

**Theorem 3.** *The map $h_{I,\infty} : B(S_{I,\infty}, \Delta_I, *) \to B(T_{I,\infty}, \Delta_I, *)$ is a homotopy equivalence.*

*Proof.* In [13], homotopy equivalences $f_I : \mathfrak{M}_\infty(\mathbb{P}^2_I, V) \to BU \times BU^I$ were described with the following property: if we let $d : BU \to BU^I$ be the diagonal inclusion, then the following diagrams

$$\begin{tikzcd} \mathfrak{M}_{\emptyset,\infty}\mathbb{H} \arrow[r, "f_\emptyset"] \arrow[d, "\pi^*_{\emptyset I}"] & BU \arrow[d, "(\mathbf{1},d)"] \\ \mathfrak{M}_{I,\infty}\mathbb{H} \arrow[r, "f_I"] & BU \times BU^I \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{M}_{x,\infty}\mathbb{H} \arrow[r, "f_x"] \arrow[d, "\pi^*_{xI}"] & BU \times BU^x \arrow[d, "(\mathbf{1},\mathbf{1},d)"] \\ \mathfrak{M}_{I,\infty}\mathbb{H} \arrow[r, "f_I"] & BU \times BU^x \times BU^{I\backslash x} \end{tikzcd}$$

are commutative. Let $\xi_I : BU \times BU^I \to BU \times BU^I$ be the map defined by

$$\xi_I\,(\,a\,,\,(a_x)_{x\in I}\,) = (\,a\,,\,(a_x - a)_{x\in I}\,)$$

and let $* \in \Omega^2 BSU$ be the basepoint. Then we have the commutative diagrams

$$\begin{tikzcd} \mathfrak{M}_{\emptyset,\infty}\mathbb{H} \arrow[r, "\xi_\emptyset \circ f_\emptyset"] \arrow[d, "\pi^*_{\emptyset I}"] & BU \arrow[d, "(\mathbf{1},*)"] \\ \mathfrak{M}_{I,\infty}\mathbb{H} \arrow[r, "\xi_I \circ f_I"] & BU \times BU^I \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{M}_{x,\infty}\mathbb{H} \arrow[r, "\xi_x \circ f_x"] \arrow[d, "\pi^*_{xI}"] & BU \times BU^x \arrow[d, "(\mathbf{1},\mathbf{1},*)"] \\ \mathfrak{M}_{I,\infty}\mathbb{H} \arrow[r, "\xi_I \circ f_I"] & BU \times BU^x \times BU^{I\backslash x} \end{tikzcd}$$

Given an $\mathscr{L}$-space $X$ we define the functor $T_I(X) : \Delta_I \to Top$ by assigning to each $[n]$ the space $X \times X^{nI} \times X^I$, and defining morphisms just as for $T_I$. Then there are weak equivalences

$$S_{I,\infty} \simeq T_I(BU) \times BU^I \qquad T_{I,\infty} \simeq T_I(BU) \times T_I(BU^I) \circ d$$

where $d : \Delta_I \to \Delta_I \times \Delta_I$ is the diagonal. $h_{I,\infty}$ induces a natural transformation

$$(\mathbf{1}, k) : T_I(BU) \times BU^I \to T_I(BU) \times T_I(BU^I) \circ d$$

that assigns to each $[n] \in \Delta_I$ the map $k(n) : BU^I \to T_I(BU^I)(n)$ given by

$$k(n) : (a_x)_{x\in I} \mapsto (\,*\,,\,*\,,\,(b_{xy})_{x,y\in I}\,) \in BU^I \times BU^{n(I\times I)} \times BU^{I\times I}$$

where

$$b_{xy} = \begin{cases} a_x & x = y \\ * & x \neq y \end{cases}$$

Consider the category $\bar{\Delta}_I$ introduced in section 3. $T_I(X)$ can be extended to $\bar{T}_I(X) : \bar{\Delta}_I \to Top$ just as $T_I$. So we get functors

$$\bar{T}_I(BU) \times BU^I, \bar{T}_I(BU) \times \bar{T}_I(BU^I) \circ d : \bar{\Delta}_I \to Top$$

For any $f \in \bar{\Delta}_I(n, -1)$, the composition

$$BU^I \xrightarrow{k_n} BU^I \times BU^{n(I\times I)} \times BU^{I\times I} \xrightarrow{f} BU^I$$

is the identity, so we can extend the natural transformation $k$ by setting $\bar{k}(-1) = \mathbf{1} : BU^I \to BU^I$. We thus have a commutative diagram (compare with the proof of theorem 4)

$$\begin{array}{ccc}
B(T_I(BU), \Delta_I, *) \times BU^I & \xrightarrow{(\mathbf{1},k)} & B(T_I(BU) \times T_I(BU^I), \Delta_I, *) \\
\downarrow \simeq & & \downarrow \simeq \\
B(\bar{T}_I(BU), \bar{\Delta}_I, *) \times BU^I & \xrightarrow{(\mathbf{1},\bar{k})} & B(\bar{T}_I(BU) \times \bar{T}_I(BU^I), \bar{\Delta}_I, *) \\
\uparrow \simeq & & \uparrow \simeq \\
\bar{T}_I(BU)(-1) \times BU^I & = & \bar{T}_I(BU)(-1) \times \bar{T}_I(BU^I)(-1)
\end{array}$$

which shows that $(1, k)$ is a homotopy equivalence. It follows that $h_{I,\infty}$ is a homotopy equivalence. □

## Appendix A. Monads

Fix $I = \{x, y\} \in \mathfrak{C}$. In this appendix we describe the open cover $\{A_x, A_y, N_2\}$ of $\mathfrak{M}_{I,2}V$, the spaces $N'_x$, $N'_y$, and the map $\boxplus : N'_x \times N'_y \to N_2$. The maps $\pi^*_{x,I}, \pi^*_{y,I}$ are open embeddings (see [12]) and we let $A_x = Im(\pi^*_{x,I})$, $A_y = Im(\pi^*_{y,I}$. To describe the other spaces we need the monad description of the moduli spaces, introduced in [4], [6], which we briefly review here.

Let $W_0, W_1$ be rank $k$ complex vector spaces. Let $\mathcal{R}$ be the space of 4-tuples $(a_1, a_2, b, c)$ where $a_i \in \mathrm{End}(W_1)$, $b \in \mathrm{Hom}(V, W_1)$ and $c \in \mathrm{Hom}(W_1, V)$, obeying the integrability condition $[a_1, a_2] + bc = 0$. Let $\mathcal{R}'$ be the space of 5-tuples $(a'_1, a'_2, d', b', c')$ where $a'_i \in \mathrm{Hom}(W_1, W_0)$, $d' \in \mathrm{Hom}(W_0, W_1)$, $b' \in \mathrm{Hom}(V, W_0)$ and $c' \in \mathrm{Hom}(W_1, V)$, such that $a'_1(W_1) + a'_2(W_1) + b'(V) = W_0$, obeying the integrability condition $a_1da_2 - a_2da_1 + bc = 0$. $Gl(W_1)$ and $Gl(W_0) \times Gl(W_1)$ act by composition on $\mathcal{R}$ and $\mathcal{R}'$ respectivelly. A 4-tuple $(a_1, a_2, b, c)$ is called nondegenerate if, for any subspace $U_1 \subset W_1$,

$$\mathrm{Im}\, b \subset U_1 \text{ and } a_i(U_1) \subset U_1 \;(i = 1, 2) \implies U_1 = W_1$$
$$U_1 \subset \mathrm{Kerc} \text{ and } a_i(U_1) \subset U_1 \;(i = 1, 2) \implies U_1 = \emptyset$$

A 5-tuple $(a'_1, a'_2, d', b', c')$ is called nondegenerate if, for any subspaces $U_0 \subset W_0$ and $U_1 \subset W_1$ such that $\dim U_0 = \dim U_1$ we have

$$\mathrm{Im}\, b' \subset U_0 \text{ and } d'(U_0) \subset U_1 \text{ and } a'_i(U_1) \subset U_0 \;(i = 1, 2) \implies U_i = W_i \;(i = 1, 2)$$
$$U_1 \subset \mathrm{Kerc}' \text{ and } d'(U_0) \subset U_1 \text{ and } a'_i(U_1) \subset U_1 \;(i = 1, 2) \implies U_i = \emptyset \;(i = 1, 2)$$

Let $\mathcal{R}_{reg}$, $\mathcal{R}'_{reg}$ denote the subspaces of nondegenerate configurations.

**Theorem** (Donaldson [4], King [6]). *The actions of $Gl(W_1)$ and $Gl(W_0) \times Gl(W_1)$ on $\mathcal{R}_{reg}$, $\mathcal{R}'_{reg}$ are free and we have isomorphisms*

$$\mathcal{R}_{reg}/Gl(W_1) \cong \mathfrak{M}_{\emptyset,k}V \qquad \mathcal{R}'_{reg}/(\,Gl(W_0) \times Gl(W_1)\,) \cong \mathfrak{M}_{x,k}V$$

*The algebraic quotients $\mathcal{R}//Gl(W_1)$, $\mathcal{R}'//Gl(W_0) \times Gl(W_1)$ are isomorphic to the Donaldson-Uhlenbeck completions $\bar{\mathfrak{M}}_{\emptyset,k}V$, $\bar{\mathfrak{M}}_{x,k}V$ of $\mathfrak{M}_{\emptyset,k}V$ and $\mathfrak{M}_{x,k}V$ respectivelly.*

Given an isometry $\alpha \in \mathcal{L}(V_1, V_2)$, we get a dual map $\alpha^* : V_2^* \cong V_2 \to V_1^* \cong V_1$. The map $\mathfrak{M}\alpha : \mathfrak{M}V_1 \to \mathfrak{M}V_2$ is induced by $b \mapsto b \circ \alpha^*$, $c \mapsto \alpha \circ c$. Whitney sum $\omega : \mathfrak{M}_{\emptyset,k}V \times \mathfrak{M}_{\emptyset,k'}V' \to \mathfrak{M}_{k+k'}(V \oplus V')$ is induced by direct sum: let $a_i \in \mathrm{End}(W_1)$, $b \in \mathrm{Hom}(V, W_1)$, $c \in \mathrm{Hom}(W_1, V)$, and let $a_i' \in \mathrm{End}(W_1')$, $b' \in \mathrm{Hom}(V', W_1')$, $c' \in \mathrm{Hom}(W_1', V')$. Then

$$\omega([a_1, a_2, b, c], [a_1', a_2', b', c']) = [a_1 \oplus a_1', a_2 \oplus a_2', b \oplus b', c \oplus c']$$

Corresponding results hold for Whitney sum on $\mathfrak{M}_x, \mathfrak{M}_y$. Now write $x = (x_1, x_2) \in \mathbb{C}^2$ and fix an isomorphism $d : W_0 \to W_1$. Then the map $\pi^*_{\emptyset,x} : \mathfrak{M}_{\emptyset,k}V \to \mathfrak{M}_{x,k}V$ is given by

$$\pi^*_{\emptyset,x}(\,[a_1, a_2, b, c]\,) = [d^{-1}(a_1 - x_1\mathbf{1}), d^{-1}(a_2 - x_2\mathbf{1}), d, d^{-1}b, c]$$

Fix $\delta > 0$ and let

$$\begin{aligned} N_x' &= \{[a_1', a_2', d', b', c'] \in \mathfrak{M}_{x,1}V \;:\; |d'a_1'| < \delta\} \\ N_\emptyset'(x) &= \{[a_1, a_2, b, c] \in \mathfrak{M}_{\emptyset,1}V\,;\, |a_1 - x_1| < \delta \end{aligned}$$

**Proposition A.1.** *There are homeomorphisms $N_\emptyset' \cong \mathfrak{M}_{\emptyset,1}V$, $N_x' \cong \mathfrak{M}_{x,1}V$.*

*Proof.* Let $M_\emptyset$ be the space of orthogonal vectors $u, v \in V$ modulo the action of $\mathbb{C}^*$ given by $g(u, v) = (gu, g^{-1}v)$ and let $M_x$ be the space of orthogonal vectors $(u, v) \in V$ modulo the action of $\mathbb{C}^* \times \mathbb{C}^*$ given by $(g_0, g_1)(u, v) = (g_0u, g_1^{-1}v)$. Then $\mathfrak{M}_{\emptyset,1}V = \mathbb{C}^2 \times M_\emptyset$ and the projection $\mathfrak{M}_{x,1} \to M_x$ given by $[a_1', a_2', d', b', c'] \to [b', c']$ is a fibre bundle with fiber $\mathbb{C}^3$. The result easily follows. $\square$

Now consider the map $\boxplus_\emptyset : N_\emptyset'(x) \times N_\emptyset'(y) \to \mathfrak{M}_{\emptyset,2}V$ given by

$$\begin{aligned} &[a_{1x}, a_{2x}, b_x, c_x] \boxplus_\emptyset [a_{1y}, a_{2y}, b_y, c_y] \\ &\qquad = \left[ \begin{pmatrix} a_{1x} & 0 \\ 0 & a_{1y} \end{pmatrix}, \begin{pmatrix} a_{2x} & \frac{b_x c_y}{a_{1y} - a_{1x}} \\ \frac{b_y c_x}{a_{1x} - a_{1y}} & a_{2y} \end{pmatrix}, \begin{pmatrix} b_x \\ b_y \end{pmatrix}, \begin{pmatrix} c_x & c_y \end{pmatrix} \right] \end{aligned}$$

and the map $\boxplus_x : N_x' \times N_\emptyset'(z) \to \mathfrak{M}_{x,2}V$ given by

$$\begin{aligned} &[a_1', a_2', d', b', c'] \boxplus_x [a_{1z}, a_{2z}, b_z, c_z] \\ &\qquad = \left[ \begin{pmatrix} a_1' & 0 \\ 0 & a_1'' \end{pmatrix}, \begin{pmatrix} a_2' & \frac{b' c''}{d'' a_1'' - d' a_1'} \\ \frac{b'' c'}{d' a_1' - d'' a_1''} & a_2'' \end{pmatrix}, \begin{pmatrix} d' & 0 \\ 0 & d'' \end{pmatrix} \begin{pmatrix} b' \\ b'' \end{pmatrix}, \begin{pmatrix} c' & c'' \end{pmatrix} \right] \end{aligned}$$

where $[a_1'', a_2'', d'', b'', c''] = \pi^*_{\emptyset,y}[a_{1y}, a_{2y}, b_y, c_y]$. Now, straightforward computations show that

**Proposition A.2.**

(1) *There is a commutative diagram*

$$\begin{tikzcd} N_\emptyset' \times N_\emptyset' \arrow[r, "\boxplus_\emptyset"] \arrow[d, "\pi^*_{\emptyset,x} \times \mathbf{1}"] & \mathfrak{M}_{\emptyset,2} \arrow[d, "\pi^*_{\emptyset,x}"] \\ N_x' \times N_\emptyset' \arrow[r, "\boxplus_x"] & \mathfrak{M}_{x,2} \end{tikzcd}$$

(2) *Let $\alpha \in \mathcal{L}(V, V')$. Then we have a commutative diagram*

$$\begin{tikzcd} \mathfrak{M}V \times \mathfrak{M}V \arrow[r, "\boxplus"] \arrow[d, "\mathfrak{M}\alpha \times \mathfrak{M}\alpha"] & \mathfrak{M}V \arrow[d, "\mathfrak{M}\alpha"] \\ \mathfrak{M}V' \times \mathfrak{M}V' \arrow[r, "\boxplus"] & \mathfrak{M}V' \end{tikzcd}$$

(3) *Let $\imath : V \to V \oplus V'$, $\imath' : V' \to V \oplus V'$ be the canonical inclusions. Then $\omega$ equals the composition*

$$\mathfrak{M}V \times \mathfrak{M}V' \xrightarrow{\imath \times \imath'} \mathfrak{M}(V \oplus V') \times \mathfrak{M}(V \oplus V') \xrightarrow{\boxplus} \mathfrak{M}(V \oplus V')$$

The maps $\boxplus_x, \boxplus_y, \boxplus_\emptyset$ are open embeddings (see [12], proposition 4.5). Let $C = \mathfrak{M}_{I,2}V \setminus (A_x \cup A_y)$. We define $N_2 = \pi^*_{x,I} Im(\boxplus_x) \cup \pi^*_{y,I} Im(\boxplus_y) \cup C$.

**Proposition A.3.** *There is a homeomorphism $\boxplus : N'_x \times N'_y \to N_2$ which extends $\boxplus_x, \boxplus_y$.*

*Proof.* The proof follows the same lines as the proof of proposition 4.9 in [12]. We sketch the proof here, refering to [12] for more details. Let $\pi_{x,I} : \mathbb{P}^2_I \to \mathbb{P}^2_x$, $\pi_{\emptyset,y} : \mathbb{P}^2_y \to \mathbb{P}^2$ be the blowup at $y$, and let $\pi_{y,I} : \mathbb{P}^2_I \to \mathbb{P}^2_y$, $\pi_{\emptyset,x} : \mathbb{P}^2_x \to \mathbb{P}^2$ be the blowup at $x$. Taking the direct image of the bundles we get maps

$$(\pi_{x,I})_* : \mathfrak{M}_{I,2}V \to \bar{\mathfrak{M}}_{x,2}V \qquad (\pi_{\emptyset,y})_* : \mathfrak{M}_{y,1}V \to \bar{\mathfrak{M}}_{\emptyset,1}V$$
$$(\pi_{y,I})_* : \mathfrak{M}_{I,2}V \to \bar{\mathfrak{M}}_{y,2}V \qquad (\pi_{\emptyset,x})_* : \mathfrak{M}_{x,1}V \to \bar{\mathfrak{M}}_{\emptyset,1}V$$

where the bar denotes the Donaldson-Uhlenbeck completion of the moduli space. Given $m_x \in N'_x$, $m_y \in N'_y$, we define $m_x \boxplus m_y$ as the only solution of the system of equations

$$(\pi_{x,I})_*(m_x \boxplus m_y) = m_x \boxplus_x (\pi_{\emptyset,y})_* m_y$$
$$(\pi_{y,I})_*(m_x \boxplus m_y) = (\pi_{\emptyset,x})_* m_x \boxplus_y m_y$$

To check existance and uniqueness we let $S_0 N'_x = N'_x \setminus \pi^*_{\emptyset,x} N'_\emptyset$, $S_0 N'_y = N'_y \setminus \pi^*_{\emptyset,y} N'_\emptyset$ and observe that

$$N'_x \times N'_y = (N'_x \times \pi^*_{\emptyset,y} N'_\emptyset(y)) \cup (\pi^*_{\emptyset,x} N'_\emptyset(x) \times N'_y) \cup (S_0 N'_x \times S_0 N'_y)$$

Outside $S_0 N'_x \times S_0 N'_y$ we can solve the equations and get

$$m_x \boxplus \pi^*_{\emptyset,y} m = \pi^*_{x,I}(m_x \boxplus_x m) \qquad \pi^*_{\emptyset,x} m \boxplus m_y = \pi^*_{y,I}(m \boxplus_y m_y)$$

If $(m_x, m_y) \in S_0 N'_x \times S_0 N'_y$ then $(\pi_{x,I})_*(m_x \boxplus m_y) \in \bar{\mathfrak{M}}_{x,2}V$ is the ideal instanton determined by $x \in \mathfrak{M}_{x,1}V$ and a delta at $y$, and $(\pi_{y,I})_*(m_x \boxplus m_y) \in \bar{\mathfrak{M}}_{y,2}$ is the ideal instanton determined by $y \in \mathfrak{M}_{y,1}V$ and a delta at $x$. By proposition 4.3 in [12], this completely determines $m_x \boxplus m_y$. Continuity is proved exactly as in [12], proposition 4.9. □

Finally we observe that the results of proposition A.2 extend to $\boxplus$ by continuity since $N'_\emptyset$ is dense in $N'_x$.

*Aknowledgements.* The author would like to thank Gustavo Granja for several fruitful discussions.

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