Stratified Models in First-Order Logic

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Abstract

We introduce *stratification* in first-order logic; then we discuss *soundness*, *conservativeness* and *completeness.*

0. Introduction.

The various nature of the mathematical objects in what concerns their complexity, our knowledge of them or the possibility to make them explicit (for example, infinitesimal or ilimited real numbers) is a strong motivation to consider their distribution into *levels* or *strata.* The *stratification* depends on the selected property (or properties) of the mathematical objects that are the subject-matter of our study.

1. Stratified Models.

Along this article $\mathcal{L}$ is a first-order language with equality, no constant symbols, no function symbols and the logical symbols:

parentheses	$), ($;
variables	$v_0, v_1, \ldots, v_n, \ldots$;
0-*ary connective*	$\perp$ (*falsity*, *falsum*, *absurdum*) ;
binary connective	$\longrightarrow$ (*implication*) ;
universal quantifier	$\forall$.

The basic definitions and conventions of $\mathcal{L}$ are as usual (see [2]); in particular, $\neg\varphi$, $\varphi \vee \psi$, $\varphi \wedge \psi$ and $(\exists v_i)\, \varphi(v_i)$ abbreviate, respectively, $\varphi \longrightarrow \perp$, $\neg\varphi \longrightarrow \psi$, $\neg(\neg\varphi \vee \neg\psi)$ and $\neg(\forall v_i)\, \neg\varphi(v_i)$.

$TERM(\mathcal{L}')$, $TERM_C(\mathcal{L}')$, $ATOM(\mathcal{L}')$, $FORM(\mathcal{L}')$, $Sent(\mathcal{L}')$, $At(\mathcal{L}')$ denote, respectively, the classes of the terms, the closed terms, the atomic formulae, the formulae, the sentences and the atomic sentences of whatever first-order language $\mathcal{L}'$ we are using.

Definition 1.1. Let P be a set and $\leq$ a total, dense, preordering relation on P.

$p < q$ and $p =_{\leq} q$ abbreviate, respectively:

$p \leq q \,\underline{\wedge}\, \underline{\neg}\; q \leq p$ and $p \leq q \,\underline{\wedge}\, q \leq p$ [1], for each $p, q \in P$.

[1] Underlined connectives, quantifiers or equality (meaning identity) belong to the metalanguage.

P has a $\leq -$ minimal element $0_{\leq}$ and no $\leq -$ maximal element; more explicitely:

$\left(\forall p \in P\right)\left(p \leq 0_{\leq} \longrightarrow p =_{\leq} 0_{\leq}\right)$,

$\neg\left(\exists q \in P\right)\left(\forall p \in P\right)\left(q \leq p \longrightarrow p =_{\leq} q\right)$.

Consider a class valued function D defined on P such that, for each $p \in P$, $D\left(p\right)$ is a non-empty class.

For $P, \leq, D, 0_{\leq}$ as described above, an informal sequence $\mathcal{F} := \left(P, \leq, D, 0_{\leq}\right)$ is called a *stratifying frame*.

The elements of P are the *nodes* of $\mathcal{F}$ and for each $p \in P$, $D\left(p\right)$ is the *domain* of $\mathcal{F}$ at p .

Definition 1.2. Let $\mathcal{F} = \left(P, \leq, D, 0_{\leq}\right)$ be a stratifying frame**.**

To each $a \in D\left(p\right)$ we associate a constant symbol $\bar{a}$ (different constant symbols for different elements of $D\left(p\right)$). If $a \in D\left(p\right)$ and $a \in D\left(q\right)$, then the constant symbol associated with a is the same.

$\mathcal{L}_*$, $\left(\mathcal{L}_*\right)_p$(for each $p \in P$) and $\left(\mathcal{L}_*\right)_+$ are first-order extensions of $\mathcal{L}$ defined as:

$\mathcal{L}_* := \mathcal{L} \cup \{\mathbf{0}, \sqsubseteq\}$, where $\mathbf{0}$ is a constant symbol and $\sqsubseteq$ is a *new* binary relation symbol called *precedence of level*;

$\left(\mathcal{L}_*\right)_p := \mathcal{L}_* \cup \left\{\bar{a} : a \in D\left(p\right)\right\}$;

$$\left(\mathcal{L}_*\right)_+ := \bigcup_{p \in P} \left(\mathcal{L}_*\right)_p .$$

$\mathcal{L}_*$ is the *stratifying language* associated with $\mathcal{L}$.

So, the class of all closed terms of $\left(\mathcal{L}_*\right)_+$ is:

$$TERM_C\left(\left(\mathcal{L}_*\right)_+\right) = \left\{\bar{a} : \left(\exists p \in P\right) a \in D\left(p\right)\right\} \cup \{\mathbf{0}\} .$$

For any terms $t_1, t_2 \in TERM\left(\left(\mathcal{L}_*\right)_+\right)$:

$$t_1 \sqsubset t_2 \text{ abbreviates } t_1 \sqsubseteq t_2 \wedge \neg\, t_2 \sqsubseteq t_1 .$$

($t_1 \sqsubseteq t_2$ and $t_1 \sqsubset t_2$ must be read, respectively, as "t_1 *precedes* t_2" and "t_1 *strictly precedes* t_2").

Consider a function $\mathbf{V}$ defined on $TERM_C\left(\left(\mathcal{L}_*\right)_+\right)$ and with values in P such that $\mathbf{V}\left(\mathbf{0}\right) = 0_{\leq}$, $a \in D\left(\mathbf{V}\left(\bar{a}\right)\right)$ and for each $p \in P$:

$a \in D\left(p\right) \longrightarrow \mathbf{V}\left(\bar{a}\right) \leq p \wedge \left(\forall q \in P\right)\left(q < \mathbf{V}\left(\bar{a}\right) \longrightarrow \neg\, a \in D\left(q\right)\right)$.

$\mathbf{V}\left(\bar{a}\right)$ must be thought as *"the"* [2] *first level of interpretation* of $\bar{a}$.

[2] The definite article refers to the binary relation $=_{\leq}$.

Consider a function Σ defined on P such that, for each $p \in P$, $\Sigma(p) \subseteq At\left((\mathcal{L}_*)_p\right)$, where $At\left((\mathcal{L}_*)_p\right)$ is the class of all atomic sentences of $(\mathcal{L}_*)_p$.

$\Sigma(p)$ establish, for each $p \in P$, the "basic truths" at p .

D, Σ and $\mathbf{V}$ satisfy the following conditions:

1) $p \leq q \underline{\longrightarrow} D(p) \subseteq D(q)$,

2) $\left(\underline{\forall} p \in P\right) \underline{\neg} \perp \in \Sigma(p)$,

3) $p \leq q \underline{\longrightarrow} \Sigma(p) \subseteq \Sigma(q)$,

4) $\bar{a}_1 = \bar{a}_2 \in \Sigma(p) \underline{\longleftrightarrow} \mathbf{V}(\bar{a}_1) \leq p \underline{\wedge} \mathbf{V}(\bar{a}_2) \leq p \underline{\wedge} a_1 \underline{=} a_2$,

5) $\bar{a}_1 \sqsubseteq \bar{a}_2 \in \Sigma(p) \underline{\longleftrightarrow} \mathbf{V}(\bar{a}_1) \leq \mathbf{V}(\bar{a}_2) \leq p$,

6) If R_i is a n_i-ary relation symbol of $\mathcal{L}$, $a_1, \ldots, a_{n_i}, b_1, \ldots, b_{n_i} \in D(p)$ and $a_1 \underline{=} b_1, \ldots, a_{n_i} \underline{=} b_{n_i}$, then:

$R_i\left(\bar{a}_1, \cdots, \bar{a}_{n_i}\right) \in \Sigma(p) \underline{\longrightarrow} R\left(\bar{b}_1, \cdots, \bar{b}_{n_i}\right) \in \Sigma(p)$.

(Evidently, if R_i is a n_i-ary relation symbol of $\mathcal{L}$ and $R_i\left(\bar{a}_1, \cdots, \bar{a}_{n_i}\right) \in \Sigma(p)$, then: $\mathbf{V}(\bar{a}_1) \leq p \underline{\wedge} \cdots \underline{\wedge} \mathbf{V}(\bar{a}_{n_i}) \leq p$, since $\Sigma(p) \subseteq At\left((\mathcal{L}_*)_p\right)$).

Definition 1.3. For $P, \leq, D, \Sigma, \mathbf{V}, 0_{\leq}$ as described in **definition 1.1** and **definition 1.2**, let $\mathcal{F} = (P, \leq, D, 0_{\leq})$ be a stratifying frame.

A *stratified model* for $\mathcal{L}_*$ is an informal sequence $\mathcal{S}_* \underline{:=} (\mathcal{F}, \Sigma, \mathbf{V}, 0)$ such that

1) $0 \in D(0_{\leq})$,

2) $\bar{0}$ is identified with $\mathbf{0}$.

The *nodes* of $\mathcal{S}_*$ and the *domain* of $\mathcal{S}_*$ at each $p \in P$ are those of $\mathcal{F}$.

Remark 1.4. If $\mathcal{S}_* \underline{=} (\mathcal{F}, \Sigma, \mathbf{V}, 0)$ is a stratified model for $\mathcal{L}_*$, then it is easy to prove that P is an infinite set and, at each $p \in P$, D and Σ determine a classical structure (see [1]) $\mathfrak{A}_p$ such that $|\mathfrak{A}_p| \underline{=} D(p)$ and:

i) $p \leq q \underline{\longrightarrow} |\mathfrak{A}_p| \subseteq |\mathfrak{A}_q|$.

ii) The interpretations $R_i^{\mathfrak{A}_p}$ of a n_i-ary relation symbol R_i of $\mathcal{L}$ and $\sqsubseteq^{\mathfrak{A}_p}$ of $\sqsubseteq$ are:

$\langle a_1, \cdots, a_{n_i} \rangle \in R_i^{\mathfrak{A}_p} \underline{:\longleftrightarrow} R_i\left(\bar{a}_1, \cdots, \bar{a}_{n_i}\right) \in \Sigma(p)$,

$\langle a_1, a_2 \rangle \in \sqsubseteq^{\mathfrak{A}_p} \underline{:\longleftrightarrow} \bar{a}_1 \sqsubseteq \bar{a}_2 \in \Sigma(p)$.

So, we have

$p \leq q \underline{\longrightarrow} R_i^{\mathfrak{A}_p} \subseteq R_i^{\mathfrak{A}_q} \underline{\wedge} \sqsubseteq^{\mathfrak{A}_p} \subseteq \sqsubseteq^{\mathfrak{A}_p}$.

iii) $\left(\underline{\forall} p \in P\right)\left(\underline{\forall} a \in D(p)\right) \bar{a}^{\mathfrak{A}_p} \underline{:=} a$; in particular, since $\mathbf{0}$ is $\bar{0}$:

iv) $\left(\underline{\forall} p \in P\right) \mathbf{0}^{\mathfrak{A}_p} \underline{:=} 0$.

Proposition 1.5. Let $\mathcal{S}_* = (\mathcal{F}, \Sigma, \mathbf{V}, 0)$ be a stratified model for $\mathcal{L}_*$.

1) If $\bar{a}_1, \cdots, \bar{a}_n \in TERM_C\left((\mathcal{L}_*)_+\right)$, then there is a $p \in P$ such that:

$(*)\ \ a_1, \cdots, a_n \in D(p) \underline{\wedge} \Big(\underline{\forall} q \in P\Big)\Big(a_1, \cdots, a_n \in D(q) \underline{\longrightarrow} p \leq q\Big)$.

2) If $\bar{a}_1, \cdots, \bar{a}_n \in TERM_C\left((\mathcal{L}_*)_+\right)$, then there is a $p \in P$ such that:

$\mathbf{V}(\bar{a}_1) \leq p \underline{\wedge} \cdots \underline{\wedge} \mathbf{V}(\bar{a}_n) \leq p \underline{\wedge} \Big(\underline{\forall} q \in P\Big)\Big(\mathbf{V}(\bar{a}_1) \leq q \underline{\wedge} \cdots \underline{\wedge} \mathbf{V}(\bar{a}_n) \leq q \underline{\longrightarrow} p \leq q\Big)$.

3) If $p_1, p_2 \in P$ satisfy $(*)$, then $p_1 =_{\leq} p_2$.

Proof.

1) We simply choose p as "the" greatest of all the $\mathbf{V}(\bar{a}_1), \cdots, \mathbf{V}(\bar{a}_n)$.

2) is a reformulation of **1)** and **3)** is a direct consequence of $(*)$. □

Definition 1.6. If $\mathcal{S}_* = (\mathcal{F}, \Sigma, \mathbf{V}, 0)$ is a stratified model for $\mathcal{L}_*$ and $\bar{a}_1, \cdots, \bar{a}_n \in TERM_C\left((\mathcal{L}_*)_+\right)$, we define $\mathbf{V}(\bar{a}_1, \cdots, \bar{a}_n)$ as "the" p , unique modulo $=_{\leq}$, satisfying $(*)$ in **proposition 1.5.**

$\big(\mathbf{V}(\bar{a}_1, \cdots, \bar{a}_n)$ is "the" *first level of interpretation of all the* $\bar{a}_1, \cdots, \bar{a}_n\big)$.

2. Stratified Semantics

Proposition 2.1. Let $\mathcal{S}_* = (\mathcal{F}, \Sigma, \mathbf{V}, 0)$ be a stratified model for $\mathcal{L}_*$. Then there exists a unique function Σ^*, defined on P, such that for each $p \in P$, $\Sigma(p) \subseteq \Sigma^*(p) \subseteq Sent\Big((\mathcal{L}_*)_p\Big)$ and

i) $\varphi \in \Big(At\Big((\mathcal{L}_*)_p\Big)\Big) \underline{\wedge}\, \underline{\neg}\, \varphi \in \Sigma(p) \underline{\longrightarrow} \underline{\neg}\, \varphi \in \Sigma^*(p)$,

ii) $\varphi \longrightarrow \psi \in \Sigma^*(p) \underline{\longleftrightarrow} \Big(\varphi \in \Sigma^*(p) \underline{\longrightarrow} \psi \in \Sigma^*(p)\Big)$,

iii) $(\forall v_i)\, \varphi(v_i) \in \Sigma^*(p) \underline{\longleftrightarrow} \Big(\underline{\forall} a \in D(p)\Big)\, \varphi(\bar{a}) \in \Sigma^*(p)$.

Proof. We simply define $\varphi \in \Sigma^*(p)$, for each $p \in P$, by induction on φ . □

Notation [3]**.** We write $p \Vdash \varphi$ for $\varphi \in \Sigma^*(p)$ (read "*p forces* φ").

So, we have, for each $p \in P$:

i) $p \Vdash \bar{a}_1 = \bar{a}_2 \underline{\longleftrightarrow} \mathbf{V}(\bar{a}_1) \leq p \underline{\wedge} \mathbf{V}(\bar{a}_2) \leq p \underline{\wedge} a_1 \underline{=} a_2$.

ii) $p \Vdash \bar{a}_1 \sqsubseteq \bar{a}_2 \underline{\longleftrightarrow} \mathbf{V}(\bar{a}_1) \leq \mathbf{V}(\bar{a}_2) \leq p$.

iii) $\underline{\neg}\, p \Vdash \bot$.

iv) $p \Vdash \varphi \longrightarrow \psi \underline{\longleftrightarrow} \Big(p \Vdash \varphi \underline{\longrightarrow} p \Vdash \psi\Big)$.

v) $p \Vdash (\forall v_i)\, \varphi(v_i) \underline{\longleftrightarrow} \Big(\underline{\forall} a \in D(p)\Big)\, p \Vdash \varphi(\bar{a})$.

[3] For a *modal view* of *forcing*, see [6] .

Corollary 2.2.
1) $p \Vdash \neg\varphi \longleftrightarrow \neg\, p \Vdash \varphi$.
2) $p \Vdash \neg\neg\varphi \longleftrightarrow p \Vdash \varphi$.
3) $p \Vdash \varphi \vee \psi \longleftrightarrow p \Vdash \varphi \vee p \Vdash \psi$.
4) $p \Vdash \varphi \wedge \psi \longleftrightarrow p \Vdash \varphi \wedge p \Vdash \psi$.
5) $p \Vdash (\exists v_i)\, \varphi(v_i) \longleftrightarrow \Big(\exists a \in D(p)\Big)\, p \Vdash \varphi(\bar{a})$.
6) $\mathbf{V}(\bar{a}_1) \Vdash \bar{a}_1 \sqsubseteq \bar{a}_2 \longleftrightarrow \mathbf{V}(\bar{a}_1) =_{\le} \mathbf{V}(\bar{a}_2)$.
7) $\mathbf{V}(\bar{a}_2) \Vdash \bar{a}_1 \sqsubseteq \bar{a}_2 \longleftrightarrow \mathbf{V}(\bar{a}_1) \le \mathbf{V}(\bar{a}_2)$.
8) $p \Vdash \bar{a}_1 \sqsubset \bar{a}_2 \longleftrightarrow \mathbf{V}(\bar{a}_1) < \mathbf{V}(\bar{a}_2) \le p$.

Proof. Quite straightforward. □

Definition 2.3. If $\mathcal{S}_* = (\mathcal{F}, \Sigma, \mathbf{V}, 0)$ is a stratified model for $\mathcal{L}_*$ and $\varphi \in FORM(\mathcal{L}_*)$ [4], we define:
$p \Vdash \varphi :\longleftrightarrow p \Vdash \mathcal{C}l(\varphi)$, where $\mathcal{C}l(\varphi)$ is the *universal closure* of φ .

Proposition 2.4. Let $\mathcal{S}_* = (\mathcal{F}, \Sigma, \mathbf{V}, 0)$ be a stratified model for $\mathcal{L}_*$. Then, if v_i and v_j are different variables of $\mathcal{L}$:
1) $p \Vdash v_i \sqsubseteq v_i$, for each $p \in P$.
2) $p \Vdash v_i \sqsubseteq v_j :\longleftrightarrow \Big(\forall a, b \in D(p)\Big)\, \mathbf{V}(\bar{a}) =_{\le} \mathbf{V}\big(\bar{b}\big)$.
3) $p \Vdash v_i \sqsubseteq \mathbf{0} :\longleftrightarrow \Big(\forall a \in D(p)\Big)\, \mathbf{V}(\bar{a}) =_{\le} 0_{\le}$.
4) $p \Vdash \mathbf{0} \sqsubseteq v_i$, for each $p \in P$.

Proof. Quite straightforward. □

Definition 2.5. Let $\mathcal{F} = (P, \le, D, 0_{\le})$ be a stratifying frame.

The classes of the *elementary progressive* and the *elementary regressive* sentences of $(\mathcal{L}_*)_+$, denoted, respectively, by $Prg_0\big((\mathcal{L}_*)_+\big)$ and $Rgr_0\big((\mathcal{L}_*)_+\big)$, are defined inductively:
P1) If $\varphi \in At\big((\mathcal{L}_*)_+\big)$, then $\varphi \in Prg_0\big((\mathcal{L}_*)_+\big)$,
P2) If $\varphi_1, \varphi_2 \in Prg_0\big((\mathcal{L}_*)_+\big)$, then $\varphi_1 \wedge \varphi_2, \varphi_1 \vee \varphi_2 \in Prg_0\big((\mathcal{L}_*)_+\big)$,
R1) $\perp \in Rgr_0\big((\mathcal{L}_*)_+\big)$ and if φ is an atomic sentence of $(\mathcal{L}_*)_+$, different from $\perp$, then $\neg\varphi \in Rgr_0\big((\mathcal{L}_*)_+\big)$,
R2) If $\varphi_1, \varphi_2 \in Rgr_0\big((\mathcal{L}_*)_+\big)$, then $\varphi_1 \wedge \varphi_2, \varphi_1 \vee \varphi_2 \in Rgr_0\big((\mathcal{L}_*)_+\big)$,

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[4] $\Big(\forall p \in P\Big) \Big(Sent(\mathcal{L}_*) \subseteq Sent\big((\mathcal{L}_*)_p\big) \wedge FORM(\mathcal{L}_*) \subseteq FORM\big((\mathcal{L}_*)_p\big)\Big)$.

PR) If $\varphi_1 \in Prg_0\left((\mathcal{L}_*)_+\right)$ and $\varphi_2 \in Rgr_0\left((\mathcal{L}_*)_+\right)$, then $\varphi_1 \longrightarrow \varphi_2 \in Rgr_0\left((\mathcal{L}_*)_+\right)$,

RP) If $\varphi_1 \in Rgr_0\left((\mathcal{L}_*)_+\right)$ and $\varphi_2 \in Prg_0\left((\mathcal{L}_*)_+\right)$, then $\varphi_1 \longrightarrow \varphi_2 \in Prg_0\left((\mathcal{L}_*)_+\right)$.

We may now define the classes of the *extended elementary progressive* and the *extended elementary regressive* sentences of $(\mathcal{L}_*)_+$, denoted, respectively, by $Prg\left((\mathcal{L}_*)_+\right)$ and $Rgr\left((\mathcal{L}_*)_+\right)$:

Pi) If $\varphi \in Prg_0\left((\mathcal{L}_*)_+\right)$, then $\varphi \in Prg\left((\mathcal{L}_*)_+\right)$,

Pii) If $\varphi(v_i) \in FORM\left((\mathcal{L}_*)_+\right)$ is such that for each $p \in P$ and $a \in D(p)$, $\varphi(\bar{a}) \in Prg_0\left((\mathcal{L}_*)_+\right)$, then $(\exists v_i)\,\varphi(v_i) \in Prg\left((\mathcal{L}_*)_+\right)$.

Ri) If $\varphi \in Rgr_0\left((\mathcal{L}_*)_+\right)$, then $\varphi \in Rgr\left((\mathcal{L}_*)_+\right)$,

Rii) If $\varphi(v_i) \in FORM\left((\mathcal{L}_*)_+\right)$ is such that for each $p \in P$ and $a \in D(p)$, $\varphi(\bar{a}) \in Rgr_0\left((\mathcal{L}_*)_+\right)$, then $(\forall v_i)\,\varphi(v_i) \in Rgr\left((\mathcal{L}_*)_+\right)$.

Proposition 2.6 (Weak Monotonicity of $\Vdash$).

Let $\mathcal{S}_* = (\mathcal{F}, \Sigma, \mathbf{V}, 0)$ be a stratified model for $\mathcal{L}_*$.

1) If $\varphi \in Prg_0(\mathcal{L}_*)$, then for each $p, q \in P$:

$$p \leq q \underset{-}{\longrightarrow} \left(p \Vdash \varphi \underset{-}{\longrightarrow} q \Vdash \varphi\right).$$

2) If $\varphi \in Rgr_0(\mathcal{L}_*)$, then for each $p, q \in P$:

$$p \leq q \underset{-}{\longrightarrow} \left(q \Vdash \varphi \underset{-}{\longrightarrow} p \Vdash \varphi\right).$$

3) If $\varphi \in Prg(\mathcal{L}_*)$ or $\varphi \in Rgr(\mathcal{L}_*)$, then we also have weak monotonicity of $\Vdash$ as in **1)** or **2)**, respectively.

Proof.

1), 2) We proceed by induction on φ.

Consider $p \leq q$.

If $\varphi \in At\left((\mathcal{L}_*)_+\right) \cap Prg_0(\mathcal{L}_*)$, then **1)** is immediate by **definition 1.2, 3)** and **proposition 2.1**.

Of course, **2)** is satisfied by $\bot$ and if $\varphi \in At\left((\mathcal{L}_*)_+\right) \setminus \{\bot\}$, then the proof of **2)** for $\neg\varphi$ is immediate by **definition 1.2, 3)** and **proposition 2.1**.

If φ is $\varphi_1 \wedge \varphi_2$, where $\varphi_1, \varphi_2 \in Prg_0\left((\mathcal{L}_*)_+\right)$, then:

$$p \Vdash \varphi_1 \wedge \varphi_2 \underset{-}{\longleftrightarrow} p \Vdash \varphi_1 \underset{-}{\wedge} p \Vdash \varphi_2 \quad \underset{\substack{-\\ Ind.\\ Hyp.}}{\longrightarrow} \quad q \Vdash \varphi_1 \underset{-}{\wedge} q \Vdash \varphi_2 \underset{-}{\longleftrightarrow} q \Vdash \varphi_1 \wedge \varphi_2 .$$

If φ is $\varphi_1 \vee \varphi_2$, where $\varphi_1, \varphi_2 \in Prg_0\left((\mathcal{L}_*)_+\right)$, then:

$$p \Vdash \varphi_1 \vee \varphi_2 \underset{-}{\longleftrightarrow} p \Vdash \varphi_1 \underset{-}{\vee} p \Vdash \varphi_2 \quad \underset{\substack{-\\ Ind.\\ Hyp.}}{\longrightarrow} \quad q \Vdash \varphi_1 \underset{-}{\vee} q \Vdash \varphi_2 \underset{-}{\longleftrightarrow} q \Vdash \varphi_1 \vee \varphi_2 .$$

If φ is $\varphi_1 \vee \varphi_2$ or φ is $\varphi_1 \wedge \varphi_2$, where $\varphi_1, \varphi_2 \in Rgr_0\left((\mathcal{L}_*)_+\right)$, then we proceed in a similar manner to prove **2).**

If φ is $\varphi_1 \longrightarrow \varphi_2$, where $\varphi_1 \in Rgr_0\left((\mathcal{L}_*)_+\right)$ and $\varphi_2 \in Prg_0\left((\mathcal{L}_*)_+\right)$, then:
$p \Vdash \varphi_1 \longrightarrow \varphi_2 \underset{-}{\wedge} q \Vdash \varphi_1 \underset{-}{\longrightarrow} p \Vdash \varphi_1 \longrightarrow \varphi_2 \underset{-}{\wedge} p \Vdash \varphi_1 \underset{-}{\longrightarrow} p \Vdash \varphi_2 \underset{-}{\longrightarrow} q \Vdash \varphi_2$.

So:

$p \Vdash \varphi \underset{-}{\longrightarrow} q \Vdash \varphi$.

Similarly, we prove **2)** for $\varphi_1 \longrightarrow \varphi_2$, where $\varphi_1 \in Prg_0\left((\mathcal{L}_*)_+\right)$ and $\varphi_2 \in Rgr_0\left((\mathcal{L}_*)_+\right)$.

3) Consider $p \leq q$.

If $\varphi \in Prg_0\left((\mathcal{L}_*)_+\right)$, then we simply use **1)**.

If φ is $(\exists v_i)\,\varphi_1(v_i)$, where $\varphi_1(v_i) \in FORM\left((\mathcal{L}_*)_+\right)$ and for each $r \in P$ and $a \in D(r)$, $\varphi_1(\bar{a}) \in Prg_0\left((\mathcal{L}_*)_+\right)$, then:

$$p \Vdash (\exists v_i)\,\varphi_1(v_i) \underset{-}{\longleftrightarrow} \left(\underset{-}{\exists} a \in D(p)\right) p \Vdash \varphi_1(\bar{a}) \underset{\substack{\mathbf{1)}\\ D(p) \subseteq D(q)}}{\underset{-}{\longrightarrow}} \left(\underset{-}{\exists} a \in D(q)\right) q \Vdash \varphi_1(\bar{a}) \underset{-}{\longleftrightarrow}$$

$$\underset{-}{\longleftrightarrow} q \Vdash (\exists v_i)\,\varphi_1(v_i) .$$

If $\varphi \in Rgr\left((\mathcal{L}_*)_+\right)$ we proceed *mutatis mutandis* in the same manner. $\square$

Definition 2.7.

1) If $\varphi \in Sent(\mathcal{L}_*)$ and $\mathcal{S}_* \underset{-}{=} (\mathcal{F}, \Sigma, \mathbf{V}, 0)$ is a stratified model for $\mathcal{L}_*$, we define $\mathcal{S}_* \Vdash \varphi$ (read "$\mathcal{S}_*$ *forces* φ" or "$\mathcal{S}_*$ *is a stratified model of* φ") as:

$$\mathcal{S}_* \Vdash \varphi :\underset{-}{\longleftrightarrow} \left(\underset{-}{\forall} p \in P\right) p \Vdash \varphi .$$

We also define $\Vdash \varphi$ (read "φ *is universally valid*" or "φ *is valid*") as:

$\Vdash \varphi :\underset{-}{\longleftrightarrow} \left(\underset{-}{\forall} \mathcal{S}_*\right) \mathcal{S}_* \Vdash \varphi$, where the possible values of the metavariable $\mathcal{S}_*$ are all stratified models for $\mathcal{L}_*$.

If Σ is a subset of $Sent(\mathcal{L}_*)$, we define $\mathcal{S}_* \Vdash \Sigma$ (read "$\mathcal{S}_*$ *forces* Σ" or "$\mathcal{S}_*$ *is a stratified model of* Σ") as:

$$\mathcal{S}_* \Vdash \Sigma :\underset{-}{\longleftrightarrow} \left(\underset{-}{\forall} \varphi \in \Sigma\right) \mathcal{S}_* \Vdash \varphi .$$

2) If $\Gamma \cup \{\varphi\} \subseteq Sent(\mathcal{L}_*)$, we define $\Gamma \Vdash \varphi$ (read "φ *is a stratified logical consequence of* Γ") as:

$$\Gamma \Vdash \varphi :\underset{-}{\longleftrightarrow} \left(\underset{-}{\forall} \mathcal{S}_*\right) (\mathcal{S}_* \Vdash \Gamma \underset{-}{\longrightarrow} \mathcal{S}_* \Vdash \varphi).$$

If $\Gamma \cup \{\varphi\} \subseteq FORM(\mathcal{L}_*)$ and $FV(\Gamma \cup \{\varphi\}) = \{v_{i_1}, \ldots, v_{i_n}\}$, where $FV(\Gamma \cup \{\varphi\})$ is the set of all free variables of $\Gamma \cup \{\varphi\}$, we define $\Gamma \Vdash \varphi$ (read "φ *is a stratified logical consequence of* Γ") as:

$\Gamma \Vdash \varphi :\underset{-}{\longleftrightarrow} \left(\underset{-}{\forall} \mathcal{S}_*\right) \left(\underset{-}{\forall} p \in P\right) \left(\underset{-}{\forall} a_1, \cdots, a_n \in D(p)\right) (p \Vdash \Gamma\left(\overrightarrow{\bar{a}}\right) \underset{-}{\longrightarrow} p \Vdash \varphi\left(\overrightarrow{\bar{a}}\right))$,

where $\overrightarrow{\bar{a}} \underset{-}{=} \langle \bar{a}_1, \cdots, \bar{a}_n \rangle$, and $p \Vdash \Gamma\left(\overrightarrow{\bar{a}}\right)$ abbreviates $\left(\underset{-}{\forall} \psi \in \Gamma\right) p \Vdash \psi\left(\overrightarrow{\bar{a}}\right)$.

3. Soundness, Conservativeness and Completeness.

Before we investigate *soundness*, let us exhibit the *rules of inference* we are using in a context of *natural deduction* (using the language $\mathcal{L}_*$) and with the usual notion of *derivation* [5]*:*

Introduction Rules

1) $(\longrightarrow I)$ $\dfrac{\begin{matrix}[\varphi]^{\,6} \\ \vdots \\ \psi\end{matrix}}{\varphi \longrightarrow \psi}$ **2)** $(\forall I)$ $\dfrac{\varphi}{(\forall v_i)\,\varphi}$

Elimination Rules

3) $(\longrightarrow E)$ $\dfrac{\varphi \quad \varphi \longrightarrow \psi}{\psi}$ **4)** $(\bot)$ $\dfrac{\bot}{\varphi}$

5) (RAA) $\dfrac{\begin{matrix}[\neg\varphi] \\ \vdots \\ \bot\end{matrix}}{\varphi}$ **6)** $(\forall E)$ $\dfrac{(\forall v_i)\,\varphi}{\varphi\,[t/v_i]}$

The axioms for $=$ and $\sqsubseteq$ are also presented as rules of inference:

Equality Rules

7) (ER_1) $\dfrac{}{v_i = v_i}$ **8)** ER_2 $\dfrac{}{v_i = v_j \longrightarrow v_j = v_i}$ **9)** (ER_3) $\dfrac{}{v_i = v_j \wedge v_j = v_k \longrightarrow v_i = v_k}$

10) (ER_4) For each n_i-ary relation symbol R_i of $\mathcal{L}$:

$$\dfrac{}{v_{j_1} = v_{k_1} \wedge \ldots \wedge v_{j_{n_i}} = v_{k_{n_i}} \wedge R_i\left(v_{j_1}, \ldots, v_{j_{n_i}}\right) \longrightarrow R_i\left(v_{k_1}, \ldots, v_{k_{n_i}}\right)}$$

11) (ER_5) $\dfrac{}{v_{i_1} = v_{j_1} \wedge v_{i_2} = v_{j_2} \wedge v_{i_1} \sqsubseteq v_{i_2} \longrightarrow v_{j_1} \sqsubseteq v_{j_2}}$

Precedence of Level Rules

12) (PLR_1) $\dfrac{}{v_i \sqsubseteq v_i}$ **13)** (PLR_2) $\dfrac{}{v_i \sqsubseteq v_j \wedge v_j \sqsubseteq v_k \longrightarrow v_i \sqsubseteq v_k}$

14) (PLR_3) $\dfrac{}{v_i \sqsubseteq v_j \vee v_j \sqsubseteq v_i}$ **15)** (PLR_4) $\dfrac{}{\mathbf{0} \sqsubseteq v_i}$

Restrictions. In **2)**, v_i does not occur free in any hypothesis on which φ depends and in **6)**, $t \in TERM\,(\mathcal{L}_*)$ is free for v_i in φ.

[5] Remember that, in particular, derivations are finite trees of formulae and Γ may have superfluous hypotheses.

[6] Hypotheses enclosed in square brackets are cancelable.

Theorem 3.1 (Soundness)**.** If $\Gamma \cup \{\varphi\} \subseteq FORM\left(\mathcal{L}_*\right)$, then:

$$\Gamma \vdash \varphi \underset{-}{\longrightarrow} \Gamma \Vdash \varphi \, .$$

Proof. We use induction on the derivation $\mathcal{D}$ of φ from Γ. During the proof, we use an arbitrary stratified model $\mathcal{S}_*$.

If $\varphi \in \Gamma$, then we certainly have, with $FV\left(\Gamma \cup \{\varphi\}\right) = \{v_{i_1}, \ldots, v_{i_n}\}$:

$\left(\underset{-}{\forall} p \in P\right)\left(\underset{-}{\forall} a_1, \cdots, a_n \in D\left(p\right)\right)\left(p \Vdash \Gamma\left(\overrightarrow{\overline{a}}\right) \underset{-}{\longrightarrow} p \Vdash \varphi\left(\overrightarrow{\overline{a}}\right)\right).$

If the end of $\mathcal{D}$ is an application of a derivation rule, then we must examine the rules of inference one by one. This is a quite uncomplicated, albeit tedious, task; for instance:

$(\forall I) : FV\left(\Gamma\right) = \{v_{i_1}, \cdots, v_{i_n}\}$ and $v_i \notin \{v_{i_1}, \cdots, v_{i_n}\}$.

Induction hypothesis:

$\left(\underset{-}{\forall} p \in P\right)\left(\underset{-}{\forall} a_1, \cdots, a_n, b \in D\left(p\right)\right)\left(p \Vdash \Gamma\left(\overrightarrow{\overline{a}}\right) \underset{-}{\longrightarrow} p \Vdash \varphi\left(\overrightarrow{\overline{a}}, \bar{b}\right)\right).$

Then:

$\left(\underset{-}{\forall} p \in P\right)\left(\underset{-}{\forall} a_1, \cdots, a_n \in D\left(p\right)\right)\left(p \Vdash \Gamma\left(\overrightarrow{\overline{a}}\right) \underset{-}{\longrightarrow} \left(\underset{-}{\forall} b \in D\left(p\right)\right) p \Vdash \varphi\left(\overrightarrow{\overline{a}}, \bar{b}\right)\right).$

Finally:

$\left(\underset{-}{\forall} p \in P\right)\left(\underset{-}{\forall} a_1, \cdots, a_n \in D\left(p\right)\right)\left(p \Vdash \Gamma\left(\overrightarrow{\overline{a}}\right) \underset{-}{\longrightarrow} p \Vdash \left(\left(\forall v_i\right) \varphi\right)\left(\overrightarrow{\overline{a}}\right)\right).$ □

Theorem 3.2. If $\Gamma \cup \{\varphi\} \subseteq FORM\left(\mathcal{L}\right)$, then:

1) (Conservativeness)

$$\Gamma \vdash_{cl} \varphi \underset{-}{\longleftrightarrow} \Gamma \vdash \varphi,$$

where $\Gamma \vdash_{cl} \varphi$ abbreviates "φ is derived from Γ, in the usual way, using the rules of inference **1)** to **10)**".

2) (Semantic Extension)

$$\Gamma \models \varphi \underset{-}{\longrightarrow} \Gamma \Vdash \varphi \, .$$

If φ is a classical logical consequence of Γ then φ is also a stratified logical consequence of Γ.

Proof.

1) Immediate**,** because the rules of inference **1)** to **10)** and the notion of derivation are the same in classical and in stratified first-order logic.

2) By classical completeness and soundness:

$$\Gamma \models \varphi \underset{-}{\longleftrightarrow} \Gamma \vdash_{cl} \varphi \, .$$

By stratified soundness:

$$\Gamma \vdash \varphi \underset{-}{\longrightarrow} \Gamma \Vdash \varphi \, .$$

Finally, using **1)**:

$$\Gamma \models \varphi \underset{-}{\longrightarrow} \Gamma \Vdash \varphi \ . \ \square$$

Definition 3.3.
$$\mathcal{P}_c\left(Sent\left(\mathcal{L}_*\right)\right) \underset{-}{:=} \left\{ \Sigma \subseteq Sent\left(\mathcal{L}_*\right) : \Sigma \text{ has a classical model} \underset{-}{\longrightarrow} \Sigma \text{ has a stratified model} \right\}.$$

Theorem 3.4 (Completeness). If Γ is a consistent [7] subset of $Sent\left(\mathcal{L}_*\right)$ and $\Gamma \cup \{\neg\varphi\} \in \mathcal{P}_c\left(Sent\left(\mathcal{L}_*\right)\right)$, then:

$$\Gamma \Vdash \varphi \underset{-}{\longrightarrow} \Gamma \vdash \varphi \ .$$

Proof. Let $\Gamma \Vdash \varphi$.
If $\underset{-}{\neg}\ \Gamma \vdash \varphi$, then:

$$\Gamma \cup \{\neg\varphi\} \ \text{ is consistent.}$$

So, $\Gamma \cup \{\neg\varphi\}$ has a classical model.
Since $\Gamma \cup \{\neg\varphi\} \in \mathcal{P}_c\left(Sent\left(\mathcal{L}_*\right)\right)$, $\Gamma \cup \{\neg\varphi\}$ has a stratified model, $\mathcal{S}_*$.
But then, $\mathcal{S}_*$ is a stratified model of $\{\varphi, \neg\varphi\}$ and that is impossible.
We conclude that:

$$\Gamma \vdash \varphi \ . \ \square$$

Conclusion.

The *stratification* presented in this work may be applied to any theory (in the usual, informal sense, of this word) formalizable in a first-order language like $\mathcal{L}$. So, we may stratify ZFC [8] or even such theories as **Nelson internal set theory**, IST, or **Hrbaček set theory**, HST, that are largely used in *nonstandard analysis* (see [5]).

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[7] Here as elsewhere in this work, "consistent" has the usual meaning in first-order logic.

[8] For a different approach, see [3] , [4] .