Theta functions, geometric quantization and unitary Schottky bundles

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Abstract. We study geometric quantization of moduli spaces of vector bundles on an algebraic curve X, and its relation to theta functions. In limits when the complex structure of X degenerates, we describe vector spaces of distributions with Verlinde dimension, associated to the quantization of some of these moduli spaces in real polarizations. We show that special bases on these spaces, defined using a trinion decomposition of X, are related by modular transformation matrices for characters of affine Lie algebras, which appear naturally in the Blattner-Kostant-Sternberg (BKS) pairing for different quantizations of the moduli space.

Contents

1. Introduction

In this article, we will relate two different bases for the quantization of the moduli space of flat connections on a compact Riemann surface X, from a trinion decomposition of X. One of these constructions relies on geometric quantization of the moduli space of flat connections on X in a real polarization and the consideration of the corresponding Bohr-Sommerfeld points, as studied in [**We, JW**] for the

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gauge group $SU(2)$. The other comes from considering spin networks on the graph associated to the given trinion decomposition. Theta functions on the moduli space are then obtained from these bases by an analytic procedure based on the coherent state transform for Lie groups. (See also [**FMN3**].)

Given a trinion decomposition of X, one obtains two different real polarizations of the moduli space for which the corresponding Blattner-Kostant-Sternberg geometric quantization pairing will be given by a discrete analogue of the Fourier transform. The space of unitary Schottky bundles on X, which embeds in the moduli space as a Lagrangian submanifold, plays a special role in this construction: it is a fiber of one of the real polarizations mentioned above, while it serves as a base space for the other.

These constructions can be carried out in detail for the cases of classical abelian theta functions and for the case of $SL(n,\mathbb{C})$ theta functions of genus 1, which were studied in [**FMN1**] and [**FMN2**] from an analytic viewpoint.

In section two, we review some aspects of vector bundles and theta functions associated with a compact Riemann surface X. In the following section, we describe the general framework of geometric quantization in a real polarization and the role of the Bohr-Sommerfeld fibers. In section 4, we describe real polarizations of the moduli space associated to a trinion decomposition of X and, in section 5, we explain the construction of the spin network basis for the quantization of the moduli space. This basis will be related to holomorphic theta functions in section 6, using the analytic machinery of the coherent state transform. In section 7, we show how the different bases for the quantization of the moduli space in real polarizations are related by the BKS pairing in geometric quantization, which is realized by the modular transformation matrix S familiar from the theory of characters of affine Lie algebras.

2. Vector bundles and theta functions

Consider the principally polarized abelian variety

$$M_\Omega = V/\Lambda_\Omega = \mathbb{C}^g/(\mathbb{Z}^g \oplus \Omega\mathbb{Z}^g), \tag{2.1}$$

where Ω belongs to the Siegel upper half space $\mathbb{H}_g$ consisting of complex symmetric $g\times g$ matrices with positive definite imaginary part. The polarization of M_Ω defines a line bundle $L \to M_\Omega$, so that $H^0(M_\Omega, L^k)$ is the space of theta functions of level k. Using the natural projection

$$p : (\mathbb{C}^*)^g \to M_\Omega = (\mathbb{C}^*)^g/\ \Omega\mathbb{Z}^g\ ,$$

theta functions of level k on M_Ω can be identified with holomorphic functions on $(\mathbb{C}^*)^g$ satisfying well known quasi-periodicity conditions. These form a vector space $\mathcal{H}_{\Omega,k}$, of dimension k^g, for which an explicit basis is [**BL**]

$$\theta_l(z,\Omega) = \sum_{n\in\mathbb{Z}^g} e^{\pi i(l+kn)\cdot\frac{\Omega}{k}(l+kn)} e^{2\pi i(l+kn)\cdot z}, \qquad l \in (\mathbb{Z}/k\mathbb{Z})^g\ . \tag{2.2}$$

In the important case when M_Ω is the Jacobian $J(X)$ of a compact Riemann surface X of genus g and period matrix Ω, L is defined by the theta divisor which is supported on the set

$$\Theta_{\mathsf{m}} = \{\mathsf{n} \in J(X) : H^0(X, \mathsf{n}\otimes\mathsf{m}) \neq 0\}.$$

Here we are identifying $M_\Omega = J(X)$ with the moduli space of holomorphic line bundles n of degree 0 on X, and fixing a line bundle m of degree $g-1$ (see [**BL**]).

This theory is extended to the non-abelian setting by considering a reductive Lie group $G_\mathbb{C}$ and by replacing the Jacobian (which is also the moduli space of $\mathbb{C}^* = GL(1,\mathbb{C})$ principal bundles over X with fixed degree) by a suitable moduli space of principal $G_\mathbb{C}$-bundles over X. In the case $G_\mathbb{C} = SL(n,\mathbb{C})$, which is our main example, one deals with the moduli space of semistable vector bundles of rank n and trivial determinant over X, $\mathcal{M}_n = \mathcal{M}_n(X)$. This is a projective variety with singularities, except in the cases of genus 1, where $\mathcal{M}_n \cong \mathbb{P}^{n-1}$, and the case $g = n = 2$, for which it is $\mathbb{P}^3$ [**NS, NR**]. Choosing a line bundle $\mathsf{m} \in J^{g-1}(X)$ as before, the set

$$\Theta_\mathsf{m} = \left\{E \in \mathcal{M}_n : H^0(X, E \otimes \mathsf{m}) \neq 0\right\}$$

can be proved to be a divisor on $\mathcal{M}_n$, whose associated line bundle $L \to \mathcal{M}_n$ is independent of m and generates the Picard group of $\mathcal{M}_n$ [**DN**]. Therefore, one naturally defines a non-abelian theta function of level k as a section of L^k,

$$\theta \in H^0(\mathcal{M}_n, L^k).$$

Of course, the non-abelian case is substantially more difficult than the classical one. For instance, since $\mathcal{M}_n$ is simply connected, one can no longer realize L as a quotient of some free action as in the abelian case. Concerning the Kähler quantization of $\mathcal{M}_n$ one obtains a bundle over Teichmüller space with fiber at X given by $H^0(\mathcal{M}_n(X), L^k)$. It is still an open problem whether the projectively flat connection naturally defined on this bundle [**AdPW, Hi**] is unitary.

The Verlinde formula [**Ve**], which computes the dimension of the space of non-abelian theta functions $H^0(\mathcal{M}_n, L^k)$, in terms of g, n and k, is by now well established from the mathematical point of view [**So**]. Also, for some particular values of these integers, relations have been found between abelian and non-abelian theta functions. (See [**Be**] for a review on some of these questions.)

A particularly interesting case is the moduli space $\mathcal{M}_n(X_\tau)$ of semistable vector bundles over an elliptic curve $X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, with $\tau \in \mathbb{H}_1$. In this case $\mathcal{M}_n(X_\tau)$ can also be described as the quotient of the (non-principally polarized) abelian variety $X_\tau \otimes \check{\Lambda}_R$, where $\check{\Lambda}_R$ is the coroot lattice of $Lie(SU(n))$, by the action of the Weyl group W (see [**L, FMN2**])

$$\mathcal{M}_n(X_\tau) = (X_\tau \otimes \check{\Lambda}_R)/W \cong \mathbb{P}^{n-1}. \tag{2.3}$$

3. Geometric quantization in a real polarization

Let us recall the general notion of a real polarization and of Bohr-Sommerfeld fibers. Consider a symplectic manifold M of dimension $2m$, with symplectic form $\omega \in H^2(M,\mathbb{Z})$, and let $L \to M$ be a Hermitian line bundle with compatible connection ∇, with curvature $F_\nabla = 2\pi i\omega$. A real polarization for the data (M, ω, L, ∇) (usually called prequantization data) is a surjective map $\pi : M \to B$ onto a manifold B of dimension m such that $\omega_{|\pi^{-1}(b)} = 0$, for all $b \in B$. Then, for a generic b, the fiber $M_b = \pi^{-1}(b)$ will be a Lagrangian submanifold and the restriction of L to M_b has a connection with zero curvature. The holonomy of this connection gives rise to a homomorphism

$$H_b : \pi_1(M_b) \to U(1).$$

For a natural number k, called the level, one defines $b \in B$ to be a k-Bohr-Sommerfeld (k-BS) point and $\pi^{-1}(b)$ to be a k-Bohr-Sommerfeld fiber if $(H_b)^k = 1$.

Let $B_{k-\mathrm{BS}} \subset B$ be the set of Bohr-Sommerfeld points and $M_{k-\mathrm{BS}} = \pi^{-1}(B_{k-\mathrm{BS}}) \subset M$ the subspace of Bohr-Sommerfeld fibers. The quantum Hilbert space for this system at level k is then provided by the vector space $\mathcal{S}_\pi(M_{k-\mathrm{BS}})$ of covariantly constant sections of the restriction of L^k to $M_{k-\mathrm{BS}}$, with its natural Hilbert space structure [**Sn**]. (For the sake of simplicity, we will not consider the metaplectic correction in this paper.) In the case when M is compact, $B_{k-\mathrm{BS}}$ is just a finite set of points and one obtains a natural identification

$$\mathcal{S}_\pi(M_{k-\mathrm{BS}}) \cong \bigoplus_{b \in B_{k-\mathrm{BS}}} \mathbb{C} \cdot s_b, \tag{3.1}$$

where for each point $b \in B_{k-\mathrm{BS}}$, s_b is a choice of nonzero covariantly constant section of the restriction of L^k to $\pi^{-1}(b)$.

As an example, consider the prequantum data $(M_\Omega, \omega, L, \nabla)$ of our abelian variety M_Ω (2.1), with symplectic form

$$\omega = \sum_{j=1}^{g} d\eta_j \wedge d\xi_j,$$

where $\eta_1, ..., \eta_j, \xi_1, ..., \xi_j$ are the periodic coordinates on V dual to the generators of the lattice Λ_Ω; they are related to the complex ones by $z = x + iy = \eta + \Omega\xi$. Above, ∇ is the Chern connection on the theta line bundle L. Then, the canonical projection

$$\begin{array}{rcl} \pi : M_\Omega & \rightarrow & B^{U(1)} := (\mathbb{R}/\mathbb{Z})^g \cong U(1)^g \\ z & \mapsto & \eta \end{array} \tag{3.2}$$

defines a real polarization, such that a connection 1-form on L^k restricted to the fiber M_η over η is $\alpha = k \sum_{j=1}^{g} \eta_j d\xi_j$. It is then easy to see that the holonomy map H_η is trivial if and only if $k\eta \in \mathbb{Z}^g$, so that the k-Bohr-Sommerfeld points are the points of order dividing k on the real torus

$$B^{U(1)}_{k-\mathrm{BS}} := \left(\frac{1}{k} \mathbb{Z}_k \right)^g \quad \subset (\mathbb{R}/\mathbb{Z})^g,$$

where $\mathbb{Z}_k$ denotes the ring of integers mod k.

We now define the *real polarization Hilbert space* for $(M_\Omega, k\omega, L^k, k\nabla)$ to be the following vector space of distributions on $U(1)^g$,

$$V^{U(1)}_{k,g} := \bigoplus_{\frac{l}{k} \in B^{U(1)}_{k-\mathrm{BS}}} \mathbb{C} \cdot \delta_l \quad \subset \quad C^\infty((\mathbb{R}/\mathbb{Z})^g)', \tag{3.3}$$

where $\delta_l(\eta) := \delta(\eta - \frac{l}{k})$ denotes the delta distribution supported on the Bohr-Sommerfeld point $\frac{l}{k} \in B^{U(1)}_{k-\mathrm{BS}}$, $l \in (\mathbb{Z}_k)^g$. We define a Hermitian structure on $V^{U(1)}_{k,g}$, so that the set $\{\delta_l,\ l \in (\mathbb{Z}_k)^g\}$ is orthogonal and, for reasons that will be apparent later on (see [**FMN1**]),

$$||\delta_l||^2 = k^{-g}. \tag{3.4}$$

Clearly, the dimension of $V^{U(1)}_{k,g}$ coincides with the Verlinde number in this situation, k^g, which is the dimension of the space of classical level k theta functions $H^0(M_\Omega, L^k)$. Note that we are simultaneously defining the real polarization Hilbert space and a particular choice of basis of this space.

As a particular instance of this example, for an elliptic curve X_τ, one defines a real polarization of the abelian variety $X_\tau \otimes \check{\Lambda}_R$ via

$$\begin{array}{rcl} \pi : X_\tau \otimes \check{\Lambda}_R & \to & B^{SU(n)} := (\mathbb{R}/\mathbb{Z}) \otimes \check{\Lambda}_R \cong (\mathbb{R}/\mathbb{Z})^{n-1} \\ z & \mapsto & \eta. \end{array} \tag{3.5}$$

Taking the quotient by the Weyl group gives a (singular) real polarization of the moduli space of semistable vector bundles $\mathcal{M}_n(X_\tau)$ in (2.3). This gives rise to the level k quantum Hilbert space (which corresponds to the quantization of the abelian variety $X_\tau \otimes \check{\Lambda}_R$ at shifted level $k+n$, see [**FMN2**]),

$$V_{k,1}^{SU(n)} := \bigoplus_{\gamma \in B^{SU(n)}_{(k+n)-\mathrm{BS}}} \mathbb{C} \cdot \delta_\gamma \quad \subset \quad C^\infty(SU(n))', \tag{3.6}$$

with

$$\delta_\gamma(\eta) := \delta(\eta - \tfrac{\gamma+\rho}{k+n}), \tag{3.7}$$

where ρ is the Weyl vector given by half the sum of the positive roots of $sl(n,\mathbb{C})$ and γ is the highest weight corresponding to an integrable representation of the affine Kac-Moody algebra $\widehat{sl(n,\mathbb{C})}_k$. That is, we have $\gamma \in D_k = \{\lambda \in \Lambda_W^+ | \langle \lambda, \hat{\alpha} \rangle \leq k\}$, where $\hat{\alpha}$ is the highest root of $sl(n,\mathbb{C})$, and $\Lambda_W^+ \subset \Lambda_W$ denotes the set of positive weights inside the weight lattice. (See [**FMN2, FMN3**].) Note that the distributions δ_γ are defined on the quotient of the maximal torus of $SU(n)$, $T := (\mathbb{R}/\mathbb{Z}) \otimes \check{\Lambda}_R$, by the Weyl group W,

$$T/W \cong SU(n)/SU(n),$$

(where $SU(n)$ acts on itself by conjugation), and extended to $SU(n)$ by Ad invariance. Again, one defines on $V_{k,1}^{SU(n)}$ a Hermitian structure for which the δ_γ are orthogonal and which will be made more precise in section 5. Also note that the dimension of $V_{k,1}^{SU(n)}$ is the Verlinde number for this case giving $\dim H^0(\mathcal{M}_n(X_\tau), L^k) = |D_k| = \binom{n+k-1}{k}$.

In good cases in the geometric quantization of symplectic manifolds, one can relate Hilbert spaces associated to two different polarizations via the so called Blattner-Kostant-Sternberg (BKS) pairing. This pairing defines a BKS map, whose unitarity is an important issue and has been established in a few interesting examples [**Ra, Wo, Ha2, FMMN**].

In the simplest case of the cotangent bundle of $\mathbb{R}^m$, $T^*\mathbb{R}^m \cong \mathbb{R}^{2m} = \{(q_i, p_i)\}_{i=1,\dots,m}$ equipped with the canonical symplectic structure, $\omega = \sum_i dp_i \wedge dq_i$, consider the two real polarizations π_p and π_q given by projection onto the p or q coordinates. Then, the corresponding BKS pairing between functions in each Hilbert space, $f \in L^2(\mathbb{R}^m, dp)$ and $g \in L^2(\mathbb{R}^m, dq)$ is just given by [**Wo**]

$$\langle f, g \rangle_{\mathrm{BKS}} = \int_{\mathbb{R}^{2m}} \overline{f(p)} g(q) e^{2\pi i p \cdot q} dq dp.$$

It is immediate to observe that the corresponding BKS map

$$\mathcal{B} : L^2(\mathbb{R}^m, dp) \to L^2(\mathbb{R}^m, dq)$$

is just the usual unitary Fourier transform.

4. Trinion decompositions and Schottky bundles

In this section, we use trinion decompositions of a Riemann surface, to describe real polarizations for some of the moduli spaces described above.

Let X be a compact Riemann surface of genus $g > 0$, and let us fix a symplectic basis of $\pi_1(X)$, i.e, an ordered set $(\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g)$ of $2g$ generators of $\pi_1(X)$ subject to the single relation

$$\prod_{j=1}^{n} [\alpha_j, \beta_j] = 1,$$

and such that the intersection pairing in homology is given by

$$(\alpha_j, \alpha_k) = (\beta_j, \beta_k) = 0, \qquad (\alpha_j, \beta_k) = -(\beta_j, \alpha_k) = \delta_{jk}. \tag{4.1}$$

This data will be called a marked Riemann surface. In what follows, we will freely identify elements of $\pi_1(X)$ with representative (oriented) curves in X.

A trinion decomposition of X is an ordered set $\gamma = (\gamma_1, ..., \gamma_E)$, where the γ_i form a maximal set of simple closed non-intersecting pairwise non-homotopic oriented curves on X. The word trinion refers to a connected region of the surface $X \backslash \{\gamma_1, ..., \gamma_E\}$ which is homeomorphic to a three-holed sphere when $g \geq 2$. When $g = 1$, we will still define a trinion decomposition which is degenerate, since the number of curves is $E = 1$ and the degenerate "trinion" $X \backslash \{\gamma\}$ is a two-holed sphere.

To every trinion decomposition $\gamma = (\gamma_1, ..., \gamma_E)$ one associates a connected (oriented) graph $\Gamma = \Gamma_\gamma$, so that to each trinion corresponds to a vertex, and to each γ_i that bounds two trinions, there corresponds an edge joining the two corresponding vertices. Γ_γ is trivalent when $g \geq 2$ and bivalent in the trivial and degenerate case $g = 1$.

Clearly, the number of edges and vertices in any such graph are, respectively,

$$E = \begin{cases} 3g-3, & g \geq 2 \\ 1, & g = 1 \end{cases}, \qquad\qquad V = \begin{cases} 2g-2, & g \geq 2 \\ 1, & g = 1. \end{cases}$$

We remark that the fundamental group of Γ is free on g generators,

$$\pi_1(\Gamma) \cong \overbrace{\mathbb{Z} * \cdots * \mathbb{Z}}^{g}.$$

Given a trinion decomposition, one obtains a 3 dimensional handlebody Y, whose boundary is X and where the γ_i are homotopically trivial. This defines the homomorphism π_γ and the short exact sequence of groups,

$$1 \to \ker(\pi_\gamma) \to \pi_1(X) \stackrel{\pi_\gamma}{\to} \pi_1(\Gamma) \cong \pi_1(Y) \to 1. \tag{4.2}$$

We will define γ to be a decomposition of type α if

$$\alpha_1, ..., \alpha_g \in \ker(\pi_\gamma),$$

and similarly define a type β decomposition. If γ is a decomposition of type α, then it is clear that the images of β_j under π_γ (also denoted by β_j for simplicity) generate the fundamental group of Γ. Conversely, if the β_j generate $\pi_1(\Gamma) \cong \pi_1(Y)$ then γ is a decomposition of type α.

We now describe a real polarization of $J(X)$, the Jacobian variety of X, associated with a trinion decomposition of type α. This is in fact equivalent to the one described above for M_Ω, when Ω is the period matrix of X. By standard results, we can make the identifications

$$\mathcal{M}(U(1)) = J(X) \cong Hom(\pi_1(X), U(1)).$$

Then the map

$$\begin{array}{rcl} \pi_\alpha : J(X) & \to & B_\alpha^{J(X)} := Hom(\ker \pi_\gamma, U(1)) \cong U(1)^g \\ \rho & \mapsto & (\rho(\alpha_1), ..., \rho(\alpha_g)) \end{array}$$

is a real polarization and the Borh-Sommerfeld fibers of level k are the points whose order divides k. Note that there is only one Bohr-Sommerfeld fiber which is a $k-$BS fiber for all $k \in \mathbb{N}$; it is

$$\pi_\alpha^{-1}(1) = Hom(\pi_1(\Gamma), U(1)).$$

This is called the space of unitary Schottky line bundles with trivial holonomy along the α cycles.

We remark that, $Hom(\pi_1(\Gamma), U(1)) \cong U(1)^g$ has two other important manifestations. By reversing the roles of α and β, this fiber is naturally the base of a real polarization corresponding to a trinion decomposition of type β

$$\begin{array}{rcl} \pi_\beta : J(X) & \to & B_\beta^{J(X)} \cong Hom(\pi_1(\Gamma, U(1))) \\ \rho & \mapsto & (\rho(\beta_1), ..., \rho(\beta_g)). \end{array} \tag{4.3}$$

On the other hand it is also the underlying space containing the support of the distributions φ_l that we will define in the next section using spin networks.

For a compact connected Lie group G, let $\mathcal{A}$ denote the space of *flat* G-connections on the trivial G-bundle over X, $X \times G$, and $\mathcal{G}$ the group of gauge transformations (the space of smooth maps from X to G). As is well known, the moduli space of flat G-connections $\mathcal{A}/\mathcal{G}$ on X can be identified with

$$\mathcal{M}(G) := Hom(\pi_1(X), G)/G.$$

We will embed the graph Γ of a trinion decomposition in the surface X, by choosing a point on each γ_j and connecting the points inside each trinion with a tree with the shape of the letter Y. Let now $\mathcal{A}_\Gamma$ denote the space of G-connections on $\Gamma \subset X$, $\mathcal{G}_\Gamma$ the group of gauge transformations on Γ, and $\mathcal{G}'_\Gamma$ the subgroup of gauge transformations on Γ which are trivial at the vertices. Notice that

$$\mathcal{G}_\Gamma / \mathcal{G}'_\Gamma \cong G^V.$$

We then obtain the following commutative diagram with natural mappings

$$\begin{array}{ccccc} G^E & \xrightarrow{\sim} & \mathcal{A}_\Gamma/\mathcal{G}'_\Gamma & & \mathcal{A} \\ \downarrow & & \downarrow & & \downarrow \\ G^E/G^V & \xrightarrow{\sim} & \mathcal{A}_\Gamma/\mathcal{G}_\Gamma \cong Hom(\pi_1(\Gamma), G)/G & \to & \mathcal{A}/\mathcal{G} \cong Hom(\pi_1(X), G)/G \end{array}$$

Note that the isomorphism of sets,

$$\phi : \frac{G^E}{G^V} \quad \xrightarrow{\sim} \quad \frac{Hom(\pi_1(\Gamma), G)}{G} \cong \frac{G^g}{G}, \tag{4.4}$$

where in the first quotient the denominator acts by gauge transformations at the vertices of the graph, while in the second quotient we have the diagonal conjugation action of G, is obtained from a choice of g generators of $\pi_1(\Gamma)$. As we saw before, this choice is canonical for a graph associated to a trinion decomposition of fixed type of a marked surface X. Generalizing the $U(1)$ case, we call the image of ϕ, $Hom(\pi_1(\Gamma), G)/G$, the space of G-Schottky bundles.

In [**We, JW**], a real polarization of $\mathcal{M}(SU(2))$ was described by taking the fibers of the moment map defined by the arcos of the traces of the holonomies along the γ_j of connections in $\mathcal{M}(SU(2))$. If the trinion decomposition is of type α, then the space of unitary Schottky representations, being the set of connections with trivial holonomy along the α cycles, is a fiber of this moment map and coincides with the image of the inclusion of $Hom(\pi_1(\Gamma), SU(2))/SU(2)$ in $Hom(\pi_1(X), SU(2))/SU(2)$. As mentioned above, in this real polarization the quantum Hilbert space for the level k quantization, has a basis labelled by the Bohr-Sommerfeld fibers which include the unitary Schottky representations.

5. Spin networks

As in the previous section, let Γ be a graph with E edges and V vertices obtained from a trinion decomposition of a compact surface X, and let G be a compact connected Lie group.

A G-spin network for the graph Γ is a set of labellings $\Lambda = (\lambda_1, ..., \lambda_E; \kappa, ..., \kappa_V)$ where each λ_i labels an irreducible representation of G, and each κ_j corresponding to the jth vertex, labels an element of a fixed orthonormal basis of the space of intertwiners from the tensor product of the representations labelling incoming edges to the tensor product of the representations labelling outgoing edges. (See [**Ba**] and references therein.)

To a graph Γ and for each choice of level k, we will associate a Hilbert space, endowed with a natural spin network basis. We will consider mainly $U(1)$-spin networks based on graphs of any genus or $SU(n)$-spin networks for $g = 1$ graphs. In both of these cases, the intertwiners are unique (up to scale), when they exist, so that a spin network can be defined by the labels of the edges only.

Example 5.1. An $SU(2)$-spin network is given by a vector

$$l = (l_1, ..., l_E) \in \mathbb{N}_0^E$$

since the set of irreducible representations of $SU(2)$ can be identified with $\mathbb{N}_0$. For $G = SU(2)$, intertwiners are unique, and they exist precisely when the three integers l_i, l_j, l_k labelling edges meeting at a single vertex satisfy

$$l_i + l_j - l_k \in 2\mathbb{N}_0$$

for every permutation of the indexes i, j, k.

Example 5.2. For the case $G = U(1)$, the set of irreducible representations can be identified with $\mathbb{Z}$, and thus a $U(1)$-spin network will be defined by a vector

$$l = (l_1, ..., l_E) \in \mathbb{Z}^E.$$

The condition for the existence of an intertwiner for $G = U(1)$ is that

$$l_i + l_j + l_k = 0, \tag{5.1}$$

whenever l_i, l_j, l_k label edges meeting at a vertex, with all three edges pointing outwards (and we replace a label by its negative if the corresponding edge points inwards).

Let us denote by Ξ^G the set of G-spin networks for Γ. We now prove that the set $\Xi^{U(1)}$, of $U(1)$-spin networks, is completely determined by integer labellings of the generators of $\pi_1(\Gamma)$.

Proposition 1. The set $\Xi^{U(1)}$, of $U(1)$-spin networks on a graph Γ associated to a trinion decomposition of fixed type on a surface, has a natural structure of a ring isomorphic to

$$Hom(\pi_1(\Gamma), \mathbb{Z}) \cong \mathbb{Z}^g.$$

Proof. We will use the singular homology of the CW complex W consisting of Γ together with the g bounded 2-simplices d_j, $j = 1, ..., g$, whose boundaries are the cycles β_j that generate $\pi_1(\Gamma)$ (we are assuming a decomposition of type α as in the previous section). This makes W into a simply connected topological space. Let $C_k(W)$, $Z_k(W)$ and $B_k(W)$ denote, respectively, the rings of k-dimensional chains, cycles and boundaries on W with coefficients in $\mathbb{Z}$, and let

$$\partial_k : C_k(W) \to C_{k-1}(W)$$

denote the usual boundary operator. Then the equation (5.1) from the example above shows that a $U(1)$-spin network is nothing more than an element of $Z_1(W) = \ker(\partial_1)$. Moreover, since the 2-simplices $C_2(W)$ consist of labellings of the 2-simplices in W (or equivalently, generators of $\pi_1(\Gamma)$), the 2-boundary map relates labellings of 2-simplices and of edges

$$\partial_2 : C_2(W) \cong Hom(\pi_1(\Gamma), \mathbb{Z}) \twoheadrightarrow B_1(W) \subset Z_1(W).$$

Also, since $H_1(W) = 0$, we conclude that $Z_1(W) = B_1(W)$, which means that there is an isomorphism

$$Hom(\pi_1(\Gamma), \mathbb{Z})/\ker(\partial_2) \cong Z_1(W).$$

Finally, since any 2-chain is of the form $\sum m_j d_j$ for some integers m_j and the $\partial d_j = \beta_j$ are linearly independent in $H_1(W) \cong \mathbb{Z}^g$, we obtain $\ker(\partial_2) = 0$, which proves the Proposition. □

This Proposition provides a useful alternative way of understanding Theorem 5.3 below in the case of the group $U(1)$. We now describe the general construction of a spin network basis of the space

$$L^2(G^g, dx)^{Ad}$$

of Ad-invariant square integrable functions on G^g, with respect to the product Haar measure. For a general G-spin network $\Lambda = (\lambda_1, ..., \lambda_E; \kappa_1, ...\kappa_V)$ for Γ, let us define the *spin network function with label* Λ to be the smooth function

$$f_\Lambda(g_1, ..., g_E) \in C^\infty\left(\frac{G^E}{G^V}\right)$$

obtained by taking the matrices corresponding to g_k in the representation λ_k and contracting with the invariant tensor κ_j at each vertex v_j. In the cases in which we are interested, more explicit formulas are available for the f_Λ. The functions f_Λ correspond under the isomorphism ϕ in (4.4) to Ad-invariant functions f_γ on G^g, which we can normalize so that $||f_\gamma|| = 1$ in $L^2(G^g, dx)^{Ad}$. We have

Theorem 5.3. [**Ba**] *For every graph* Γ*, the set* $\{f_\Lambda, \Lambda \in \Xi^G\}$ *forms an orthonormal basis for* $L^2(G^g, dx)^{Ad}$.

For $G = U(1)$, let $l = (l_1, ..., l_E) \in \mathbb{Z}^E$ be a spin network for a graph Γ. Then, the associated spin network functions are

$$f_l(\theta_1, ..., \theta_E) = e^{2\pi i \sum_{j=1}^{E} l_j \theta_j} \quad \in C^\infty(\tfrac{U(1)^E}{U(1)^V}).$$

Proposition 2. The functions f_l on $\frac{U(1)^E}{U(1)^V}$ are pullbacks under ϕ (4.4) of

$$\hat{f}_m(\xi_1, ..., \xi_g) := e^{2\pi i \sum_{j=1}^{g} m_j \xi_j} \quad \in C^\infty(Hom(\pi_1(\Gamma), U(1)))$$

for the integers $m = (m_1, ..., m_g)$ such that $\partial_2 m = l$ as in the proof of Proposition 1.

Proof. For any function $g \in \mathbb{C}^\infty(Hom(\pi_1(\Gamma), U(1)))$, its pullback under ϕ is given by $(\phi^* g)(\theta) = g(\xi)$, where the variables θ and ξ are related by

$$\xi_j = \sum_{k=1}^{E} \varepsilon_{jk}\theta_k, \;\; j = 1, ..., g,$$

where

$$\varepsilon_{jk} = \begin{cases} 0, & \beta_j \text{ does not contain } e_k \\ \pm 1, & \beta_j \text{ contains } e_k. \end{cases}$$

Above, the sign is chosen according to whether e_k is oriented in the same way as β_j or in the opposite way. On the other hand, if the labels l and m are related by $\partial m = l$, it is easy to check that $l_k = \sum_{j=1}^{g} \varepsilon_{jk} m_j$, so the proof follows after a simple computation, which, with an appropriate interpretation of the variables θ and ξ, is an instance of Stokes Theorem. □

For the cases $G = U(1)$ with X a Riemann surface of genus g, and $G = SU(n)$ with X an elliptic curve X_τ, we will construct Hilbert spaces of special distributions defined on $Hom(\pi_1(\Gamma), G)/G$, with dimensions given the Verlinde dimension for the space of theta functions on the corresponding moduli space. These distributions are linear combinations of Dirac delta distributions supported over Bohr-Sommerfeld points of the real polarization, defined by the moment map associated to a dual trinion decomposition of type β, which we described in section 3.

Fix a non-negative integer level k. Using the spin network functions we will now define the level k spin network Hilbert space. We say that a $U(1)$-spin network l is of level k if all labels l_j of the edges lie between 0 and $k-1$, that is they label integrable representations of the affine Lie algebra $\widehat{u(1)_k}$. Then the space of level k spin networks is of dimension k^g, which is also the Verlinde number for $U(1)$. Using the same argument in the proof of Proposition 1, but now with coefficients in the ring of integers mod k, $\mathbb{Z}_k$, we find that this space is isomorphic with the set $Hom(\pi_1(\Gamma), \mathbb{Z}_k)$ of labellings of the generators of $\pi_1(\Gamma)$ with integers from $\{0, ..., k-1\}$.

For any level k and $l = (l_1, \ldots, l_g) \in (\mathbb{Z}_k)^g$, consider the subset of $Hom(\pi_1(\Gamma), \mathbb{Z}) \cong \mathbb{Z}^g$ given by

$$[l]_k := l + k\mathbb{Z}^g,$$

and the following distributions on $U(1)^g$

$$\varphi_l = \sum_{m \in [l]_k} \hat{f}_m. \tag{5.2}$$

We then define the *level k spin network Hilbert space* to be the following k^g dimensional space of distributions on $Hom(\pi_1(\Gamma), U(1)) \cong U(1)^g$

$$W_{k,g}^{U(1)} = \bigoplus_{l\in(\mathbb{Z}_k)^g} \mathbb{C}\cdot\varphi_l \quad \subset \quad C^\infty(U(1)^g)'. \tag{5.3}$$

With a view towards the relation to classical theta functions on X, we consider on $W_{k,g}^{U(1)}$ an Hermitian structure for which the distributions φ_l are orthonormal. (See [**FMN1**].)

For the elliptic curve X_τ and $G = SU(n)$, consider now the affine Weyl group $W^{\mathrm{aff}} := W \rhd \Lambda_R$ for $\widehat{sl(n,\mathbb{C})}_k$, where $\rhd$ denotes the semi-direct product. Consider also the level $k+n$ action of W^{aff} on $\mathfrak{h}_{\mathbb{R}}^*$ given by

$$\tilde{w}\cdot\lambda = w\cdot(\lambda + (n+k)\alpha), \quad \tilde{w} = (w,\alpha)\in W^{\mathrm{aff}}.$$

We denote by $[\lambda]_k$ the orbit of $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ under this action.

The spin network function for the $g = 1$ graph associated to X_τ, where the unique edge is labelled by the highest weight for an irreducible representation of $SU(n)$, $\lambda \in \Lambda_W^+$, is just the character of the representation, χ_λ. Note that for the $g = 1$ graph, it is clear that $Hom(\pi_1(\Gamma), SU(n))/SU(n) = SU(n)/SU(n)$, where $SU(n)$ acts by conjugation.

For an integrable representation of level k, $\gamma \in D_k$, define the following distribution on $SU(n)$,

$$\varphi_\gamma = \sum_{\lambda+\rho\in[\gamma+\rho]_k\cap\Lambda_W^+} \epsilon_\lambda \chi_\lambda, \tag{5.4}$$

with $\epsilon_\lambda = \epsilon(\tilde{w})$, where $\tilde{w}$ is the unique element of W^{aff} such that $\lambda+\rho = \tilde{w}\cdot(\gamma+\rho)$. Note that φ_γ is defined on the quotient $((\mathbb{R}/\mathbb{Z})\otimes\check{\Lambda}_R)/W$ and extended by Ad-invariance to $SU(n)$.

The spin network Hilbert space for this situation will be the space of distributions on $SU(n)$,

$$W_{k,1}^{SU(n)} = \bigoplus_{\gamma\in D_k} \mathbb{C}\cdot\varphi_\gamma \quad \subset \quad C^\infty(SU(n))'. \tag{5.5}$$

Note that $\dim W_{k,1}^{SU(n)}$ is the Verlinde number for $H^0(\mathcal{M}_n(X_\tau), L^k)$.

The natural inner product on $H^0(\mathcal{M}_n(X_\tau), L^k)$ induced from the canonical Hermitian structure on L, leads us to equip $W_{k,1}^{SU(n)}$ with an Hermitian structure for which the distributions φ_γ are orthonormal [**FMN2, FMN3**].

We now describe the relation between the vector spaces of distributions corresponding to the real polarization Hilbert spaces $V_{k,g}^G$ in (3.3), and (3.6)-(3.7), and the spin network Hilbert space $W_{k,g}^G$ defined above in (5.2)-(5.5).

Considering first the case $G = U(1)$, let S be the unitary modular transformation matrix for the characters of the affine Lie algebra $\widehat{u(1)}_k^g$ [**Ka**],

$$S_{ll'} := k^{-\frac{g}{2}} e^{2\pi i \frac{l'\cdot l}{k}}, \; l, l' \in (\mathbb{Z}_k)^g. \tag{5.6}$$

For the abelian variety M_Ω, consider the real polarization given by the projection to the η coordinates, as in (3.2). Recall, PROPOSITION 3. [**FMN1**] We have

$$\varphi_l(\eta) = \sum_{l' \in (\mathbb{Z}_k)^g} k^{g/2} S_{ll'} \delta_{l'}(\eta).$$

Note that S can be seen as a discrete analogue of the Fourier transform. The Proposition also explains the choice of norm for the distributions δ_l in (3.4), since the φ_l are orthonormal. As an immediate corollary to Proposition 3, we obtain

THEOREM 5.4. *We have $V_{k,g}^{U(1)} = W_{k,g}^{U(1)}$, as subspaces of $C^\infty(U(1)^g)'$.*

A similar situation occurs for the case of the elliptic curve X_τ and $G = SU(n)$, where this time we consider the unitary modular transformation matrix for the characters of the affine Lie algebra $\widehat{sl(n,\mathbb{C})}_k$ given by [**Ka**],

$$S_{\gamma\gamma'} := c(k,n)\ \bar\sigma(\tfrac{\gamma+\rho}{k+n})\ \chi_{\gamma'}(-\tfrac{\gamma'+\rho}{k+n}), \gamma, \gamma' \in D_k, \tag{5.7}$$

where $c(k,n) = i^{|\Delta_+|} |\Lambda_W/(k+n)\Lambda_R|^{-1/2}$ is a constant, with Δ_+ the set of positive roots, Λ_R is the root lattice, and σ is the denominator in the Weyl character formula. Recall that $T = (\mathbb{R}/\mathbb{Z}) \otimes \check\Lambda_R$ is a maximal torus of $SU(n)$ and $\mathcal{M}_n(X_\tau)$ is the quotient of the abelian variety $M := X_\tau \otimes \check\Lambda_R$ by the Weyl group W.

PROPOSITION 4. [**FMN3**] We have on $T/W \cong SU(n)/SU(n)$,

$$\delta_\gamma = \frac{1}{c(k,n)\bar\sigma(\frac{\gamma+\rho}{k+n})} \sum_{\gamma' \in D_k} S_{\gamma\gamma'} \varphi_{\gamma'}.$$

As an immediate corollary we obtain

THEOREM 5.5. *We have $V_{k,1}^{SU(n)} = W_{k,1}^{SU(n)}$, as subspaces of $C^\infty(SU(n))'$.*

These Theorems show that the real polarization Hilbert space and the spin network Hilbert space coincide when viewed as abstract Hilbert spaces in $C^\infty(G)'$. However, from the discussion above it is clear that when we consider also the explicit dependence on variables, the following corollary to Theorems 5.4 and 5.5 is valid.

COROLLARY 1. Let X be a marked Riemann surface and let Γ be the graph of a trinion decomposition γ of type α.

(i) The spin network Hilbert space for $J(X)$, $W_{k,g}^{U(1)}$, constructed from γ, equals, as a subspace of $C^\infty(Hom(\pi_1(\Gamma), U(1)))'$, the Hilbert space $V_{k,g}^{U(1)}$ of the real polarization of $J(X) \cong Hom(\pi_1(X), U(1))$ given by a trinion decomposition of type β as in (4.3).

(ii) If $X = X_\tau$ is an elliptic curve, the spin network Hilbert space $W_{k,1}^{SU(n)}$, associated to the (degenerate) trinion decomposition of type α, equals, as a subspace of $C^\infty(Hom(\mathbb{Z}\cdot\beta, SU(n)))'$, the space $V_{k,1}^{SU(n)}$ of the real polarization of the moduli space $\mathcal{M}_n(X_\tau) \cong Hom(\pi_1(X_\tau), SU(n))/SU(n) \cong (T \times T)/W$ obtained by projecting to the coordinates corresponding to the cycle β.

6. Holomorphic polarization

When we have a Kähler manifold M together with a positive holomorphic line bundle L on M, there is an alternative path for the level k quantization of M. One uses a holomorphic polarization and defines the quantum Hilbert space as $H^0(M, L^k)$. As for the case of two real polarizations, it is expected that, in favorable cases, there should be a natural unitary isomorphism between Hilbert spaces corresponding to different complex structures on M. Moreover, it is natural to expect that the cardinality of the k-Bohr-Sommerfeld set $B_{k-\mathrm{BS}} \subset B$ coincides with the dimension of this space. (See [**AdPW, Hi, We, JW, Ty**].).

In this section we will present a procedure for relating the quantizations of the moduli space in a real and in holomorphic polarizations. This relies on extensions of the so-called coherent state transform (CST), or Segal-Bargmann-Hall transform, for Lie groups [**Ha1**]. Let G be a compact connected Lie group, with a fixed Ad-invariant bilinear form $\langle\cdot,\cdot\rangle$ on its Lie algebra. The CST for G is defined as a map from $L^2(G, dx)$, where dx is the Haar measure on G, to the space of holomorphic functions on the complexification $G_{\mathbb{C}}$. This map was extended to the space of distributions on G, $C^\infty(G)'$ in [**FMN1, FMN2**].

For a distribution $f \in C^\infty(G)'$ of the form

$$f = \sum_{\lambda \in \Lambda_W^+} \mathrm{tr}(\mathrm{R}_\lambda A_\lambda), \tag{6.1}$$

where we label irreducible representations of G by the highest weights λ in the set of dominant weights Λ_W^+, the CST is

$$C_t(f) := \mathcal{C} \circ e^{\pi t \Delta_G} f = \sum_{\lambda \in \Lambda_W^+} e^{-\pi t c_\lambda} \mathrm{tr}(\mathrm{R}_\lambda A_\lambda), \tag{6.2}$$

where $t > 0$, $\mathcal{C}$ denotes analytic continuation from G to $G_{\mathbb{C}}$ and Δ_G is the invariant Laplace operator on G defined by $\langle\cdot,\cdot\rangle$.

In the case of the abelian group $U(1)^g$, for which the complexification is $(\mathbb{C}^*)^g$, consider a complex valued inner product on $Lie(G)$ given by

$$\langle a, b\rangle := -ia \cdot \Omega b, \qquad a, b \in Lie(G),$$

where $\Omega \in \mathbb{H}_g$.

The CST above extends to this case, and equipping the space of theta functions with the natural inner product induced from the usual Hermitian structure on $L \to M_\Omega$, we obtain

Theorem 6.1. [**FMN1**] *For any natural number k, the restriction of the CST $C_{1/k}$ to $W_{k,g}^{U(1)}$ is an isometric isomoprhism from $W_{k,g}^{U(1)}$ onto $H^0(M_\Omega, L^k)$. Moreover, $C_{1/k}$ sends the special basis φ_l onto the special basis of theta functions with characteristics in (2.2).*

Therefore we have the following diagram where the top and bottom horizontal arrows are unitary isomorphisms.

$$\begin{array}{ccc}
L^2(U(1)^g, dx) & \xrightarrow[\sim]{C_{1/k}} & \mathcal{H}((\mathbb{C}^*)^g) \bigcap L^2((\mathbb{C}^*)^g, d\nu_{1/k}) \\
\downarrow & & \downarrow \\
C^\infty(U(1)^g)' & \xhookrightarrow{C_{1/k}} & \mathcal{H}((\mathbb{C}^*)^g) \\
\uparrow & & \uparrow \\
W_{k,g}^{U(1)} & \xrightarrow[\sim]{C_{1/k}} & H^0(M_\Omega, L^k) \ .
\end{array}$$

Above, $d\nu_t$ stands for the so called averaged heat kernel measure on $G_{\mathbb{C}}$ which is determined by the Laplacian Δ_G [**Ha1, FMN1**].

For the study of non-abelian theta functions on the moduli space of semistable vector bundles on an elliptic curve X_τ consider now $G = SU(n)$. On $Lie(G)$ consider the complex inner product defined by $\langle\cdot,\cdot\rangle = i\tau K$, where K is the Killing form. We have,

Theorem 6.2. [**FMN2**] *The restriction of the CST $C_{1/(k+n)}$ to $W_{k,1}^{SU(n)}$ is an isometric isomorphism onto the space of level k non-abelian theta functions for X_τ, $H^0(\mathcal{M}_n(X_\tau), L^k)$.*

Therefore, the CST provides the analytic link between the quantization space for a real polarization of the moduli space, associated to a trinion decomposition of X, and the space of holomorphic theta functions. This correspondence is established for classical theta functions and for non-abelian ones in $g = 1$. We hope to address more general cases in [**FMNT**]. Note that the trinion decomposition of X, which led to the spin network Hilbert spaces, leads also to a choice of space of unitary Schottky bundles on X. The complexification of this space is the space of complex Schottky bundles, which maps naturally to the moduli space. This map plays a crucial role in the analytic theory of theta functions that we described above [**Fl, FMN1, FMN2**]. We also remark that the distributions above can be regarded as "boundary values" of holomorphic theta functions when the complex structure of X degenerates to $\Omega \to 0$.

7. The BKS pairing and the matrix S

The real polarization Hilbert spaces $V_{k,g}^G$ and the spin network Hilbert space $W_{k,g}^G$ that we described above are, as we have seen, naturally related to each other. In fact, the special bases for these spaces that were described in the previous sections, are related by the BKS pairing for the two real polarizations of the moduli space defined by the trinion decomposition of X.

In the case of $V_{k,g}^{U(1)}$ and $W_{k,g}^{U(1)}$, the special bases are related by the BKS pairing map between the Hilbert spaces associated to the transverse real polarizations of M_Ω given by projection to the η or ξ coordinates. In fact, define

$$\langle\delta_l, \delta_{l'}\rangle_{\mathrm{BKS}} = \int_{M_\Omega} \overline{\delta_l(\eta)}\delta_{l'}(\xi) e^{2\pi i k\langle\eta,\xi\rangle} d\eta d\xi. \tag{7.1}$$

Note that this formula is the generalization to the abelian variety M_Ω of the formula for the BKS pairing between the two canonical real polarizations of $\mathbb{R}^{2n}$, which is the Fourier transform.

This gives

$$\langle \delta_l, \delta_{l'} \rangle_{\text{BKS}} = e^{\frac{2\pi i}{k}\langle l, l' \rangle} = k^{g/2} S_{ll'}, \tag{7.2}$$

where S is the modular transformation matrix in (5.6), which, using Proposition 3, proves the following Theorem for the corresponding BKS pairing map $\mathcal{B}$.

THEOREM 7.1. *The BKS pairing map $\mathcal{B}$ between the quantization spaces for the above real polarizations of M_Ω is given by*

$$(\mathcal{B}\delta_l)(\eta) = k^g \varphi_l(\eta). \tag{7.3}$$

Note that, when $M_\Omega = J(X)$, the modular transformation matrix S from the theory of affine Lie algebras gives the BKS map between the quantization spaces corresponding to the two real polarizations for the moduli space, associated to the trinion decomposition of X. As we mentioned above, the spaces of unitary Schottky bundles associated to the α and to the β cycles on X play dual roles in this construction.

A similar result holds for non-abelian theta functions in genus 1. From Proposition 4, we obtain for $v \in M = X_\tau \otimes \check{\Lambda}_R$,

$$\delta^+_\gamma(v) := \frac{1}{|W|} \sum_{w \in W} \delta(w(v) - \tfrac{\gamma+\rho}{k+n}) = \frac{1}{c(k,n)\, \bar{\sigma}(\frac{\gamma+\rho}{k+n})} \sum_{\gamma' \in D_k} S_{\gamma\gamma'}\ \varphi_{\gamma'}(v), \tag{7.4}$$

where S is the modular transformation matrix in (5.7).

Since the basis $\{\varphi_\gamma, \gamma \in D_k\}$ for $W^{SU(n)}_{k,1}$ is orthonormal, it follows that the basis

$$\{\hat{\delta}^+_\gamma := c(k,n)\ \sigma(-\tfrac{\gamma+\rho}{k+n})\ \delta^+_\gamma\} \tag{7.5}$$

is also orthonormal.

Consider now the two transverse real polarizations associated to the η and ξ coordinates in M. Recalling that the moduli space is $\mathcal{M}_n(X_\tau) = M/W$, where $M = X_\tau \otimes \check{\Lambda}_R$, we define the BKS pairing between the two corresponding Hilbert spaces by

$$\langle \hat{\delta}^+_\gamma, \hat{\delta}^+_{\gamma'} \rangle_{\text{BKS}} = \frac{1}{|W|} \int_M \overline{\hat{\delta}^+_\gamma(\eta)} \hat{\delta}^+_{\gamma'}(\xi) e^{2\pi i (k+n)\langle \eta, \xi \rangle} \bar{\sigma}(\eta)\sigma(\xi) d\eta d\xi, \tag{7.6}$$

where $\langle \eta, \xi \rangle$ denotes the pairing on M induced from the Killing form via $\check{\Lambda}_R \subset Lie(SU(n))$.

The exponential factor is the analog of the same factor in the $U(1)^g$ case above, but for quantization at the appropriate shifted level $k+n$. (See [**FMN2**].) The factors of $\bar{\sigma}(\eta)$ and $\sigma(\xi)$ arise from the correspondence between the pull-back of non-abelian theta functions from $\mathcal{M}_n(X_\tau)$ to M and then to $SU(n)/SU(n)$ as in [**FMN2**]. They appear essentially due to the Jacobian in the Weyl integration formula.

THEOREM 7.2. *We have* $\langle \hat{\delta}^+_\gamma, \hat{\delta}^+_{\gamma'} \rangle_{\text{BKS}} = c(k,n) \bar{S}_{\gamma\gamma'}$.

Proof. The result follows by direct computation. Since these distributions are defined on the group $SU(n)$, we should factor out the Jacobian in the Weyl integral formula

$$J(\eta,\xi)=\frac{|\sigma(\eta)|^2|\sigma(\xi)|^2}{|W|^2},$$

to express $\langle\hat\delta^+_\gamma,\hat\delta^+_{\gamma'}\rangle_{\mathrm{BKS}}$ as the following integral

$$\tfrac{|c(k,n)|^2}{|W|}\sigma(\tfrac{\gamma+\rho}{k+n})\bar\sigma(\tfrac{\gamma'+\rho}{k+n})\sum_{w,w'\in W}\int_M\frac{e^{2\pi i(k+n)\langle\eta,\xi\rangle}}{\sigma(\eta)\bar\sigma(\xi)}\overline{\delta_\gamma(w(\eta))}\delta_{\gamma'}(w'(\xi))J(\eta,\xi)d\eta d\xi.$$

This integral localizes on the support of the Dirac delta distributions, and using the fact that σ is W anti-invariant, we obtain

$$\langle\hat\delta^+_\gamma,\hat\delta^+_{\gamma'}\rangle_{\mathrm{BKS}}=\frac{|c(k,n)|^2}{|W|}\sum_{w,w'\in W}\epsilon(w)\epsilon(w')e^{2\pi i(k+n)\langle w(\frac{\gamma+\rho}{k+n}),w'(\frac{\gamma'+\rho}{k+n})\rangle}$$

which simplifies to

$$\langle\hat\delta^+_\gamma,\hat\delta^+_{\gamma'}\rangle_{\mathrm{BKS}}=|c(k,n)|^2\sum_{w\in W}\epsilon(w)e^{2\pi i\langle w(\gamma'+\rho),\gamma+\rho\rangle}, \tag{7.7}$$

which from (5.7) gives the Theorem. □

The following Theorem then follows immediately from Proposition 4.

Theorem 7.3. *The BKS pairing map $\mathcal{B}$ between the Hilbert spaces associated to the real polarizations of $\mathcal{M}_n(X_\tau)$ is given by*

$$(\mathcal{B}\hat\delta^+_\gamma)(\eta)=c(k,n)\varphi_\gamma(\eta). \tag{7.8}$$

Again, the modular transformation matrix relates the two quantization spaces for the real polarizations associated to the trinion decomposition of X_τ. The corresponding spaces of unitary Schottky bundles are associated to the choice of the α or β cycle in $\pi_1(X_\tau)$.

Note that applying the CST of section 6 to any of these two Hilbert spaces produces holomorphic theta functions. For the elliptic curve X_τ, the spaces $H^0(\mathcal{M}_n(X_\tau),L^k)$ and $H^0(\mathcal{M}_n(X_{-\frac{1}{\tau}}),L^k)$ are naturally related by the matrix S in (5.7), since the theta functions in this case are essentially given by the characters for the integrable representations of $\widehat{sl(n,\mathbb{C})}_k$. The inverse CST then extends this natural correspondence to the Hilbert spaces of the two real polarizations above, and this is reflected in Theorem 7.3 in the form of the BKS pairing for these polarizations.

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