DEGENERATING KÄHLER STRUCTURES AND GEOMETRIC QUANTIZATION

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Abstract. We review some recent results on the problem of the choice of polarization in geometric quantization. Specifically, we describe the general philosophy, developed by the author and collaborators, of treating real polarizations as limits of degenerating families of holomorphic polarizations. We first review briefly the general framework of geometric quantization, with a particular focus on the problem of the dependence of quantization on the choice of polarization. The problem of quantization in *real* polarizations is emphasized. We then describe the relation between quantization in real and Kähler polarizations in some families of symplectic manifolds, that can be explicitly quantized and that constitute an important class of examples: cotangent bundles of Lie groups, abelian varieties and toric varieties. Applications to theta functions and moduli spaces of vector bundles on curves are also reviewed.

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1. Introduction

Quantum theory has been a major source of rich mathematical ideas ever since the early XXth century (see [Hal5] for a recent textbook written for mathematicians). Mathematically speaking, the problem of quantization is realized in terms of a symplectic manifold (X, ω), which plays the role of classical phase space. Recall that the space of real valued smooth functions on X, $C^\infty(X)$, is equipped with a Lie algebra structure given by the Poisson bracket $\{,\} : C^\infty(X) \times C^\infty(X) \to C^\infty(X)$,

$$\{f, g\} = \omega(X_f, X_g), \quad f, g \in C^\infty(X),$$

where X_f is the hamiltonian vector field associated to f and defined by $i_{X_f}\omega = df$.

The quantization of (X, ω) would correspond to a linear map $q : C^\infty(X) \to Op(\mathcal{H})$, from the space of smooth functions on the classical phase space to the space of linear operators on some Hilbert space of "quantum states", $\mathcal{H}$. Ideally, this map should satisfy some natural requirements (see, for example, Section 2 of [Ki]):

i) $q(1) = id_{\mathcal{H}}$
ii) $q(\{f, g\}) = \frac{i}{\hbar}[q(f), q(g)]$
iii) $q(f) = q(f)^\dagger$
iv) If $\{f_1, \ldots, f_m\}$ is a complete set in $C^\infty(X)$ (that is $\{g, f_j\} = 0, j = 1, \ldots, m$ implies g is constant) then $\{q(f_1), \ldots, q(f_m)\}$ is also complete (that is $[A, q(f_j)] = 0, j = 1, \ldots, m$ implies $A = \lambda\, id_{\mathcal{H}}$ for some $\lambda \in \mathbb{C}$),

where $\hbar$ is Planck's constant. Condition iv) corresponds to the irreducibility of the representation $\mathcal{H}$.

Suppose that $\omega = d\theta$ is exact. (This happens, for example, in the important case when $X = T^*M$, θ is the canonical one-form and ω the canonical symplectic form.) Then, for $f \in C^\infty(X)$,

$$q(f) = f + \frac{\hbar}{i}X_f - \theta(X_f), \tag{1}$$

acting on $L^2(X, \frac{\omega^n}{n!})$, satisfies i), ii), iii), but not iv). For example, for $X = \mathbb{R}^{2n} \cong T^*\mathbb{R}^n$, with canonical symplectic form $\omega = \sum_{i=1}^n dp^i \wedge dx^i$, one has $\theta = \sum_{i=1}^n p^i dx^i$ and $q(x_i) = x^i + \frac{\hbar}{i}\frac{\partial}{\partial p^i}$, $q(p^i) = -\frac{\partial}{\partial x^i}$. Then, the operators $\frac{\partial}{\partial p^j}$ commute with $q(x^i), q(p^i), i = 1, \dots, n$ and one does not have irreducibility. Still for this simplest, paradigmatic, example, one can show that given $q(x^i), q(p^i), i = 1, \dots, n$, obeying canonical commutation relations,

$$[q(x^i), q(x^j)] = 0,\ [q(p^i), q(p^j)] = 0,\ [q(x^i), q(p^j)] = i\hbar\delta_{ij},\ \ i, j = 1, \dots, n,$$

there is no linear quantization map $q : C^\infty(X) \to Op(\mathcal{H})$ such that

$$q(1) = id_{\mathcal{H}}, q(fg) = \frac{1}{2}(q(f)q(g) + q(g)q(f)), [q(f), q(g)] = -i\hbar q(\{f, g\}).$$

(See, for example, Section 4 of [Got] and references therein.)

Therefore, the general problem of quantization does not have a solution if one imposes all of these requirements. However, the ideas of quantization still lead to many interesting mathematical topics and problems. For instance, in deformation quantization, condition ii) is relaxed and is imposed only asymptotically as $\hbar \to 0$ (this is the *semi-classical limit*). One is then faced with associative (noncommutative) algebras whose product is a deformation of the ordinary pointwise multiplication of functions in $C^\infty(X)$. In another approach, one can consider Berezin-Toeplitz quantization (see, for example, [BMS, Sc, Hal4]) where condition ii) is also valid asymptotically in the semi-classical limit (here meaning quantization of $(X, k\omega)$, $k \in \mathbb{N}$, $k \to \infty$), and realized in terms of Toeplitz operators acting on holomorphic sections of the prequantum bundle (see Section 2).

Geometric quantization is a general geometric framework in which issues related to the quantization of a symplectic manifold can be addressed. In particular, it gives a generalization of (1). Of course, as already mentioned, geometric quantization cannot give rise to a quantization map as described above. However, it does lead to spaces of quantum states $\mathcal{H}$ of rich geometric flavour and to an arena where ideas of symplectic, algebraic and complex geometry (and also some analysis) interact very strongly. If one wants, one can regard quantization as an excuse for the study of this rich geometry. In the following we will set $\hbar = 1$. The bibliography gives an indicative, but by no means exhaustive, list of references.

2. Geometric Quantization

2.1. **General set up.** Let (X, ω) be a symplectic manifold. It is said to be quantizable if there exists a hermitian complex line bundle $L \to X$ with compatible connection ∇ of curvature $-i\omega$. This is the case if ω is exact, which occurs for cotangent bundles with canonical symplectic structure, or, in the case when X is compact, if $\frac{[\omega]}{2\pi} \in H^2(X, \mathbb{Z})$. Let h denote the hermitian structure on L. One calls (L, ∇, h) the *prequantization data*.

For $f \in C^\infty(X)$, equation (1) is then generalized to

$$f \to f - i\nabla_{X_f},$$

where this operator acts on the space of smooth sections of L, $C^\infty(L)$, and extends to a self-adjoint operator on $\Gamma_{L^2}(L, \frac{\omega^n}{n!})$. As we have seen, however, this will not satisfy the irreducibility requirement. In order to define a smaller space of quantum states, with the usual Schrödinger quantization serving as a guide, the concept of polarization is introduced. A *polarization* $\mathcal{P}$ of (X, ω) is an integrable, Lagrangian distribution in $TX \otimes \mathbb{C}$.

We will be mainly interested in two cases:

- $\mathcal{P}$ is a *real polarization* if $\bar{\mathcal{P}} = \mathcal{P}$. In this case, X is foliated by the Lagrangian leaves of $\mathcal{P}$. Often this defines a (possibly singular) Lagrangian fibration of X.
- $\mathcal{P}$ is a *Kähler polarization* (which sometimes we will just call a *holomorphic polarization*) if $\bar{\mathcal{P}} \cap \mathcal{P} = \{0\}$ and if the hermitian form
$$i\omega(\cdot,\cdot) : \bar{\mathcal{P}} \times \mathcal{P} \to C^\infty_{\mathbb{C}}(X)$$
is positive definite[1]. In this case, there are local complex coordinates $\{z^i\}_{i=1,\dots,n}$ such that $\mathcal{P}$ is spanned pointwise by $\{\partial/\partial z^i\}_{i=1,\dots,n}$. The $(0,1)$-piece of the connection ∇ then defines an holomorphic structure on L.

A section $s \in C^\infty(L)$ is said to be $\mathcal{P}$-polarized if it is covariantly constant along $\bar{\mathcal{P}}$, $\nabla_{\bar{\mathcal{P}}} s = 0$. The Hilbert space of quantum states associated to the polarization $\mathcal{P}$ *could* then be defined by the L^2 closure of the space of polarized sections:

$$\mathcal{H}_{\mathcal{P}} = \{s \in \Gamma_{L^2}(L, \frac{\omega^n}{n!}) : \nabla_{\bar{\mathcal{P}}} s = 0\}. \tag{2}$$

However, the space of quantum states is often taken to be "corrected by half-forms". (For example, in the noncompact case, the space in (2) could be trivial for a real polarization. The half-form correction allows one to integrate along the space of leaves of the polarization, instead of integrating over X, in order to obtain states with finite L^2-norm.) We will comment on the half-form correction below.

When $\mathcal{P}$ is a Kähler polarization, $\mathcal{H}_{\mathcal{P}} = H^0_{L^2}(X, L)$ is the space of (finite norm) holomorphic sections of L. This is an object of great interest in algebraic and complex geometries. If X is compact then $\mathcal{H}_{\mathcal{P}}$ will be finite-dimensional. If $k \in \mathbb{N}$, the quantization of $(X, k\omega)$ is called *quantization at level* k and is given by $H^0_{L^2}(X, L^k)$.

When $\mathcal{P}$ is a real polarization the quantum states have quite a different flavour. Here, $\mathcal{H}_{\mathcal{P}}$ is given by sections of the prequantum bundle which are covariantly constant along the leaves of $\mathcal{P}$. Since these are Lagrangian, ∇ is flat along such leaves and there are no local obstructions to the existence of covariantly constant sections. However, global obstructions may occur. Covariantly constant sections can only be supported along leaves where ∇ has trivial holonomy. These are called *Bohr-Sommerfeld* leaves.

[1] This is consistent with the definition of $\mathcal{H}_{\mathcal{P}}$ in (2) below, in the sense that for a "positive" Kähler polarization $\mathcal{P}$ there will be nonzero holomorphic sections of L, and $\mathcal{H}_{\mathcal{P}} = H^0_{L^2}(X, L)$.

In the compact case and for sufficiently nice polarizations, an open dense subset of X will be fibered by Arnold-Liouville tori. Only a finite number of these will satisfy the Bohr-Sommerfeld condition and the equations of covariant constancy will therefore admit no smooth solutions. The author and collaborators have considered, in different classes of examples, a weak version of these equations and have studied their distributional solutions. The space of quantum states in this case will be a finite-dimensional space of *distributional sections* of L. We note that a theorem of Śniatycki for the case of non-singular fibrations, see [Sn2], shows that the cohomology of the sheaf of *smooth* covariantly constant local sections, is concentrated in mid dimension and has rank given precisely by the number of Bohr-Sommerfeld fibers. In our approach, due to the fact that we consider the sheaf of distributional solutions of L, the quantum states appear already in degree zero cohomology. Of course, this allows for a more direct relation with the holomorphic polarizations, where also H^0 appears, and opens the door to the study of the degeneration of holomorphic sections into distributional ones. Also, note that in [BFMN], where toric varieties were considered, one deals with singular Lagrangian fibrations. The general definition of quantization in singular real polarizations remains an open problem. For a non-toric example see [HM].

2.2. **Weak equations of covariant constancy.** Assume, for simplicity, that X is compact.[2] In order to consider distributional solutions to the equations of covariant constancy, one takes the injection $\iota : C^\infty(L) \hookrightarrow C^{-\infty}(L) = (C^\infty(L^{-1}))'$ given by

$$s \mapsto \iota s(\phi) = \int_X s\phi \frac{\omega^n}{n!}, \ \phi \in C^\infty(L^{-1}).$$

Given the prequantum connection ∇ and a section $\xi \in \mathcal{P}$ we define the operator $\nabla'_{\bar\xi}$ to act on $C^{-\infty}(L)$ by demanding that its action commutes, via the injection ι, with the action of $\nabla_{\bar\xi}$ on $C^\infty(L)$. This gives,

$$(\nabla'_{\bar\xi}\sigma)(\phi) = -\sigma(\operatorname{div}\xi\phi + \nabla^{-1}_{\bar\xi}\phi), \ \ \sigma \in C^{-\infty}(L), \phi \in C^\infty(L^{-1}), \tag{3}$$

where ∇^{-1} is the connection induced on L^{-1} by ∇.

Weakly covariant constant sections are then in the intersection of the kernels of the operators $\nabla'_{\bar\xi}$ for all sections ξ of $\mathcal{P}$.

Note that when $\mathcal{P}$ is a Kähler polarization, the weak solutions to the equations of covariant constancy are just the holomorphic sections of L, due to the regularity of the Cauchy-Riemann equations (see Chapter 6 of [Gun]).

2.3. **The half-form correction.** The process of geometric quantization described above can only rarely be implemented explicitly. To begin with, real polarizations and interesting families of Kähler strutures can be hard to find or to describe explicitly. Also, in the noncompact case and for a real polarization, one would like the inner product in $\mathcal{H}_\mathcal{P}$ to be given by integration along the space of leaves of the Lagrangian foliation defined by $\mathcal{P}$, since the sections will be covariantly constant

[2] Otherwise, just take test sections with compact support.

along the (possibly noncompact) leaves. But a natural measure on the space of leaves would need to be introduced. Also, even in the compact case, particular care needs to be taken in the definition of the inner product since the sections will be distributional in nature. On the other hand, if one succeeds in quantizing (X,ω) in two different polarizations $\mathcal{P},\mathcal{P}'$, then one would also like to find a natural (preferably unitary) pairing map between the Hilbert spaces of quantum states $\mathcal{H}_{\mathcal{P}},\mathcal{H}_{\mathcal{P}'}$, making these quantizations equivalent. In the classes of examples where geometric quantization can be performed a (at least partial) solution to some of these problems is achieved by the so-called half-form correction. (See, for example, Chapter 10 of [Wo], Section 4 of [Ki], Chapters 4 and 5 of [Sn1] and Chapter 23 of [Hal5]).

In the paradigmatic case ($X = \mathbb{R}^{2n}, \omega = \sum_{i=1}^n dp^i \wedge dx^i$), for the so-called Schrödinger polarization $\mathcal{P}_{Sch}$ spanned pointwise by $\{\frac{\partial}{\partial p^i}\}_{i=1,\dots n}$, half-form quantization produces quantum states of the form

$$\psi(x_1,\dots,x_n)\sqrt{dx^1\wedge\cdots\wedge dx^n},$$

so that the inner product of two of these states becomes the ordinary inner product in $L^2(\mathbb{R}^n, dx)$,

$$\langle\psi_1,\psi_2\rangle = \int_{\mathbb{R}^n}\psi_1(x)\bar{\psi}_2(x)dx,$$

where $dx = dx^1\wedge\cdots\wedge dx^n$. In the compact case, real polarizations are often singular, that is the corresponding Lagrangian foliation of X will contain singular leaves, so that the definition of half-forms may become problematic at the singular set. In the present review, as described below, we will describe instances of quantization in real polarizations viewed as (degenerate) limits of holomorphic polarizations. For holomorphic polarizations, by contrast with general real polarizations, the definition of half-forms can be given clearly in terms of the canonical bundle.

If $\mathcal{P}$ is a Kähler polarization, let K_J be the canonical bundle of X associated to the complex structure J defining $\mathcal{P}$. Suppose that K_J admits a square root $\delta_J\to X$, that is $\delta_J^{\otimes 2}\cong K_J$. Lie derivation along $\mathcal{P}$ provides a partial connection ∇^{K_J} on K_J. Then $\nabla\otimes 1 + 1\otimes\frac{1}{2}\nabla^{K_J}$ defines a partial connection on $L\otimes\delta_J$. The (half-form corrected) space of quantum states is then defined to be

$$\mathcal{H}_{\mathcal{P}} = H^0_{L^2}(X, L\otimes\delta_J).$$

Note that, in general for the compact case, $\dim H^0(X, L\otimes\delta_J)\neq\dim H^0(X,L)$.

Let $\mathcal{P},\mathcal{P}'$ be two Kähler polarizations, associated to two complex structures J, J' on X, so that $K_J, K_{J'}$ admit square roots $\delta_J,\delta_{J'}$. There is a natural pairing[3] $H^0(X,\delta_J)\times H^0(X,\delta_{J'})\to C^\infty_{\mathbb{C}}(X)$,

$$(\nu,\nu') = \sqrt{\frac{\nu^2\wedge\bar{\nu}'^2}{(2i)^n(-1)^{n(n+1)}\omega^n/n!}},\ \nu\in C^\infty(\delta_J), \nu'\in C^\infty(\delta_{J'}). \tag{4}$$

[3]For $\mathcal{P}=\mathcal{P}'$ we can choose the plus sign of the square root and a consistent sign can be chosen, for instance, if the complex structures vary on a simply connected space.

The *Blattner-Kostant-Sternberg (BKS) pairing* between $\mathcal{H}_{\mathcal{P}}$ and $\mathcal{H}_{\mathcal{P}'}$ is then defined by

$$\langle s \otimes \nu, s' \otimes \nu' \rangle = \int_X h(s,s')(\nu,\nu')\frac{\omega^n}{n!},$$

for $s \otimes \nu \in \mathcal{H}_{\mathcal{P}}$, $s' \otimes \nu' \in \mathcal{H}_{\mathcal{P}'}$, where h is the hermitian structure on L. When this pairing is nondegenerate, it gives rise to a *BKS pairing map*, $B_{\mathcal{P}\mathcal{P}'} : \mathcal{H}_{\mathcal{P}'} \to \mathcal{H}_{\mathcal{P}}$. It is an important question in general to determine whether these maps are unitary and whether they are transitive, that is whether $B_{\mathcal{P}\mathcal{P}'} \circ B_{\mathcal{P}'\mathcal{P}''} = B_{\mathcal{P}\mathcal{P}''}$.

When (X, ω) is compact and Kähler and $\mathcal{P}$ is the holomorphic polarization associated to the complex structure J, the square of (4) defines the hermitian structure on K_J determined by the Kähler Riemannian metric on X. In this case, ∇^{K_J} is the Chern-Levi-Civita connection on K_J and its curvature is $i\rho_J$, where $\rho_J = Ric(J\cdot, \cdot)$ is the Ricci form, with $c_1(X) = -c_1(K_J) = -[\frac{\rho_J}{2\pi}] \in H^2(X, \mathbb{Z})$.

In the following sections, the half-form correction plays a crucial role in the definition of quantization in real polarizations. On one hand, it ensures the nice convergence of holomorphic sections to distributional ones. On the other hand, it provides a natural definition of the inner product on the space of states for certain real polarizations obtained by degeneration of holomorphic ones.

2.4. **Degeneration of complex structures.** Some interesting classes of symplectic manifolds can be explicitly quantized in both real and Kähler polarizations. One unifying principle that seems to be very interesting to adopt in the study of these examples, is to view real polarizations as limits of degenerating Kähler polarizations. (For earlier work on this idea, wih a focus on the counting of the dimension of the space of quantum states, see [An].) Of course, finding a nice workable family of Kähler polarizations that degenerate to a given interesting real polarization is very hard in general. However, for some interesting families of symplectic manifolds, namely cotangent bundles of Lie groups, Abelian varieties and toric varieties this turns out to be possible. These examples will be detailed in the next sections.

For these examples, one can actually follow the behavior of the holomorphic sections of L along the degeneration and study their convergence to distributional sections. For cotangent bundles of Lie groups and for Abelian varieties, the degeneration process can be described analytically in a very nice way, in terms of the so-called coherent state transform for Lie groups of Hall [Hal1, Hal2, Hal3].

In all of these examples, and also for the flat case $X = \mathbb{R}^{2n}$ (see Section 9.9 and Chapter 10 of [Wo], Chapter 22 of [Hal5] and [KW]), the inclusion of the half-form correction turns out to be crucial in ensuring the correct behavior of the sections as the complex structure degenerates. Moreover, one can study the unitarity (or lack of unitarity) of the BKS pairing maps relating quantizations in different polarizations whithin the families considered. Note that unitarity of the BKS pairing maps is to be expected only in special examples. For example, it does not hold for different quantizations of symplectic toric manifolds [KMN]. For earlier examples, see also [R].

The inclusion of the half-form correction is not always straightforward, however. For instance, the hermitian structure on the space of half-forms degenerates when applied to a real polarization. The family of degenerating complex structures then proves crucial in the definition of a natural hermitian structure on the space of quantum states for the real polarization. (See, for example, [FMMN1, FMMN2, BMN].) In [KMN], the degenerating family is also instrumental in defining the half-form corrected connection on $L \otimes \delta$ for the toric real polarization.

In nice examples, like the ones that will be studied in the following sections, there is explicit control over a family of complex structures $\mathcal{J}$, where some points on the boundary of $\mathcal{J}$ correspond to real polarizations. Collecting the Hilbert spaces of quantum states for all $J \in \mathcal{J}$ then gives a bundle of Hilbert spaces $\mathcal{H} \to \mathcal{J}$. The question of the transivity of the BKS pairing maps can then be formulated in terms of the relation of these maps to the parallel transport of a flat, or projectively flat, connection on $\mathcal{H}$. The unitarity of this connection, corresponding to the unitarity of the BKS pairing maps, would then correspond to the equivalence of quantizations in the different polarizations parametrized by $\mathcal{J}$.

3. Cotangent Bundles of Lie Groups

The results in this Section are mostly contained in [FMMN1, FMMN2]. Cotangent bundles of Lie groups are a source for other families of interesting examples. When we take $(\mathbb{C}^*)^n \cong T^*\mathbb{T}^n$, taking a quotient by an appropriate lattice produces an abelian variety whose quantization leads to theta functions (Section 4). On the other hand, $(\mathbb{C}^*)^n$ also plays the role of the open dense orbit in a toric manifold (Section 5).

3.1. **The coherent state transform for Lie groups.** A very interesting relation between functions on a Lie group K of compact type (a product of $\mathbb{R}^k \times H$ where H is compact, semi-simple) and certain functions on its complexification $K_\mathbb{C}$ has been described by Hall in [Hal1, Hal2, Hal3]. Let K be equipped with a bi-invariant metric with (negative definite) Laplacian Δ. Let $\rho_t, t > 0$ be the heat kernel on K, for the heat equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$. Hall proves that ρ_t has a unique analytic continuation to $K_\mathbb{C}$, also denoted by ρ_t.

The Ad_K-invariant inner product $\langle \cdot, \cdot \rangle$, on $LieK$ defines a real valued Ad_K-invariant inner product on $LieK_\mathbb{C} = LieK \otimes \mathbb{C} = LieK \oplus iLieK$, by $\langle A+iB, A'+iB' \rangle = \langle A, A' \rangle + \langle B, B' \rangle$. Let $\tilde{\Delta}$ be the Laplace operator associated to the resulting K-bi-invariant and left $K_\mathbb{C}$-invariant metric on $K_\mathbb{C}$ and let μ_t be its heat kernel.

The averaged heat kernel $\nu_t \in C^\infty(K_\mathbb{C})$ is then defined by

$$\nu_t(g) = \int_K \mu_t(xg)dx,$$

where dx is the Haar measure on K. This function has gaussian decay along the fibers of $T^*K \cong K_\mathbb{C}$. Let $\mathcal{C}$ denote analytic continuation from K to $K_\mathbb{C}$. Consider the following map, for $t > 0$, which is called the *coherent state transform* (CST),

$$\begin{aligned} C_t : L^2(K, dx) &\to \mathcal{H}L^2(K_{\mathbb{C}}, d\nu_t) \\ f &\mapsto C_t(f) = \mathcal{C} \circ e^{\frac{t}{2}\Delta} f, \end{aligned}$$

where $\mathcal{H}L^2(K_{\mathbb{C}}, d\nu_t)$ denotes the space of holomorphic functions on $K_{\mathbb{C}}$ which are L^2 with respect to the measure $d\nu_t = \nu_t dg$. Hall proves:

Theorem 3.1 (Hall, [Hal1]). *For all $t > 0$, C_t is a unitary isomorphism of Hilbert spaces.*

As will be described in the following, the CST plays an important role in quantization. In the context of Lie groups it is an explicit analytical tool that transforms "real data" into "holomorphic data", while preserving unitarity. For more general cotangent bundles a similar role is played by the FBI transform of [LGS] which, however, in general does not have the nice properties of the CST.

3.2. **Quantization of T^*K.** In [Hal3], Hall related the quantization of T^*K to the CST. Let $\dim K = n$ and $\{X^i\}_{1,\ldots,n}$ be an orthonormal basis of left invariant vector fields on K and let $\{y^i\}_{i=1,\ldots,n}$ be the corresponding coordinates on $(LieK)^* \cong LieK$. Let $\{w^i\}_{i=1,\ldots,n}$ be the basis of left-invariant 1-forms on K dual to the vector fields X^i. We will denote also by w^i their pull-backs to T^*K by the canonical projection. Consider the symplectic structure on T^*K given by

$$\omega = -d\theta,$$

where $\theta = \sum_{i=1}^n y^i w^i$ is the canonical 1-form. The diffeomorphism

$$\begin{aligned} T^*K \cong K \times LieK &\to K_{\mathbb{C}} \\ (x, Y) &\mapsto xe^{iY} \end{aligned}$$

can be used to pull-back the canonical complex structure from $K_{\mathbb{C}}$ to T^*K, so that (T^*K, ω) becomes a Kähler manifold. The prequantum bundle is the trivial bundle $T^*K \times \mathbb{C}$, so that its sections are just functions on T^*K. The half-form quantization of T^*K in the vertical polarization

$$\mathcal{P}_0 = \langle \frac{\partial}{\partial y^i}, i = 1, \ldots, n \rangle_{\mathbb{C}}$$

then produces

$$\mathcal{H}_0 = \{f \otimes \sqrt{dx}, f \in L^2(K, dx)\} \cong L^2(K, dx),$$

where $dx = w^1 \wedge \cdots \wedge w^n$, $\sqrt{dx}$ represents a half-form whose square is dx, and where we denote the pull-back of $f \in L^2(K, dx)$ to T^*K also by f. In the following, for simplicity, we will omit the symbol $\otimes$ and will write simply $f\sqrt{dx}$ whenever we are considering the tensor product with half-forms.

Hall has studied the quantization of T^*K in the Kähler polarization, to find the Hilbert space of quantum states

$$\mathcal{H}_1 = \{\sigma = F(xe^{iY}) e^{-\frac{|Y|^2}{2}} \sqrt{dZ}, F \text{ holomorphic}, \langle \sigma, \sigma \rangle < \infty\},$$

where dZ is the trivializing section of the canonical bundle of T^*K, obtained by wedging a basis of holomorphic 1-forms on T^*K obtained by pull-back of holomorphic left $K_{\mathbb{C}}$-invariant 1-forms on $K_{\mathbb{C}}$.

One has the isomorphism of Hilbert spaces

$$\begin{array}{rcl} \mathcal{H}_1 & \cong & \mathcal{H}L^2(K_{\mathbb{C}}, d\nu_1) \\ F(xe^{iY})e^{-\frac{|Y|^2}{2}}\sqrt{dZ} & \mapsto & F(xe^{iY}). \end{array}$$

Hall proves:

Theorem 3.2. [Hal3] *The BKS pairing map $\mathcal{H}_0 \to \mathcal{H}_1$ coincides with the CST up to a multiplicative constant.*

Therefore, the quantizations of T^*K in the vertical real polarization and in the holomorphic polarization are related by the CST. That is, holomorphically polarized states are essentially obtained by applying the heat operator to an L^2 function on K followed by analytic continuation to T^*K. Of course, we are free to rescale the inner product in $\mathcal{H}_0$ by the appropriate constant to obtain unitarity. However, this procedure seems ad hoc unless there is some justification for it. In fact, this justification is provided by a continuous family of Kähler structures on T^*K whose holomorphic polarizations degenerate to the vertical one.

3.3. **A family of equivalent quantizations of T^*K.** Consider the following one-parameter family of diffeomorphisms, for $s > 0$,

$$\begin{array}{rcl} T^*K \cong K \times LieK & \stackrel{\psi_s}{\to} & K_{\mathbb{C}} \\ (x, Y) & \mapsto & xe^{isY} \end{array}$$

Let J_s denote the complex structure on T^*K obtained by pull-back of the canonical complex struture on $K_{\mathbb{C}}$ by ψ_s. Then, J_s is still compatible with ω and one obtains a one-parameter family of Kähler structures on T^*K labeled by s. Let Ω_s be the corresponding trivializing section of the canonical bundle of T^*K. Let $\mathcal{P}_s, s > 0$ be the Kähler polarization of T^*K associated to J_s.

Proposition 3.3. [FMMN1] *One has $\lim_{s\to 0}\mathcal{P}_s = \mathcal{P}_0$, pointwise in the Grassmannian of Lagrangian subspaces on T^*K.*

and

Theorem 3.4. [FMMN1] *The Hilbert space of quantum states for the half-form quantization of T^*K in the polarization $\mathcal{P}_s, s > 0$ is*

$$\mathcal{H}_s = \{\sigma_s = F(xe^{isY})e^{-s\frac{|Y|^2}{2}}\sqrt{\Omega_s}, F \text{ holomorphic}, \langle\sigma_s, \sigma_s\rangle < \infty\}. \tag{5}$$

As before, there is an isomorphism of Hilbert spaces

$$\begin{array}{rcl} \mathcal{H}_s & \cong & \mathcal{H}L^2(K_{\mathbb{C}}, d\nu_s) \\ F(xe^{isY})e^{-s\frac{|Y|^2}{2}}\sqrt{\Omega_s} & \mapsto & F(xe^{isY}). \end{array}$$

Let $s, s' > 0$, $F \in \mathcal{H}L^2(K_{\mathbb{C}}, d\nu_s), F' \in \mathcal{H}L^2(K_{\mathbb{C}}, d\nu_{s'})$ where $F = C_s(f)$ and $F' = C_{s'}(f'), f, f' \in L^2(K, dx)$. Let $\sigma_s \in \mathcal{H}'_s, \sigma_{s'} \in \mathcal{H}_{s'}$ be given by

$$\begin{aligned} \sigma_s &= F(xe^{isY})e^{-s\frac{|Y|^2}{2}}\sqrt{\Omega_s} \\ \sigma'_{s'} &= F'(xe^{is'Y})e^{-s'\frac{|Y|^2}{2}}\sqrt{\Omega_{s'}}. \end{aligned}$$

We then have

Theorem 3.5. [FMMN2] *The BKS pairing between $\mathcal{H}_s, \mathcal{H}_{s'}, s, s' > 0$ is given by*

$$\langle \sigma_s, \sigma'_{s'} \rangle = a_{\frac{s+s'}{2}} \langle f, f' \rangle_{L^2(K,dx)}, \tag{6}$$

where $a_s = \pi^{\frac{n}{2}} e^{s|\rho^2|}$ and ρ is the Weyl vector given by half the sum of the positive roots of LieK. (If K is abelian one can set $\rho = 0$.)

The BKS pairing between the vertically polarized Hilbert space $\mathcal{H}_0$ and $\mathcal{H}_s$, $s > 0$, is obtained by taking the $s' \to 0$ limit of the right hand side of (6).

Let the *quantum bundle*, $\mathcal{H} \to \mathbb{R}^+$ be the bundle of Hilbert spaces with fiber $\mathcal{H}_s$ over $s > 0$. We can extend this bundle to $\mathbb{R}_0^+$ by including the fiber $\mathcal{H}_0$ over $s = 0$, representing the vertical polarization. Since $a_0 = \pi^{\frac{n}{2}}$, we see that Theorem 3.5 suggests that we rescale the inner product in $\mathcal{H}_0$ by a factor of $\pi^{\frac{n}{2}}$ in order to obtain a continuous hermitian structure on the extended bundle $\bar{\mathcal{H}} \to \mathbb{R}_0^+$. Let us redefine the inner product in $\mathcal{H}_0$ in this way.

Consider the unitary isomorphisms, for $s > 0$,

$$\begin{aligned} S_s : \mathcal{H}L^2(K_{\mathbb{C}}, d\nu_s) &\to \mathcal{H}_s \\ F &\mapsto F a_s^{-\frac{1}{2}} e^{-s\frac{|Y|^2}{2}} \sqrt{\Omega_s}. \end{aligned}$$

Unitarity of the CST and the fact that $\sqrt{a_s a_{s'}} = a_{\frac{s+s'}{2}}$ then imply

Theorem 3.6. [FMMN2] *For $s, s' \geq 0$ the BKS pairing map $B_{ss'} : \mathcal{H}_{s'} \to \mathcal{H}_s$ is unitary and for $s, s' > 0$ it is given by $B_{ss'} = S_s \circ C_s \circ C_{s'}^{-1} \circ S_{s'}^{-1}$.*

Therefore, we have a one-parameter family of equivalent holomorphic quantizations of T^*K which are connected, and are also unitarily equivalent, to the Schrödinger quantization in the vertical polarization. Moreover, essentially, the holomorphically polarized sections are generated from vertically polarized ones by applying the CST.

3.4. **The connection on the quantum bundle.** Let $\eta_s \to T^*K$ be the half-form bundle for the complex structure J_s, which has trivializing section $\sqrt{\Omega_s}$. Let $L = T^*K \times \mathbb{C}$ be the trivial bundle. Elements of the space of smooth sections of $L \otimes \eta_s \to T^*K$, $C^\infty(L \otimes \eta_s)$, are then of the form $\sigma_s = f\sqrt{\Omega_s}$, where $f \in C^\infty(T^*K)$. Let

$$V_s = \{\sigma_s \in C^\infty(L \otimes \eta_s), \langle \sigma_s, \sigma_s \rangle < \infty\}.$$

The "prequantum" bundle $\mathcal{H}^{pQ} \to \mathbb{R}^+$ is the bundle of Hilbert spaces with fiber over $s > 0$ given by the norm completion of V_s, $\mathcal{H}_s^{pQ} = \bar{V}_s$. One defines a smooth structure on $\mathcal{H}^{pQ}$ by declaring that the sections of the form $\frac{f}{\sqrt{|\Omega_s|}}\sqrt{\Omega_s}$, $f \in C^\infty(T^*K)$ are smooth. Note that these sections have s-independent norm.

A hermitian connection δ on $\mathcal{H}^{pQ}$ is defined by letting

$$\delta\left(\frac{f}{\sqrt{|\Omega_s|}}\sqrt{\Omega_s}\right) = 0,$$

for all $f \in L^2(T^*K, \frac{\omega^n}{n!})$.

From (5), we see that the quantum bundle $\mathcal{H}$ of the previous Section is a Hilbert sub-bundle of $\mathcal{H}^{pQ}$. One can therefore define a hermitian connection on $\mathcal{H}$, δ^Q by setting $\delta^Q = P \circ \delta$, where P denotes the fiberwise orthogonal projection $\mathcal{H}^{pQ} \to \mathcal{H}$.

The BKS pairing on $\mathcal{H}$ also defines a hermitian connection δ^{BKS} by

$$\langle \sigma_s, \delta^{BKS}_{\frac{\partial}{\partial s}} \sigma'_{s'} \rangle = \left.\frac{\partial}{\partial s'}\right|_{s=s'} \langle \sigma_s, \sigma'_{s'} \rangle \tag{7}$$

where $\sigma_s \in \mathcal{H}_s, \sigma'_{s'} \in \mathcal{H}_{s'}$.

From the definitions there follows,

Theorem 3.7. [FMMN2] *The connections δ^Q and δ^{BKS} on $\mathcal{H}$ are equal. Moreover, the parallel transport between two fibers of $\mathcal{H}$ defined by δ^Q coincides with the corresponding BKS pairing map.*

Note that different pairings defined on $\mathcal{H}$ can define the same connection δ^Q via (7). In general, the parallel transport for that connection will therefore not coincide with the pairing one starts with. It is remarkable that in this case the parallel transport of the connection defined by the BKS pairing *is* given by the BKS pairing maps. This is reflected in the fact that the BKS pairing maps in Theorem 3.6 are transitive, that is, $B_{ss'} \circ B_{s's''} = B_{ss''}$ for all $s, s', s'' \geq 0$.

Finally, let us note that horizontal sections of $\mathcal{H}$ satisfy a heat equation. Let $\Delta_{\mathbb{C}}$ be the quadratic Casimir operator on $K_{\mathbb{C}}$ and consider a section of $\mathcal{H}$ of the form $\sigma(s) = F(s, xe^{isY}) a_s^{-\frac{1}{2}} e^{-s\frac{|Y|^2}{2}} \sqrt{\Omega_s}$, where F is holomorphic in the variable xe^{isY}.

Theorem 3.8. [FMMN1] $\delta^Q_{\frac{\partial}{\partial s}} \sigma = 0$ *iff*

$$\frac{\partial F}{\partial s} = \frac{1}{4}\Delta_{\mathbb{C}} F.$$

That is, the connection form for δ^Q takes the form of a second order differential operator along the fibers of $\mathcal{H}$. This is in close analogy with the form of the stress energy tensor in certain conformal field theories (through the Sugawara construction), as described in [AdPW, Hi], and to the heat equation satisfied by theta functions to which we now turn our attention.

4. ABELIAN VARIETIES

The results in this section are mostly contained in [FMN1, FMN2, BMN]. Let $\Lambda \subset \mathbb{Z}^{2g}$ be a maximal rank lattice such that $X = \mathbb{C}^g/\Lambda$ is an abelian variety, that is a complex torus that can be holomorphically embedded in projective space. Assume that X is principally polarized so that for a convenient choice of basis one has $\Lambda = \mathbb{Z}^g \oplus \Omega\mathbb{Z}^g$, where the (symmetric) period matrix Ω satisfies $\operatorname{Im}\Omega > 0$, that is $\Omega \in \mathbb{H}_g$ where $\mathbb{H}_g$ is the Siegel upper half-space. Then, one has $X = (\mathbb{C}^*)^g/\Omega\mathbb{Z}^g$ where $(\mathbb{C}^*)^g \cong \mathbb{C}^g/\mathbb{Z}^g$.

Classical theta functions on X are holomorphic sections of holomorphic line bundles on X. The following sequence of maps

$$U(1)^g \hookrightarrow (\mathbb{C}^*)^g \to X \tag{8}$$

allows one to pull-back such sections to the Stein space $(\mathbb{C}^*)^g$ where they become holomorphic functions with appropriate quasi-periodicity conditions. This is the classical description of theta functions. It is interesting to note, however, that $(\mathbb{C}^*)^g$ is the complexification of the compact Lie group $U(1)^g$. It turns out that theta functions are in the image of an extension of the CST to an appropriate space of distributions on $U(1)^g$. More importantly, the natural inner product[4] on the space of theta functions can also be described very naturally in terms of the analytical ingredients of the CST.

4.1. **Theta functions and the CST.** Let $\Lambda = \mathbb{Z}^g \oplus \Omega\mathbb{Z}^g$, with generators $\{\lambda_i, \lambda_{i+g} = \sum_{j=1}^g \Omega_{ji}\lambda_j\}_{i=1,\ldots,g}$, and let $L \to X$ be the holomorphic line bundle defined by the automorphy factors

$$e_{\lambda_i}(z) = 1, e_{\lambda_{i+g}}(z) = e^{-2\pi i z_i - \pi i \Omega_{ii}}, z \in \mathbb{C}^g, i = 1, \ldots, g.$$

Holomorphic sections of L are identified with functions on $\mathbb{C}^g$ that satisfy the quasi-periocidity conditions

$$\theta(z+\lambda_i, \Omega) = \theta(z, \Omega), \;\; \theta(z+\lambda_{g+i}, \Omega) = e^{-2\pi i z_i - \pi i \Omega_{ii}}\theta(z, \Omega), \tag{9}$$

for $z \in \mathbb{C}^g, i = 1, \ldots, g$.

Note that these holomorphic functions are in fact well defined on $(\mathbb{C}^*)^g \cong \mathbb{C}^g/\mathbb{Z}^g$. If k is a positive integer, one calls $H^0(X, L^k)$ the space of *level k theta functions* on X. (See, for example, Chapter 5 of [Ke] for the classical theory of theta functions.)

One has $\dim H^0(X, L^k) = k^g$. A basis for this space is given by $\{\theta_l(z, \Omega)\}_{l \in \mathbb{Z}^g/k\mathbb{Z}^g}$, where

$$\theta_l(z, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i (l+kn)\cdot \frac{\Omega}{k}(l+kn) + 2\pi i (l+kn)\cdot z}.$$

Let $e^{2\pi i z}$ be coordinates on $(\mathbb{C}^*)^g$, $z = x - \Omega y$, with $x \in [0,1]^g$ periodic coordinates along the compact subgroup $U(1)^g \subset (\mathbb{C}^*)^g$. Consider the complex (non self-adjoint) Laplace operator on $U(1)^g$

$$\Delta_\Omega = -\sum_{i,j=1}^g \frac{i}{2\pi}\Omega_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

[4]This is defined naturally in terms of the Appel-Humbert data for the line bundle in question.

Note that the real part of Δ_Ω, which is determined by $\operatorname{Im}\Omega > 0$, is an elliptic operator. It is easy to verify that the restriction of $\theta_l(z,\Omega)$ to $U(1)^g$ satisfies a heat equation:

$$\frac{\partial}{\partial t}\theta_l(x,t\Omega) = \frac{1}{2}\Delta_\Omega\theta_l(x,t\Omega).$$

This is a classical and well known fact that is, however, interesting to look at from the point of view of the CST.

4.2. **Degeneration of complex structure.** The generalization of the coherent state transform of Hall to the complex Laplace operator Δ_Ω is straightforward, since the exponentiation of the imaginary part of Δ_Ω produces a unitary operator.

Proposition 4.1. [FMN1] *For any $\Omega \in \mathbb{H}_g$ and $t > 0$ the transform*

$$C_t^\Omega = \mathcal{C} \circ e^{\frac{t}{2}\Delta_\Omega} : L^2(U(1)^g, dx) \to \mathcal{H}L^2((\mathbb{C}^*)^g, d\nu_t)$$

is unitary.

The application to theta function needs, however, further generalization. Indeed, for theta functions the initial condition for the heat equation, at $t = 0$, reads:

$$\theta_l(x,0) = \sum_{n\in\mathbb{Z}^g} e^{2\pi i(l+kn)\cdot x} = \sum_{l'\in\mathbb{Z}^g/k\mathbb{Z}^g} k^{-g}e^{2\pi i\frac{l\cdot l'}{k}}\delta_{l'}(x),$$

where

$$\delta_l(x) = \delta\left(x - \frac{l}{k}\right) = \sum_{n\in\mathbb{Z}^g} e^{2\pi i n\cdot(x-\frac{l}{k})},$$

is the Dirac delta distribution on $U(1)^g$ supported at $x = \frac{l}{k}$. Therefore, this initial condition is distributional (and not L^2). Recall that the space of distributions on $U(1)^g$, $C^\infty(U(1)^g)'$ is the space of Fourier series of the form

$$\sum_{n\in\mathbb{Z}^g} a_n e^{2\pi i x\cdot n},$$

where the coefficients a_n grow slower than some power of n. By linearity, one can extend the CST to distributions obtaining

Lemma 4.2. [FMN1] *Let $\sum_{n\in\mathbb{Z}^g} a_n e^{2\pi i x\cdot n} \in C^\infty(U(1)^g)'$. Then, for $t > 0$,*

$$C_t^\Omega\left(\sum_{n\in\mathbb{Z}^g} a_n e^{2\pi i x\cdot n}\right) := \sum_{n\in\mathbb{Z}^g} a_n e^{t i\pi n\cdot\Omega n}e^{2\pi i z\cdot n}$$

is a holomorphic function on $(\mathbb{C}^*)^g$.

Note that $\theta_l(z,\Omega) = C_{1/k}^\Omega(\theta_l(x,0)), \forall l \in \mathbb{Z}^g/k\mathbb{Z}^g$. Consider the k^g-dimensional subspace of $C^\infty(U(1)^g)'$, $\mathcal{F}_k$, with basis $\{\delta_l(x)\}_{l\in\mathbb{Z}^g/k\mathbb{Z}^g}$. Then, clearly, there is an isomorphism

$$\mathcal{F}_k \overset{C_{1/k}^\Omega}{\cong} H^0(X, L^k). \tag{10}$$

The natural (geometrically defined) inner product on $H^0(X, L^k)$ gives that $\{\theta_l(z,\Omega)\}_{l\in\mathbb{Z}^g/k\mathbb{Z}^g}$ is an orthonormal basis. In particular, the inner product between the θ_l's is Ω-independent. It is remarkable that this inner product is precisely the one that it is naturally defined on $H^0(X, L^k)$ also from the point of view of the (generalized) CST, as we describe next.

Since theta functions require distributional conditions, lying therefore outside of $L^2(U(1)^g, dx)$, one loses the unitarity of the generalized CST. On the other hand, even though we are viewing theta functions as holomorphic functions on $(\mathbb{C}^*)^g$, since they are really pull-backs of holomorphic sections of L^k over X, we would like to integrate them not over the whole of $(\mathbb{C}^*)^g$ but only over a fundamental domain for the projection $(\mathbb{C}^*)^g \to X$.

Remarkably, this is possible because independence on the choice of fundamental domain follows from the fact that the averaged heat kernel defines a hermitian structure on (the pull-back of) L^k:

Lemma 4.3. [FMN1] *The function $\nu_{1/k}(z)$ on $(\mathbb{C}^*)^g$ is the pull-back of a hermitian structure on L^k. Moreover, $\{\theta_l(z,\Omega)\}_{l\in\mathbb{Z}^g/k\mathbb{Z}^g}$ is an orthonormal basis of $H^0(X, L^k)$ for this hermitian structure.*

Defining on $\mathcal{F}_k$ the inner product for which the basis $\{\theta_l(x,0)\}_{l\in\mathbb{Z}^g/k\mathbb{Z}^g}$ is orthonormal, the isomorphism in (10) becomes unitary.

Given $\Omega, \Omega' \in \mathbb{H}_g$, let $X_\Omega, X_{\Omega'}$ be the two abelian varieties defined by the corresponding complex structures on the smooth manifold underlying X. We see that level k quantizations of (the underlying smooth symplectic manifold of) X for the corresponding complex structures are therefore related by unitary maps $C^{\Omega}_{1/k} \circ (C^{\Omega'}_{1/k})^{-1} : \mathcal{H}_{\Omega'} \to \mathcal{H}_\Omega$, where $\mathcal{H}_\Omega = H^0(X_\Omega, L^k_\Omega)$ is the space of quantum states for the holomorphic quantization of X in the complex structure defined by Ω.

The interpretation of the distributions δ_l in terms of Bohr-Sommerfeld fibers and also of the unitary matrix $S_{ll'} = k^{-\frac{g}{2}} e^{2\pi i \frac{l\cdot l'}{k}}$, which is known to describe the action of the modular transformation $\tau \mapsto -\frac{1}{\tau}$ on the characters of integrable representations of the affine Lie algebra $\widehat{u(1)}_k$, is described in the next section.

4.3. **Quantization of tori.** In this section we will describe theta functions as the product of the geometric quantization of a symplectic torus in a holomorphic polarization. The distributions described in the previous section appear in the quantization associated to the real polarization where $\Omega \to 0$.

Let $\mathbb{T}^{2g} = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ be the standard even dimensional torus of dimension $2g$, equipped with the standard symplectic form $\omega = \sum_{i=1}^g dy_i \wedge dx_i$, where $x, y \in \mathbb{R}^g$ are periodic coordinates.

Consider the $\mathbb{Z}^{2g}$ action on $\mathbb{R}^{2g} \times \mathbb{C}$ given by

$$\lambda \cdot (u, \zeta) = (u + \lambda, \alpha(\lambda) e^{-\pi i \omega(u,\lambda)} \zeta), \lambda \in \mathbb{Z}^{2g}, u \in \mathbb{R}^{2g}, \tag{11}$$

where $\alpha : \mathbb{Z}^{2g} \to \pm 1 \subset U(1)$ is the "canonical semi-character" $\alpha(\lambda) = (-1)^{\sum_{i=1}^g \lambda_i \lambda_{i+g}}$.

The line bundle $L \to \mathbb{T}^{2g}$ given by $L = \mathbb{R}^{2g} \times_{\mathbb{Z}^{2g}} \mathbb{C}$ defines a prequantum line bundle, with hermitian structure $h((u,\zeta),(u',\zeta')) = \zeta\bar{\zeta}'$ and compatible connection

$$\nabla s = ds - \pi i s \sum_{j=1}^{g} (y_j dx^j - x_j dy_j).$$

Here, sections of L are identified with quasiperiodic functions on $\mathbb{R}^{2g}$ satisfying

$$s(u+\lambda) = \alpha(\lambda) e^{-\pi i \omega(u,\lambda)} s(u), \lambda \in \mathbb{Z}^{2g}.$$

If $s \in C^\infty(L^k)$ then $e^{k\pi i x\cdot y} s(x,y)$ is periodic in x and admits a Fourier decomposition, leading to the Weil-Brezin [Fo] isomorphism,

$$\begin{array}{rcl} C^\infty(L^k) & \to & \Pi_{l\in(\mathbb{Z}/k\mathbb{Z})^g} \mathcal{S}(\mathbb{R}^g) \\ s & \mapsto & (s)_l(y) = \int_{[0,1]^g} s(x, y + \frac{l}{k}) e^{k\pi i x\cdot(y+\frac{l}{k})} e^{-2\pi i l\cdot x} d^g x, \end{array} \tag{12}$$

where $\mathcal{S}(\mathbb{R}^g)$ is the Schwarz space of rapidly decreasing functions. The inverse form of this decomposition is

$$\begin{array}{rcl} \Pi_{l\in(\mathbb{Z}/k\mathbb{Z})^g} \mathcal{S}(\mathbb{R}^g) & \to & C^\infty(L^k) \\ \{(s)_l\}_{l\in(\mathbb{Z}/k\mathbb{Z})^g} & \mapsto & s(x,y) = e^{k\pi i x\cdot y} \sum_{l\in(\mathbb{Z}/k\mathbb{Z})^g, m\in\mathbb{Z}^g} (s)_l(y - m - \frac{l}{k}) e^{2\pi i (l+km)\cdot x}. \end{array} \tag{13}$$

This map is an isomorphism between the topological space of smooth sections of L^k and a product of Schwarz spaces and extends to an isomorphism of the dual spaces of distributional sections of L^k and of a product of k^g spaces of tempered distributions on $\mathbb{R}^g$. For L^2 sections s, s' we have

$$\langle s, s' \rangle = \sum_{l\in(\mathbb{Z}/k\mathbb{Z})^g} \langle (s)_l, (s')_l \rangle = \sum_{l\in(\mathbb{Z}/k\mathbb{Z})^g} \int (s)_l (\bar{s}')_l.$$

Let now $\Omega = \Omega_1 + i\Omega_2 \in \mathbb{H}_g$ and let $X_\Omega = \mathbb{C}^g/\mathbb{Z}^g \oplus \Omega\mathbb{Z}^g$ be the corresponding principally polarized abelian variety, as described in the previous Section. The smooth isomorphism $\phi_\Omega : \mathbb{T}^{2g} \to X_\Omega$, given by[5] $\phi_\Omega(x,y) = z_\Omega = x - \Omega y \in \mathbb{C}^g$, defines on $\mathbb{T}^{2g}$ a structure of abelian variety and, in particular, defines a Kähler structure on it.

Let F_Ω be the bilinear form on $\mathbb{C}^g \times \mathbb{C}^g$, $F_\Omega(z,w) = -z \cdot \Omega_2^{-1} \mathrm{Im}\, w$. Then, Ω defines a holomorphic structure on L, identifying it with the holomorphic line bundle of the previous Section, as follows. $L_\Omega = \mathbb{C}^g \times_{\Lambda_\Omega} \mathbb{C}$, where $\Lambda_\Omega = \mathbb{Z}^g \oplus \Omega\mathbb{Z}^g$ acts as in (11) combined with the isomorphism

$$\mathbb{R}^{2g} \times \mathbb{C} \ni ((x,y),\zeta) \mapsto (x - \Omega y, e^{\pi i F_\Omega(z_\Omega, z_\Omega)}\zeta).$$

(This corresponds to a trivialization where the conditions (11) correspond to (9).)

As mentioned in the previous Section, $H^0(X_\Omega, L_\Omega^k)$ has a basis given by

$$\{\theta_l(z,\Omega)\}_{l\in\mathbb{Z}^g/k\mathbb{Z}^g}.$$

[5]The minus sign will be convenient below.

In terms of the original trivialization, these classical theta functions correspond to sections

$$\vartheta^l_\Omega(x,y) = e^{-k\pi i F_\Omega(z_\Omega, z_\Omega)}\theta_l(z,\Omega), \tag{14}$$

which have Weil-Brezin coefficients given by gaussians

$$(\vartheta^l_\Omega)_{l'}(y) = \delta_{ll'}e^{k\pi i y\cdot\Omega y}.$$

For $\Omega \in \mathbb{H}_g$, let K_Ω be the (trivial) canonical bundle of X_Ω with trivializing section $d^g z_\Omega = dz^1_\Omega \wedge \cdots dz^g_\Omega$. We can therefore choose for half-form bundle δ_Ω the trivial holomorphic bundle on X_Ω. The pairing (4) gives

$$\langle\sqrt{d^g z_\Omega}, \sqrt{d^g z_{\Omega'}}\rangle = \det\left(\frac{1}{2ki}(\Omega - \bar\Omega')\right)^{\frac{1}{2}}, \Omega, \Omega' \in \mathbb{H}_g,$$

where a consistent choice of square root is given by defining

$$\det\left(\frac{1}{2ki}(\Omega - \bar\Omega')\right)^{-\frac{1}{2}} := \int_{\mathbb{R}^g} e^{-\frac{\pi}{2ki}\xi\cdot(\Omega-\bar\Omega')\xi}d^g\xi.$$

We then have

Theorem 4.4. [BMN] *For $\Omega, \Omega' \in \mathbb{H}_g$, the BKS pairing is given by*

$$\langle\vartheta^l_\Omega \otimes \sqrt{d^g z_\Omega}, \vartheta^{l'}_{\Omega'} \otimes \sqrt{d^g z_{\Omega'}}\rangle = 2^{-\frac{g}{2}}k^{-g}\delta_{ll'}.$$

In order to study points at infinity in the boundary of $\mathbb{H}_g$, it is convenient to perform a Cayley transform and consider $\tau = (i-\Omega)(i+\Omega)^{-1}$, so that $\tau \in \mathbb{D}_g$ where the Siegel disk $\mathbb{D}_g$ is the closure of the image of $\mathbb{H}_g$ under the Cayley transform. The inverse transform is $\Omega = i(1-\tau)(1+\tau)^{-1}$. Translation invariant polarizations of $(\mathbb{T}^{2g}, \omega)$ are then given by

$$\mathcal{P}_\tau = \mathrm{span}_{\mathbb{C}}\left\{\sum_{l=1^g} -i(1-\bar\tau)_{lk}\frac{\partial}{\partial x_l} + (1+\bar\tau)_{lk}\frac{\partial}{\partial y_l}\right\}_{k=1,\ldots,g}.$$

Note that $\mathcal{P}_\tau$ is a real polarization iff $\tau \in \partial\mathbb{D}_g$ is unitary.

The study of covariant constant sections for τ in the boundary of $\mathbb{D}_g$ is much simplified by the existence of the Weil representation of the metaplectic group $Mp(2g,\mathbb{R})$, that is the connected double cover of $Sp(2g,\mathbb{R})$. (See [An, Fo, dG].) This gives an action of $Mp(2g,\mathbb{R})$ on $L^2(\mathbb{R}^g)$ which implies an action on sections of L^k covering the natural action of $Sp(2g,\mathbb{R})$ on $\mathbb{D}_g$. Let us describe this construction.

Consider the natural (transitive) action of $Sp(2g,\mathbb{R})$ on $\mathbb{H}_g$

$$M(\Omega) = (A\Omega+B)(C\Omega+D)^{-1},\ M = \begin{bmatrix} A & B \\ C & D\end{bmatrix} \in Sp(2g,\mathbb{R}), \tag{15}$$

which corresponds to the action by symplectomorphisms on $\mathbb{T}^{2g}$ given by

$$\mathbb{T}^{2g} \ni \begin{bmatrix} x \\ y\end{bmatrix} \mapsto M\begin{bmatrix} x \\ y\end{bmatrix} \in \mathbb{T}^{2g}.$$

(The action written in holomorphic coordinates is simply $z_\Omega \mapsto ((C\Omega+D)^t)^{-1}z_\Omega$.) The action on the Lagrangian Grassamannian is simply given by push-forward $M_*\mathcal{P}_\Omega = \mathcal{P}_{M(\Omega)}$.

The metaplectic group can then be defined in terms of an open subset $U \subset Sp(2g,\mathbb{R})$, such that $U \cdot U = Sp(2g,\mathbb{R})$, where U is the set of triples (P, L, Q) of $g \times g$ matrices, with L invertible, P, Q symmetric, with

$$(P, L, Q) \mapsto M_{(P,L,Q)} = \begin{bmatrix} PL^{-1} & PL^{-1}Q - L^t \\ L^{-1} & L^{-1}Q \end{bmatrix}. \tag{16}$$

The preimage of U in the double cover $Mp(2g,\mathbb{R})$ consists of two disjoint copies of U, and $Mp(2g,\mathbb{R})$ can be realized as a group of unitary operators on $L^2(\mathbb{R}^g)$, as follows. Let $m \in \mathbb{Z}/4\mathbb{Z}$ index a choice of square root i^m of the sign of $\det L$. Consider the integral operator $S(P, L, Q)_m : \mathcal{S}(\mathbb{R}^g) \to \mathcal{S}(\mathbb{R}^g)$ given by

$$S(P, L, Q)_m f(u) = i^{-\frac{g}{2}+m} k^{\frac{g}{2}} \sqrt{|\det L|} \int_{\mathbb{R}^g} e^{k\pi i(u\cdot Pu - 2u\cdot L^t v + v\cdot Qv)} f(v) d^g v.$$

These operators are continuous on $\mathcal{S}(\mathbb{R}^g)$, therefore also on $(\mathcal{S}(\mathbb{R}^g))'$, and are unitary isomorphisms when restricted to $L^2(\mathbb{R}^g)$. The metaplectic group can then be defined as the group generated by operators of this form. The $2:1$ projection to $U \subset Sp(2g,\mathbb{R})$ is given by $S(P, L, Q)_m \mapsto (P, L, Q)$.

Due to the Weil-Brezin decomposition $C^\infty(L^k) \cong \Pi_{l\in(\mathbb{Z}/k\mathbb{Z})^g}\mathcal{S}(\mathbb{R}^g)$, we will therefore get an action of these operators on sections of L^k.

Lemma 4.5. *Let $\Omega \in \mathbb{H}_g$. Then,*

$$(S(P, L, Q)_m e^{k\pi i v\cdot\Omega v})(u) = i^m \left(\frac{|detL|}{\det(\Omega + Q)}\right)^{\frac{1}{2}} e^{k\pi i u\cdot\Omega' u},$$

where $\Omega' = P - L^t(\Omega + Q)^{-1}L = M_{(P,L,Q)}(\Omega)$.[6]

The natural action by pull-back of $Sp(2g,\mathbb{R})$ on the half-form bundle

$$(M^*)^{-1} d^g z_\Omega = \det(C\Omega + D) d^g z_{\Omega'},$$

then lifts to an action of $Mp(2g,\mathbb{R})$ given by

$$S(P, L, Q)_m \sqrt{d^g z_\Omega} = i^{-m} |\det L|^{-\frac{1}{2}} \sqrt{\det(\Omega + Q)} \sqrt{d^g z_{\Omega'}},$$

where $\Omega' = M_{(P,L,Q)}(\Omega)$. We therefore have

Proposition 4.6. [BMN] *The product of the action of $Mp(2g,\mathbb{R})$ on $C^{-\infty}(L^k)$ with its action on the half-form bundle, descends to an action of $Sp(2g,\mathbb{R})$ on $C^{-\infty}(L^k) \otimes \sqrt{d^g z_\Omega}$ that lifts its action on $\mathbb{H}_g$.*

For a given polarization $P_\tau, \tau \in \mathbb{D}_g$, the equations of covariant constancy translate into a system of partial differential equations $\mathcal{D}_\tau$ for each of the Weil-Brezin coefficients separately. Furthermore, the action of the symplectic group preserves the spaces of solutions of these PDE's in the following sense:

[6] The sign of the square root of $\det(\Omega + Q)$ is determined by a Gaussian integral, just following the definition of $S(P, L, Q)$.

Proposition 4.7. [BMN] *Let $M_{(P,L,Q)} \in Sp(2g,\mathbb{R})$ be the matrix with $g\times g$ blocks A,B,C,D as in (15) and (16). For $\tau \in \mathbb{D}_g$, let $\tau' = M(\tau)$ be its transformation under $M_{(P,L,Q)}$ corresponding to (15). Consider the $g\times g$ matrix*

$$R_{\tau'} = \frac{1}{2}((\tau'+1)(A+iB) + (\tau'-1)(iC-D)),$$

acting on vectors of g elements of $(\mathcal{S}(\mathbb{R}^g))'$. Then,

$$\mathcal{D}_{\tau'} \circ M_{(P,L,Q)} = R_{\tau'} \circ M_{(P,L,Q)} \circ \mathcal{D}_{\tau}.$$

In particular the kernels of the family of operators $\mathcal{D}_\tau$ are preserved by symplectic transformations.

This means that through the transitive action of $Sp(2g,\mathbb{R})$ on $\mathbb{D}_g$, we can generate the space of half-form corrected covariantly constant distributional sections for all τ from a given fixed τ. When we approach points on $\partial\mathbb{D}_g$, the norm of $\sqrt{d^g z_\Omega}$ degenerates and it is convenient to define

$$d^g(x,y)_\tau = \wedge^g[(1+\tau)dx - i(1-\tau)dy] = \det(1+\tau)d^g z_{\Omega(\tau)},$$

where the left-hand side is defined for all $\tau \in \mathbb{D}_g$. We get

$$\sqrt{d^g(x,y)_\tau} = (\det(1+\tau))^{-\frac{1}{2}}\sqrt{d^g z_{\Omega(\tau)}},$$

with the branch of the square root defined such that it is 1 for $\tau = 0$.

For $\tau \in \mathbb{D}_g$ let

$$\mathcal{H}_\tau = \{s \otimes \sqrt{d^g(x,y)_\tau} : s \in (C^\infty(L^k))',\ \mathcal{D}_\tau(s)_l = 0, \forall l \in (\mathbb{Z}/k\mathbb{Z})^g\}$$

be the space of half-form corrected sections polarized with respect to $\mathcal{P}_\tau$. For τ in the interior of $\mathbb{D}_g$, H_τ is isomorphic to the k^g-dimensional space of theta functions $H^0(X_{\Omega(\tau)}, L^k_{\Omega(\tau)})$. On the other hand, Proposition (4.7) guarantees that over $\partial\mathbb{D}_g$ we also get a k^g space of (distributional) solutions of the equations for polarized sections. Therefore, we get a rank k^g "quantum" vector bundle $\mathcal{H} \to \mathbb{D}_g$ with fibers $\mathcal{H}_\tau$.

Theorem 4.4 and Proposition 4.7 then imply that

$$\{\sigma^l_\tau = 2^{\frac{g}{4}} k^{\frac{g}{2}} \vartheta^l_\Omega(\tau) \otimes \sqrt{d^g z_{\Omega(\tau)}}\}_{l\in(\mathbb{Z}/k\mathbb{Z})^g}$$

gives a orthonormal trivializing frame for $\mathcal{H}$ over the interior of $\mathbb{D}_g$ which extends continuously to a trivialization of $\mathcal{H}$ over $\mathbb{D}_g$. Moreover, this frame is preserved by the (unitary, transitive) BKS pairing maps induced from Theorem 4.4 and which therefore extend continuously to the boundary. We note that at the level of Weil-Brezin coefficients the BKS pairing maps correspond to the transformations in Lemma 4.5 where the symplectic group takes gaussians into gaussians.

Note that, for example, as $\tau \to -1 \in \partial\mathbb{D}_g$, the norm of the sections ϑ^l_τ goes to zero while the norm of $\sqrt{d^g z_\Omega(\tau)}$ blows up, such that σ^l_τ remains of norm 1. This limit corresponds to going to infinity in $\mathbb{H}_g$ along a ray $\lim_{t\to+\infty} t\Omega$. The limiting polarization P_{-1} is the horizontal polarization, generated by $\{\partial_{x_j}, j = 1,\ldots,g\}$, and one can check directly that the frame $\{\sigma^l_{-1}\}_{l\in(\mathbb{Z}/k\mathbb{Z})^g}$ is given by Dirac delta distributions supported along Bohr-Sommerfeld fibers at $y = \frac{l}{k}, l \in (\mathbb{Z}/k\mathbb{Z})^g$. In this case, since elements in the frame $\{\sigma^l_{-1}\}$ are supported along a single (connected) Bohr-Sommerfeld fiber, we say that this is a *Bohr-Sommerfeld basis* [Ty] for $\mathcal{H}_{-1}$. As τ evolves along $\partial\mathbb{D}_g$, passing through real and mixed polarizations, the basic elements σ^l_τ become linear combinations of Dirac delta distributions supported on Bohr-Sommerfeld leaves of P_τ. (Note that, when $-i\tau$ has non-rational entries $\mathcal{P}_\tau$ no longer defines a regular fibration of $\mathbb{T}^{2g}$, the leaves being no longer compact.)

The vertical polarization corresponds to $\tau = 1$. Elements of the frame $\{\sigma^l_1\}$ are linear combinations of Dirac delta distributions supported on Bohr-Sommerfeld fibers. These correspond naturally to the distributions on $U(1)^g$ that were described in Section 4.2 and which generated holomorphic sections via the CST. In fact, the coefficients of the decomposition in terms of Dirac delta distributions are precisely given by the matrix S described at the end of Section 4.2.

Quantization of tori in real polarizations has been considered in [Ma], where the BKS pairing was defined in terms of an intersection pairing between Bohr-Sommerfeld fibers. In the above construction, the pairing bewteen the quantization spaces associated to two real polarizations is naturally described in terms of the parallel frame $\{\sigma^l_\tau\}$ and by continuity from the BKS pairing for holomorphic sections in Theorem 4.4. In fact, using the fact that Dirac delta ditributions supported along transverse fibers can be multiplied by each other (see Chapter VIII of [Ho], in particular Example 8.2.11), we can show that our BKS pairing is also an intersection pairing for real transverse polarizations:

Proposition 4.8. [BMN] *Let $\tau, \tau' \in \partial\mathbb{D}_g$ be such that $\mathcal{P}_\tau, \mathcal{P}_{\tau'}$ are transverse real polarizations with compact leaves. Let BS and BS' be the union of the Bohr-Sommerfeld leaves of $\mathcal{P}_\tau$ and $\mathcal{P}_{\tau'}$ respectively.*

If $\sigma \in \mathcal{H}_\tau, \sigma' \in \mathcal{H}_{\tau'}$, then the BKS pairing $\langle\sigma, \sigma'\rangle_{BKS}$ is obtained by evaluating a distribution supported along $BS \cap BS'$ on the constant function 1.

5. Toric Varieties

The results in this section are mostly contained in [BFMN, KMN]. Symplectic toric manifolds/toric varieties provide a family of examples for both symplectic geometry and algebraic geometry which is at the same time rich and amenable to explicit computations. Let $P \subset \mathbb{R}^n$ be a polytope[7] determined by d linear conditions

$$l_i(x) = \langle x, \nu_i\rangle - \lambda_i \geq 0, i = 1, \dots, d, \;\; x \in \mathbb{R}^n,$$

where $\lambda_i \in \mathbb{R}$ and $\nu_i \in \mathbb{R}^n$ for $i = 1, \dots, d$. Thus, P is bounded by d facets (that is, faces of codimension one) described by $l_i(x) = 0$ for some i. The ith facet of P will have ν_i as inner pointing normal. We will assume that the vectors $\nu_i, i = 1 \dots, d$, are primitive elements in $\mathbb{Z}^n \subset \mathbb{R}^n$ and that there are exactly n facets meeting at each vertex of P and such that the corresponding normals

[7] We will use the term "polytope" even if P is not compact.

ν_i form a basis for $\mathbb{Z}^n \subset \mathbb{R}^n$. To such polytope one can associate a symplectic $2n$-dimensional manifold, (X_P, ω), realized as a symplectic quotient of $\mathbb{C}^d$ by a $(d-n)$-dimensional subtorus of $\mathbb{T}^d$, see, for instance, [De, dS, Gui]. X_P comes with an Hamiltonian action by $\mathbb{T}^n$ and moment map $\mu : X_P \to \mathbb{R}^n$, such that $\mu(X_P) = P$. The action of $\mathbb{T}^n$ is free on the pre-image of the interior of P, $\breve{P}$, so that there is an open dense set $X_P \supset \breve{X}_P \cong \breve{P} \times \mathbb{T}^n$. In action-angle coordinates on $\breve{X}_P$, $(x, \theta) \in \breve{P} \times \mathbb{T}^n$ one has $\omega = \sum_{i=1}^n dx^i \wedge d\theta^i$ and $\mu(x, \theta) = x$. On the other hand, X_P can also be realized as a quotient of an open subset of $\mathbb{C}^d$ by a $(d-n)$-dimensional complex torus. The resulting complex structure is compatible with ω and gives X_P a Kähler structure [Gui]. In this realization, the open dense set $\breve{X}_P$ coincides with the top dimensional orbit for a global holomorphic action of $(\mathbb{C}^*)^n$ on X_P.

From the algebro-geometric point of view X_P is constructed by means of a fan whose n-dimensional cones correspond to the vertices of P (see, for example, [Co, Da, Od], or Chapter 4 of [V]). In this way, X_P has an open cover $\{U_v\}_{v\in V}$ where V is the set of vertices of P and where $U_v \cong \mathbb{C}^n, \forall v \in V$. The dense open orbit $\breve{X}_P$ is then the open set U_0 corresponding to the cone $\{0\}$ in the fan defining X_P. We will shortly describe toric complex structures on X_P in terms of these neighbourhoods.

The polytope P also defines on X_P a $(\mathbb{C}^*)^n$-invariant divisor defining a holomorphic line bundle L, such that $H^0(X_P, L)$ has a basis of $(\mathbb{C}^*)^n$ "invariant" holomorphic sections whose elements are in one-to-one correspondence with the integral points of P (see, for example, Chapter 4 of [V]). Note that (integral) translations of P correspond to shifting the divisor defined by P by a principal divisor. (Here, we take the usual assumption that the vertices of P are integral points. In particular, $\lambda_i \in \mathbb{Z}$ for $i = 1, \dots, d$.) Toric invariant divisors are generated by the preimages of the facets of P under the moment map μ. Note that the normals ν_i to these facets correspond to the one-dimensional cones in the fan for X_P. Let those divisors be $D_1, \dots, D_d$.[8] Then, P determines a section σ_P of L with divisor

$$\mathrm{div}\,(\sigma_P) = -\sum_{i=1}^d \lambda_i D_i.$$

Moreover, the Chern class of L is given by [Gui]

$$c_1(L) = \frac{[\omega]}{2\pi} \in H^2(X_P, \mathbb{Z}) \cong Pic(X_P).$$

Holomorphic quantization of (X_P, ω) is then given by $H^0(X_P, L)$ with $h^0(X_P, L) = \#P \cap \mathbb{Z}^n$. As we will describe next, one can study explicit families of toric Kähler structures on X_P such that the corresponding holomorphic polarizations of X_P converge to the real toric polarization. Moreover, it is possible to describe the

[8]Since principal divisors have the form $\sum_{i=1}^d \langle \alpha, \nu_i \rangle D_i$, for some $\alpha \in \mathbb{Z}^n$, the linear equivalence relations between the D_i's are given by $\sum_{j=1}^n (\nu_j)^i [D_j] = 0, i = 1, \dots, n$.

convergence of holomorphic sections of L to distributional sections which are covariantly constant for the real toric limit polarization.

5.1. **Toric complex structures.** Let $g_P : \check{P} \to \mathbb{R}$ be the symplectic potential defined by Guillemin [Gui]

$$g_P(x) = \frac{1}{2}\sum_{i=1}^{d} l_i(x)\log l_i(x),\ \ x \in \check{P}. \tag{17}$$

This is smooth in $\check{P}$, with logarithmic singularities in the first derivatives and inverse linear singularities in the second derivatives along ∂P. In [Gui], it is shown that g_P determines a toric Kähler structure on X_P. Moreover, Abreu [Ab1, Ab2] classified (for the compact case) all such toric complex structures on X_P compatible with ω, in terms of symplectic potentials $g : \check{P} \to \mathbb{R}$ of the form

$$g = g_P + \varphi,$$

where φ is smooth in $\check{P}$ and Hess $(g_P + \varphi)$ is positive definite on $\check{P}$ and satisfies the regularity conditions $\det(\mathrm{Hess}\,(g_P + \varphi)) = [\alpha(x)\Pi_{i=1}^{d} l_i(x)]^{-1}$ along ∂P, where α is smooth and strictly positive on the whole of P.

Such a symplectic potential defines a diffeomorphism $\check{P} \leftrightarrow \mathbb{R}^n$ given by

$$\check{P} \ni x \mapsto y = \frac{\partial g}{\partial x} \in \mathbb{R}^n,$$

which determines a $\mathbb{T}^n$ equivariant biholomorphism

$$\check{P} \times \mathbb{T}^n \ni (x,\theta) \mapsto w = e^{y+i\theta} := (w^1,\dots,w^n) = (e^{y^1+i\theta^1},\dots,e^{y^n+i\theta^n}) \in (\mathbb{C}^*)^n,$$

where w is a system of holomorphic coordinates on the open dense orbit $\check{X}_P$. The pull-back of the standard complex structure on $(\mathbb{C}^*)^n$ by this diffeomorphism then defines a ($\mathbb{T}^n$-invariant) complex structure on $\check{X}_P$ which extends to the whole of X_P [Gui]. The inverse transformation to symplectic coordinates (x,θ) is given by the inverse Legendre transform $x = \frac{\partial h}{\partial y}$ where $h(y) = x(y)\cdot y - g(x(y))$.

Let us now describe the holomorphic coordinates on the vertex charts $U_v, v \in V$. Let v be a vertex of P such that, after reordering the indices if necessary, v is defined by the intersection of the n facets

$$l_1(v) = \cdots = l_n(v) = 0.$$

Let $A_v \in GL_n(\mathbb{Z})$ be given by the normals to the n facets of P meeting at v

$$(A_v)_{ij} = (\nu_i)^j,\ \ i,j = 1,\dots,n.$$

Define new coordinates on[9]

$$U_v = \mu^{-1}\left(\{v\} \cup \bigcup_{\text{faces } F \text{ of } P \text{ adjacent to } v} \check{F}\right)$$

by setting

$$x_v = A_v x - \lambda_v, \quad \theta_v = {}^t A_v^{-1}\theta,$$

[9]By "faces of P" we mean faces of all dimensions including the n-dimensional face $\check{P}$.

where $\lambda_v = (\lambda_1, \ldots, \lambda_n)$. Note that these coordinates have singularities, similar to polar coordinate singularities on the plane, along the faces of positive codimension which are adjacent to v. Over U_v, we have $\omega = \sum_{i=1}^n dx_v^i \wedge d\theta_v^i$. Letting $y_v = \frac{\partial g}{\partial x_v}$, one has that

$$w_v = e^{y_v + i\theta_v}$$

is a (global) system of holomorphic coordinates on U_v. Coordinate transformations then have the form[10]

$$w_{v'} = w_v^{A_v A_{v'}^{-1}}, \; w = w_v^{A_v},$$

on the corresponding intersections of coordinate neighbourhoods. Notice that while the coordinates w_v themselves depend on the λ_i's, the structure of complex manifold on X_P, as shown by the previous glueing conditions, depends only on the normals $\nu_i, i = 1, \ldots, d$.

5.2. **Sections, connection and hermitian structure on L.** Let us now describe holomorphic sections of L explicitly. For $m \in \mathbb{Z}^n$, by slight abuse of notation, denote also by w^m the (unique) meromorphic funtion on X_P which on $\check{X}_P$ is given by w^m. Its divisor is given by (Chapter 4 of [V])

$$\text{div}\,(w^m) = \sum_{i=1}^d \langle m, \nu_i \rangle D_i. \tag{18}$$

Recall that P determines a section σ_P of L with divisor $-\sum_{i1}^d \lambda_i D_i$ and that $H^0(X_P, L)$ is in bijective correpondence with $P \cap \mathbb{Z}^n$. Explicitly, one easily obtains that this bijection is given by

$$m \in P \cap \mathbb{Z}^n \mapsto \sigma^m = w^m \sigma_P \in H^0(X_P, L).$$

Let $1_v \in H^0(X_P, L)$ be the section corresponding to a vertex $v \in V$. Since $\text{div}\,(1_v) \cap U_v = 0$, 1_v defines a holomorphic trivialization of L over U_v. We thus obtain a system of local holomorphic trivializations for L. Note that σ_P also gives a local holomorphic trivialization of L over $U_0 = \check{X}_P$, so that we also set $1_0 = \sigma_P$. The associated transition functions for L are then given by

$$g_{v0} = w^{-A_v^{-1}\lambda_v} = w^{-v}, \; g_{v'v} = w_{v'}^{-\lambda_{v'} + A_{v'} A_v^{-1} \lambda_v}. \tag{19}$$

In these local frames, the expressions for $\sigma^m, m \in P \cap \mathbb{Z}^n$ read

$$\sigma_v^m = w_v^{l_v(m)} 1_v,$$

where $l_v(x) = A_v x - \lambda_v$, $v \in V$.

A hermitian structure can be defined on L by setting $||1_v|| = e^{-h_v(x)}$, $||1_0|| = e^{-h(x)}$, where $h(x) = x \cdot y - g(x)$ and $h_m(x) = (x - m) \cdot \frac{\partial g}{\partial x} - g(x)$. Then, one can define a system of normalized trivializations $\{1_v^{U(1)}, 1_0^{U(1)}\}$ for L, with

$$1_v^{U(1)} = \frac{1_v}{||1_v||}, \; 1_0^{U(1)} = \frac{1_0}{||1_0||}, \; v \in V.$$

[10]Here, w^A means $(w^A)^j = \Pi_{i=1}^n (w^i)^{A_{ij}} = (w^1)^{A_{1j}} \cdots (w^n)^{A_{nj}}$.

Note that in this system of unitary trivializing frames for L, the transition functions are $U(1)$ valued and read

$$g_{v0}^{U(1)} = e^{-iA_v^{-1}\lambda_v\cdot\theta} \quad g_{v'v} = e^{-i(\lambda_{v'}-A_{v'}A_v^{-1}\lambda_v)\cdot\theta_{v'}}. \tag{20}$$

In particular, these transition functions define on X_P a line bundle with $U(1)$ structure which is independent of the choice of symplectic potential g and also of the corresponding choice of toric Kähler structure on X_P.

The holomorphic and hermitian structures on L then determine a unique metric connection ∇, which has curvature $-i\omega$, defined by

$$\nabla 1_v^{U(1)} = -ix_v\cdot d\theta_v\, 1_v^{U(1)},\ \nabla 1_0^{U(1)} = -ix\cdot d\theta\, 1_0^{U(1)}\ v\in V.$$

5.3. **The real toric polarization.** Let $\mathcal{P}_{\mathbb{R}}$ denote the real toric polarization

$$\mathcal{P}_{\mathbb{R}} = \operatorname{Ker} d\mu\otimes\mathbb{C} = \langle\frac{\partial}{\partial\theta^i}, i=1,\dots,n\rangle_{\mathbb{C}}.$$

$\mathcal{P}_{\mathbb{R}}$ is a singular polarization since over ∂P the $\mathbb{T}^n$-orbits over the interior of a face of P of codimension k are isomorphic to $\mathbb{T}^{n-k}$.

Recall that in [Sn2], Śnyaticki defines quantization for a real polarization in terms of the Cech cohomology groups of the sheaf of polarized smooth local sections of the prequantum bundle. For nonsingular polarizations, that is defining a regular Lagrangian fibration of the symplectic manifold with fiber $\mathbb{T}^k\times\mathbb{R}^{n-k}$, he proves that the cohomology is concentrated in degree k. In the compact case, one gets cohomology concentrated in degree n, with rank given by the (finite) number of BS fibers. Of course, it is of great interest to see how this result generalizes to the case of real polarizations with singularities.

In the toric case, except for $P=\mathbb{R}^n, X_P=(\mathbb{C}^*)^n$, since $\mathcal{P}_{\mathbb{R}}$ has singular fibers, Śnyaticki's result does not apply and it is very interesting to study the quantization of X_P in this polarization. In [Ham], it is shown that in the toric case the computation of the Śnyaticki cohomology groups is still concentrated in degree n with rank given by the number of *non-singular* BS leaves. As we will see below, BS leaves in X_P correspond to integral points in P. Therefore, this result is not satisfactory since one expects quantization in the real polarization to give a space of quantum states of the same dimension has obtained in the holomorphic polarizations, that is $h^0(X_P,L)=\#(P\cap\mathbb{Z}^n)$, so that singular BS fibers, corresponding to integral points along ∂P, *should* contribute to quantization. Indeed, if we consider distributional solutions of the equations of covariant constancy such contribution is correctly captured.

Recall from Section 2.2 and (3) that the space of polarized sections for $\mathcal{P}_{\mathbb{R}}$ is defined by

$$\mathcal{H}_{\mathbb{R}} = \{\sigma\in C^{-\infty}(L) : \forall U\subset X_P \text{ open}, \forall\xi\in C^\infty(\mathcal{P}_{\mathbb{R}|_U}), \nabla'_\xi\sigma_{|_U}=0\}.$$

Note that a natural inner product on this space can be defined only after we have described it in terms of degenerating holomorphic sections, for which there is a well defined hermitian structure.

From the explicit form of the connection ∇ on L it is easily verified that BS leaves of $\mathcal{P}_{\mathbb{R}}$ are precisely the ones in $\mu^{-1}(P\cap\mathbb{Z}^n)$. On the other hand, the equations of weak covariantly constancy show that each such connected leaf supports only one linear independent solution, which consists of a Dirac delta distribution supported on the leaf and a Fourier transform along the leaf.

Theorem 5.1. [BFMN] *Let $W\subset X_P$ be a $\mathbb{T}^n$-invariant open set. Then,*

a) *Sections in $(C_c^\infty(L^{-1}_{|W}))'$ covariantly constant along $\mathcal{P}_{\mathbb{R}}$ are supported on BS leaves contained in W.*

b) *Let $m\in P\cap\mathbb{Z}^n$. The distribution*

$$\delta^m(\tau)=\frac{1}{(2\pi)^n}\int_{\mu^{-1}(m)} e^{im\cdot\theta}\tau(m,\theta)d\theta,\ \tau\in C_c^\infty(L^{-1}_{|W})$$

is covariantly constant.

c) *The set $\{\delta^m\}_{m\in P\cap\mathbb{Z}^n}$ is a basis for the space of covariantly constant sections on W.*

Obviously, these distributional sections supported on BS leaves can be extended globally and we obtain $\dim\mathcal{H}_{\mathbb{R}}=\#P\cap\mathbb{Z}^n=h^0(X_P,L)$ as expected. Note that this theorem is local (in the space of fibers) in nature and since every hamiltonian $\mathbb{T}^n$-space has a local normal form in a neighborhood of an orbit, this result describes, up to equivariant symplectomorphism, more general situations.

We will now describe how these distributions can be approximated by (normalized) holomorphic sections of L in a family of degenerating complex structures.

5.4. **Degenerate toric Kähler structures.** As we have seen, a choice of symplectic potential g determines a toric Kähler structure on X_P with symplectic form ω. We will now describe a family of such potentials such that the corresponding holomorphic polarizations degenerate to the real toric polarization. Let ψ be a strictly convex function on a neighborhood of P, that is the Hessian $H_\psi>0$ on P. Then, the symplectic potentials

$$g_s=g_P+s\psi \tag{21}$$

define toric Kähler structures on X_P for all $s\geq 0$. With $y_s=\frac{\partial g_s}{\partial x}$, let $w_s=e^{y_s+i\theta}$ be the corresponding holomorphic coordinates on $\check{X}_P$. The associated holomorphic polarization is

$$\mathcal{P}_s=\langle\frac{\partial}{\partial w_s^i}, i=1,\dots,n\rangle_{\mathbb{C}}.$$

Note that $\frac{\partial y_s}{\partial x}=H_{g_s}$ becomes more and more dominated by sH_ψ as $s\to+\infty$. However, recall that the Hessian of $g_P=g_0$ becomes singular as we approach ∂P. In fact, if we approach a face of codimension k of P given by $\{l_{i_j}=0, j=1,\dots,k\}$, the corresponding terms $\sum_{j=1}^k\frac{1}{2}l_{i_j}\log l_{i_j}$ in (17) for g_P become non-smooth. Therefore, the closer we are to a face the harder it is for sH_ψ to dominate these singular terms. However, pointwise for $x\in\check{P}$, sH_ψ dominates in the limit and a short calculation gives

Proposition 5.2. [BFMN] *Over $\check{X}_P$,*

$$\lim_{s\to+\infty}\mathcal{P}_s = \mathcal{P}_{\mathbb{R}}.$$

It is interesting to observe that, since $\check{X}_p \cong (\mathbb{C}^*)^n$, this corresponds to the "horizontal" polarization of $T^*U(1)^n$ in the context of Section 3. The family of Kähler structures considered in Section 3 connects the vertical polarization (in the $s\to 0$ limt) to the "horizontal polarization" (in the $s\to+\infty$ limit). In the present case however, note that the $s\to 0$ limit is described by $g_0 = g_P$ and corresponds to the "canonical" Kähler structure on X_P described in [Gui] (as long as P has enough facets so that H_{g_P} is non-degenerate, so that the y^i's can be defined).

Note that, as we vary the symplectic potential g, the phases of the holomorphic coordinates w, w_v do not change but only their modules are varying functions of g. Therefore, it is easy to check that over a face where, for some index i, $w_v^i = 0$, $\langle\frac{\partial}{\partial w_v^i}\rangle_{\mathbb{C}}$ is independent of g. Note that w_v^i can be zero only for $y_v^i \to -\infty$ and this can only happen over ∂P, where the linear function $l_i(x)$ vanishes. Then, over a face where $w_v^i \neq 0$, one has $\lim_{s\to+\infty}\frac{\partial}{\partial w_v^i} = \frac{\partial}{\partial\theta_v^i}$. Therefore, over P, that is including ∂P, we have

$$\lim_{s\to+\infty}\mathcal{P}_s = \mathcal{P}_{\mathbb{R}} \oplus \langle\frac{\partial}{\partial w_v^j} : w_v^j = 0\rangle_{\mathbb{C}}.$$

Smooth sections of $\lim_{s\to+\infty}\mathcal{P}_s$ cannot have non-vanishing components along the holomorphic directions over ∂P and we have

Theorem 5.3. [BFMN]

$$C^\infty(\lim_{s\to\infty}\mathcal{P}_s) = C^\infty(\mathcal{P}_{\mathbb{R}}).$$

We will now describe how (properly normalized) holomorphic sections of L converge to the distributional sections of the quantization in $\mathcal{P}_{\mathbb{R}}$, as $s\to+\infty$. Note that the norm of a holomorphic section $\sigma^m \in H^0(X_P, L)$, is given by $e^{-h_m\circ\mu}$, where $h_m(x) = (x-m)\cdot\frac{\partial g}{\partial x} + g$. Therefore, for a fixed $x\in\check{P}$, as $s\to+\infty$, the behavior of the norm of σ^m is determined by $f_m = (x-m)\cdot\frac{\partial\psi}{\partial x} - \psi(x)$. Strict convexity of ψ gives that f_m has a unique minimum at $x = m$ and

Lemma 5.4. [BFMN] *In the sense of distributions,*

$$\lim_{s\to+\infty}\frac{e^{-sf_m}}{||e^{-sf_m}||_1} = \delta(x-m).$$

This implies

Theorem 5.5. [BFMN] *For $s>0$, let σ_s^m be the section of L determined by $m\in p\cap\mathbb{Z}^n$ and holomorphic with respect to the holomorphic strcuture determined by the symplectic potential g_s. Then, for the L^1-normalized family of holomorphic sections,*

$$\lim_{s\to+\infty}\frac{\sigma_s^m}{||\sigma_s^m||_1} = \delta^m \in C^{-\infty}(L).$$

Therefore, we obtain a satisfactory description of the convergence of states in the holomorphic quantization to (distributional) sates for the quantization in the real toric polarization. Note, however, that this convergence is achieved with L^1, and not L^2, normalized sections. Just like for abelian varieties, the L^2 normalization will be recovered when the half-form correction is considered. We note that this family of degenerating toric Kähler structures has also been used in [HK] to relate the quantizations of flag manifolds in real and Kähler polarizations [GS], via their toric degenerations. For a different approach relating the quantizations of X_P in real and Kähler polarizations, where holomorphic sections and Dirac delta distributional sections supported on BS fibers are related by projection, see [BGU, SeD].

5.5. **The canonical bundle of X_P.** In order to study the half-form correction in the quantization of toric varieties, let us summarize some properties of the canonical bundle K_{X_P} corresponding to some fixed toric complex structure on X_P, determined by a symplectic potential g. Let $z^i = y^i + i\theta^i$ so that $w = e^z$ is the system of holomorphic coordinates on $\check{X}_P$ described above. In the open orbit $U_0 = \check{X}_P$, $dW = dw^1 \wedge \cdots \wedge dw^n$ is a trivializing holomorphic section of K_{X_P}. The section[11] $dZ = w^{-1}dW$ then defines a global meromorphic section of K_{X_P} with divisor

$$\mathrm{div}\,(dZ) = -D_1 - \cdots - D_d, \tag{22}$$

where the toric invariant divisors $D_i = \{p \in X_P : l_i(\mu(p)) = 0\}$ were described above. The Kähler metric on X_P induces metrics on the tensor bundles over X_P, so that K_{X_P} comes with the metric described in Section 2.3, which for an $(n,0)$-form η reads

$$||\eta||^2_{K_{X_P}} = \frac{\eta \wedge \bar{\eta}}{(2i)^n(-1)^{n(n+1)/2}\omega^n/n!}.$$

The corresponding metric connection $\nabla^{K_{X_P}}$ has connection form $\partial \log ||dZ||^2_{K_{X_P}}$, in the frame dZ. A straightforward computation gives

$$||dZ||^2_{K_{X_P}} = \det H_g.$$

The curvature is then $F_{\nabla^{K_{X_P}}} = \bar{\partial}\partial \log \det H_g = i\rho$, where ρ is the Ricci form[12]. One has

$$c_1(K_{X_P}) = -c_1(X_P) = [iF_{\nabla^{K_{X_P}}}/2\pi] \in H^2(X_P, \mathbb{Z}).$$

5.6. **Inclusion of the half-form.** The half-form corrected quantization of (X_P, ω) in a holomorphic polarization should be given by $H^0(X_P, L \otimes K^{\frac{1}{2}}_{X_P})$ for some square root $K^{\frac{1}{2}}_{X_P}$ of K_{X_P}. However, not all toric varieties admit square roots of their canonical bundles. Obvious examples which do not admit a square root of the

[11] Note that, to simplify notation, here $w = w^{\mathbf{1}} = w^1 \cdots w^n$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{Z}^n$.

[12] For a Kähler metric $\gamma = \omega(I\cdot, \cdot)$ where I is the complex structure, the Ricci form is given by $\rho = Ric(I\cdot, \cdot)$ where Ric is the Ricci tensor.

canonical bundle are the projective spaces $\mathbb{P}^{2k}_{\mathbb{C}}$, for which $K_{\mathbb{P}^{2k}_{\mathbb{C}}} = \mathcal{O}(-(2k+1))$. In fact, using (18) and (22), we have that $c_1(X)$ is even iff for all the normals $\nu_i, i = 1, \dots, d$, to the facets of P one has that $\sum_{j=1}^{n} (\nu_i)^j$ is odd; here, $(\nu_i)^j$ are the coordinates of ν_i in any of the vertex basis $\{\nu_{i_1}, \dots, \nu_{i_n}\}$ where these are the normals at the facets meeting at the vertex.

Therefore, to perform quantization with half-forms in the context of toric varieties, we need to start with a possibly non-integral symplectic form, so that after the half-form correction we have a well defined line bundle over the symplectic manifold and can then pursue the quantization process. Let us then consider on the smooth toric variety X_P a symplectic form ω such that

$$\frac{[\omega]}{2\pi} - \frac{c_1(X_P)}{2} \in H^2(X_P, \mathbb{Z}).$$

Let $L \to X_P$ be now a hermitian line bundle with $c_1(L) = \frac{[\omega]}{2\pi} - \frac{c_1(X_P)}{2}$ with a metric connection ∇ of curvature

$$F_\nabla = -i\omega + \frac{i}{2}\rho,$$

where ρ is the Ricci form. Note that ∇ is the sum of a "would be" connection with curvature $-i\omega$ with the "would be" connection $\frac{1}{2}\nabla^{K_{X_P}}$. Of course, if $c_1(X_P)$ is even, then $[\omega]/2\pi$ is integral and $K_{X_P}^{\frac{1}{2}}$ exists, so that $L \cong l \otimes K_{X_P}^{\frac{1}{2}}$ where $c_1(l) = [\omega]/2\pi$. In this case, we are back to the usual setting for half-form quantization and the two "would be" connections are true connections on well defined line bundles.

We will define half-form corrected quantization of (X_P, ω) to be the quantization associated to the "prequantization" data defined by (X_P, ω, L, ∇). Note that a choice of complex structure on X_P, and hence also of a Kähler polarization, is implicit in this construction. Of course, this definition can be applied to more general situations than toric varieties. Essentially, in this sense, the half-form correction selects only appropriate cohomology classes for ω and imposes a piece of the "prequantum" connection to be given by one-half the metric connection on the canonical bundle. As we will see, this will have as consequence that holomorphic quantization is given by the holomorphic sections of L, labeled by the integral points of P_L which should be viewed inside a polytope P associated to (X_P, ω), as described below. The integral points of P_L will then correspond to shifted *non-singular* BS leaves in X_P.

In fact, as above, $L \to X_P$ is defined by a polytope P_L given by d linear conditions

$$l_i^L(x) = \langle x, \nu_i \rangle - \lambda_i^L \geq 0, \ i = 1, \dots, d,$$

where the λ_i^L's are integers. However, since we will be using symplectic coordinates, it is more useful to describe X_P by a polytope P associated to the (possibly non-integral) class $[\omega]/2\pi$. Such P can be defined by the linear conditions

$$l_i(x) = \langle x, \nu_i \rangle - \lambda_i \geq 0, \ i = 1, \dots, d,$$

where $\lambda_i^L = \lambda_i + \frac{1}{2}, i = 1, \ldots, d$. Therefore, $P_L \subset P$. Note that P never has integral vertices, since the λ_i^L's are integers so that $\langle v, \nu_i \rangle = \lambda_i^L - \frac{1}{2}$ can never have integral solutions. The polytope P_L, which has integral vertices, has facets which are given by inner-pointing $\frac{1}{2}$ shifts of the facets of P along the normals.[13]

The line bundle L, as seen in Section 5.2, has a $U(1)$-structure that is independent of the choice of toric complex structure on X_P. This structure is particularly suited to study the convergence of holomorphic sections to distributional ones, since we can regard holomorphic sections as varying in a space which is independent of the complex structure. However, as we have seen in several instances already, an important consequence of the half-form correction is that it ensures a correct normalization of sections enabling their nice behavior as the complex structure degenerates. In order to take this into account and to achieve a unified description of the half-forms for $\mathcal{P}_\mathbb{R}$ and for $\mathcal{P}_s, s > 0$, which is necessary to study convergence as $s \to +\infty$, we will then treat the "$U(1)$-part" of the canonical bundle separately from its "norm" part. That is, we will decompose $K_{\mathcal{P}_s}$ as a product $K_{\mathcal{P}_s} = K_{\mathcal{P}_s}^{U(1)} \otimes |K_{\mathcal{P}_s}|$, where the line bundle $K_{\mathcal{P}_s}^{U(1)}$ has (complex structure independent) unitary transition functions of the form (20) and $|K_{\mathcal{P}_s}|$ is a trivializable line bundle with $\mathbb{R}_+$ valued transition functions given by the modulus of (19). Note that, even if $K_{\mathcal{P}_s}$ does not admit a square root, the trivializable line bundle $|K_{\mathcal{P}_s}|$ always admits a square root, denoted by $|K_{\mathcal{P}_s}|^{\frac{1}{2}}$.

Sections of $|K_{\mathcal{P}_s}|$ can be described by viewing n-forms as real-valued functions of vector fields. Consider, for simplicity, the real toric polarization, $\mathcal{P}_\mathbb{R} = \langle \frac{\partial}{\partial \theta^i}, i = 1 \ldots n \rangle_\mathbb{C}$. It's "canonical" bundle corresponds to the bundle of top forms on the space of leaves. Thus, the fibers of $K_{\mathcal{P}_\mathbb{R}}$ are generated by[14] $dX = dx^1 \wedge \cdots \wedge dx^n$. We will let $|dX|$ be a map $|dX| : \mathcal{X}(\check{X}_P)^n \to C^0(\check{X}_P)$, where $\mathcal{X}(\check{X}_P)$ is the space of smooth vector fields on X_P. This is defined simply as

$$|dX|(A_1, \ldots, A_n) = |dX(A_1, \ldots, A_n)|.$$

We define $\sqrt{|dX|}$ by setting $\sqrt{|dX|}(A_1, \ldots, A_n) = |dX(A_1, \ldots, A_n)|^{\frac{1}{2}}$. We can similarly define $\sqrt{|dZ_s|}$, where $dZ_s = dz_s^1 \wedge \cdots \wedge dz_s^n$ are the generators of fibers of the canonical bundle of X_P defined by the symplectic potential g_s in (21). A global trivializing section of $|K_{\mathcal{P}_s}|^{\frac{1}{2}}$ is then given by $\frac{\sqrt{|dZ_s|}}{||dZ_s||^{\frac{1}{2}}}$. The connection $\nabla^{K_{\mathcal{P}_s}}$ on $K_{\mathcal{P}_s}$ splits naturally into a connection on $K_{\mathcal{P}_s}^{U(1)}$ and a connection on $|K_{\mathcal{P}_s}|$, such that this global trivializing section of $|K_{\mathcal{P}_s}|^{\frac{1}{2}}$ is covariantly constant. (See Section 2.1 of [KMN].)

[13]When $[\omega]/2\pi$ is integral, one normally fixes the moment map so that the moment polytope P has integral vertices and the holomorphic sections of the prequantum line bundle are labeled by the integral points of P. Here, we are taking the moment polytope P to have vertices with some of the coordinates half-integral. This corresponds to a shift, or translation, of the usual moment polytope. The holomorphic sections of the line bundle L will then be labeled by integral points of the polytope $P_L \subset P$. The corresponding fibers of the moment map are called *shifted BS leaves*, since the moment polytope P has been shifted with respect to the usual one in the case when $[\omega]/2\pi$ is integral.

[14]Note that dX is smooth and globally defined on X_P, vanishing on $\mu^{-1}(\partial P)$.

The definition for the half-form corrected space of states for the holomorphic quantization of (X_P, ω) corresponding to the symplectic potential $g_s, s > 0$ is then

$$\mathcal{H}_s = \left\{ \sigma \otimes \frac{\sqrt{|dZ_s|}}{||dZ_s||^{\frac{1}{2}}} : \sigma \in H^0(X_P, L) \right\},$$

where the holomorphic structure on L is determined by the symplectic potential g_s. We have therefore that

$$\dim \mathcal{H}_s = h^0(X_P, L) = \# P_L \cap \mathbb{Z}^n = \# P \cap \mathbb{Z}^n.$$

Again, recall that, even if $c_1(X_P)$ is even, P now denotes a shifted polytope with non-integral vertices.

Let us now study the half-form corrected quantization in the real toric polarization. Since (4) degenerates and the Lie derivative of dX along $\mathcal{P}_{\mathbb{R}}$ vanishes, there is no fruitful direct way of defining a half-form correction for the prequantum connection in this polarization. Since we have the family of holomorphic polarizations $\mathcal{P}_s$ converging to $\mathcal{P}_{\mathbb{R}}$, as described in Section 5.4, we will use the correspondig family of connections to define half-form corrected quantization in the (singular) real polarization $\mathcal{P}_{\mathbb{R}}$.

For $s > 0$, let ∇^s be the metric connection on L, as described above, corresponding to the symplectic potential g_s in (21). ∇^s is described over $\breve{X}_P$ by the connection form (relative to the unitary frame of Section 5.2)

$$-ix \cdot d\theta + \frac{i}{2} \mathrm{Im}\, \partial \log \det H_{g_s} = -ix \cdot d\theta + \frac{i}{4} \left(\frac{\partial}{\partial x} \log \det H_{g_s} \right) H_{g_s}^{-1} d\theta,$$

where the second term "corresponds" to $\frac{1}{2} \nabla^{K_{X_P}}$ written in the unitary frame $||dZ_s||^{-\frac{1}{2}}_{K^{\frac{1}{2}}_{X_P}} \sqrt{dZ_s}$. As we have already seen, pointwise in $\breve{X}_P$ we have $H_{g_s} \to sH_\psi$ as $s \to +\infty$. Therefore, on $\breve{X}_P$, we have

$$-2i\nabla^s_{\frac{\partial}{\partial \bar{z}^i_s}} \to \nabla^{\mathbb{R}}_{\frac{\partial}{\partial \theta^i}} := \frac{\partial}{\partial \theta^i} - ix^i, \text{as } s \to +\infty,$$

in the sense that for distributional sections

$$-2i\nabla^s_{\frac{\partial}{\partial \bar{z}^i_s}} \sigma(\tau) \to \nabla^{\mathbb{R}}_{\frac{\partial}{\partial \theta^i}} \sigma(\tau),$$

as $s \to +\infty$, for $\tau \in C^\infty(L^{-1})$. Moreover, repeating this over the vertex charts produces

$$\nabla^{\mathbb{R}}_{\frac{\partial}{\partial \theta^i_v}} := \frac{\partial}{\partial \theta^i_v} - ix^i_v + \frac{1}{2}, v \in V, i = 1, \ldots, n,$$

where the "extra" $\frac{1}{2}$ arises from the shift $\lambda^L_i = \lambda_i + \frac{1}{2}$ in the transition functions for L. These operators define a partial connection along $\mathcal{P}_{\mathbb{R}}$ on L (viewed as a smooth bundle, with an s-independent $U(1)$-structure).

With this motivation, the space for the half-form corrected quantization in the toric real polarization is then defined by

$$\mathcal{H}_{\mathbb{R}} = \{\sigma \otimes \sqrt{|dX|} : \sigma \in \cap_{i=1}^{n} \mathrm{Ker}\, \nabla^{\mathbb{R}}_{\frac{\partial}{\partial \theta^i}}, i = 1, \ldots, n\}.$$

The factor of $\sqrt{|dX|}$ will become relevant when we relate $\mathcal{H}_{\mathbb{R}}$ to the family of Hilbert spaces $\mathcal{H}_s = \mathcal{H}_{g_s}$ in the limit $s \to +\infty$.

In a way that is similar to the uncorrected quantization, we obtain

Theorem 5.6. [KMN] *The space $\mathcal{H}_{\mathbb{R}}$ is finite-dimensional with basis given by $\{\delta^m \otimes \sqrt{|dX|}\}_{m \in P_L \cap \mathbb{Z}^n}$.*

Recall that integral points in P, which are also the integral points in $P_L \subset P$, correspond to shifted BS fibers of (X_P, ω) since the polytope P has been shifted with respect to its more conventional location (even in the case when $c_1(X_P)$ is even). Note also that there are no contributions from ∂P to the quantization in the polarization $\mathcal{P}_{\mathbb{R}}$. In fact, $\nabla^{\mathbb{R}}$ has non-trivial holonomy on the leaves of $\mathcal{P}_{\mathbb{R}}$ over ∂P. This is a reflection of the fact that as $s \to +\infty$ the connections ∇^s develop curvature singularities along $\mu^{-1}(\partial P)$.

Let us now study the convergence of the $\mathcal{P}_s$-polarized states to the $\mathcal{P}_{\mathbb{R}}$-polarized states. As remarked above, due to the half-form correction, convergence is achieved for L^2-normalized sections. This compares favorably to the "uncorrected" case of Section 5.4 where L^1-normalization was needed.

Recall that $||dZ_s|| = (\det H_{g_s})^{\frac{1}{2}}$. We then have

Proposition 5.7. [KMN]

$$\lim_{s \to +\infty} \frac{\sqrt{|dZ_s|}}{||dZ_s||} = \sqrt{|dX|}.$$

Let σ_s^m be the holomorphic section of L corresponding to the integral point $m \in P_L \cap \mathbb{Z}^n$ and to the symplectic potential g_s.

Proposition 5.8. [KMN] *As distributional sections in $C^{-\infty}(L)$,*

$$\lim_{s \to +\infty} \frac{\sigma_s^m}{||\sigma_s^m||_{L^2}} ||dZ_s||^{\frac{1}{2}} = 2^{n/2} \pi^{n/4} \delta^m.$$

Define $\hat{\sigma}_s^m = \sigma_s^m \otimes \frac{\sqrt{|dZ_s|}}{||dZ_s||^{\frac{1}{2}}}$. Then we obtain, for the half-form corrected holomorphic sections,

Theorem 5.9. [KMN] *We have,*

$$\lim_{s \to +\infty} \frac{\hat{\sigma}_s^m}{||\sigma_s^m||_{L^2}} = 2^{n/2} \pi^{n/4} \delta^m \otimes \sqrt{|dX|},$$

in the sense that

$$\lim_{s \to +\infty} \frac{\hat{\sigma}_s^m}{||\sigma_s^m||_{L^2}} (\tau, A_1, \ldots, A_n) = 2^{n/2} \pi^{n/4} \delta^m(\tau) \sqrt{|dX|}(A_1, \ldots, A_n),$$

$\forall \tau \in C_c^\infty(L^{-1})$ *and vector fields* $A_i, \ldots, A_n \in \chi(X_P)$.

The trivializing section $\frac{\sqrt{|dZ_s|}}{||dZ_s||^{\frac{1}{2}}}$ is unitary for the natural hermitian structure on $|K_{\mathcal{P}_s}|^{\frac{1}{2}}$. Then, $||\hat{\sigma}_s^m||_{L^2} = ||\sigma_s^m||_{L^2}$ and it is natural to define on $\mathcal{H}_{\mathbb{R}}$ an inner product where the basis $\{2^{n/2}\pi^{n/4}\delta^m \otimes \sqrt{|dX|}\}_{m\in P_L\cap\mathbb{Z}^n}$ is orthonormal.

5.7. **The BKS pairing.** In order to pair quantum states for half-form corrected quantizations of X_P, we first need to define a pairing for the objects $\sqrt{|dZ_s|}$. In the toric case we have been describing, we noticed that L has a $U(1)$-structure which is independent of complex structure. This is due to the fact that changes in the symplectic potential do not change the definitions of the angular coordinates θ, θ_v but only the modulus of w, w_v. Thus, in the toric case, when we pair dZ_s with $dZ_{s'}$ the angular coordinates do not contribute. Hence, the pairing of half-forms is entirely captured by the behavior of norms. This justifies the definition,

$$\langle\sqrt{|dZ_s|}, \sqrt{|dZ_{s'}|}\rangle = \langle dZ_s, dZ_{s'}\rangle^{\frac{1}{2}} = \left(\frac{dZ_s \wedge d\bar{Z}_{s'}}{(2i)^n(-1)^{n(n+1)}\omega^n/n!}\right)^{\frac{1}{2}} \quad s, s' > 0,$$

following the standard definition in (4)[15]. One obtains

$$\langle dZ_s, dZ_{s'}\rangle = \det\left(\frac{H_{g_s} + H_{g_{s'}}}{2}\right) > 0.$$

We then have,

Theorem 5.10. [KMN] *For $s, s' > 0$ and $m, m' \in P \cap \mathbb{Z}^n$,*

$$\langle\hat{\sigma}_s^m, \hat{\sigma}_{s'}^{m'}\rangle = \delta_{mm'} \int_P e^{-h_m^s} e^{-h_{m'}^{s'}} \det\left(\frac{H_{g_s} + H_{g_{s'}}}{2}\right)^{\frac{1}{2}} dx.$$

Therefore, the corresponding BKS pairing maps act diagonally along the one-dimensional subspaces of $H^0(X_P, L)$ corresponding to a given integral point in P. Moreover, note that $\langle\hat{\sigma}_s^m, \hat{\sigma}_{s'}^{m'}\rangle$ is a real number, which is a consequence of the fact that the "phase" part of the complex coordinates on X_P is independent of the choice of symplectic potential. Unitarity of these maps would therefore imply

$$\langle\hat{\sigma}_s^m, \hat{\sigma}_{s'}^m\rangle = ||\hat{\sigma}_s^m||_{L^2}||\hat{\sigma}_{s'}^m||_{L^2}.$$

From the previous Theorem, $\langle\hat{\sigma}_s^m, \hat{\sigma}_{s'}^m\rangle = \alpha(s + s')$. It is then easy to show that unitarity would require $||\hat{\sigma}_s^m||_{L^2} = ||\hat{\sigma}_0^m||_{L^2}e^{sb}$, for some constant b. An estimate at large s then shows that for the toric case the BKS pairing maps are not unitary, but with unitarity holding to leading order in $\frac{1}{s}$.

5.8. **Example:** $X_P = S^2$**.** In this section, we will describe explicitly the simple, but paradigmatic, case when $P \subset \mathbb{R}$ is an interval and $X_P \cong S^2$, with the standard S^1-action given by rotation about an axis. (Note that in this example, once a complex structure is introduced, $X_P \cong \mathbb{P}^1_{\mathbb{C}}$, with $K_{X_P} \cong \mathcal{O}(-2)$ and $\sqrt{K_{X_P}} \cong \mathcal{O}(-1)$.) Let $N \in \mathbb{N}$ and consider $P = [-\frac{1}{2}, N + \frac{1}{2}]$, described by the linear

[15]Note that with this definition we have indeed $\langle\sqrt{|dZ_s|}, \sqrt{|dZ_s|}\rangle = ||dZ_s||$ further justifying the choice of inner product on $\mathcal{H}_{\mathbb{R}}$.

inequalities

$$l_1(x) = x - \lambda_1 = x + \frac{1}{2} > 0, \quad l_2(x) = -x - \lambda_2 = -x + N + \frac{1}{2} > 0,$$

so that $P_L = [0, N]$. The Guillemin symplectic potential (17) is then

$$g_P(x) = \frac{1}{2}\left(l_1(x)\log l_1(x) + l_2(x)\log l_2(x)\right).$$

Consider now the strictly convex function $\psi(x) = \frac{1}{2}x^2$, and the family of symplectic potentials (21)

$$g_s = g_P + \frac{s}{2}x^2, \ \ s > 0.$$

The corresponding holomorphic coordinate on the open dense $(\mathbb{C})^*$-orbit $\check{X}_P =]-\frac{1}{2}, N+\frac{1}{2}[\times S^1 \subset X_P$ is then given by

$$w_s = e^{y_s + i\theta},$$

where

$$y_s = \frac{\partial g_s}{\partial x} = \frac{1}{2}\log\left(\frac{x+\frac{1}{2}}{-x+N+\frac{1}{2}}\right) + sx.$$

Thus,

$$w_s = \sqrt{\frac{x+\frac{1}{2}}{-x+N+\frac{1}{2}}}e^{sx+i\theta}.$$

It is straightforward to write down explicitly the holomorphic vertex coordinates w_{v_1}, w_{v_2}, say for $s = 0$, as described in Section 5.1. The two vertices of P (v_1, v_2 given by $x = -\frac{1}{2}$ and $x = N + \frac{1}{2}$ respectively) correspond to the two fixed points for the action of S^1 on S^2 by rotations about an axis. The holomorphic vertex coordinates are then related, on $\check{X}_P$, by

$$w_{v_1} = (w_{v_2})^{-1},$$

so that in the two vertex coordinate charts we recognize the usual holomorphic coordinate system for the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$. The space of holomorphic sections of L then has a basis $\{\sigma^m\}_{m=0,\dots,N}$, labeled by the integral points of P_L, $m = 0, \dots, N$, which is just the standard monomial basis of $H^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}(N))$. Explicitly, from Section 5.2 and Lemma 3.3 of [KMN], for $m = 0, \dots, N$, $s > 0$,

$$\sigma_s^m = e^{-h_m(x)+im\theta}(\det H_{g_s})^{\frac{1}{4}} 1_0^{U(1)},$$

where H_{g_s} is the Hessian of g_s,

$$h_m(x) = (x-m)y_s - g_s(x),$$

and $1_0^{U(1)}$ is a unitary trivialization of L on $\check{X}_P$. Thus, as $s \to +\infty$,

$$h_m(x) \sim \frac{s}{2}(x-m)^2 - \frac{s}{2}m^2,$$

so that the holomorphic sections behave asymptotically as gaussian sections

$$\sigma_s^m \sim s^{\frac{1}{4}} e^{\frac{s}{2}m^2} e^{-\frac{s}{2}(x-m)^2 + im\theta} 1_0^{U(1)}.$$

As $s \to +\infty$, the gaussian becomes more and more concentrated around $x = m$ and $||\sigma_s^m||_{L^2} \sim e^{\frac{s}{2}m^2}\pi^{\frac{1}{4}}$. Also, we have, with $z_s = \log w_s$, as $s \to +\infty$,

$$dz_s \sim sdx + id\theta,$$

so that, $||dz_s||^{\frac{1}{2}} \sim s^{\frac{1}{4}}$. As $s \to +\infty$, we thus obtain,

$$\frac{\sigma_s^m}{||\sigma_s^m||_{L^2}}||dz_s||^{\frac{1}{2}} \sim s^{\frac{1}{2}}\pi^{-\frac{1}{4}}e^{-\frac{s}{2}(x-m)^2+im\theta}1_0^{U(1)}.$$

From the well-known Dirac sequence

$$\lim_{k\to+\infty}(2\pi)^{-\frac{1}{2}}k^{\frac{1}{2}}e^{-\frac{k}{2}(x-m)^2} = \delta(x-m),$$

we then obtain, explicitly, the result described in Proposition 5.8.

6. Moduli Spaces of Vector Bundles on Curves

Let X be a Riemann surface of genus g. Classical theta functions associated to X are sections of line bundles over the Jacobian, $J(X)$. On the other hand, $J(X)$ can be realized as the moduli space of holomorphic line bundles (say, of degree zero) on X. This construction generalizes when one considers the moduli space of (semistable) vector bundles on X of fixed rank and degree. Holomorphic line bundles over this moduli space are obtained as tensor powers of a generating line bundle called the *determinant line bundle.* Their holomorphic sections are known as *non-abelian theta functions.* Of course, the study of these sections is much more involved that in the classical case and, in particular, their anaytical properties are very much unknown [B]. However, in some simple(r) situations, namely in the case when X is an elliptic curve, a detailed description of non-abelian theta functions can be given.

The results in this section are mostly contained in [FMN2, FMN3, FMN4].

6.1. **Non-abelian theta functions.**

Let $M_n(X)$ be the moduli space of degree zero, rank n semistable holomorphic vector bundles on X with trivial determinant. The celebrated theorem of Narasimhan-Seshadri [NS] identifies $M_n(X)$ with the moduli space of flat $SU(n)$ connections on the trivial principal bundle over X.[16] Therefore, points in $M_n(X)$ are given by representations of $\pi_1(X)$ in $SU(n)$, up to conjugation. A choice of complex structure on X gives $M_n(X)$ a Kähler structure. In the approach of [Do], this can be understood by obtaining $M_n(X)$ from an (infinite-dimensional) symplectic Kähler reduction from the affine space of $SU(n)$ connections on X; the $(0,1)$-piece of the flat connection, with respect to the chosen complex structure on X, defines a holomorphic structure on the trivial vector bundle over X.

Generalizing the theta divisor on $J(X)$, which originates the classical theory of theta functions, on $M_n(X)$ there is a naturally defined divisor which corresponds to a holomorphic line bundle $L \to M_n(X)$ which in fact generates $Pic(M_n(X))$.

[16]Note that any principal bundle on X with simply connected structure group is topologically trivial.

Level k non-abelian theta functions are then elements in $H^0(M_n(X), L^k)$. A remarkable fact is that the dimension of this space is computed by the Verlinde formula which originated in conformal field theory.

From the point of view of quantization, it is clear that associated to a choice of complex structure on X we have a holomorphic polarization of $M_n(X)$ whose quantization is given by the corresponding space $H^0(M_n(X), L^k)$. In this way, one gets a vector bundle over Teichmüller space, of rank given by the Verlinde number, $\mathcal{H} \to \mathcal{T}_g$. This was one of the original contexts where the dependence of quantization on the choice of complex structure was studied. In [AdPW, Hi] a projectively flat connection on $\mathcal{H}$ was constructed. The unitarity of this connection, for $g > 1$, remains an important open problem.

Two special cases, however, can be examined thoroughly from the point of view of the geometric quantization framework that we described in the previous sections. The first, is the case of classical theta functions, studied in Section 4, which corresponds to $n = 1$. The other, is when X is an elliptic curve. The fact that $\pi_1(X)$ is abelian in that case, allows for explicit treatments from different points of view.

Below, we will describe an analytical treatment of non-abelian theta functions in genus one which is based on the CST. As in Section 4, the idea is to establish independence of quantization with respect to the complex structure on X, by giving unitary isomorphisms between the quantization spaces $H^0(M_n(X), L^k)$ and a fixed Hilbert space associated to a real polarization (viewed as a degenerate limit of a family of holomorphic polarizations). This defines a unitary connection on $\mathcal{H} \to \mathcal{T}_{g=1}$.

As already mentioned, $M_n(X)$ and non-abelian theta functions have deep relations to conformal field theory. In many formulas in conformal field theory (and in the very much related representation theory of affine Kac-Moody algebras), the level k appears shifted to $k + n$. Often, this shift is justified by formal manipulations of infinite-dimensional Feynman integrals. An interesting point of the approach based on the CST is that it provides a finite-dimensional "explanation" of the shift in the level.

The results below for elliptic curves are very hard to generalize to the higher genus case. However, even the $g > 1$ case can be fitted in the same strategy for studying the dependence of non-abelian theta functions on complex structure. This is a generalization of the ideas described in the beginning of Section 4.

Let X have genus g and let $\tilde{X}$ be its universal cover. For appropriate representations $\rho \in Hom(\pi_1(X), SL_n(\mathbb{C}))$, the vector bundle $E_\rho = \tilde{X} \times_\rho \mathbb{C}^n$ will be in $M_n(X)$. Let $\{\alpha_i, \beta_i\}_{i=1\dots g}$ be a symplectic basis for $H_1(X, \mathbb{Z})$. Consider the subset of representations ρ such that $\rho(\alpha_i) = 1$. Such representations are in one to one correspondence with $SL_n(\mathbb{C})^g$, since the cycles β_j can be assigned any elements of $SL_n(\mathbb{C})$. It is conjectured that the resulting *Schottky map*

$$S : SL_n(\mathbb{C})^g \dashrightarrow M_n(X),$$

which is defined on some open subset of $SL_n(\mathbb{C})$, is surjective. But this remains an open problem. In fact, from the Narasimhan-Seshadri theorem, it is known that S is well defined, and is a submersion, on an open neighborhood of the subset of irreducible unitary representations in $SU(n)^g \subset SL_n(\mathbb{C})$ [Fl]. (Note that S factors through the quotient by conjugation and that $\dim SL_n(\mathbb{C})^g/SL_n(\mathbb{C}) = \dim M_n(X)$.) Therefore we have a sequence of maps analog to (8),

$$SU(n)^g \hookrightarrow SL_n(\mathbb{C})^g \overset{S}{\dashrightarrow} M_n(X),$$

where $SL_n(C)^g$ is a Stein space and is the complexification of the compact Lie group $SU(n)^g$. One can now consider the pull-back of the line bundle $L \to M_n(X)$ to $SL_n(\mathbb{C})$ and try to describe holomorphic sections as functions on $SL_n(\mathbb{C})^g$ obeying appropriate "quasi-periodicity conditions". Since the image of S contains an open set, non-abelian theta functions would then be determined by their preimages on $SL_n(\mathbb{C})^g$. It is now, of course, tempting to conjecture that non-abelian theta functions can be described by holomorphic functions on $SL_n(\mathbb{C})^g$ which are the image by a CST type of transform of some set of distributions on $SU(n)^g$. These distributions would correspond to the quantization of $M_n(X)$ in a real polarization obtained as a limit of degenerating holomorphic polarizations.

While this program can be completed in full detail for X an elliptic curve [FMN2], it is remarkable that for $n = 2, g > 1$, there are natural real polarizations of $M_2(X)$ [We1, We2]. Moreover, the Verlinde number in this case can be obtained form counting admissibe labelings of trivalent graphs associated to X by integrable representations (of level k) of $SU(2)$ [JW]. Again, one is tempted to use these labelings to generate a vector space of distributions on $SU(n)^g$ with dimension given by the Verlinde number [FMN3]. Unfortunately, the conjecture that these distributions can be obtained from non-abelian theta functions in limits of degenerating complex structure has been, to date, too hard to prove.

6.2. **Vector bundles on elliptic curves.** The fundamental group of a surface of genus one, X, has two generators and is abelian. Therefore, elements in $Hom(\pi_1(X), SU(n))$, up to conjugation, correspond to elements in $(T \times T)/W$, where $T \subset SU(n)$ is a maximal torus and $W \cong S_n$ is the Weyl group, acting by simultaneous conjugation on $T \times T$. Pairs in $T \times T$ represent the holonomy of a flat connection along the two generators of $\pi_1(X)$. Therefore, the moduli space of flat $SU(n)$ connections on the trivial bundle over X is given by $M_n(X) \cong (T \times T)/W$. As mentioned above, by the theorem of Narasimhan-Seshadri [NS], a choice of complex structure on X induces a complex structure on $M_n(X)$, which is then identified with an appropriate space of holomorphic vector bundles on X. One wants to study the variation in the quantization of $M_n(X)$ as this complex structure is varied and, most importantly, describe unitary equivalences between these quantizations. Let then $\tau \in \mathbb{H}$ and consider X equipped with the structure of elliptic curve $X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$. Semistable vector bundles over X_τ are (S-equivalent to) direct sums of line bundles [At, Tu]. Therefore, one gets a map

$$\begin{array}{ccc} M_n(X_\tau) & \to & Sym^n(X_\tau) \\ E \cong L_1 \oplus \cdots \oplus L_n & \mapsto & \{L_1, \ldots, L_n\}, \end{array}$$

where $L_i \in Pic^o(X_\tau) \cong X_\tau, i = 1, \ldots, n$. The fact that the determinant of E is trivial gives $divL_i = [z_i] - [0], z_i \in X_\tau$ where $\sum_{i=1}^n z_i = 0$. Therefore, $M_n(X_\tau)$ is the fiber over zero of the Abel-Jacobi map $AJ : Sym^n(X_\tau) \to J(X_\tau) \cong X_\tau$, so that $M_n(X_\tau) \cong \mathbb{P}^{n-1}$.

These observations allow one to describe $M_n(X_\tau)$ in terms of an abelian variety. Indeed, the condition $\sum_{i=1}^n z_i = 0$, interpreted as a zero trace condition, leads to the consideration of the abelian variety $M = X_\tau \otimes \check{\Lambda}_R$ where $\check{\Lambda}_R$ is the co-root lattice of $sl_n(\mathbb{C})$. If H denotes the usual Cartan subalgebra of $sl_n(\mathbb{C})$, we have then[17]

$$M = X_\tau \otimes \check{\Lambda}_R \cong H/(\check{\Lambda}_R \oplus \tau\check{\Lambda}_R),$$

with a projection $M \to M_n(X_\tau) \cong M/W \cong H/(W \triangleright \check{\Lambda}_R \oplus \tau\check{\Lambda}_R)$. Note that if $T \subset SU(n)$ is the maximal torus corresponding to H, we have

$$T \cong U(1)^{n-1} \subset T_\mathbb{C} \cong (\mathbb{C}^*)^{n-1} \cong H/\check{\Lambda}_R \to M,$$

and also the corresponding sequence obtained after quotienting by conjugation,

$$T/W \subset T_\mathbb{C}/W \cong H/W \triangleright \check{\Lambda}_R \to M/W \cong M_n(X_\tau).$$

This is a W-invariant analog of the sequence (8). We therefore have a sequence of maps

$$SU(n)/SU(n) \cong T/W \hookrightarrow SL_n(\mathbb{C})/SL_n(\mathbb{C}) \cong T_\mathbb{C}/W \to M/W \cong M_n(X_\tau),$$

where the quotients correspond to the conjugation action. One could now study the pull-back of non-abelian theta functions on $M_n(X_\tau)$ to M and then to $T_\mathbb{C}$ and view them as W-invariant functions, proceeding as in Section 4. However, as described in the introduction to the present Section, in order to pursue a strategy that could conceivably be generalized to higher genus, one should try to work at the level of the non-abelian groups $SU(n) \hookrightarrow SL_n(\mathbb{C})$, that is one should try to describe non-abelian theta functions as class functions on $SL_n(\mathbb{C})$. However, the issue of unitarity, that is the question of the unitary equivalence of quantizations of $M_n(X_\tau)$ for different values of τ, leads us to consider W-*anti-invariant* functions instead, since in the Weyl integration formula for class functions

$$\int_{SU(n)} f = \frac{1}{|W|} \int_T f_{|T} |\sigma|^2,$$

the denominator of Weyl's character formula, σ, is a W-anti-invariant function on T. As described below, this is behind the shift in the level $k \to k + n$ that was mentioned above, and should be seen as a nice consequence of trying to use coherent state transforms in the description of non-abelian theta functions.

[17]Note that, as a smooth manifold, $M \cong T \times T$, as described in the beginning of this section.

Next, let us describe in detail how the CST for class functions on $SU(n)$ relates to the CST for Weyl invariant functions on T.[18] Let Δ and Δ^T be the Laplace operators on $SU(n)$ and T respectively, corresponding to the bi-invariant metric for which the simple roots have squared length 2. For $t > 0$, let ν_t and $\nu_t^{T_{\mathbb{C}}}$ be the corresponding averaged heat kernels on $SL_n(\mathbb{C})$ and $T_{\mathbb{C}}$ respectively. Finally, let $C_t : L^2(SU(n), dx) \to \mathcal{H}L^2(SL_n(\mathbb{C}), d\nu_t)$ and $C_t^T : L^2(T, dh) \to \mathcal{H}L^2(T_{\mathbb{C}}, d\nu_t^{T_{\mathbb{C}}})$ be the associated coherent state transforms.

The Weyl integration formula gives the isometric isomorphism

$$\begin{array}{rcl} \varphi : L^2(SU(n), dx)^{Ad} & \to & L^2(T, dh)^W_- \\ f & \mapsto & \frac{\sigma}{|W|^{\frac{1}{2}}} f_{|T}, \end{array}$$

where $L^2(SU(n), dx)^{Ad}$ is the space of L^2 class functions on $SU(n)$ and $L^2(T, dh)^W_-$ is the space of L^2 W-anti-invariant functions on T. Analogously, defining

$$\begin{array}{rcl} \varphi_{\mathbb{C}} : \mathcal{H}L^2(SL_n(\mathbb{C}), d\nu_t)^{Ad} & \to & \mathcal{H}L^2(T_{\mathbb{C}}, d\nu_t^{T_{\mathbb{C}}})^W_- \\ f & \mapsto & e^{-t\pi||\rho||^2} \frac{\sigma_{\mathbb{C}}}{|W|^{\frac{1}{2}}} f_{|T_{\mathbb{C}}}, \end{array}$$

where ρ is the Weyl vector[19] and $\sigma_{\mathbb{C}}$ is the analytic continuation of σ, gives

Theorem 6.1. [FMN2] *The map $\varphi_{\mathbb{C}}$ is an isometric isomorphism and*

$$\varphi_{\mathbb{C}} \circ C_t = C_t^T \circ \varphi,$$

for all $t > 0$.

As described above, the CST's can be extended to injective maps defined on distributions and the Laplace operators can be generalized to $\Delta_\tau = -i\tau\Delta$ and $\Delta_\tau^T = -i\tau\Delta^T$, for $\tau \in \mathbb{H}$. Let $C_{t,\tau}$ and $C_{t,\tau}^T$ be the corresponding coherent state transforms. Extending φ to distributions and $\varphi_{\mathbb{C}}$ to holomorphic sections which are not necessarily L^2, gives isomorphisms $\varphi : (C^\infty(SU(n))')^{Ad} \to (C^\infty(T)')^W_-$ and $\varphi_{\mathbb{C}} : \mathcal{H}(SL_n(\mathbb{C}))^{Ad} \to \mathcal{H}(T_{\mathbb{C}})^W_-$. One has,

Theorem 6.2. [FMN2] *Let $\tau \in \mathbb{H}, t > 0$. Then, $\varphi_{\mathbb{C}} \circ C_{t,\tau} = C_{t,\tau}^T \circ \varphi$.*

The relation between the averaged heat kernel measures on $SL_n(\mathbb{C})$ and on $T_{\mathbb{C}}$ is given by

Proposition 6.3. [FMN2] *Let $q : SL_n(\mathbb{C}) \to T_{\mathbb{C}}/W$ be the projection given by the adjoint action. Then, for $t > 0, \tau \in \mathbb{H}$,*

$$q_* d\nu_{t\tau 2} = e^{-2\pi t\tau_2||\rho||^2} |\sigma_{\mathbb{C}}|^2 d\nu_{t\tau_2}^{T_{\mathbb{C}}}.$$

As before, with unitarity in mind, we should look for distributions on $SU(n)$ which after application of the CST become holomorphic functions on $SL_n(\mathbb{C})$

[18]Many of the results below hold for more general compact groups $K \hookrightarrow K_{\mathbb{C}}$. Here, we will keep to $SU(n)$ and $SL_n(\mathbb{C})$ since the main application will be to the moduli space $M_n(\mathbb{C})$.

[19]ρ is half the sum of the positive roots. The eigenvalues of Δ are of the form $c_2(\lambda) = ||\lambda + \rho||^2 - ||\rho||^2$, for highest weights λ, while the eigenvalues of Δ^T are given by $||\lambda + \rho||^2$.

with appropriate quasi-periodicity conditions. In this case, it should be possible to integrate such functions on a fundamental domain for the projection

$$SL_n(\mathbb{C}) \to M_n(X_\tau) = T_{\mathbb{C}}/W \rhd \tau\check{\Lambda}_R.$$

Let Λ_W be the weight lattice for $sl_n(\mathbb{C})$ and let

$$D_k = \Lambda_W/W \rhd k\Lambda_R$$

be the (finite) set of integrable representations of the affine Kac-Moody algebra $\widehat{sl_n(\mathbb{C})}_k$. The set of such represention[20] coincides with the Verlinde number for $g = 1$ and group $SU(n)$,

$$\#D_k = \begin{pmatrix} n+k-1 \\ k \end{pmatrix}.$$

Theorem 6.4. [FMN2] *The sets of distributions on* $SU(n)$,

$$\mathcal{F} = \{\psi \in (C^\infty(SU(n))')^{Ad} : |C_{t,\tau}(\psi)|^2 |\sigma_{\mathbb{C}}|^2 \nu_{t\tau_2}^{T_{\mathbb{C}}} \text{ is } \tau\check{\Lambda}_R \text{ invariant as a function on } H\},$$

are non-trivial iff $t = \frac{1}{k+n}$ *for* $k \in \mathbb{N}_0$. *For* $t = \frac{1}{k+n}$, $\mathcal{F}_k$ *is a (*τ*-independent) finite-dimensional subspace of* $(C^\infty(SU(n))')^{Ad}$, *with dimension given by the Verlinde number*

$$\dim \mathcal{F}_k = \#D_k$$

and with a basis given by

$$\psi_{\gamma,k}(h) = \frac{1}{\sigma(h)} \sum_{w\in W} \varepsilon(w)\theta^0_{\gamma+\rho,k+n}(w(h)), h \in H_{\mathbb{R}}, \gamma \in D_k, \tag{23}$$

where

$$\theta^0_{\gamma+\rho,k+n}(h) = \sum_{\alpha\in\Lambda_R} e^{2\pi i(\gamma+\rho+(k+n)\alpha)(h)} \in (C^\infty(T)').$$

Therefore, unitarity arguments lead us automatically to a set of distributions on $SU(n)$ with the correct (Verlinde) dimension.

Let us now describe in more detail the pull-back of the determinant line bundle $\mathcal{L} \cong \mathcal{O}(1)$ on $M_n(X_\tau) \cong M/W \cong \mathbb{P}^{n-1}$ to M. By looking explicitly at the automorphy factors, one finds that the only line bundle in $Pic^0(M)$ which is W-invariant is the trivial bundle. This, combined with a result of Looijenga [L], gives that points in the W-invariant part of $Pic(M)$ are given by W-invariant symmetric integral bilinear forms on $\check{\Lambda}_R$. If $n \geq 3$, up to an integral multiplicative constant, the only such bilinear form is the Killing metric, so that $Pic(M)^W$ is infinite cyclic, with generator L.[21] Comparing the dimensions of the W-invariant part of H^0 given in [L], one concludes that if $p : M \to M_n(X_\tau)$ is the projection, then $p^*\mathcal{L} = L$. Therefore, the space of non-abelian theta functions of level k is identified

[20]These correspond to irreducible representations of $SU(n)$ such that the Young tableau has less than $k+1$ boxes in the first row.

[21]The case $n = 2$ is special as there are only 2 roots, $\pm\alpha$. We will proceed with the $n \geq 3$ case since the results for $n = 2$ are similar, although one has to study the odd and even level cases separately.

with the space of W-invariant holomorphic sections of L^k, that is

$$H^0(M_n(X_\tau), \mathcal{L}^k) \cong H^0(M, L^k)^W.$$

More explicitly,

Proposition 6.5. [FMN2] *The space $H^0(M_n(X_\tau), \mathcal{L}^k)$ of non-abelian theta functions of level k is identified with the space of Weyl invariant holomorphic functions on H, satisfying the quasi-peridicity conditions*

$$\theta(h + \check{\alpha} + \tau\check{\beta}) = e^{-2\pi i k\beta(h) - \pi i k\tau\langle\beta,\beta\rangle}\theta(h), h \in H, \check{\alpha}, \check{\beta} \in \check{\Lambda}_R.$$

An explicit basis for $H^0(M, L^k)^W$ is given by $\{\theta^+_{\gamma,k}\}_{\gamma \in D_k}$, where

$$\theta^+_{\gamma,k} = \sum_{w \in W} \theta_{w(\gamma),k},$$

with

$$\theta_{\gamma,k}(h) = \sum_{\alpha \in \check{\Lambda}_R} e^{\pi i \frac{\tau}{k}\langle\gamma + k\alpha, \gamma + k\alpha\rangle + 2\pi i\langle\gamma + k\alpha, h\rangle}, \gamma \in \lambda_W / k\Lambda_R, h \in H.$$

As we have seen above, in Theorem 6.4, however, CST considerations lead us to focus on W-anti-invariant holomorphic functions on H. Moreover, as already mentioned, the CST also leads us to consider the shift in the level $k \to k + n$. In fact, Looijenga proved that the space of W-anti-invariant sections of L^n is one-dimensional and is generated by $\theta^-_{\rho,n}$, where

$$\theta^-_{\gamma,k} = \sum_{w \in W} \varepsilon(w)\theta_{w(\gamma),k}, \gamma \in D_k.$$

Moreover, he proves that W-anti-invariant theta functions of level $k+n$ on M are divisible by $\theta^-_{\rho,n}$. Therefore, one has a basis for $H^0(M, L^k)^W$ different from the above, and more natural from the CST point of view, given by $\{\hat{\theta}^+_{\gamma,k}\}_{\gamma \in D_k}$, with

$$\hat{\theta}^+_{\gamma,k} = \frac{\theta^-_{\gamma+\rho,k+n}}{\theta^-_{\rho,n}}.$$

Define on the spaces of distributions $\mathcal{F}_k \subset (C^\infty(SU(n))')^{Ad}$ of Theorem 6.4 the hermitian structure for which the basis $\{\psi_{\gamma,k}\}_{\gamma \in D_k}$ in (23) is orthonormal. On $H^0(M, L^k)^W$ define the hermitian structure[22]

$$(24) \qquad \langle\theta^+, \theta^{+'}\rangle = \frac{1}{|W|}\int_M \bar{\theta}^+\theta^{+'}|\theta^-_{\rho,n}|^2 d\nu^{t_\mathbb{C}}_{\tau_2/(k+n)}.$$

Noting that

$$C_{\tau, \frac{1}{k+n}}(\psi_{\gamma,k})_{|T_\mathbb{C}} = \frac{e^{-\frac{i\pi\tau}{k+n}||\rho||^2}}{\sigma_\mathbb{C}}\theta^-_{\gamma+\rho,k+n},$$

and using Theorem 6.2, we have

Theorem 6.6. [FMN2] *The CST $C_{\tau,\frac{1}{k+n}}$ is a unitary isomorphism between $\mathcal{F}_k$ and the space of level k non-abelian theta functions $H^0(M_n(X_\tau), \mathcal{L}^k) \cong H^0(M, L^k)^W$.*

[22]The fact that $\nu^{t_\mathbb{C}}_{\tau_2/(k+n)}$ defines a hermitian structure on L^{k+n} is the result Lemma 4.3, with the slight generalization that M is not principally polarized.

Therefore, for any $\tau \in \mathbb{H}$, we have an unitary isomorphism between $H^0(M_n(X_\tau), \mathcal{L}^k)$ and a fixed (τ-independent) Hilbert space of distributions on $SU(n)$. Moreover, the distributions in $\mathcal{F}_k$ can be described in terms of the behavior of non-abelian theta functions in the limit of degenerating complex structure, $\tau \to 0$. It should be noted that the hermitian structure for which the frame $\{\hat{\theta}^+_{\gamma,k}\}_{\gamma \in D_k}$ is orthonormal, and which is the one to be naturally considered in the context of the CST, is the one of interest in conformal field theory [AdPW, EMSS], where the factor of $|\theta^-_{\rho,n}|^2$ in the hermitian structure arises in the (regularized) evaluation of an infinite-dimensional determinant when performing the path integral.

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