# Coherent State Transforms and Vector Bundles on Elliptic Curves

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TO THE MEMORY OF ANDREI TYURIN

We extend the coherent state transform (CST) of Hall to the context of the moduli spaces of semistable holomorphic vector bundles with fixed determinant over elliptic curves. We show that by applying the CST to appropriate distributions, we obtain the space of level $k$, rank $n$ and genus one non-abelian theta functions with the unitarity of the CST transform being preserved. Furthermore, the shift in the level $k \to k + n$ appears in a natural way in this finite-dimensional framework.

**CONTENTS**



## 1. INTRODUCTION

In [Ha1] Hall proposed a generalization of the Segal-Bargmann or coherent state transform (CST) [Se1, Se2, Ba] in which $\mathbb{R}^n$ is replaced by an arbitrary compact connected Lie group $K$ and $\mathbb{C}^n$ by the complexification $K_{\mathbb{C}}$ of $K$. This Segal-Bargmann-Hall CST was further generalized to gauge theories with applications to gravity in the context of Ashtekar variables in [ALMMT] and to Yang-Mills theories in two space-time dimensions in [DH]. For reviews and further developments see [Ha2, Ha3] and [Th].

In the present paper we continue the project started in [FMN] of the application of CST techniques to the study of theta functions.

For a Riemann surface $X$ of genus $g$, $K_{\mathbb{C}}$-theta functions are sections of holomorphic line bundles $\mathcal{L}$ over the moduli space of semistable $K_{\mathbb{C}}$-bundles on $X$

$$\begin{array}{rcl} \mathcal{L} & \to & \mathcal{M}_{K_{\mathbb{C}}}(X) \\ \theta & \in & H^0(\mathcal{M}_{K_{\mathbb{C}}}(X), \mathcal{L}). \end{array}$$

The study of theta functions motivates considerable interest both from the mathematical and physical points of view. In physics, the spaces $H^0(\mathcal{M}_{K_{\mathbb{C}}}(X), \mathcal{L})$ correspond both to spaces of conformal blocks in WZW conformal field theories and to Hilbert spaces of states of Chern-Simons theories. It was in this context that the Verlinde formula for the dimensions of these vector spaces of holomorphic sections was discovered, see for example [Bea2, So] for a review.

In the case when $K_{\mathbb{C}}$ is $SL(n,\mathbb{C})$, this moduli space can be interpreted, via the well-known theorem of Narasimhan and Seshadri [NS], as the moduli space $\mathcal{M}_n(X) := \mathcal{M}_{SL(n,\mathbb{C})}(X)$ of semistable rank $n$ vector bundles with trivial determinant over $X$, and the corresponding non-abelian theta functions were already subject of study by Weil. The conformal blocks are then represented by holomorphic sections of powers of determinant line bundles over $\mathcal{M}_n(X)$ which is also the moduli space of flat $SU(n)$-connections on $X$. These non-abelian theta functions have been widely studied since the nineties, mainly from the point of view of algebraic geometry. However, an analytic theory of these functions is not yet fully developed, and there are many open questions related to them [Bea1, Fa].

In this work we will consider the case when $X$ is an elliptic curve, $X = X_\tau, \mathrm{Im}\tau > 0$. Non-abelian theta functions of level $k$ on the moduli space of holomorphic bundles on elliptic curves have been studied mainly from two points of view. Expressions for orthonormal frames for these theta functions have been obtained by physical (formal) functional integral methods in conformal field theory, e.g. in [Ber, EMSS, FG, G]. These expressions were also obtained from geometric quantization in infinite dimensionions and symplectic reduction of affine spaces in the context of Chern-Simons theory [AdPW]. In both approaches, one observes a shift in the level $k \to k+h$, with $h$ being the dual Coxeter number of $K$ ($h = n$ for $SU(n)$), which arises from a regularization of infinite determinants of differential operators on bundles over $X$ (see also point 4) below). Note that the shift in the level can also be given a cohomological interpretation in the context of geometric quantization [Hi]. We will show that by extending the CST of Hall for $SU(n)$ to appropriate finite-dimensional spaces of distributions, we obtain the spaces of non-abelian theta functions and that the averaged heat kernel measure descends to a hermitean structure on $\mathcal{L}^k$ making the CST transform unitary, with the correct shift of level $k \to k+n$.

More precisely, to obtain the relation between the CST for $SU(n)$ and genus one non-abelian theta functions we adopt a strategy similar to that of section 4 of [FMN]:

1) In propositions 2.3 and 2.4 of subsection 2.3 we consider the extension of the CST to complexified Laplacians and then to the space of distributions $C^\infty(SU(n))'$. By taking the second step we of course lose unitarity of the CST. The restriction of the CST, for a simple, compact, connected and simply connected group $K$, to $Ad_K$-invariant functions and distributions is intimately related with the CST for the maximal torus $T \subset K$, but for Weyl anti-invariant functions and distributions as we show in theorems 2.2

and 2.4. This simple fact will play a crucial role and it is behind the success of the CST in reproducing the shift $k \to k+h$.

2) The relation between holomorphic functions on $SL(n,\mathbb{C})$ (obtained from this extended CST) and sections of line bundles over $\mathcal{M}_n(X_\tau)$, is provided by pull-back via the Schottky map, see section 4 (specially equations (74), (75) and proposition 4.1),

$$S : SL(n,\mathbb{C}) \to \mathcal{M}_n(X_\tau).$$

In analogy with the abelian case, this gives a description of non-abelian theta functions in genus one as holomorphic functions on a space whose complex structure is canonical (independent of $\tau$).

3) The theorems of subsection 5.2 show that the restriction of the CST to appropriate finite dimensional subspaces $\mathcal{F}_k$ of Ad-invariant distributions (see (94) and (96)) leads to a vector bundle over the Teichmüller space of genus one curves

$$\widetilde{\mathcal{H}}_k \to \mathcal{T}_1 = \{\tau \in \mathbb{C} : \mathrm{Im}\tau > 0\} \tag{1}$$

which is isomorphic to the vector bundle $\mathcal{H}_k \to \mathcal{T}_1$ of conformal blocks, see (86), with simple unitary (see the point 4) below) isomorphism $\Phi_k$ given by

$$\begin{aligned} \Phi_{k,\tau} : \widetilde{\mathcal{H}}_{k,\tau} &\to \mathcal{H}^+_{k,\tau} \\ \Psi &\mapsto \theta^+ = e^{\frac{||\rho||^2}{k+n}\pi i\tau} \frac{\sigma_{\mathbb{C}}}{\theta^-_{\rho,n}} \Psi \end{aligned} \tag{2}$$

where $\widetilde{\mathcal{H}}_{k,\tau} = \widetilde{\mathcal{H}}_{k|_\tau}$, $\mathcal{H}^+_{k,\tau} = \mathcal{H}_{k|_\tau}$ and $\Phi_{k,\tau} = \Phi_{k|_{\widetilde{\mathcal{H}}_{k,\tau}}}$. Here, $\theta^\pm$ denote Weyl invariant or anti-invariant theta functions, $\rho$ is the Weyl vector and $\sigma_{\mathbb{C}}$ (correspondingly $\sigma$) is the denominator in the Weyl character formula for the maximal torus $T_{\mathbb{C}} \subset K_{\mathbb{C}}$ ($T \subset K$). Note that $\sigma_{\mathbb{C}}$ is the analytic ocntinuation of $\sigma$. The image of $\mathcal{F}_k$ under the CST selects a trivialization of the bundle of conformal blocks corresponding to the frame (90).

4) The natural hermitean structure defined on $\mathcal{H}^+_{k,\tau}$ would seem to be the one for which the frame (87) of Weyl invariant theta functions of level $k$ is orthonormal. In fact, however, the relevant hermitean structure for conformal field theory is the one for which the orthonormal frame is defined in terms of level $(k+h)$ Weyl anti-invariant theta functions as in (90). This is what is known as "shift of level" in the conformal field theory literature, where it can obtained with the help of infinite dimensional Feynman path integral methods [AdPW]. As we will show, the CST on $SU(n)$ selects the correct hermitean structure and thus leads to the shift of level $k \to k+n$.

5) The Hall averaged heat kernel measure on $SL(n,\mathbb{C})$ defines a hermitian structure (97) on $\widetilde{\mathcal{H}}_k$ for which the unitarity of the extended CST is recovered.

This strategy provides an intrinsically finite-dimensional framework for the study of non-abelian theta functions in genus one.

The organization of the paper is the following. In section 2, we extend the CST to class functions on compact Lie groups and extend the results to $Ad$-invariant distributions on the group. In section 3, we extend the results of [FMN] to abelian varieties with a general polarization. In the following sections, we restrict ourselves to the context of elliptic curves and $K = SU(n)$ ($K_{\mathbb{C}} = SL(n,\mathbb{C})$) corresponding to the moduli space $\mathcal{M}_n(X_\tau)$. (We expect that the corresponding results for other Lie groups should also apply.) In section 4, we describe the Schottky map from the space of Schottky representations of $\pi_1(X_\tau)$ in $K_{\mathbb{C}}$ to the moduli space $\mathcal{M}_n(X_\tau)$. By considering the Schottky map, we show in section 5 that the results in [FMN] extend to non-abelian theta functions in genus 1. Namely, applying the CST to appropriate finite dimensional spaces of distributions $\mathcal{F}_k \subset C^\infty(SU(n))'$ yields the spaces of level $k$ non-abelian theta functions (Theorem 5.3). In this case however, as we show in Theorems 5.3 and 5.4, not only the Hall averaged heat kernel measure makes the CST unitary, but also the hermitean structure that it defines on the corresponding vector bundle over the Teichmüller space is the "correct" one in the sense that it contains the level shift $k \mapsto k+n$. Surprisingly, the case of $SU(2)$ is special and is treated in subsection 5.3, where the main results are formulated in theorem 5.5.

Extensions of this work and applications to the moduli space of semistable vector bundles on higher genus curves will appear in [FMNT].

## 2. EXTENSIONS OF THE COHERENT STATE TRANSFORM

### 2.1. Coherent State Transform for Lie Groups

Let $K$ be a compact connected Lie group of rank $l$, $K_{\mathbb{C}}$ its complexification (see [Ho]) and let $\rho_t, t > 0$, be the heat kernel for the Laplacian $\Delta_K$ on $K$ associated to an Ad-invariant inner product on its Lie algebra $Lie(K)$. If $\{X_i,\, i = 1, ..., \dim K\}$ is a corresponding orthonormal basis for $Lie(K)$ viewed as the space of left-invariant vector fields on $K$, then $\Delta_K = \sum_{i=1}^{\dim K} X_i X_i$. As proved in [Ha1], $\rho_t$ has a unique analytic continuation to $K_{\mathbb{C}}$, also denoted by $\rho_t$. The $K$-averaged coherent state transform

(CST) is defined as the map

$$\begin{aligned} C_t \quad &: \quad L^2(K, dx) \mapsto \mathcal{H}(K_{\mathbb{C}}) \\ C_t(f)(g) \quad &= \quad \int_K f(x)\rho_t(x^{-1}g)dx, \qquad f \in L^2(K, dx),\ g \in K_{\mathbb{C}} \end{aligned} \tag{3}$$

where $dx$ is the normalized Haar measure on $K$ and $\mathcal{H}(K_{\mathbb{C}})$ is the space of holomorphic functions on $K_{\mathbb{C}}$. For each $f \in L^2(K, dx)$, $C_t f$ is just the analytic continuation to $K_{\mathbb{C}}$ of the solution of the heat equation,

$$\frac{1}{\pi}\frac{\partial u}{\partial t} = \Delta_K u, \tag{4}$$

with initial condition given by $u(0, x) = f(x)$. Therefore, $C_t(f)$ is given by

$$C_t f(g) = (\mathcal{C} \circ \rho_t \star f)(g) = \left(\mathcal{C} \circ e^{t\pi\Delta_K} f\right)(g), \tag{5}$$

where $\star$ denotes the convolution in $K$ and $\mathcal{C}$ denotes analytic continuation from $K$ to $K_{\mathbb{C}}$. Let $d\nu_t$ be the $K$-averaged heat kernel measure on $K_{\mathbb{C}}$ defined in [Ha1]. Then the following result holds.

THEOREM 2.1. [Hall] *For each $t > 0$, the mapping $C_t$ defined in (3) is a unitary isomorphism from $L^2(K, dx)$ onto the Hilbert space $L^2(K_{\mathbb{C}}, d\nu_t) \bigcap \mathcal{H}(K_{\mathbb{C}})$.*

To obtain a more explicit description of this CST, consider the expansion of $f \in L^2(K, dx)$ given by the Peter-Weyl theorem,

$$f(x) = \sum_{\mathrm{R}} \mathrm{tr}(\mathrm{R}(x)A_{\mathrm{R}}), \tag{6}$$

where the sum is taken over the set of (equivalence classes of) irreducible representations of $K$, and $A_{\mathrm{R}} \in End\ V_{\mathrm{R}}$ is given by

$$A_{\mathrm{R}} = (\dim V_{\mathrm{R}}) \int_K f(x)\mathrm{R}(x^{-1})dx, \tag{7}$$

$V_{\mathrm{R}}$ being the representation space for R. Then one obtains:

$$C_t f(g) = \sum_{\mathrm{R}} e^{-t\pi c_{\mathrm{R}}}\, \mathrm{tr}(\mathrm{R}(g)A_{\mathrm{R}}), \tag{8}$$

where $c_{\mathrm{R}} \geq 0$ is the eigenvalue of $-\Delta_K$ on functions of the type

$$\mathrm{tr}(A\mathrm{R}(x)), \quad A \in End(V_{\mathrm{R}}).$$

### 2.2. Coherent State Transform for Class Functions on Lie groups

From now on, let $K$ be a compact connected and simply connected simple Lie group and let $<,>$ be the Ad-invariant inner product on $Lie(K)$ for which the longest root has squared length 2. Here, we will study the restriction of the CST to class (i.e. $Ad_K$-invariant) functions and its relation to the CST transform on a maximal torus $T \subset K$. The main results in this section are theorems 2.2 and 2.3. Let $K/Ad_K$ be the quotient space for the adjoint action of $K$ on itself. As we will show in section 5.2, for $K = SU(n)$, it turns out that the image of appropriately chosen distributions on $K/Ad_K \cong T/W$ (where $W$ is the Weyl group), related with Bohr-Sommerfeld conditions in geometric quantization, with respect to a natural extension of $C_t$ in (5) and (8), gives functions satisfying quasi-periodicity conditions in the imaginary directions of $K_{\mathbb{C}} = SL(n, \mathbb{C})$. These functions correspond to holomorphic sections of the pull-back of line bundles over the moduli space of holomorphic vector bundles with trivial determinant over an elliptic curve $X_\tau$. Here, the appropriate metric to be considered on $K_{\mathbb{C}}$ is related to the complex structure $\tau$ on $X_\tau$, where $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$.

Let $\mathfrak{h}$ be the Cartan subalgebra for $K_{\mathbb{C}}$ corresponding to $T$, and let $\Lambda_R, \Lambda_W \subset \mathfrak{h}^*_{\mathbb{R}}$, $\check{\Lambda}_R, \check{\Lambda}_W \subset \mathfrak{h}_{\mathbb{R}}$ be the root, weight, coroot and coweight lattices respectively. We consider fixed a choice of positive roots. Denote also by $<,>$ the inner product induced on $\mathfrak{h}^*_{\mathbb{R}}$ by the inner product $<,>$ on $\mathfrak{h}$.

From the Peter-Weyl expansion (6) and Schur's lemma, one sees that the space $L^2(K, dx)^{Ad_K}$ of Ad-invariant functions on $L^2(K, dx)$ corresponds to chosing all the endomorphisms $A_{\mathrm{R}}$ proportional to the identity $A_R = a_R I_R$. Therefore, any $f \in L^2(K, dx)^{Ad_K}$ can be expressed as

$$f = \sum_{\lambda \in \Lambda_W^+} a_\lambda \chi_\lambda, \tag{9}$$

where we labelled irreducible representations of $K$ by the highest weights $\lambda$ in the set of dominant weights $\Lambda_W^+ \subset \Lambda_W$ and $\chi_\lambda = \mathrm{tr}(\mathrm{R}_\lambda)$ is the character corresponding to $\lambda$.

By restricting the coherent state transform (3) and (8) to the closed subspace of Ad-invariant functions on $K$ we obtain

Proposition 2.1. *The restriction $C_t^{\mathrm{Ad}}$ of the CST (3) to the Hilbert space $L^2(K, dx)^{Ad_K}$ is an isometric isomorphism onto the Hilbert space $L^2(K_{\mathbb{C}}, d\nu_t)^{Ad_{K_{\mathbb{C}}}} \bigcap \mathcal{H}(K_{\mathbb{C}})$.*

*Proof.* From (9) and (8) we see that for $f \in L^2(K, dx)^{Ad_K}$ we have

$$
\begin{aligned}
C_t^{\mathrm{Ad}} f(g) = C_t f(g) &= \sum_{\lambda \in \Lambda_W^+} a_\lambda \, e^{-t\pi c_\lambda} \, \chi_\lambda(g) = \\
&= \sum_{\lambda \in \Lambda_W^+} a_\lambda \, e^{-t\pi c_\lambda} \, \mathrm{tr}(\mathrm{R}_\lambda(g)), \quad g \in K_{\mathbb{C}}. \qquad (10)
\end{aligned}
$$

where $c_\lambda = c_{\mathrm{R}_\lambda}$, and therefore the image of an $Ad_K$-invariant function on $K$ is an $Ad_{K_{\mathbb{C}}}$-invariant function on $K_{\mathbb{C}}$. On the other hand, a non $Ad_K$-invariant $f$ has in its Peter-Weyl expansion at least one $A_R$ which is not proportional to the identity and therefore its image will not be $Ad_{K_{\mathbb{C}}}$-invariant. Let now $F \in L^2(K_{\mathbb{C}}, d\nu_t)^{Ad_{K_{\mathbb{C}}}} \bigcap \mathcal{H}(K_{\mathbb{C}})$. From the ontoness of $C_t$ and from Schur's lemma as above, we see that $F$ has the form (10) and is therefore the image under $C_t$ of an $Ad_K$-invariant function on $K$. ▮

In view of the isomorphism of $C^\infty(K)^{Ad_K} \cong C^\infty(T)^W$ it is interesting to relate the CST $C_t^{\mathrm{Ad}}$ for Ad-invariant functions on $K$ with the "abelian" CST $C_t^T$ for functions on $T$. Amazingly, the interesting relation is with $W$-anti-invariant and not $W$-invariant functions on $T$. This fact will have important consequences for the application to non-abelian theta functions that we pursue in the next sections. In particular, it will lead to the well known shifts of level $k \to k + h$ and of weights $\lambda \to \lambda + \rho$, where $h$ is the dual Coxeter number for $Lie(K)$ and $\rho \in \Lambda_W$ is the Weyl vector given by half the sum of the positive roots.

First of all notice that from the Weyl integration formula (see for example [Kn]) we see that there exists an isometric isomorphism into the space of $W$-invariant functions on $T$[1],

$$
\begin{array}{rcl}
L^2(K, dx)^{Ad_K} & \longrightarrow & L^2(T, |\sigma|^2 dh/|W|)^W \\
f & \mapsto & f_{|T}
\end{array} \qquad (11)
$$

where $dh$ denotes the normalized Haar measure on $T$, $|W|$ is the order of $W$ and $\sigma$ is the denominator of the Weyl character formula given by

$$
\sigma(e^{2\pi i h}) = \sum_{w \in W} \epsilon(w) e^{2\pi i w(\rho)(h)}, \quad \text{for } h \in \mathfrak{h}_{\mathbb{R}} \, . \qquad (12)
$$

Here, $\epsilon(w) = \det(w)$ with $w \in W$ viewed as an orthogonal transformation in $\mathfrak{h}_{\mathbb{R}}^*$. Notice that $\sigma$ is Weyl anti-invariant, that is if $w \in W$ then $\sigma \circ w = \epsilon(w)\sigma$.

[1] Notice that, strictly speaking, this map is defined first on continuous class functions and then extended uniquely to $L^2(K, dx)^{Ad_K}$ by continuity.

Let $\alpha_1, ..., \alpha_l$ be a set of simple roots and $\lambda_1, ..., \lambda_l$ the corresponding set of fundamental weights such that $\frac{2<\lambda_i, \alpha_j>}{<\alpha_j, \alpha_j>} = \delta_{ij}$. The coordinates on $\mathfrak{h}_{\mathbb{R}}$ corresponding to the coroots $\check{\alpha}_j$ will be denoted by $x_j$. The invariant Laplacian on $T$ corresponding to the inner product $<,>$ on $\mathfrak{h}$ is given by

$$\Delta_T = \sum_{i,j=1}^{l} \frac{1}{4\pi^2} C^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \tag{13}$$

where $C^{ij} =< \lambda_i, \lambda_j >$ is the inverse of the Cartan matrix $C_{ij} =< \alpha_i, \alpha_j >$ for $Lie(K)$.

As in the case of the group $K$, there is a unique analytic continuation $\rho_t^T$ to $T_{\mathbb{C}}$ of the heat kernel for the Laplacian $\Delta_T$ on $T$, and we define for each $t > 0$ the "abelian CST" as the map:

$$\begin{aligned} C_t^T \quad &: \quad L^2(T, dh) \to \mathcal{H}(T_{\mathbb{C}}) \\ C_t^T(f)(z) \quad &= \quad \int_T f(h) \rho_t^T(h^{-1}z) dh, \qquad f \in L^2(T, dh),\ z \in T_{\mathbb{C}}, \end{aligned} \tag{14}$$

where $\mathcal{H}(T_{\mathbb{C}})$ is the space of holomorphic functions on $T_{\mathbb{C}} \cong \mathfrak{h}/\check{\Lambda}_R \cong (\mathbb{C}^*)^l$. It is easy to see that if $f \in L^2(T, dh)$ is given by

$$f(h) = \sum_{\lambda \in \Lambda_W} b_\lambda \, e^{2\pi i \lambda}(h), \quad \text{for } h \in T, \tag{15}$$

then

$$C_t^T(f)(z) = \sum_{\lambda \in \Lambda_W} b_\lambda \, e^{-t\pi ||\lambda||^2} e^{2\pi i \lambda}(z), \text{ where } z \in T_{\mathbb{C}}. \tag{16}$$

As before, Hall's result applies to this CST and we have

Corollary 2.1. [Hall] *For each $t > 0$, the mapping $C_t^T$ is an isometric isomorphism onto the Hilbert space $L^2(T_{\mathbb{C}}, d\nu_t^{T_{\mathbb{C}}}) \bigcap \mathcal{H}(T_{\mathbb{C}})$, where $d\nu_t^{T_{\mathbb{C}}} = \nu_t^{T_{\mathbb{C}}} dz$ is the averaged heat kernel measure on $T_{\mathbb{C}}$.*

The explicit expression for $\nu_t^{T_{\mathbb{C}}}$ is

$$\nu_t^{T_{\mathbb{C}}}(e^{2\pi i v}) = \left(\frac{2}{t}\right)^{l/2} \sqrt{\det[C_{ij}]} \, e^{\frac{\pi}{2t} < v - \bar{v}, v - \bar{v} >}, \tag{17}$$

where $v \in \mathfrak{h}$ and $T_{\mathbb{C}} \cong \mathfrak{h}/\check{\Lambda}_R$, and conjugation $v \mapsto \bar{v}$ is the anti-linear involution in $\mathfrak{h}$ preserving $\mathfrak{h}_{\mathbb{R}}$.

Functions in (10) are the analytic continuation of Ad-invariant solutions of the heat equation on $K$ while those in (16) are the analytic continuation of solutions of the heat equation on $T$ and which, if Weyl invariant, can in turn be extended to $Ad_K$-invariant functions on $K$. As noticed by Fegan [Fe] two facts make it particularly simple to relate Ad-invariant solutions of the heat equation on $K$ with Weyl *anti-invariant* solutions of the heat equation on $T$. The first is the Weyl character formula

$$\chi_\lambda = \frac{1}{\sigma}\sum_{w\in W}\epsilon(w)e^{2\pi i w(\lambda+\rho)} \tag{18}$$

and the second is the identity

$$c_\lambda = ||\lambda+\rho||^2 - ||\rho||^2 \ . \tag{19}$$

Let $L^2(T,dh)^W_-$ and $L^2(T_\mathbb{C},d\nu_t^{T_\mathbb{C}})^W_-$ denote the $W$-anti-invariant subspaces. Since the actions of $W$ and $\Delta_T$ on $L^2(T,dh)$ commute, we have a result analogous to Proposition 2.1 (with similar proof which we omit):

Proposition 2.2. *The restriction of the abelian CST (14) (which we will denote by the same symbol $C_t^T$) to the space $L^2(T,dh)^W_-$ is an isometric isomorphism onto the Hilbert space $L^2(T_\mathbb{C},d\nu_t^{T_\mathbb{C}})^W_-\bigcap\mathcal{H}(T_\mathbb{C})$.*

From (10), (16), (18) and (19) we see that by multiplying an Ad-invariant solution of the heat equation on $K$ by $\sigma e^{-t\pi||\rho||^2}$ we obtain a $W$-anti-invariant solution of the heat equation on $T$. Moreover, this map takes $C_t^{\mathrm{Ad}}$ to $C_t^T$ restricted to $W$ anti-invariant functions. More precisely, consider the maps $\varphi$ and $\varphi_\mathbb{C}$ given respectively by

$$\begin{array}{rccc} \varphi: & L^2(K,dx)^{Ad_K} & \to & L^2(T,dh)^W_- \\ & f & \mapsto & \frac{\sigma}{\sqrt{|W|}}f_{|T} \end{array}$$

and

$$\begin{array}{rccc} \varphi_\mathbb{C} & : L^2(K_\mathbb{C},d\nu_t)^{Ad_{K_\mathbb{C}}}\bigcap\mathcal{H}(K_\mathbb{C}) & \to & L^2(T_\mathbb{C},d\nu_t^{T_\mathbb{C}})^W_-\bigcap\mathcal{H}(T_\mathbb{C}) \\ & f & \mapsto & e^{-t\pi||\rho||^2}\frac{\sigma_\mathbb{C}}{\sqrt{|W|}}\,f_{|T_\mathbb{C}} \end{array}$$

where $\sigma_\mathbb{C}$ denotes the analytic continuation of (12). We then have the following

Theorem 2.2. *The maps $\varphi$ and $\varphi_\mathbb{C}$ are isometric isomorphisms and the following diagram is commutative*

$$\begin{array}{ccc} L^2(K,dx)^{Ad_K} & \overset{C_t^{\mathrm{Ad}}}{\to} & L^2(K_\mathbb{C},d\nu_t)^{Ad_{K_\mathbb{C}}}\bigcap\mathcal{H}(K_\mathbb{C}) \\ \downarrow\varphi & & \downarrow\varphi_\mathbb{C} \\ L^2(T,dh)^W_- & \overset{C_t^T}{\to} & L^2(T_\mathbb{C},d\nu_t^{T_\mathbb{C}})^W_-\bigcap\mathcal{H}(T_\mathbb{C}) \end{array} \tag{20}$$

*Proof.* From the Weyl integration formula we see that $\varphi$ is an isometry. It is easy to check that if $f \in L^2(T, dh)^W_-$, then its expansion (15) receives contributions only from non-singular weights $\lambda \in \Lambda_W$, i.e. such that $< \lambda, \alpha_i > \neq 0$ for all simple roots $\alpha_i$. This implies that such $f$ is of the form

$$f(h) = \sum_{\substack{\lambda \in \Lambda_W \\ \lambda \text{ non-singular}}} b_\lambda e^{2\pi i \lambda(h)}.$$

Using Weyl anti-invariance and the fact that any regular $\lambda' \in \Lambda^+_W$ is of the form $\lambda' = \lambda + \rho, \lambda \in \Lambda^+_W$, we obtain

$$f(h) = \sum_{w \in W} \epsilon(w) \sum_{\lambda \in \Lambda^+_W} b_{\lambda+\rho} e^{2\pi i w(\lambda+\rho)(h)} = \sigma(h) \sum_{\lambda \in \Lambda^+_W} b_{\lambda+\rho} \chi_\lambda(h),$$

so that $f/\sigma$ can be extended to $L^2(K, dx)^{Ad_K}$, and $\varphi$ is an isomorphism. On the other hand, from (10), (16), (18) and (19) we see that the diagram commutes. It then follows from Propositions 2.1 and 2.2 that $\varphi_{\mathbb{C}}$ is also an isometric isomorphism. ∎

The extension of the map $\varphi_{\mathbb{C}}$ to all $Ad_{K_{\mathbb{C}}}$ invariant $L^2$ functions for the heat kernel measure, gives also an isometric isomorphism between $L^2(K_{\mathbb{C}}, d\nu_t)^{Ad_{K_{\mathbb{C}}}}$ and $L^2(T_{\mathbb{C}}, d\nu_t^{T_{\mathbb{C}}})^W_-$. This is a consequence of the following integration formula which is an analog of the Weyl integral formula for the heat kernel measures[2].

THEOREM 2.3. *If $f \in L^1(K_{\mathbb{C}}, d\nu_t)$ is a class function, then, for all $t > 0$*

$$\int_{K_{\mathbb{C}}} f(g)\nu_t(g)dg = \frac{e^{-2t\pi\|\rho\|^2}}{|W|} \int_{T_{\mathbb{C}}} f(h)|\sigma_{\mathbb{C}}(h)|^2 \nu_t^{T_{\mathbb{C}}}(h)dh. \qquad (21)$$

*Proof.* Let $f(g)$ be any integrable function on $K_{\mathbb{C}}$. Its integral over $K_{\mathbb{C}}$ can be computed as a double integral, by the following formula of Harish-Chandra [HC]

$$\int_{K_{\mathbb{C}}} f(g)dg = \frac{1}{|W|} \int_{T_{\mathbb{C}}} \left( \int_{T_{\mathbb{C}} \backslash K_{\mathbb{C}}} f(u^{-1}hu)du \right) |D(h)|^2 dh, \qquad (22)$$

[2]We thank the referee for emphasising this analogy, and for pointing out that the related proof of proposition 5.2 was incorrect in the first version of the paper.

where

$$D(h) = \prod_{\alpha \in \Delta} (1 - e^{-2\pi i \alpha}(h))$$

with $\Delta$ being the set of roots of the complexification of $Lie(K_{\mathbb{C}})$, considered as a real Lie algebra (see for example [Kn], Ch. VIII). It is not difficult to verify that since $Lie(K_{\mathbb{C}})$ is already complex, $D(h)$ is equal to $|\sigma_{\mathbb{C}}(h)|^2$. (see for instance, [Ga]). Applying (22) to the product $f(g)\nu_t(g)$, where $f(g)$ is now a class function, we obtain

$$\begin{aligned} \int_{K_{\mathbb{C}}} f(g)\nu_t(g)dg &= \frac{1}{|W|}\int_{T_{\mathbb{C}}} f(h)\left(\int_{T_{\mathbb{C}}\setminus K_{\mathbb{C}}} \nu_t(u^{-1}hu)du\right)|\sigma_{\mathbb{C}}(h)|^4 dh \\ &= \frac{1}{|W|}\int_{T_{\mathbb{C}}} f(h)|\sigma_{\mathbb{C}}(h)|^2 F_{\nu_t}(h)dh, \end{aligned} \tag{23}$$

where the orbit integral

$$F_{\nu_t}(h) = |\sigma_{\mathbb{C}}(h)|^2 \int_{T_{\mathbb{C}}\setminus K_{\mathbb{C}}} \nu_t(u^{-1}hu)du$$

has been computed by Gangolli ([Ga], (3.18) and (5.11)), for regular $h = e^{2\pi i v}$,

$$F_{\nu_t}(h) = e^{-2t\pi\|\rho\|^2}\left(\frac{2}{t}\right)^{\frac{l}{2}} \sqrt{\det C_{ij}}\, e^{\frac{\pi}{2t}||v-\bar{v}||^2} = e^{-2t\pi\|\rho\|^2}\nu_t^{T_{\mathbb{C}}}(h). \tag{24}$$

To compare (24) with Gangolli's expression note that the vector $\rho_*$ in [Ga] equals $2\rho$ in our notation. Also, if $|\cdot|$ denotes both the norm induced in $\mathfrak{h}_{\mathbb{R}}$ and in $\mathfrak{h}_{\mathbb{R}}^*$ by the Killing inner product then there exists a positive constant $c$ (depending only on $Lie(K)$) such that

$$\begin{aligned} |u| &= c||u|| \quad , \quad u \in \mathfrak{h}_{\mathbb{R}} \\ |\lambda| &= \frac{1}{c}||\lambda|| \quad , \quad \lambda \in \mathfrak{h}_{\mathbb{R}}^*. \end{aligned}$$

Up to a constant multiplicative factor (depending on the normalization of the Haar measure on $K_{\mathbb{C}}$) $\nu_t$ coincides with $g_{c^2\pi t/2}$ of [Ga]. The multiplicative factor in (24) can be determined by choosing $f = 1$ in (22) and by using the isometricity of $\varphi_{\mathbb{C}}$ in (20).

Since the set of all regular $h \in T_{\mathbb{C}}$ is a full (Haar) measure set we can substitute (24) in (22) to obtain (21). ∎

COROLLARY 2.2. *For every* $t > 0$, *the map* $\varphi_{\mathbb{C}}$ *defined in (20) establishes an isometric isomorphism between* $L^2(K_{\mathbb{C}}, d\nu_t)^{Ad_{K_{\mathbb{C}}}}$ *and* $L^2(T_{\mathbb{C}}, d\nu_t^{T_{\mathbb{C}}})^W_-$.

### 2.3. Extension to Distributions

In order to apply later the CST to the study of non-abelian theta functions on an elliptic curve with modular parameter $\tau$ in the Teichmuller space of genus 1 curves $\mathcal{T}_1 = \{\tau \in \mathbb{C} : \mathrm{Im}\tau > 0\}$, let us consider the complex non-self-adjoint Laplacian on $K$

$$\Delta_K^{(-i\tau)} = -i\tau\Delta_K = -i\tau \sum_{j=1}^{\dim K} X_j X_j \ . \tag{25}$$

where $\{X_i,\ i = 1, ..., \dim K\}$ is an orthonormal basis for $Lie(K)$ viewed as the space of left-invariant vector fields on $K$ as in section 2.1. Then we have

PROPOSITION 2.3. *For each* $\tau \in \mathcal{T}_1$ *and each* $t > 0$, *the mapping* $C_t^\tau$

$$C_t^\tau = \mathcal{C} \circ e^{t\pi\Delta_K^{(-i\tau)}} \ : \ L^2(K, dx) \to \mathcal{H}(K_{\mathbb{C}}) \tag{26}$$

*is a unitary isomorphism onto the Hilbert space*

$$L^2(K_{\mathbb{C}}, d\nu_{t\tau_2}) \bigcap \mathcal{H}(K_{\mathbb{C}}),$$

*where* $\tau_2 = \mathrm{Im}\tau$ *and* $d\nu_{t\tau_2}$ *is the averaged heat kernel measure corresponding to the Laplacian* $\Delta_K$. *The restriction of the CST (26) to the space* $L^2(K, dx)^{Ad_K}$, *also denoted by* $C_t^\tau$, *is an isometric isomorphism onto the Hilbert space*

$$L^2(K_{\mathbb{C}}, d\nu_{t\tau_2})^{Ad_{K_{\mathbb{C}}}} \bigcap \mathcal{H}(K_{\mathbb{C}}).$$

*Proof.* Let $\tau_1 = \mathrm{Re}(\tau)$ and decompose the transform (26) as

$$\mathcal{C} \circ e^{t\pi\Delta_K^{(\tau_2)}} \circ e^{t\pi\Delta_K^{(-i\tau_1)}} = C_t^{i\tau_2} \circ e^{-i\tau_1 t\pi\Delta_K}.$$

The Laplace operator $\Delta_K$ is self-adjoint on $L^2(K, dx)$ and therefore the operator

$$e^{-i\tau_1 t\pi\Delta_K} : L^2(K, dx) \longrightarrow L^2(K, dx)$$

is unitary. The unitarity of $C_t^{i\tau_2}$ follows from Theorem 2.1. To obtain the second statement notice that the image under the CST (26) of $f$ in the form (9) is given by

$$C_t^\tau f(z) = \sum_{\lambda \in \Lambda_W^+} a_\lambda \, e^{i\pi\tau t c_\lambda} \chi_\lambda(z). \tag{27}$$

The proof then follows from the proof of proposition 2.1 with obvious changes. ∎

We will now extend the CST transform of proposition 2.3 to the space of distributions $C^\infty(K)'$ [V]. This is the space of Fourier series of the form [Sc]

$$f = \sum_{\lambda \in \Lambda_W^+} \mathrm{tr}(\mathrm{R}_\lambda A_\lambda) \tag{28}$$

for which there exists an integer $N > 0$ such that

$$\lim_{||\lambda|| \to \infty} \frac{||A_\lambda||}{(1 + ||\lambda||^2)^N} = 0, \tag{29}$$

where $||A_\lambda||$ is the operator norm of the endomorphism $A_\lambda \in End(\mathrm{R}_\lambda)$. Consider the extension of the adjoint action of $K$ on $C^\infty(K)$ to $C^\infty(K)'$ defined by

$$x \cdot f = \sum_{\lambda \in \Lambda_W^+} \mathrm{tr}(\mathrm{R}_\lambda A_\lambda^x) \ ,$$

where $A_\lambda^x = R_\lambda(x) A_\lambda R_\lambda(x^{-1}), \forall x \in K$. If $f$ is in the space of $Ad_K$-invariant distributions $f \in (C^\infty(K)')^{Ad_K}$ then Schur's lemma and (28) again imply that it has a unique representation of the form

$$f = \sum_{\lambda \in \Lambda_W^+} a_\lambda \chi_\lambda. \tag{30}$$

The Laplace operator and its powers act as continuous linear operators on the space $C^\infty(K)'$ and for $\tau_2 > 0$ define the action of the operator $e^{\pi t \Delta_K^{(-i\tau)}}$ on it. For $f$ of the form (28) we have

$$e^{\pi t \Delta_K^{(-i\tau)}} f = \sum_{\lambda \in \Lambda_W^+} e^{i\pi t\tau c_\lambda} \mathrm{tr}(\mathrm{R}_\lambda A_\lambda). \tag{31}$$

Proposition 2.4. *If $f = \sum_{\lambda \in \Lambda_W^+} \mathrm{tr}(\mathrm{R}_\lambda A_\lambda) \in C^\infty(K)'$ then the series*

$$\sum_{\lambda \in \Lambda_W^+} e^{i\pi t\tau c_\lambda} \mathrm{tr}(\mathrm{R}_\lambda(g) A_\lambda) \tag{32}$$

*where $g \in K_{\mathbb{C}}$, defines a holomorphic function on $K_{\mathbb{C}} \times \mathcal{T}_1$ which we denote by $\left(\mathcal{C} \circ e^{\pi t \Delta_K^{(-i\tau)}}\right)(f)$.*

*Proof.* The growing condition (29), together with (19), implies that there exists a $c > 0$ such that

$$||A_\lambda|| e^{-t\pi\tau_2 c_\lambda} \leq e^{-c||\lambda||^2},$$

for $||\lambda||$ sufficiently large. On the other hand, writing $g = x\exp(iY)$ for $x \in K, Y \in Lie(K)$, one has $||R_\lambda(g)|| \leq \exp(M||Y||||\lambda||)$, for some constant $M > 0$. Therefore, the series (32) is uniformly convergent on compact subsets of $K_{\mathbb{C}} \times \mathcal{T}_1$ and its sum defines a holomorphic function there. ∎

COROLLARY 2.3. *If the distribution $f$ is $Ad_K$-invariant of the form (30) then the series*

$$\sum_{\lambda \in \Lambda_W^+} a_\lambda e^{i\pi t\tau c_\lambda} \chi_\lambda(g) \tag{33}$$

*defines an $Ad_{K_{\mathbb{C}}}$-invariant holomorphic function on $K_{\mathbb{C}} \times \mathcal{T}_1$.*

DEFINITION 2.1. The $K$-coherent state transform for the elliptic curve $X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, $\tau \in \mathcal{T}_1$, and $t > 0$ is the map

$$C_t^\tau = \mathcal{C} \circ e^{\pi t \Delta_K^{(-i\tau)}} : (C^\infty(K)')^{Ad_K} \longrightarrow \mathcal{H}(K_{\mathbb{C}})^{Ad_{K_{\mathbb{C}}}}. \tag{34}$$

*Remark 2. 1.* The role of the elliptic curve $X_\tau$ will become clear in section 5.

We will also need the extension of the abelian coherent state transform in (14) to the present case. This will be related later on in definition 3.1 to a CST associated to the matrix $\widetilde{\Omega} = \tau C^{-1}$ and defined by

$$C_t^{T(\tau)} = \mathcal{C} \circ e^{-i\pi\tau\Delta_T} : \; C^\infty(T)' \to \mathcal{H}(T_{\mathbb{C}}).$$

Consider the maps $\varphi$ and $\varphi_{\mathbb{C}}$ generalizing the maps in (20) given by

$$\begin{aligned} \varphi \; : \; (\mathbb{C}^\infty(K)')^{Ad_K} \;&\to\; (\mathbb{C}^\infty(T)')^W_- \\ \sum_{\lambda \in \Lambda_W^+} a_\lambda \chi_\lambda \;&\mapsto\; \frac{1}{\sqrt{|W|}} \sum_{w \in W} \epsilon(w) \sum_{\lambda \in \Lambda_W^+} a_\lambda e^{2\pi i w(\lambda+\rho)} \end{aligned} \tag{35}$$

and

$$\begin{array}{rcl} \varphi_{\mathbb{C}} \; : \; \mathcal{H}(K_{\mathbb{C}}) & \to & \mathcal{H}(T_{\mathbb{C}})^W_- \\ f & \mapsto & e^{it\tau\pi||\rho||^2} \dfrac{\sigma_{\mathbb{C}}}{\sqrt{|W|}} \, f_{|T_{\mathbb{C}}} . \end{array} \tag{36}$$

We then have the following

THEOREM 2.4. *The maps $\varphi$ and $\varphi_{\mathbb{C}}$ are isomorphisms, and the following diagram is commutative*

$$\begin{array}{ccc} (\mathbb{C}^\infty(K)')^{Ad_K} & \stackrel{C_t^\tau}{\to} & \mathcal{H}(K_{\mathbb{C}})^{Ad_{K_{\mathbb{C}}}} \\ \downarrow \varphi & & \downarrow \varphi_{\mathbb{C}} \\ (\mathbb{C}^\infty(T)')^W_- & \stackrel{C_t^{T(\tau)}}{\to} & \mathcal{H}(T_{\mathbb{C}})^W_- \end{array} \tag{37}$$

*and the maps $C_t^\tau$ and $C_t^{T(\tau)}$ are injective.*

*Proof.* A similar argument to the one in the proof of theorem (2.2), the Weyl character formula (18) and (30) shows that $\varphi$ is an isomorphism. On the other hand, from (27), (30) and (16) we see that $C_t^\tau$ and $C_t^{T(\tau)}$ are injective. Indeed, two distributions $f_1$ and $f_2$ with representations (30) are different if and only if there exists $\lambda \in \Lambda_W^+$ such that the corresponding coefficients $a_\lambda^1$, $a_\lambda^2$ are different. Then, the two holomorphic functions $C_t^\tau f_1$ and $C_t^\tau f_2$ have also different coefficients with respect to $\chi_\lambda$ and are therefore different, as can be readily seen by restriction to $K$. The injectivity of $C_t^{T(\tau)}$ is proved in a similar way. The fact that $\varphi_{\mathbb{C}}$ is also an isomorphism, follows from lemma 9 of [Ha1]. Indeed, any $f \in \mathcal{H}(K_{\mathbb{C}})$ has a unique Peter-Weyl decomposition which corresponds to the unique analytic continuation to $K_{\mathbb{C}}$ of $f|_K \in L^2(K, dx)$. (The same applies to $T_{\mathbb{C}}$.) Therefore, again, an analogous argument to the one in the proof of theorem (2.2) shows that $\varphi_{\mathbb{C}}$ is an isomorphism. Finally, the diagram commutes due to the Weyl character formula, (19) and the expressions for the CST in (27), (16) and (3.1). ▌

## 3. COHERENT STATE TRANSFORM AND THETA FUNCTIONS ON POLARIZED ABELIAN VARIETIES

In this section, we extend the results of [FMN] to abelian varieties with general polarization. This extension is necessary because non-abelian theta functions on the moduli space of holomorphic vector bundles over elliptic curves are naturally related with abelian theta functions on certain non principally polarized abelian varieties (see (73), (77), (80), the Appendix and also [L]). The main result of [FMN] continues to hold (see theorem 3.1) with small modifications. The information about the polarization is reflected for instance in the support of the distributions in (61) and in the dimensionality of the space of level $k$ theta functions (see (60)).

Let $V$ be an $l$-dimensional complex vector space and $\Lambda \cong \mathbb{Z}^{2l}$ a maximal lattice in $V$ such that the quotient

$$M = V/\Lambda \tag{38}$$

is an abelian variety, i.e. a complex torus which can be holomorphically embedded in projective space. For later convenience we will assume that $M$ is endowed with a polarization $H$, not necessarily principal [GH, BL, Ke]. By definition, $H$ is a positive definite hermitean form on $V$ whose imaginary part is integral on $\Lambda$. Let $E = -\mathrm{Im}H$, so that $E$ is an integral alternating bilinear form on $\Lambda$. According to the elementary divisor theorem, there is a canonical basis of $\Lambda$, $\beta_1, \ldots, \beta_l, \tilde{\beta}_1, \ldots, \tilde{\beta}_l$ characterized by

$$\begin{aligned} E(\beta_i, \beta_j) &= E(\tilde{\beta}_i, \tilde{\beta}_j) = 0 \\ E(\beta_i, \tilde{\beta}_j) &= \delta_i \delta_{ij}, \qquad i, j = 1, \ldots, l, \end{aligned} \tag{39}$$

where $\delta_1 | \delta_2 | ... | \delta_l$ are positive integers, depending only on $E$, and $\delta_{ij}$ is Kronecker's delta symbol. Define $E_1 = -\mathrm{Im}H_1$ to be the form with $\delta_1 = 1$.

Now, let us decompose $V$ into isotropic subspaces with respect to $E_1$, $V_1 = \oplus_{i=1}^{l} \mathbb{R}\beta_i$, $V_2 = \oplus_{i=1}^{l} \mathbb{R}\tilde{\beta}_i$, and decompose the lattice in the same way $\Lambda_i = V_i \cap \Lambda$, $i = 1, 2$, so that $\Lambda = \Lambda_1 \oplus \Lambda_2$. Let $\alpha$ be the semicharacter for $H$ which is trivial on $\Lambda_1$ and $\Lambda_2$, i.e, $\alpha$ is the unique map $\alpha : \Lambda \to U(1)$ satisfying

$$\begin{cases} \alpha(\lambda + \lambda') = \alpha(\lambda)\alpha(\lambda') e^{-\pi i E_1(\lambda, \lambda')} \\ \qquad\qquad \alpha|_{\Lambda_1} = \alpha|_{\Lambda_2} = 1. \end{cases}$$

To this particular Appell-Humbert pair $(\alpha, H_1)$, we can associate a line bundle which we will denote by $L_1 = L(\alpha, H_1)$ over $M$ via the following (canonical) factors of automorphy

$$a(v, \lambda) = \alpha(\lambda) e^{\pi H_1(v, \lambda) + \frac{\pi}{2} H_1(\lambda, \lambda)}.$$

Recall that, via the canonical identification of $H^2(M, \mathbb{Z})$ with the space of integral alternating bilinear forms on $\Lambda$, the first Chern class of $L_1$, $c_1(L_1)$, corresponds to the form $E_1$.

Level $k$ theta functions on $M$ are holomorphic sections, $\tilde{\theta}$, of $L_1^k = L(\alpha^k, kH_1)$, $\tilde{\theta} \in H^0(M, L_1^k)$.

For convenience, we will consider the following different but equivalent factors of automorphy for $L_1$. Let $S$ be the $\mathbb{C}$-bilinear extension of $H|_{V_1 \times V_1}$

to $V \cong \mathbb{C} \cdot V_1$, and define

$$F = \frac{1}{2i}(H - S) \tag{40}$$

which is a form $\mathbb{C}$-linear in the first variable. Then, $L_1^k$ is also given by the following (classical) factors of automorphy $e(\lambda, v) = \alpha(\lambda)e^{2\pi ikF(v,\lambda)+\pi ikF(\lambda,\lambda)}$, which can be rewritten as

$$e(\lambda_1 + \lambda_2, v) = e^{2\pi ikF(v,\lambda_2)+\pi ikF(\lambda_2,\lambda_2)}, \qquad \lambda_i \in \Lambda_i. \tag{41}$$

To write down explicit expressions for the theta functions, consider the lattice dual to $\Lambda$ with respect to $E_1$,

$$\widehat{\Lambda} = \{v \in V : E_1(v, \lambda) \in \mathbb{Z}, \forall \lambda \in \Lambda\} = \widehat{\Lambda_1} \oplus \widehat{\Lambda_2} \supset \Lambda \tag{42}$$

where $\widehat{\Lambda_i} = V_i \bigcap \widehat{\Lambda} \supset \Lambda_i$, for $i = 1, 2$. A basis for $\widehat{\Lambda_1}$ and $\widehat{\Lambda_2}$ is given respectively by

$$\begin{aligned} \beta_j' &= \frac{1}{\delta_j}\beta_j, \\ \tilde{\beta}_j' &= \frac{1}{\delta_j}\tilde{\beta}_j = \sum_{i=1}^{l} \Omega_{ij}\beta_i, \quad j = 1, ..., l, \end{aligned}$$

where $\Omega = (\Omega_{ij})$ is a matrix in the Siegel upper half space $\mathbb{H}_l$ of symmetric $l \times l$ matrices with positive imaginary part.

One computes,

$$F(\tilde{\beta}_i', \tilde{\beta}_j') = -\Omega_{ij}. \tag{43}$$

Using the automorphy factors (41), we see that the space $H^0(M, L_1^k)$, is isomorphic to the space of holomorphic functions on $V/\Lambda_1 \cong (\mathbb{C}^*)^l$ satisfying quasi-periodicity conditions in the directions of $\Lambda_2$ given by

$$\theta(v + b) = e^{2\pi ikF(v,b)+\pi ikF(b,b)}\theta(v), \, b \in \Lambda_2. \tag{44}$$

Let us denote the latter space by $\mathcal{H}_{k,\Omega}$,

$$\mathcal{H}_{k,\Omega} \subset \mathcal{H}((\mathbb{C}^*)^l),$$

where $\mathcal{H}((\mathbb{C}^*)^l)$ denotes the space of holomorphic functions on $(\mathbb{C}^*)^l$. Conditions (44) then imply that there is one independent theta function $\theta \in$

$\mathcal{H}_{\Omega,k}$, for every $C \in (k^{-1}\widehat{\Lambda}_2)/\Lambda_2$ given by

$$\begin{aligned} \theta_C(v) &= \sum_{b\in\Lambda_2} e^{-\pi ikF(c_0+b,c_0+b)-2\pi ikF(c_0+b,v)} \\ &= \sum_{c\in C} e^{-\pi ikF(c,c)-2\pi ikF(c,v)}, \end{aligned} \tag{45}$$

where $[c_0] = C \in (k^{-1}\widehat{\Lambda}_2)/\Lambda_2$. In coordinates,

$$\begin{array}{ll} c = \sum_j m_j \frac{\tilde{\beta}'_j}{k}, & m = (m_1, ..., m_l) \in \mathbb{Z}^l, \\ b = \sum_j p_j \tilde{\beta}_j = \sum_j p_j \delta_j \tilde{\beta}'_j, & p = (p_1, ..., p_l) \in \mathbb{Z}^l, \\ v = \sum_j z_j \beta_j, & z = (z_1, ..., z_l) \in \mathbb{C}^l, \end{array} \tag{46}$$

the theta functions take the form

$$\theta_m(z, \Omega) = \sum_{p\in\mathbb{Z}^l} e^{\pi i(m+k\delta p)\cdot\frac{\Omega}{k}(m+k\delta p)+2\pi i(m+k\delta p)\cdot z}, \tag{47}$$

where $m \in \mathbb{Z}^l/k(\delta_1\mathbb{Z} \oplus \cdots \oplus \delta_l\mathbb{Z})$, with $\delta_1 = 1$, $\delta = \mathrm{diag}(\delta_1, ..., \delta_l)$ and $z \cdot z' = z_1 z'_1 + \cdots + z_l z'_l$. In these coordinates the automorphy factors (41) take the form

$$e(\tilde{\beta}_j, z) = e^{-2\pi ikz_j\delta_j - \pi ik\Omega_{jj}\delta_j^2}. \tag{48}$$

For the benefit of section 5 let us consider a more general basis $\{\gamma_j\}_{j=1}^{2l}$ (not necessarily canonical) for $\Lambda_1 \oplus \Lambda_2$ and its dual basis $\{\gamma'_j\}_{j=1}^{2l}$ for $\widehat{\Lambda}_1 \oplus \widehat{\Lambda}_2$. Notice that the basis of $\Lambda_1$ given by $\{\gamma_j\}_{j=1}^{l}$ can be extended to a canonical basis of $\Lambda_1 \oplus \Lambda_2$ if and only if $\delta_j\gamma'_{j+l}$ is a basis of $\Lambda_2$. Let $\{\beta_j, \tilde{\beta}_j\}$ continue to denote a canonical basis of $\Lambda_1 \oplus \Lambda_2$ and

$$\begin{array}{llllll} \beta_j &=& \sum_{i=1}^l \gamma_i P_{ij} & \tilde{\beta}_j &=& \tilde{\beta}'_j \delta_j \\ \beta_j &=& \beta'_j \delta_j & \tilde{\beta}'_j &=& \sum_{i=1}^l \gamma'_{i+l} Q_{ij} \\ \gamma_{j+l} &=& \sum_{i=1}^l \gamma'_{i+l} R_{ij} & & & \end{array} \tag{49}$$

where $P, Q \in SL(l, \mathbb{Z})$ and duality demands that $P^t Q = Id$. We then have

$$\gamma'_{j+l} = \sum_{i=1}^{l} \gamma_i \widetilde{\Omega}_{ij} \tag{50}$$

where

$$\widetilde{\Omega} = P\Omega P^t \in \mathbb{H}_l, \tag{51}$$

and also $F(\gamma_j, \gamma'_{i+l}) = -\delta_{ji}$ and $F(\gamma'_{j+l}, \gamma'_{i+l}) = -\widetilde{\Omega}_{ji}$, where $\delta_{ji}$. Considering then instead of (46) the coordinates

$$\begin{array}{ll} c = \sum_j \tilde{m}_j \frac{\gamma'_{j+l}}{k}, & \tilde{m} = (\tilde{m}_1, ..., \tilde{m}_l) \in \mathbb{Z}^l, \\ b = \sum_j \tilde{p}_j \gamma_{j+l} = \sum_j \gamma'_{j+l} R_{ij} \tilde{p}_i, & \tilde{p} = (\tilde{p}_1, ..., \tilde{p}_l) \in \mathbb{Z}^l, \\ v = \sum_j \tilde{z}_j \gamma_j, & \tilde{z} = (\tilde{z}_1, ..., \tilde{z}_l) \in \mathbb{C}^l, \end{array} \tag{52}$$

we obtain for the theta functions the expressions

$$\theta_{\tilde{m}}(\tilde{z}, \widetilde{\Omega}) = \theta_m(z, \Omega) = \sum_{\tilde{p} \in \mathbb{Z}^l} e^{\pi i (\tilde{m} + kR\tilde{p}) \cdot \frac{\tilde{\Omega}}{k} (\tilde{m} + kR\tilde{p}) + 2\pi i (\tilde{m} + kR\tilde{p}) \cdot \tilde{z}}. \tag{53}$$

In these coordinates the automorphy factors take the form

$$e(\gamma_{l+j}, \tilde{z}) = e^{-2\pi i k (R^t \tilde{z})_j - \pi i k (R^t \tilde{\Omega} R)_{jj}}. \tag{54}$$

Returning to the canonical coordinates we see from (47) that the theta functions are the analytic continuation to $T_{\mathbb{C}} = (\mathbb{C}^*)^l \cong V/\Lambda_1$ of solutions (with $t = 1/k$) of the heat equation on $T = U(1)^l$

$$\frac{1}{\pi} \frac{\partial u}{\partial t} = \Delta_T^{(-i\Omega)} u \ , \tag{55}$$

where

$$\Delta_T^{(-i\Omega)} = - \sum_{j,j'}^{l} \frac{i}{4\pi^2} \Omega_{jj'} \frac{\partial^2}{\partial x_j \partial x_{j'}} \ , \tag{56}$$

and the $x_i \in [0, 1]$ are angular coordinates on $U(1)^l$. Of course, that in the coordinates (52) the functions $\theta_{\tilde{m}}$ are the analytic continuations of solutions of the heat equation of (56) with $\widetilde{\Omega}$ in the place of $\Omega$.

Following [FMN] we extend the CST to distributions on $(S^1)^l$.

Definition 3.1. The CST for the matrix $\Omega \in \mathbb{H}_l$ is the transform

$$\begin{array}{rcl} C_t^{T(\Omega)} \ : \left( C^\infty((S^1)^l) \right)' & \to & \mathcal{H}((\mathbb{C}^*)^l) \\ f & \mapsto & \mathcal{C} \circ e^{\pi t \Delta_T^{(-i\Omega)}} f \ , \end{array} \tag{57}$$

The fact that the map (57) is well defined follows from Lemma 4.1 of [FMN]. The averaged heat kernel function $\nu_t^{T_{\mathbb{C}}}$ in this case reads

$$\nu_t^{T_{\mathbb{C}}}(z) = \left( \frac{2}{t} \right)^{\frac{l}{2}} (\det W)^{\frac{1}{2}} \, e^{\frac{\pi}{2t} \sum_{ij} (z_i - \overline{z}_i) W_{ij} (z_j - \overline{z}_j)} \ , \tag{58}$$

where $W = \Omega_2^{-1} = \mathrm{Im}(\Omega)^{-1}$. In periodic coordinates $(\eta, \xi)$ dual to the canonical basis $\{\beta_j, \tilde{\beta}_j\}$ and related to $z$ in (46) by $z = \eta + \Omega\delta\xi$, the heat kernel measure reads

$$\nu_t^{T_{\mathbb{C}}}(\eta, \xi) d\eta d\xi = \left(\frac{2}{t}\right)^{\frac{l}{2}} (\det \Omega_2)^{\frac{1}{2}} (\det \delta) e^{-2\pi t \sum_{ij} \xi_i \delta_i \Omega_{2_{ij}} \xi_j \delta_j} d\eta d\xi. \quad (59)$$

Consider the distributions on $(S^1)^l$ given by

$$\theta_m^0(x) = \theta_m(x, 0) = \sum_{p \in \mathbb{Z}^l} e^{2\pi i (m + k\delta p)\cdot x} \ , \quad (60)$$

and let $\mathcal{I}_{k,\delta}$ denote the $(\delta_1 \cdots \delta_l k^l)$-dimensional subspace of $\left(C^\infty((S^1)^l)\right)'$ generated by these distributions with inner product $(\cdot, \cdot)$ for which the distributions in (60) form an orhonormal basis. The distributions in $\mathcal{I}_{k,\delta}$ are linear combinations of Dirac delta distributions supported on points arising from Bohr-Sommerfeld conditions [Sn, Ty]. In fact,

$$\delta_{m'}(x) = \delta(x - \frac{\delta^{-1} m'}{k}) = \sum_{0 \le m_\alpha < k\delta_\alpha} e^{-2\pi i m' \cdot \frac{\delta^{-1}}{k} m} \theta_m^0(x). \quad (61)$$

The extension of the results of [FMN] to general polarizations can be summarized in

Theorem 3.1.

*1.The image under the CST $C_{t=1/k}^{T(\Omega)}$ of $\mathcal{I}_{k,\delta}$ is the space $\mathcal{H}_{k,\Omega}$ of all level $k$ theta functions on the abelian variety $M$ with polarization given by $\delta$.*

*2.The function $\nu_t^{T_{\mathbb{C}}}$ is, for $t = 1/k$, the pull-back from $M$ to $(\mathbb{C}^*)^l$ of a hermitean structure on $L_1^k$.*

*3.Consider on $\mathcal{H}_{k,\Omega}$ the inner product induced by the CST transform*

$$< \theta, \theta' > = \int_{[0,1]^l \times [0,1]^l} \overline{\theta} \theta' \nu_{1/k}^{T_{\mathbb{C}}}(\eta, \xi) d\eta d\xi \ . \quad (62)$$

*The CST transform of definition 3.1 is, for $t = 1/k$, a unitary transform between $(\mathcal{I}_{k,\delta}, (\cdot, \cdot))$ and $(\mathcal{H}_{k,\Omega}, < \cdot, \cdot >)$.*

*Proof.*
1. This follows immediately from the definition of $\mathcal{I}_{k,\delta}$.
2. This means that $\nu_{\frac{1}{k}}^{T_{\mathbb{C}}}$ is the pull-back of a section of $(L_1^k \otimes \bar{L}_1^k)^*$ and so

should satisfy the quasi-periodicity conditions

$$\nu_{1/k}^{T_{\mathbb{C}}}(z+\tilde{\beta}_j) = |e(\tilde{\beta}_j, z)|^{-2}\nu_{1/k}^{T_{\mathbb{C}}}(z) = |e^{-2\pi ikz_j\delta_j - \pi ik\Omega_{jj}\delta_j^2}|^{-2}\nu_{1/k}^{T_{\mathbb{C}}}(z), \quad (63)$$

which are easy to check by taking $\xi_j \to \xi_j + 1$ in the expression (59).
3. This follows from a direct computation:

$$\int_{[0,1]^l\times[0,1]^l} \bar{\theta}_m\theta_{m'}\nu_{1/k}^{T_{\mathbb{C}}}(\eta,\xi)d\eta d\xi = \delta_{mm'}(2k)^{l/2}(\det\Omega_2)^{1/2}\det(\delta)\times$$
$$\times\sum_{p\in\mathbb{Z}^l}\int_{[0,1]^l} e^{-2\pi k(\delta\xi+\frac{1}{k}(m+k\delta p))\cdot\Omega_2(\delta\xi+\frac{1}{k}(m+k\delta p))}d\xi = \delta_{mm'}, \quad (64)$$

for all $m, m' \in \mathbb{Z}^l/k(\delta_1\mathbb{Z}\oplus\cdots\oplus\delta_l\mathbb{Z})$. ∎

We end this section by noting that the $(\delta_1\cdots\delta_l k^l)\times(\delta_1\cdots\delta_l k^l)$ matrix

$$A_{m'm} = \left(e^{-2\pi im'\cdot\frac{\delta^{-1}}{k}m}\right)$$

in (61) satisties $\bar{A}^tA = \delta_1\cdots\delta_l k^l I$, where $I$ is the $(\delta_1\cdots\delta_l k^l)$-dimensional identity matrix. This means that the distributions $(1/\sqrt{\delta_1\cdots\delta_l k^l})\delta_m(x)$ in (61) are orthonormal in $(\mathcal{I}_{k,\delta},(\cdot,\cdot))$.

## 4. VECTOR BUNDLES ON ELLIPTIC CURVES

In this section we recall the existence of a moduli space $\mathcal{M}_n = \mathcal{M}_n(X_\tau)$, parametrizing $S$-equivalence classes of semistable bundles of rank $n$ and trivial determinant over an elliptic curve $X_\tau$, $\tau\in\mathcal{T}_1$. Non-abelian theta functions of genus one are then defined to be holomorphic sections of the line bundles over $\mathcal{M}_n$. We then define the Schottky map associated to a general complex linear group $G$ and a Riemann surface $X$, compute it explicitly for the case $G=\mathbb{C}^*$, and relate it to a map used in [FMN] to study abelian theta functions. We compute the Schottky map for the group $SL(n,\mathbb{C})$ over the elliptic curve $X_\tau$ and use it in section 5 to pull-back to $SL(n,\mathbb{C})$ sections of line bundles over $\mathcal{M}_n$ and compare them to the CST considered before. This will relate the CST for the elliptic curve $X_\tau$, in definition 2.1, to non-abelian theta functions of genus one.

### 4.1. The Moduli Space of Semistable Vector Bundles

Let $X$ be a smooth complex projective algebraic curve of genus $g$. Here we will consider *holomorphic vector bundles* over $X$, which we will denote simply by the term *bundle.* For details of the constructions and proofs of the results in this section, we refer to [A], [Tu].

To construct a moduli space for bundles over $X$ one introduces the following notions. A bundle $E$ is called *stable* (resp. *semistable*) if for every proper subbundle $F \subset E$ we have $\mu_F < \mu_E$ (resp. $\mu_F \leq \mu_E$), where $\mu_E$ denotes the *slope* of a bundle $E$, defined by $\mu_E = \mathrm{deg}E/\mathrm{rk}E$. Two semistable bundles are called *$S$-equivalent* if their associated graded bundles

$$\mathrm{Gr}(E) = \oplus_{i=1}^m E_i/E_{i-1}$$

are isomorphic, where

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = E,$$

is the so-called Jordan-Holder filtration in which the successive quotients $E_i/E_{i-1}$ are stable of the same slope and are uniquely defined up to permutation. By the theorem of Narasimhan and Seshadri [NS], the space of $S$-equivalence classes of semistable bundles of rank $n$ and trivial determinant has the structure of a projective algebraic variety, which we denote here by $\mathcal{M}_n(X)$.

In the case when $X = X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ is an elliptic curve, and $E$ has trivial determinant, one can show that the Jordan-Holder quotients $L_i := E_i/E_{i-1}$ are line bundles, necessarily of degree 0. Then $E$ is $S$-equivalent to $L_1 \oplus \cdots \oplus L_n$ and

$$\mathrm{det}E = L_1 \otimes \cdots \otimes L_n = \mathcal{O}_{X_\tau},$$

where $\mathcal{O}_{X_\tau}$ denotes the structure sheaf of $X_\tau$ (corresponding to the trivial line bundle on $X_\tau$). Let $J(X_\tau) \cong X_\tau$ be the Jacobian variety of $X_\tau$ and $M$ be the kernel of the group homomorphism $t : J(X_\tau)^n \to J(X_\tau)$ given by the tensor product of line bundles. Consider the following natural maps

$$\begin{array}{rclcl} M \equiv \ker t & \to & \mathcal{M}_n(X_\tau) & \to & \mathrm{Sym}^n(J(X_\tau)) \\ (L_1, \ldots, L_n) & \mapsto & E & \mapsto & \{L_1, \ldots, L_n\} \end{array} \tag{65}$$

where the last space is the symmetric product of the Jacobian. One can prove that the second map is injective, and that the image of the composition is a projective space. One has

THEOREM 4.1. [Tu] *For an elliptic curve $X_\tau$, the moduli space $\mathcal{M}_n = \mathcal{M}_n(X_\tau)$ is isomorphic to the complex projective space of dimension $n-1$,*

$$\mathcal{M}_n \cong \mathbb{P}^{n-1}.$$

*Therefore, assuming $n \geq 2$, the Picard group $Pic(\mathcal{M}_n)$ of isomorphism classes of line bundles over $\mathcal{M}_n$ is isomorphic to $\mathbb{Z}$, and letting $L_\Theta \cong \mathcal{O}(1)$*

*denote the ample generator, we have*

$$dimH^0(\mathcal{M}_n, L_\Theta^k) = \binom{n+k-1}{k}.$$

Although the analogous moduli spaces for higher genus curves do not admit such a simple description, their Picard groups are all isomorphic to $\mathbb{Z}$ [DN]. In analogy with this case, a non-abelian theta function (for genus 1) of level $k$ is defined to be a section $\theta$ of the line bundle $L_\Theta^k$,

$$\theta \in H^0(\mathcal{M}_n, L_\Theta^k).$$

### 4.2. The Schottky Map

We now define the Schottky map for a complex linear subgroup $G$ of $GL(n,\mathbb{C})$ and a general compact Riemann surface $X$ of genus $g$. Let $\mathcal{G}$ be the sheaf of germs of holomorphic functions from $X$ to $GL(n,\mathbb{C})$. The inclusion $G \hookrightarrow \mathcal{G}$ (where $G$ is identified with its constant sheaf on $X$), defines a map

$$\mathcal{E} : H^1(X, G) \to H^1(X, \mathcal{G}), \tag{66}$$

that sends a flat $G$-bundle into the corresponding (isomorphism class of) holomorphic vector bundle of rank $n$ over $X$ (necessarily of degree 0).

There is a well know bijection between the space of flat $G$-bundles over $X$ and the space of $G$-representations of the fundamental group of $X$, $\pi_1(X)$, modulo overall conjugation $Hom(\pi_1(X), G)/G$; it is given explicitely by:

$$V : Hom(\pi_1(X), G)/G \to H^1(X, G), \qquad \rho \mapsto V_\rho := \tilde{X} \times_\rho G, \tag{67}$$

where the notation means that $\pi_1(X)$ acts diagonally through $\rho$ on the trivial $G$-bundle over the universal cover $\tilde{X}$ of $X$ and $V_\rho$ is the quotient with respect to that action.

Let us fix a canonical basis of $\pi_1(X)$: elements $a_1, ..., a_g, b_1, ..., b_g$ that generate $\pi_1(X)$, subject to the single relation $\prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} = 1$. Let $F_g$ be a free group on $g$ generators $B_1, ..., B_g$, and $q : \pi_1(X) \to F_g$ be the homomorphism given by $q(a_i) = 1, q(b_i) = B_i, i = 1, ..., g$. Then we can form the exact sequence of groups:

$$\pi_1(X) \xrightarrow{q} F_g \to 1, \tag{68}$$

which in turn defines the inclusion $i : Hom(F_g, G) \hookrightarrow Hom(\pi_1(X), G)$.

DEFINITION 4.1. The Schottky map is the composition $S = \mathcal{E} \circ V \circ i$,

$$G^g \cong Hom(F_g, G) \overset{i}{\hookrightarrow} Hom(\pi_1(X), G) \xrightarrow{V} H^1(X, G) \xrightarrow{\mathcal{E}} H^1(X, \mathcal{G}).$$

Intuitively, this map sends a $g$-tuple of $n \times n$ invertible matrices $(N_1, ..., N_n) \in G^g \subset GL(n, \mathbb{C})^g$ to the flat rank $n$ holomorphic vector bundle determined by the holonomies $(1, ..., 1, N_1, ..., N_g)$ around the loops $(a_1, ..., a_g, b_1, ..., b_g)$, respectively. To have a good description of this map however, we need to substitute the space of all isomorphism classes of vector bundles $H^1(X, \mathcal{G})$, by a nicer space such as the moduli spaces of semistable bundles of section 4.1. If $\rho \in Hom(F_g, U(n))$ and is irreducible, it is known that the Schottky map is locally bi-holomorphic onto a neighbourhood in the moduli space of semistable bundles [Fl], however it is conjectured that its image is dense in $\mathcal{M}_n(X)$. In sections 4.3 and 4.4 we will consider two cases where this map can be given explicitly.

### 4.3. The rank one Schottky Map

In the case of line bundles the situation is simple, since the group $GL(1, \mathbb{C}) = \mathbb{C}^*$ is an abelian group and the degree 0 line bundles in $H^1(X, \mathcal{O}^*)$ form an abelian variety, the Picard variety of $X$,

$$Pic^0(X) = \frac{H^1(X, \mathcal{O})}{H^1(X, \mathbb{Z})} \cong \frac{H^1(X, \mathbb{C})}{H^0(X, K_X) \oplus H^1(X, \mathbb{Z})},$$

where $K_X$ is the canonical bundle on $X$. The last expression is obtained from the short exact sequence (see [Gu])

$$0 \to \mathbb{C} \to \mathcal{O} \to K_X \to 0. \tag{69}$$

Our choice of basis for $\pi_1(X)$ induces an isomorphism $H^1(X, \mathbb{C}) \cong Hom(\pi_1(X), \mathbb{C}) \cong \mathbb{C}^{2g}$, and allows a very explicit description of the Schottky map as follows

$$\begin{array}{rccc} S: & Hom(F_g, \mathbb{C}^*) \cong (\mathbb{C}^*)^g & \to & Pic^0(X) \cong \mathbb{C}^{2g} \big/ H^0(X, K_X) \oplus H^1(X, \mathbb{Z}) \\ & (e^{2\pi i z_1}, ..., e^{2\pi i z_g}) & \mapsto & [(0, ..., 0, z_1, ..., z_g)]. \end{array} \tag{70}$$

We can still be more concrete and at the same time relate the Schottky map with the construction of [FMN], by using the Jacobian variety $J(X)$ instead of the Picard variety of $X$. By definition

$$J(X) = H^0(X, K_X)^* / H_1(X, \mathbb{Z}) \cong \mathbb{C}^g / \left( \mathbb{Z}^g \oplus \Omega \mathbb{Z}^g \right),$$

where the last expression arises from considering $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ as the basis of the lattice $H_1(X, \mathbb{Z})$ and $\{a_1, \ldots, a_g\}$ as the basis of the complex vector space $H^0(X, K_X)^*$, via the natural action of $H_1(X, \mathbb{Z})$ on $H^0(X, K_X)^*$. As is well known, the Jacobian and Picard varieties of $X$ are canonically isomorphic, as follows. Let

$$\Pi : H^1(X, \mathbb{C}) \cong \mathbb{C}^{2g} \to H^1(X, \mathcal{O}) \cong \mathbb{C}^g$$

be any linear map with kernel equal to $H^0(X, K)$. Simple computations using (69), show that $\Pi$ is represented by a $g \times 2g$ matrix, also denoted $\Pi = [\Pi_1|\Pi_2]$, such that $\Pi_1 + \Pi_2\Omega = 0$. The isomorphism between the Picard and Jacobian varieties of $X$ is then given by $\Pi_2 : J(X) \to Pic^0(X)$, (see [Gu]), hence

$$\begin{array}{ccccc} H^1(X, \mathbb{C}) & \stackrel{\Pi}{\to} & Pic^0(X) & \stackrel{\Pi_2^{-1}}{\to} & J(X) \\ (w, z) & \mapsto & \Pi_1 w + \Pi_2 z & \mapsto & -\Omega w + z. \end{array} \tag{71}$$

Therefore, we obtain

THEOREM 4.2. *With our choices of basis of $H_1(X, \mathbb{Z})$ and $H^0(X, K_X)^*$, we have*

$$(\Pi_2^{-1} \circ S)(e^{2\pi i z_1}, \cdots, e^{2\pi i z_g}) = z \, (\mathrm{mod} \Lambda). \tag{72}$$

*Proof.* This follows immediately from the maps (70) and (71). ∎

Because of this result, the composition $\Pi_2^{-1} \circ S : (\mathbb{C}^*)^g \to J(X)$, is independent of the actual isomorphism between $J(X)$ and $Pic^0(X)$, and will henceforth be called the abelian Schottky map, and denoted by $s$. This map was used in [FMN], in order to identify, via pullback, classical theta functions with holomorphic functions on $(\mathbb{C}^*)^g$. We will use it later to relate the CST for the elliptic curve $X_\tau$, in definition 2.1, to non-abelian theta functions of genus one.

### 4.4. The Schottky Map for genus one

Let us now consider the Schottky map for the case of an elliptic curve $X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, and for semistable bundles of rank $n$ with trivial determinant over $X_\tau$. This will be a map

$$S : SL(n, \mathbb{C}) \to \mathcal{M}_n \cong \mathbb{P}^{n-1}.$$

From geometric invariant theory (see [FKM, N]), we know that under the adjoint action, $SL(n, \mathbb{C})$ has a good quotient, which is a map $SL(n, \mathbb{C}) \to T_\mathbb{C}/W$, where $T_\mathbb{C}$ is the maximal torus of $SL(n, \mathbb{C})$ and $W$ the Weyl group; this map sends a matrix to the unordered set of its eigenvalues. Therefore

$SL(n,\mathbb{C})$ satisfies a universal property, which in this case translates into the statement that the Schottky map factors through $T_{\mathbb{C}}/W$, as in the following diagram.

$$\begin{array}{ccc} SL(n,\mathbb{C}) & \stackrel{Q}{\to} & T_{\mathbb{C}}/W \\ & \searrow S & \downarrow f \\ & & \mathcal{M}_n \end{array}$$

We are therefore reduced to describing $f$. Let us fix the following Cartan subalgebra of $sl(n,\mathbb{C})$

$$\mathfrak{h} = \{A \in sl(n,\mathbb{C}) : A \text{ is diagonal}\},$$

and using the corresponding coroot lattice $\check{\Lambda}_R$, define $M$ to be the abelian variety

$$M = \check{\Lambda}_R \otimes X_\tau = \check{\Lambda}_R \otimes (\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}) = \mathfrak{h}/\left(\check{\Lambda}_R \oplus \tau\check{\Lambda}_R\right),$$

From the explicit form of $\mathfrak{h}$ it is not difficult to see that $M$ is isomorphic to the kernel of the map $t : J\left(X_\tau\right)^n \to J\left(X_\tau\right)$, obtained by tensoring the entries, as in (65), with explicit isomorphism as follows.

$$\begin{array}{ccc} \mathfrak{h}/\left(\check{\Lambda}_R \oplus \tau\check{\Lambda}_R\right) & \to & \ker t \subset J\left(X_\tau\right)^n \\ [(z_1, ..., z_n)] & \mapsto & \left(L_{z_1}, ..., L_{z_n}\right), \end{array}$$

where $z_i \in \mathbb{C}$, , $z_1 + \cdots + z_n = 0$, and $L_z$ denotes the line bundle over $X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ corresponding to the divisor $[z] - [0]$. We will use these two different representations interchangeably.

The Weyl group $W$ acts naturally on $M$, via the usual action on $\check{\Lambda}_R$, and, as shown in [FM],[FMW],[L],[M], $\mathcal{M}_n$ is isomorphic to the $l = (n-1)$-dimensional complex projective space $\mathbb{P}^l(\mathbb{C})$ obtained as the quotient under this action,

$$\mathcal{M}_n \cong M/W.$$

In our case, this quotient is given explicitely by the map [Tu]

$$\begin{array}{cccc} \pi : & M & \to & \mathcal{M}_n \\ & \left(L_{z_1}, ..., L_{z_n}\right) & \mapsto & L_{z_1} \oplus \cdots \oplus L_{z_n}. \end{array} \tag{73}$$

Let now

$$s : \mathbb{C}^* \to J(X_\tau) \cong X_\tau, \qquad e^{2\pi i z} \mapsto L_z$$

be the abelian Schottky map for genus one, where we identify, as usual the elliptic curve with its Jacobian. It is easy to see that we have the following commutative diagram, where the vertical arrows are inclusions

$$\begin{array}{ccccc} \mathfrak{h} & \to & T_{\mathbb{C}} & \overset{s^n|_{T_{\mathbb{C}}}}{\to} & M \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}^n & \to & (\mathbb{C}^*)^n & \overset{s^n}{\to} & J(X_\tau)^n . \end{array}$$

The map $f$ is then the Weyl invariant restriction of $s^n$ to $T_{\mathbb{C}}$. More precisely,

PROPOSITION 4.1. *The following diagram is commutative*

$$\begin{array}{ccccccc} SL(n,\mathbb{C}) & \overset{Q}{\to} & T_{\mathbb{C}}/W & \overset{\tilde{\pi}}{\leftarrow} & T_{\mathbb{C}} & \hookrightarrow & (\mathbb{C}^*)^n \\ & S \searrow & \downarrow f & & \downarrow s^n|_{T_{\mathbb{C}}} & & \downarrow s^n \\ & & \mathcal{M}_n & \overset{\pi}{\leftarrow} & M & \hookrightarrow & J(X_\tau)^n \end{array}$$

*Proof.* The commutativity of the right square is clear, so let us consider the middle one. By the construction of the good quotient, $f$ coincides with $S$ when evaluated on diagonal matrices

$$f \circ \tilde{\pi}(w_1, ..., w_n) = f(\{w_1, ..., w_n\}) = S(\mathrm{diag}(w_1, ..., w_n)), \qquad w \in T_{\mathbb{C}}.$$

Since the diagonal matrices correspond to vector bundles which are direct sums of line bundles of degree 0, we have

$$S(\mathrm{diag}(w_1, ..., w_n)) = s(w_1) \oplus \cdots \oplus s(w_n) = L_{z_1} \oplus ... \oplus L_{z_n} \tag{74}$$

where $e^{2\pi i z_j} = w_j$, $j = 1, ..., n$, and we identify once again $X_\tau$ with $J(X_\tau)$. Therefore, by (73),

$$f \circ \tilde{\pi}(w_1, ..., w_n) = \pi(L_{z_1}, ..., L_{z_n}) = \pi \circ s^n(w_1, ..., w_n),$$

which proves the proposition. ▮

For the benefit of the next section we write explicitly the map $s^n|_{T_{\mathbb{C}}}$ as

$$\begin{array}{rcl} s^n|_{T_{\mathbb{C}}} : \mathfrak{h}/\check{\Lambda}_R \cong T_{\mathbb{C}} & \longrightarrow & M = \mathfrak{h}/(\check{\Lambda}_R \oplus \tau\check{\Lambda}_R) \\ v + \check{\Lambda}_R & \mapsto & v + \check{\Lambda}_R + \tau\check{\Lambda}_R . \end{array} \tag{75}$$

## 5. NON-ABELIAN THETA FUNCTIONS IN GENUS ONE

### 5.1. The Structure of Non-abelian Theta Functions in Genus One

Consider again our elliptic curve $X_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and the moduli space $\mathcal{M}_n(X_\tau)$ of semistable holomorphic vector bundles of rank $n$ and trivial determinant over $X_\tau$. We are therefore restricting ourselves in this and next subsection to the case when $K = SU(n)$, with the extra assumption that $n \geq 3$. The case of $SU(2)$ is special and will be treated in subsection 5.3. We expect form [L] that the case of general compact, connected, simply connected semi-simple group $K$ can be treated by analogous techniques. As above, let the Cartan subalgebra be $\mathfrak{h} = \{A \in sl(n,\mathbb{C}) \ : \ A \text{ is diagonal}\}$, and consider the abelian variety

$$M = \check{\Lambda}_R \otimes X_\tau = \mathfrak{h}/(\check{\Lambda}_R \oplus \tau\check{\Lambda}_R).$$

Let us start with the following result which is a consequence of [L],

Proposition 5.1. *The space $H^0(\mathcal{M}_n(X_\tau), L_\Theta^k)$ of level $k$ non-abelian theta functions on the moduli space of semistable rank $n$ holomorphic vector bundles with trivial determinant, $\mathcal{M}_n(X_\tau)$ is naturally identified with the space $\mathcal{H}^+_{k,\tau}$ of Weyl invariant holomorphic functions on $\mathfrak{h}$ satisfying the quasiperiodicity conditions*

$$\theta(v+\check{\alpha}+\tau\check{\beta}) = e^{-2\pi i k\beta(v)-\pi i k\tau<\beta,\beta>}\theta(v), \tag{76}$$

*i.e,*

$$H^0(\mathcal{M}(X_\tau), L_\Theta^k) \xrightarrow{\sim} \{\theta \in \mathcal{H}(\mathfrak{h}),\ \theta \text{ satisfies } (76), w\theta = \theta, \forall w \in W\}.$$

To prove this propositon we will need first two lemmata. Since our aim is to describe theta functions as $Ad_{SL(n,\mathbb{C})}$-invariant holomorphic functions on $SL(n,\mathbb{C})$ it will be usefull to recall their definition as $W$-invariant sections of appropriate holomorphic line bundles on $M$. Therefore, we apply the results of section 3 to the abelian variety $M$. All Weyl invariant anti-symmetric integral forms $E$ on $\Lambda = \check{\Lambda}_R \oplus \tau\check{\Lambda}_R$ are integral multiples of the form $E_1$ given by [L]

$$\begin{aligned} E_1(\check{\alpha},\tau\check{\beta}) &= <\check{\alpha},\check{\beta}> = <\alpha,\beta> \\ E_1(\check{\alpha},\check{\beta}) &= E_1(\tau\check{\alpha},\tau\check{\beta}) = 0, \qquad \forall \alpha,\beta \in \Lambda_R, \end{aligned} \tag{77}$$

where, as in section 2.2, $\check{\alpha}, \check{\beta}$ denote the coroots corresponding to $\alpha, \beta \in \Lambda_R$ and recall that $<,>$ is the inner product on $\mathfrak{h}$ for which the roots have squared length 2.

We are now interested in finding a classification of line bundles on $M$ which are Weyl invariant. Recall that there exists a one to one correspondence between W-invariant antisymmetric integral bilinear forms on $\Lambda$ which are compatible with the complex structure, and elements of the lattice of integral symmetric bilinear forms on $\check{\Lambda}_R$, denoted by $S^2\check{\Lambda}_R$ [L]. Consider the familiar exact sequence

$$0 \to Pic^0(M) \to Pic(M) \xrightarrow{c_1} H^2(M, \mathbb{Z}). \tag{78}$$

LEMMA 5.1. *[Looijenga [L]][3] Let $n \geq 3$. For the abelian variety $M = X_\tau \otimes \check{\Lambda}_R$ the sequence (78) becomes*

$$0 \to \mathfrak{h}^*/(\Lambda_W \oplus \tau\Lambda_W) \to Pic(M) \to S^2\check{\Lambda}_R \to 0, \tag{79}$$

*for which the Weyl invariant part is*

$$0 \to 0 \to Pic(M)^W \to (S^2\check{\Lambda}_R)^W \to 0. \tag{80}$$

*Proof.* We only show that $Pic^0(M)^W = 0$ since this is the only difference with respect to [L]. The automorphy factors of the line bundle corresponding to $x \in \mathfrak{h}^*$, with $x = x_1 + \tau x_2$ and $x_1, x_2 \in \Lambda_R \otimes \mathbb{R} = \mathfrak{h}^*_{\mathbb{R}}$ are given by

$$e(\check{\alpha} + \tau\check{\beta}) = e^{2\pi i(x_1(\check{\alpha}) + x_2(\check{\beta}))}, \quad \check{\alpha}, \check{\beta} \in \check{\Lambda}_R. \tag{81}$$

So, $x$ leads to a Weyl invariant point in $Pic^0(M)$ if and only if for all the elementary Weyl reflections $w_j$ one has

$$x - w_j(x) =< \alpha_j, x > \alpha_j \in \Lambda_W \oplus \tau\Lambda_W. \tag{82}$$

From the Cartan matrix for the algebras $A_l$ for $l \geq 2$ we conclude that this implies that $< \alpha_j, x > \in \mathbb{Z} \oplus \tau\mathbb{Z}$, for all simple roots $\alpha_j$, and therefore $x \in \Lambda_W \oplus \tau\Lambda_W$ and leads to the trivial line bundle $0 \in Pic^0(M)$. ∎

Since all symmetric Weyl invariant integral bilinear forms on $\check{\Lambda}_R$ are integral multiples of $<,>$, we see from (80) that $Pic(M)^W$ is infinite cyclic *i.e.* there is a Weyl invariant line bundle $L_1 \to M$ (this corresponds to the

[3]The $Pic^0(M)^W$ part of the sequence (3.2.1) of [L] does not hold for $n \geq 3$ as we show in our proof. In subsection 5.3 we show that it does hold for $n = 2$.

line bundle $L_1$ of section 3) such that

$$Pic(M)^W = \{L_1^k, k \in \mathbb{Z}\}. \tag{83}$$

Recall that we have the projection $\pi : M \to \mathcal{M}_n(X_\tau)$, and let $L_\Theta$ be the theta bundle over $\mathcal{M}_n(X_\tau)$, as in section 4. From theorem (3.4) of [L], we conclude that $\pi^* L_\Theta^k \cong L_1^k$. More precisely, we have

Lemma 5.2. *Let $\pi : M \to M/W \cong \mathbb{P}^{n-1}$ be the canonical projection and $\mathcal{L} = \pi^* L_\Theta$. Then $\mathcal{L} = L_1$, for $n \geq 3$.*

*Proof.* Since $\mathcal{L}$ is a $W$-invariant line bundle on $M$, its polarization $E = -\mathrm{Im} H$ is a multiple of $E_1$, which means that $\mathcal{L} = L_1^p$, for some $p \in \mathbb{Z}$, $p \geq 1$. It is shown in [L] that

$$H^0(M, L_1)^W \cong \mathbb{C}^n,$$

and that $\dim H^0(L_1^m)^W > n$ for $m > 1$. Since $L_\Theta \cong \mathcal{O}(1)$ we have $n = h^0(\mathbb{P}^{n-1}, L_\Theta) = \dim H^0(\mathcal{L})^W$ which implies $\mathcal{L} = L_1$. ▮

*Proof.* (of proposition 5.1) In the notation of section 3 we have that $\Lambda_1 = \check{\Lambda}_R$, $\Lambda_2 = \tau \check{\Lambda}_R$ and $\widehat{\Lambda} = \check{\Lambda}_W \oplus \tau \check{\Lambda}_W$. The form $F$ (40) on $V \times V_2$ is given by

$$F(.,.) = -\tau^{-1} < .,. > . \tag{84}$$

The space of theta functions $H^0(M, L_1^k)$ is then isomorphic to a subspace of the space of holomorphic functions on $\mathfrak{h}/\check{\Lambda}_R \cong (\mathbb{C}^*)^l \cong T_\mathbb{C}$, satisfying the quasiperiodicity conditions (54), with a basis $\{\theta_{\gamma,k}\}_{\gamma \in \Lambda_W / k\Lambda_R}$ given by (see (45)),

$$\theta_{\gamma,k}(v) = \sum_{\alpha \in \Lambda_R} e^{\pi i k \tau < \alpha + \frac{\gamma}{k}, \alpha + \frac{\gamma}{k} > + 2\pi i k (\alpha + \frac{\gamma}{k})(v)}, \quad v \in \mathfrak{h}. \tag{85}$$

The pull-back under $\pi \circ s^n|_{T_\mathbb{C}}$ (see (75)) of $H^0(\mathcal{M}_n(X_\tau), L_\Theta^k)$ to $\mathfrak{h}/\check{\Lambda}_R$, corresponds the space $\mathcal{H}^+_{k,\tau}$ of Weyl invariant linear combinations of elements of the form (85). ▮

As shown in section 3, these functions are the image under an abelian CST of certain distributions in $U(1)^l$. In order to apply theorem 3.1 to (85), we show in the appendix that the basis of simple coroots $\check{\alpha}_i$, for $i = 1, ..., l$ cannot be completed to a canonical basis of $\Lambda_1 \oplus \Lambda_2 = \check{\Lambda}_R \oplus \tau \check{\Lambda}_R$ so that we are in the situation described in (49)-(54). The role of $\gamma'_{i+l}$ is being played by $\tau \check{\lambda}_i$, where $\check{\lambda}_i$ are the fundamental coweights. The matrix $\tilde{\Omega}$ is then given by $\tilde{\Omega} = \tau C^{-1}$ where $C^{-1}$ is the inverse Cartan matrix and $R = C$ (see (49)).

The spaces of level $k$ non-abelian theta functions $\mathcal{H}^+_{k,\tau}$ are the fibers of a vector bundle over the Teichmüller space of genus one curves (called the bundle of genus one, level $k$, $SU(n)$ conformal blocks in conformal field theory)

$$\begin{array}{rcl} \mathcal{H}_k & \longrightarrow & \mathcal{T}_1 = \{\tau \in \mathbb{C} \ : \ \mathrm{Im}\tau = \tau_2 > 0\} = \mathbb{H}_1 \\ \mathcal{H}_{k_{|_\tau}} & = & \mathcal{H}^+_{k,\tau} \ . \end{array} \tag{86}$$

From (85) we see that, for every $\tau$, a basis of $\mathcal{H}^+_{k,\tau}$ is given by Weyl invariant theta functions of the form

$$\theta^+_{\gamma,k} = \sum_{w\in W} \theta_{w(\gamma),k}, \qquad \gamma \in \Lambda_W/(W \rhd k\Lambda_R), \tag{87}$$

where $W \rhd k\Lambda_R$ denotes de semi-direct product of $W$ and $k\Lambda_R$.

Taking into account the $\tau$ dependence, $\{\theta^+_{\gamma,k}\}$ defines a global moving frame of sections of $\mathcal{H}_k \to \mathcal{T}_1$ and therefore fixes a trivialization of the bundle of conformal blocks. A different trivialization is obtained as follows. Let $\mathcal{H}^-_{k,\tau}$ be the space of Weyl anti-invariant theta functions of level $k$ with basis given by

$$\theta^-_{\gamma,k} = \sum_{w\in W} \epsilon(w) \ \theta_{w(\gamma),k}, \qquad \gamma \in \Lambda_W/(W \rhd k\Lambda_R) \ . \tag{88}$$

Notice that $\theta^-_{\gamma,k} = 0$ if $\gamma$ is singular, i.e. if $< \gamma, \alpha_i >= 0$ for some simple root $\alpha_i$. A non-singular dominant weight $\gamma \in \Lambda^+_W$ can always be written in the form $\gamma = \gamma' + \rho$ with $\gamma'$ being a, possibly singular, dominant weight.

Recall the following

Theorem 5.1. *[Looijenga [L]]*

*a) The space of Weyl anti-invariant theta functions of level $n$, $\mathcal{H}^-_{n,\tau}$ is one-dimensional and $\mathcal{H}^-_{n,\tau} =< \theta^-_{\rho,n} >_{\mathbb{C}}$.*

*b) The map*

$$\begin{array}{rcl} \mathcal{H}^+_{k,\tau} & \to & \mathcal{H}^-_{k+n,\tau} \\ \theta^+ & \mapsto & \theta^- = \theta^-_{\rho,n} \ \theta^+, \end{array} \tag{89}$$

*is an isomorphism between the space of level $k$ Weyl invariant theta functions and the space $\mathcal{H}^-_{k+n,\tau}$ of Weyl anti-invariant theta functions of level $k+n$.*

Let $D_k$ be the set of integrable representations of level $k$ of the Kac-Moody algebra $\widehat{sl(n,\mathbb{C})}_k$, $D_k = \{\lambda \in \Lambda_W^+ | \ <\lambda, \hat{\alpha}> \ \leq k\} \cong \Lambda_W/(W \triangleright k\Lambda_R)$, where $\hat{\alpha} = \alpha_1 + \cdots + \alpha_{n-1}$ is the highest root for $sl(n,\mathbb{C})$. The previous theorem leads to a basis of $\mathcal{H}^+_{k,\tau}$, different from (87), given by

$$\hat{\theta}^+_{\gamma,k} = \frac{\theta^-_{\gamma+\rho,k+n}}{\theta^-_{\rho,n}}, \qquad \gamma \in D_k. \tag{90}$$

As we will see in the next subsection from the point of view of the heat equation the trivialization of the bundle of conformal blocks corresponding to (90) is more convenient than the one corresponding to (87).

### 5.2. CST and Non-abelian Theta Functions in Genus One

We now continue to follow the strategy indicated in the introduction. As we mentioned before, by extending the $SU(n)$-CST $C_t^\tau$ of definition 2.1 to distributions we of course lose unitarity. From propositions 2.3 and 2.4 it follows that the image under $C_t^\tau$ of a distribution on $SU(n)$ with infinite $L^2$ norm is a holomorphic function on $SL(n,\mathbb{C})$ with infinite norm with respect to the heat kernel measure $d\nu_{t\tau_2}$.

Consider again the projection

$$Q : SL(n,\mathbb{C}) \to T_{\mathbb{C}}/W \cong \mathfrak{h}/(W \triangleright \check{\Lambda}_R) \cong SL(n,\mathbb{C})/Ad_{SL(n,\mathbb{C})}$$

of section 4.4. In order to recover unitarity we will, for every $\tau$, restrict integration on $SL(n,\mathbb{C})$ to the region $Q^{-1}([\mathfrak{h}_0])$ with $\mathfrak{h}_0 \subset \mathfrak{h}$ a fundamental domain with respect to the group $W \triangleright (\check{\Lambda}_R \oplus \tau\check{\Lambda}_R)$ and $[\mathfrak{h}_0] = W\mathfrak{h}_0 + \check{\Lambda}_R$. Of course this will be meaningful only for those holomorphic funtions on $SL(n,\mathbb{C})$ for which the integral will not depend on the choice of $\mathfrak{h}_0$. As we will see, this simple condition gives a precise analytic characterization of the pull-back of non-abelian theta functions to $SL(n,\mathbb{C})$ with respect to the Schottky map $S$ (theorems 5.2 and 5.3). The relation between the heat kernel measures $d\nu_{t\tau_2}$ on $SL(n,\mathbb{C})$ and $d\nu^{T_{\mathbb{C}}}_{t\tau_2}$ on $T_{\mathbb{C}}$ is given by

Proposition 5.2. *The push-forward of the measure $d\nu_{t\tau_2}$ on $SL(n,\mathbb{C})$ with respect to the projection $Q$ is given by*

$$Q_* d\nu_{t\tau_2} = e^{-2t\pi\tau_2||\rho||^2} |\sigma_{\mathbb{C}}|^2 d\nu^{T_{\mathbb{C}}}_{t\tau_2} \tag{91}$$

*where on the right-hand side we denote the restriction of $d\nu^{T_{\mathbb{C}}}_{t\tau_2}$ to a fundamental domain of $W \triangleright \check{\Lambda}_R$ in $\mathfrak{h}$, by the same symbol.*

*Proof.* This is essentially a restatement of theorem 2.3 with $t$ replaced by $t\tau_2$. ∎

Let us state the main results of the present section (theorems 5.2, 5.3 and 5.4) which will be proved in the end of the section. We will use the same notation for $(W \triangleright \check{\Lambda}_R)$-invariant functions on $\mathfrak{h}$, the corresponding $Ad$-invariant functions on $SL(n,\mathbb{C})$ and also their restrictions to $T_{\mathbb{C}}$.

Theorem 5.2. *The sets of distributions on $SU(n)$,*

$$\begin{aligned} \mathcal{F} = \quad & \{\psi \in (C^\infty(SU(n))')^{Ad_{SU(n)}} : |C_t^\tau(\psi)|^2|\sigma_{\mathbb{C}}|^2\nu_{t\tau_2}^{T_{\mathbb{C}}}, \\ & \text{is } \ \tau\check{\Lambda}_R \ \text{ invariant as a function on } \mathfrak{h}\}. \end{aligned} \tag{92}$$

*are nontrivial ($\{0\} \subsetneq \mathcal{F}$) if and only if $t = \frac{1}{k+n}$ with $k \in \mathbb{N} \cup \{0\}$. For $t = \frac{1}{k+n}$, these are $\tau$-independent finite dimensional subspaces $\mathcal{F}_k$ of $(C^\infty(SU(n))')^{Ad_{SU(n)}}$ with dimensions given by the Verlinde numbers*

$$\dim \mathcal{F}_k = \dim \mathcal{H}^-_{k+n,\tau} = \binom{n+k-1}{k} \tag{93}$$

*and have basis formed by*

$$\psi_{\gamma,k}(v) = \frac{1}{\sigma}\sum_{w\in W}\epsilon(w)\theta^0_{\gamma+\rho,k+n}(w(v)), \quad v \in \mathfrak{h}_{\mathbb{R}}, \tag{94}$$

*where $\gamma = 0$ if $k = 0$ and $\gamma \in \Lambda_W/(W \triangleright k\Lambda_R)$ if $k > 0$ and*

$$\theta^0_{\gamma+\rho,k+n}(v) = \sum_{\alpha\in\Lambda_R} e^{2\pi i(\gamma+\rho+(k+n)\alpha)(v)}. \tag{95}$$

In terms of characters of irreducible representations of $SU(n)$, the distributions $\psi_{\gamma,k}, \gamma \in D_k = \{\lambda \in \Lambda_W^+ | \ < \lambda, \hat{\alpha} > \ \leq k\} \cong \Lambda_W/(W \triangleright k\Lambda_R)$, where $\hat{\alpha}$ is the highest root for $sl(n,\mathbb{C})$, have the form

$$\psi_{\gamma,k} = \sum_{\substack{\lambda\in\Lambda_W^+ \\ \lambda+\rho\in[\gamma+\rho]}} \epsilon_\lambda \chi_\lambda, \tag{96}$$

where $[\gamma+\rho]$ is the $(W \triangleright (k+n)\Lambda_R)$-orbit of $(\gamma+\rho)$ and $\epsilon_\lambda = \epsilon(w)$ for the unique $w \in W$ such that $\lambda + \rho = w(\gamma+\rho) \bmod (k+n)\Lambda_R$. This follows from the Weyl character formula (18) and the fact that the group $W \triangleright (k+n)\Lambda_R$ acts freely on the set of dilated Weyl alcoves [Ber, Fr, PS].

For $\Psi$ and $\Psi'$ $Ad$-invariant holomorphic functions on $SL(n,\mathbb{C})$, and for a choice of a fundamental domain $\mathfrak{h}_0 \subset \mathfrak{h}$ of the group $W \triangleright (\check{\Lambda}_R \oplus \tau\check{\Lambda}_R)$, define the modified Hall inner product as

$$\langle\langle\, \Psi, \Psi' \rangle\rangle := \int_{Q^{-1}([\mathfrak{h}_0])} \overline{\Psi}\Psi' d\nu_{t\tau_2} = \int_{\mathfrak{h}_0} \overline{\Psi}\Psi' e^{-2t\pi\tau_2||\rho||^2}|\sigma_{\mathbb{C}}|^2 d\nu_{t\tau_2}^{T_{\mathbb{C}}}, \tag{97}$$

where $[\mathfrak{h}_0] = W\mathfrak{h}_0 + \check{\Lambda}_R$.

We now consider the images of the spaces $\mathcal{F}_k$ under the CST. We will see that, remarkably, any two distributions $\psi, \psi' \in \mathcal{F}_k$ in theorem 5.2 lead to functions $C_t^\tau(\psi)\sigma_{\mathbb{C}}$ such that $\langle\langle\, C_t^\tau(\psi), C_t^\tau(\psi')\, \rangle\rangle$ is independent of $\tau$. These functions are automatically the pull-backs to $\mathfrak{h}/\check{\Lambda}_R$ of holomorphic sections of line bundles over $\mathfrak{h}/(\check{\Lambda}_R \oplus \tau\check{\Lambda}_R)$.

Let $\widetilde{\mathcal{H}}_{k,\tau}$ be the image of $\mathcal{F}_k$ under $C^\tau_{t=1/(k+n)}$,

$$\widetilde{\mathcal{H}}_{k,\tau} = C^\tau_{t=1/(k+n)}\left(\mathcal{F}_k\right). \tag{98}$$

On $\widetilde{\mathcal{H}}_{k,\tau}$ we have the hermitean inner product $\langle\langle\cdot,\cdot\rangle\rangle$ (97) induced by the Hall CST inner product. On the other hand, the hermitean structure on $\mathcal{H}_k$ in (86), which is of interest in Conformal Field Theory, is [AdPW]

$$<\theta^+, \theta^{+'} >= \frac{1}{|W|}\int_M \overline{\theta^+}\theta^{+'}|\theta^-_{\rho,n}|^2 d\nu^{T_{\mathbb{C}}}_{\tau_2/(k+n)}, \tag{99}$$

for $\theta^+, \theta^{+'} \in \mathcal{H}^+_{k,\tau}$.

We then have,

Theorem 5.3. *The family $\{\widetilde{\mathcal{H}}_{k,\tau}\}_{\tau\in\mathcal{T}_1}$ in (98) forms a vector bundle over $\mathcal{T}_1$ isomorphic to the bundle of conformal blocks $\mathcal{H}_k \to \mathcal{T}_1$ (86) with isomorphism given by $\Phi_k : \widetilde{\mathcal{H}}_k \to \mathcal{H}_k$ in (2).*

*The hermitean structure defined by (97), with $t = 1/(k+n)$, does not depend on $\mathfrak{h}_0$ and the map $\Phi_k$ is a unitary isomorphism of vector bundles.*

*Remark 5. 1.* Since the functions $|C_t^\tau(\psi)|^2|\sigma_{\mathbb{C}}|^2\nu^{T_{\mathbb{C}}}_{t\tau_2}$ in (92) are automatically $W \triangleright \check{\Lambda}_R$ invariant, being also $\tau\check{\Lambda}_R$ invariant means that they are the pull-back of functions on $\mathcal{M}_n = \mathfrak{h}/(W \triangleright (\check{\Lambda}_R \oplus \tau\check{\Lambda}_R))$.

Finally, we recover the unitarity of the CST with

Theorem 5.4. *With respect to the $\tau$-independent inner product $(\cdot,\cdot)$ on $\mathcal{F}_k$, for which the basis $\{\psi_{\gamma,k}\}$ in (94) and (96) is orthonormal, and the inner product (97) the CST*

$$C^\tau_{t=1/(k+n)}: \quad (\mathcal{F}_k, (\cdot,\cdot)) \to \left(\widetilde{\mathcal{H}}_{k,\tau}, \langle\langle\,\cdot,\cdot\,\rangle\rangle\right)$$

*is a unitary isomorphism $\forall\tau \in \mathcal{T}_1$ and $k \in \mathbb{N}_0$.*

We know from lemma 4.2 of [FMN] and section 3 above that if $t = 1/k$ with $k \in \mathbb{N}$ then $\nu^{T_{\mathbb{C}}}_{t\tau_2}$ defines an hermitean structure on $L_1^k$. We then have the following result,

LEMMA 5.3. *For $t \neq 1/k'$ with $k' \in \mathbb{N}$, the sets $\mathcal{F}$ in (92) are trivial.*

*Proof.* The key observation is that $\nu^{T_{\mathbb{C}}}_{t\tau_2}$ satisfies the quasi-periodicity conditions

$$\begin{aligned} \nu^{T_{\mathbb{C}}}_{t\tau_2}(v+\check{\alpha}) &= \nu^{T_{\mathbb{C}}}_{t\tau_2}(v), \\ \nu^{T_{\mathbb{C}}}_{t\tau_2}(v+\tau\check{\alpha}) &= |e^{2\pi i \frac{1}{t}\alpha(v)}|^2 |e^{\pi i \frac{1}{t}\tau <\alpha,\alpha>}|^2 \nu^{T_{\mathbb{C}}}_{t\tau_2}(v), \text{ for all } \check{\alpha} \in \check{\Lambda}_R, \end{aligned} \quad (100)$$

which follow from (17). On the other hand, from corollary 2.3 $C_t^\tau(\psi)\sigma_{\mathbb{C}}$ is a holomorphic function on $\mathfrak{h} \times \mathcal{T}_1$ and verifies

$$(C_t^\tau(\psi)\sigma_{\mathbb{C}})(v+\check{\alpha}) = (C_t^\tau(\psi)\sigma_{\mathbb{C}})(v), \quad (101)$$

for all $\check{\alpha} \in \check{\Lambda}_R$ and for all $\psi \in (C^\infty(SU(n))')^{Ad_{SU(n)}}$.

If the function $|C_t^\tau(\psi)|^2|\sigma_{\mathbb{C}}|^2\nu^{T_{\mathbb{C}}}_{t\tau_2}$ is $(\check{\Lambda}_R \oplus \tau\check{\Lambda}_R)$-invariant then from (100) and the holomorphicity in $\mathfrak{h} \times \mathcal{T}_1$ it follows that $C_t^\tau(\psi)\sigma_{\mathbb{C}}$ must also satisfy the following quasi-periodicity conditions

$$(C_t^\tau(\psi)\sigma_{\mathbb{C}})(v+\tau\check{\alpha}) = e^{-2\pi i <d,\alpha>} e^{-2\pi i \frac{1}{t}\alpha(v)} e^{-\pi i \frac{1}{t}\tau <\alpha,\alpha>} (C_t^\tau(\psi)\sigma_{\mathbb{C}})(v), \quad (102)$$

where $d \in \mathfrak{h}^*_{\mathbb{R}}$. Non-zero holomorphic functions satisfying (101) and (102) do not exist if $1/t \notin \mathbb{N}$. This results from the fact that if $1/t \notin \mathbb{N}$ then the automorphy factors in (102) are not invariant under $v \mapsto v + \check{\beta}$, $\check{\beta} \in \check{\Lambda}_R$, which makes it impossible to solve (101) and (102). ∎

Note that functions satisfying (101) and (102) with $t = 1/k'$ are level $k'$ theta functions with automorphy factors (compare with (54) and (76))

$$e(\check{\alpha} + \tau\check{\alpha}', v) = e^{-2\pi i k' \alpha'(v) - \pi i k' \tau <\alpha',\alpha'> - 2\pi i <d,\alpha'>}. \quad (103)$$

To prove theorem 5.2, let us now find the nontrivial set of distributions in $\mathcal{F}_k$. Since $\sigma_{\mathbb{C}}$ is $W$-anti-invariant and $C^\tau_{1/k'}(\psi)$ is $W$-invariant, we conclude that the functions $C^\tau_{1/k'}(\psi)\sigma_{\mathbb{C}}$ satisfying (101) and (102) are $W$-anti-invariant. As in the proof of lemma 5.1, this implies that $d$ in (103) belongs to $\Lambda_W$. Therefore, we can take $d = 0$.

*Proof.* (of Theorem 5.2) From lemma 5.3 we can consider $1/t \in \mathbb{N}$. On the other hand, from theorem 5.1 it follows that non-zero $W$-anti-invariant theta functions exist only for level $k' \geq n$. So if $0 \neq \psi \in$

$(C^\infty(SU(n))')^{Ad_{SU(n)}}$ satisfies the condition (92) then $t = 1/(k+n)$, $k \geq 0$ and $\varphi_{\mathbb{C}} \circ C^\tau_{1/(k+n)}(\psi)$ (see (36)) is a Weyl anti-invariant theta function of level $k+n$. Conversely, from theorems 2.4 and 3.1 it follows that every Weyl anti-invariant theta function $\theta \in \mathcal{H}^-_{k+n,\tau}$ is the image under $\varphi_{\mathbb{C}} \circ C^\tau_{1/(k+n)}$ of a unique $Ad_{SU(n)}$-invariant distribution from $\mathcal{F}_k$. It is easy to check that the inverse images with respect to the map $\varphi_{\mathbb{C}} \circ C^\tau_{1/(k+n)}$ of the theta functions $\theta^-_{\gamma,k}$ in (88) are given by the distributions $\sqrt{|W|}\psi_{\gamma,k}$ in the theorem. The $\tau$-independence of $\mathcal{F}_k$ follows from (94). ▮

*Proof.* (of Theorem 5.3) We have shown in theorem 5.2 that the restriction of $\varphi_{\mathbb{C}} \circ C^\tau_{1/(k+n)}$ to $\mathcal{F}_k \subset (C^\infty(SU(n))')^{Ad_{SU(n)}}$ is an isomorphism to $\mathcal{H}^-_{k+n,\tau}$. From theorem 5.1 it then follows that the bundle of conformal blocks $\mathcal{H}_k \to \mathcal{T}_1$ in (86) is isomorphic to the bundle

$$\widetilde{\mathcal{H}}_k = \left\{\widetilde{\mathcal{H}}_{k,\tau}\right\}_{\tau\in\mathcal{T}_1} \to \mathcal{T}_1 \tag{104}$$

with simple isomorphism $\Phi_k$ given by (2).

The identities (100), (101) and (102) imply that the hermitean structure (97) does not depend on $\mathfrak{h}_0$. From theorems 3.1 and 5.1 it follows that (99) defines a shifted hermitean structure on $\mathcal{H}_k$ for which the frame

$$\{\widehat{\theta}^+_{\gamma,k} = \frac{\theta^-_{\gamma+\rho,k+n}}{\theta^-_{\rho,n}}\}$$

is orthonormal. (Notice that the same is not true for the “unshifted” frame $\{\theta^+_{\gamma,k}\}$.) We see from formula (97) that with these hermitean structures on $\widetilde{\mathcal{H}}_k$ and $\mathcal{H}_k$ the isomorphism $\Phi_k$ in (2) is unitary. ▮

*Proof.* (of Theorem 5.4) From theorem 2.4 we see that

$$C^\tau_{1/(k+n)}(\psi_{\gamma,k}) = \frac{e^{-\frac{i\pi\tau}{k+n}||\rho||^2}}{\sigma_{\mathbb{C}}}\theta^-_{\gamma+\rho,k+n}. \tag{105}$$

Unitarity then follows from (97) and theorem 3.1. ▮

In terms of characters of $SU(n)$, from (31), (96) and (105) we obtain

$$C^\tau_{1/(k+n)}(\psi_{\gamma,k}) = \sum_{\substack{\lambda\in\Lambda^+_W \\ \lambda+\rho\in[\gamma+\rho]}} \epsilon_\lambda e^{\frac{i\pi\tau}{k+n}c_\lambda}\chi_\lambda. \tag{106}$$

### 5.3. The Case of $SU(2)$

Somewhat surprisingly, the case of $K = SU(2)$ is special for several reasons. To begin with, the Weyl invariant antisymmetric integral bilinear forms on

$$\Lambda = \check{\Lambda}_R \oplus \tau\check{\Lambda}_R = \mathbb{Z}\check{\alpha}_1 \oplus \tau\mathbb{Z}\check{\alpha}_1$$

are integral multiples of $E_1$, where $E_1(\check{\alpha}_1, \tau\check{\alpha}_1) = \frac{1}{2} < \alpha_1, \alpha_1 >= 1$ which is different from (77) by a factor of 1/2. The form $F$ is now given by $F(.,.) = -\frac{1}{2}\tau^{-1} < .,. >$. As in (41) this defines the line bundles $L_1^k$ on the abelian variety $M = X_\tau \otimes \check{\Lambda}_R = \mathbb{C}\check{\alpha}_1/(\mathbb{Z}\check{\alpha}_1 \oplus \tau\mathbb{Z}\check{\alpha}_1) \cong X_\tau, \mathbb{C}\check{\alpha}_1 = \mathfrak{h}$.

The space of theta functions $H^0(M, L_1^k)$ is isomorphic to a subspace of the space of holomorphic functions on $\mathfrak{h}/\check{\Lambda}_R \cong \mathbb{C}^* \cong T_\mathbb{C}$ with a basis $\{\theta_{\gamma,k}\}_{\gamma\in\Lambda_W/(k/2)\Lambda_R}$ given by (see (45)),

$$\theta_{\gamma,k}(v) = \sum_{\alpha\in\Lambda_R} e^{\pi ik\tau\frac{1}{2}<\frac{2}{k}\gamma+\alpha,\frac{2}{k}\gamma+\alpha>+\pi ik(\frac{2}{k}\gamma+\alpha)(v)}, \tag{107}$$

or in more explicit classical notation, $\gamma = m\lambda_1 = (m/2)\alpha_1$, $v = z\check{\alpha}_1$ and $\alpha = p\check{\alpha}_1$

$$\theta_{\gamma,k}(v) = \theta_{m,k}(z) = \sum_{p\in\mathbb{Z}} e^{\pi i\frac{\tau}{k}(m+kp)^2+2\pi i(m+kp)z}, \tag{108}$$

with $0 \le m < k$, $m \in \mathbb{N}$. In this case we have $(S^2\check{\Lambda}_R)^W = S^2\check{\Lambda}_R$ and

Proposition 5.3. *[Looijenga [L]] The Weyl invariant part of the sequence (79) is*

$$0 \to [1/2\,(\Lambda_W \oplus \tau\Lambda_W)]\,/\,(\Lambda_W \oplus \tau\Lambda_W) \to (Pic(M))^W \to S^2\check{\Lambda}_R \to 0 \tag{109}$$

*Proof.* As in lemma 5.1 it suffices to determine $Pic^0(M)^W$. The condition (82) has a solution if and only if $x = (m_1 + \tau m_2)\alpha_1/4$, $m_1, m_2 \in \mathbb{Z}$. Therefore,

$$(Pic^0(M))^W \cong [1/2\,(\Lambda_W \oplus \tau\Lambda_W)]\,/\,(\Lambda_W \oplus \tau\Lambda_W) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

∎

The second main difference is that, in contrast with lemma 5.2, not all line bundles in $Pic(M)^W$ are the pull-backs of line bundles on $\mathcal{M}_2(X_\tau)$. Denote line bundles in $(Pic^0(M))^W$ by $N_\eta$, $\eta \in [1/2\,(\Lambda_W \oplus \tau\Lambda_W)]\,/\,(\Lambda_W \oplus \tau\Lambda_W)$ and let $L_1$ be the line bundle on $M$ with automorphy factors defined by the form $F$ above,

$$e(\check{\alpha} + \tau\check{\beta}, v) = e^{-\pi i\beta(v)-\frac{1}{2}\pi i\tau<\beta,\beta>}. \tag{110}$$

We see from (109) that

$$(Pic(M))^W = \left\{N_\eta \otimes L_1^k, k \in \mathbb{Z}, \eta \in [1/2\,(\Lambda_W \oplus \tau\Lambda_W)]\,/\,(\Lambda_W \oplus \tau\Lambda_W)\right\}. \tag{111}$$

Lemma 5.4. *Let $\pi : M \to M/W \cong \mathcal{M}_2(X_\tau) \cong \mathbb{P}^1$ be the canonical projection and $\mathcal{L} = \pi^* L_\Theta$. Then $\mathcal{L} \cong L_1^2$.*

*Proof.* In this case, $M \cong X_\tau$ and the map $\pi : M \to \mathcal{M}_2$ becomes the two-fold ramified covering of $\mathbb{P}^1$, $X_\tau \to X_\tau/\mathbb{Z}_2 \cong \mathbb{P}^1$. The pull-back to $X_\tau$ of the the $k$th power of the theta bundle, $\mathcal{O}(k) \to \mathcal{M}_2$, will be a line bundle of degree $2k$. On the other hand, $\pi^{-1}([\pi((\frac{1}{2}+\tau\frac{1}{2})\check{\alpha}_1)]) = 2[(\frac{1}{2}+\tau\frac{1}{2})\check{\alpha}_1]$ as an element in $Div(X_\tau)$ and therefore $\pi^*\mathcal{O}(1) \cong L_1^2$, since the zero of the Riemann theta function ($k=1, m=0$ in (108)) is $(1/2+1/2\tau)\check{\alpha}_1$. ▮

As a consequence of the above results, we obtain as the third significative difference with the case $n \geq 3$, the fact that the CST applied to the spaces $\mathcal{F}_k$ as defined in theorem 5.2 leads not only to non-abelian theta functions on $\mathcal{M}_2(X_\tau)$, but also to Weyl invariant theta functions on $M$ which do not descend to sections of bundles on the moduli space. We have in place of lemma 5.3

Lemma 5.5. *The spaces $\mathcal{F}_k$ for $SU(2)$ are trivial if $t \neq 2/k'$ with $k' \in \mathbb{N}$.*

*Proof.* This follows immediately from the proof of lemma 5.3 and from $< \alpha_1, \alpha_1 >= 2$. ▮

From (102), (110) and (111) we see that the possible automorphy factors of the functions $C^\tau_{2/k'}(\psi)\sigma$ for distributions satisfying (92) are

$$e(\alpha+\tau\alpha', v) = e^{-\pi i k'\alpha'(v) - \pi i\tau\frac{k'}{2}<\alpha',\alpha'> - 2\pi i<d,\alpha'>}, \tag{112}$$

with $d=0$ or $d=(1/2)\lambda_1 = (1/4)\alpha_1$.

Functions $\theta^{(1/2)}$ with automorphy factors corresponding to $d=(1/2)\lambda_1$ in (112) can be obtained from those with $d=0$ through a translation of $L_1^{k'}$ by $N_{(\tau\lambda_1/2)}$,

$$\theta(v) \mapsto \theta^{(1/2)}(v) = \theta(v+\lambda_1/k'). \tag{113}$$

A basis for $H^0(M, N_{(\tau\lambda_1/2)} \otimes L_1^{k'})$ is then given by

$$\theta^{(1/2)}_{m,k'}(z) = \sum_{p\in\mathbb{Z}} (-1)^p e^{\pi i\frac{\tau}{k'}(m+k'p)^2 + 2\pi i(m+k'p)z}, \tag{114}$$

for $m=0,\dots,k'-1$. The action of the Weyl group $W \cong \mathbb{Z}_2$ is $\theta(v) \mapsto \theta(-v)$ on the elements of the basis in (108) and (114) is

$$\begin{aligned} \theta_{m,k'}(-z) &= \theta_{k'-m,k'}(z) \\ \theta^{(1/2)}_{m,k'}(-z) &= \theta^{(1/2)}_{k'-m,k'}(z). \end{aligned} \tag{115}$$

Let $\mathcal{H}^+_{k',\tau}$ and $\mathcal{H}^-_{k',\tau}$ be the spaces of, respectively, Weyl invariant and anti-invariant sections of $L_1^{k'}$. We have the following

LEMMA 5.6.

*1.If $k' = 2k$, with $k \in \mathbb{N}$, then $\mathcal{H}^+_{2k,\tau}$ has dimension $k+1$ and is generated by $\{\theta_{0,2k}, \theta_{k,2k}, \theta_{j,2k} + \theta_{2k-j,2k}, j = 1, ..., k-1\}$. The space $\mathcal{H}^-_{2k,\tau}$ has dimension $k-1$ and a basis is formed by $\{\theta_{j,2k} - \theta_{2k-j,2k}, j = 1, ..., k-1\}$.*

*2.If $k' = 2k+1$, with $k \in \mathbb{N}$, then $\mathcal{H}^+_{2k+1,\tau}$ has dimension $k+1$ and is generated by $\{\theta_{0,2k+1}, \theta_{j,2k+1} + \theta_{2k+1-j,2k+1}, j = 1, ..., k\}$.*

*The space $\mathcal{H}^-_{2k+1,\tau}$ has dimension $k$ and a basis is formed by $\{\theta_{j,2k+1} - \theta_{2k+1-j,2k+1}, j = 1, ..., k\}$.*

*Proof.* The result follows immediately from (114). ▮

The corresponding similar result holds for the decomposition of $H^0(M, N_{(\tau\lambda_1/2)} \otimes L_1^{k'})$ under the action of the Weyl group. The analog of theorem 5.1 becomes,

COROLLARY 5.1.

*a)The space $\mathcal{H}^-_{4,\tau}$ of Weyl anti-invariant sections of $L_1^4$ is one-dimensional and $\mathcal{H}^-_{4,\tau} =< \theta_4^- >_{\mathbb{C}}$, where $\theta_4^- = \theta_{1,4} - \theta_{3,4}$.*

*b)The map*

$$\begin{array}{rcl} \mathcal{H}^+_{2k,\tau} & \to & \mathcal{H}^-_{2k+4,\tau} \\ \theta^+ & \mapsto & \theta^- = \theta_4^- \ \theta^+, \end{array} \tag{116}$$

*is an isomorphism between the space of Weyl invariant sections of $L_1^{2k}$ and the space of Weyl anti-invariant sections of $L_1^{2k+4}$.*

*Proof.* This is an immediate consequence of the previous lemma. ▮

Again, the corresponding result holds if we replace $L_1^{2k}$ by $N_{(\tau\lambda_1/2)} \otimes L_1^{2k}$. We note that the space $\mathcal{H}^-_{3,\tau}$ is also one-dimensional and also provides an isomorphism from $\mathcal{H}^+_{2k,\tau}$ to $\mathcal{H}^-_{2k+3,\tau}$, which are both of dimension $k+1$.

Consider the distributions on $SU(2)$ defined by

$$
\begin{aligned}
\theta^0_{m,k'}(x) &= \sum_{p\in\mathbb{Z}} e^{2\pi i(m+k'p)x}, \\
\theta^{(1/2)0}_{m,k'}(x) &= \sum_{p\in\mathbb{Z}} (-1)^p e^{2\pi i(m+k'p)x},
\end{aligned}
\tag{117}
$$

for $m = 0, ..., k'-1$. We then have,

THEOREM 5.5. *The set $\mathcal{F}$ in (92) is nontrivial if and only if $t = 2/k'$, where $k'$ is of the form $k' = 2k+3$ or $k' = 2k+4$ with $k \in \mathbb{N}_0$. For every such $k'$, the set $\mathcal{F}_k$ of distributions satisfying (92) is the union of two subspaces $\mathcal{F}^{(0)}_k$ and $\mathcal{F}^{(1/2)}_k$ intersecting only in $0$ and with dimensions equal to $k+1$. A basis for $\mathcal{F}^{(0)}_k$ is formed by*

$$
\psi_{m,k}(v) = \frac{1}{\sigma} \sum_{w\in W} \epsilon(w) \theta^0_{m+1,k'}(w(v)),
\tag{118}
$$

*with $m = 0, ..., k$. A basis for $\mathcal{F}^{(1/2)}_k$ is formed by distributions with identical expressions, only with $\theta^{(1/2)0}_{m+1,k'}$ replacing $\theta^0_{m+1,k'}$.*

*Proof.* Follows immediately from the proof of theorem 5.2, with obvious changes coming from lemmas 5.4, 5.5 and 5.6. ∎

When $k' = 2k+4$ the image of $\mathcal{F}^{(0)}_k$ under the CST $C^\tau_{t=2/(2k+4)}$, for $\tau \in \mathcal{T}_1$, gives a vector bundle $\widetilde{\mathcal{H}}_k \to \mathcal{T}_1$, which is isomorphic to the bundle of conformal blocks. The analogs of theorems 5.3 and 5.4 now follow from corollary 5.1 by adapting the proofs of theorems 5.3 and 5.4.

The image under the CST of the space of distributions $\mathcal{F}^{(1/2)}_k$, does not lead to pull-backs of holomorphic sections of line bundles on $\mathcal{M}_2(X_\tau)$. Similarly, when $k' = 2k+3$, the image by the CST of the space $\mathcal{F}_k$ leads to holomorphic functions on $SL(n,\mathbb{C})$ which are $W \triangleright (\check{\Lambda}_R \oplus \tau\check{\lambda}_R)$ invariant but which, nevertheless, descend to $\mathcal{M}_2(X_\tau)$ only as sections of orbifold line bundles.

Therefore, while in the case of $\mathcal{M}_n(X_\tau)$ for $n \geq 3$ the CST led automatically to non-abelian theta functions, for $SU(2)$ this happens only for a subset of the solutions of (92).

## 6. GENERAL COMMENTS AND CONCLUSIONS

The vector bundle of conformal blocks $\mathcal{H}_k$ has been the subject of much interest. In [AdPW], Chern-Simons topological quantum field theory was analysed from the point of view of geometric quantization on infinite dimensional affine spaces of connections. In that context, $\mathcal{H}_k$ is endowed with a hermitian structure (see eq. 5.49 in [AdPW]) which coincides with the one described in section 5. In that case, the shift in the level $k \to k+n$ which gives rise to the correct unitary structure on $\mathcal{H}_k$, comes from (not fully rigorous) Feynman path integral calculations, leading to regularized determinants of Laplace operators on adjoint bundles. On the other hand, in the present work the shift follows naturally from the generalized CST and the Schottky map. In fact, the consideration of class functions or distributions on $K$, and the factor of $\sigma$ in the Weyl integration formula, lead to Weyl anti-invariance and ultimately to the hermitean structure (97) or equivalently (99). This is the same hermitean structure obtained in [AdPW] and, as noted already, is different from the, *a priori*, most natural one, for which the frame $\theta^+_{\gamma,k}$ in (87) would be orthonormal. The present work then provides a finite dimensional framework for explaining the shift in the level.

The CST (extended to $(C^\infty(SU(n)))'$ and restricted to $\mathcal{F}_k$) induces a parallel transport on the bundle $\mathcal{H}_k$ which coincides with the one associated to the Knizhnik-Zamolodchikov-Bernard [KZ, Ber] connection introduced in Wess-Zumino-Witten conformal field theories [Wi1, Wi2] or also the connection considered in the context of Chern-Simons theories [AdPW, FG, G, Hi].

We believe that our results also contribute to clarifying the relations between the heat equation on the compact group $K$ and the representations of the loop group $LK$, a point raised in [PS] (see page 286). The genus 1 non-abelian theta functions appear in the Weyl-Kac character formula for the affine algebra $\widehat{Lie(LK)}$, and as we have seen these theta functions, including the unitary structure, can be naturally studied with CST techniques.

Similar techniques are also useful for the (much harder) study of non-abelian theta functions for curves of genus greater than 1 [FMNT]. We note that in the present work and also in the case of classical (abelian) theta functions [FMN], the choice of distributions to which the CST should be applied to produce non-abelian theta functions is dictated by naturality conditions related to the unitarity of the extended CST. Alternatively, in both cases the same distributions can be determined by Bohr-Sommerfeld conditions in geometric quantization [Ty]. In fact, the distributions in (94) are combinations of Dirac delta distributions supported on Bohr-Sommerfeld points. This should be related to the work of [We]. We expect that this continues to be true in the higher genus non-abelian case as well.

## APPENDIX

In this appendix, for convenience of the reader, we obtain explicit expressions for the canonical bases of $\check{\Lambda}_R \oplus \tau\check{\Lambda}_R$ with respect to the form $E_1$ in (77), for the case of the Lie algebra $sl(n,\mathbb{C})$. We also write down the corresponding period matrices.

We start with a lemma valid for a simple Lie algebra $\mathfrak{g}$, of rank $l$. To simplify the notation, instead of the coroot lattice, we will work with the root lattice $\Lambda_R \subset \mathfrak{h}$. Let $E$ be the antisymmetric $\mathbb{Z}$-bilinear form on $\Lambda_R \oplus \tau\Lambda_R$ given by

$$E\left(\alpha_i, \alpha_j\right) = E\left(\tau\alpha_i, \tau\alpha_j\right) = 0, \qquad E\left(\alpha_i, \tau\alpha_j\right) = -E\left(\tau\alpha_j, \alpha_i\right) = C_{ij},$$

where $C = [C_{ij}]$ is the Cartan matrix of $\mathfrak{g}$. Recall that a canonical basis $(\beta_1, ..., \beta_l, \tilde{\beta}_1, ...., \tilde{\beta}_l)$ of $\Lambda_R \oplus \tau\Lambda_R$ with respect to $E$ satisfies by definition

$$E(\beta_i, \beta_j) = E(\tilde{\beta}_i, \tilde{\beta}_j) = 0, \qquad E(\beta_i, \tilde{\beta}_j) = -E(\tilde{\beta}_i, \beta_j) = \delta_i \delta_{ij},$$

where $\delta_1 | \cdots | \delta_l$ are integers, with $\delta_1 \cdots \delta_l = \det C$.

To find such a canonical basis, let us write the (ordered) basis of simple roots in $\Lambda_R$ as a row vector $\alpha = (\alpha_1, ..., \alpha_l)$. In particular, any other basis $\beta = (\beta_1, ..., \beta_l)$ can be written as $\beta = \alpha A$ for some integer matrix $A$ with unit determinant. In this notation, it is easy to see that the pair $(\beta, \tilde{\beta})$ forms a canonical basis of $\Lambda_R \oplus \tau\Lambda_R$ if and only if there are matrices $A, \tilde{A} \in SL(l, \mathbb{Z})$ such that

$$A^t C \tilde{A} = \Delta, \tag{*}$$

where $\Delta = diag(\delta_1, ..., \delta_n)$, and $A^t$ denotes the transpose of $A$. In fact, the bases are related by $\beta = \alpha A$ and $\tilde{\beta} = \tau\alpha\tilde{A}$.

The solutions of (*) can be naturally identified with bases of $\Lambda_W$, the weight lattice. More precisely, let us say that a basis $\beta$ of $\Lambda_R$ is *completable* if there exists a basis $(\beta, \tilde{\beta})$ of $\Lambda_R \oplus \tau\Lambda_R$, which is canonical with respect to $E$, or equivalently, if there exists a solution of (*) with $\beta = \alpha A$. Then, we have

Lemma A.1. *$\beta$ is completable if and only if $\beta\Delta^{-1}$ is a basis of $\Lambda_W$.*

*Proof.* Let $\beta = \alpha A$ and let $A^t C \tilde{A} = \Delta$ for some matrices $A, \tilde{A} \in SL(l, \mathbb{Z})$. Since the relation between roots and weights is given by $\alpha = \lambda C$, we have

$$\beta\Delta^{-1} = \alpha A \Delta^{-1} = \lambda C A \Delta^{-1},$$

which means that $\beta\Delta^{-1}$ is a basis of $\Lambda_W$, because $CA\Delta^{-1} = (\tilde{A}^t)^{-1}$ is a unimodular matrix. ∎

Now let us consider the case of $sl(n,\mathbb{C})$, where the simple roots $\{\alpha_1, ..., \alpha_l\}$ form a basis of $\Lambda_R$, $n = l+1$. From the above lemma, we see that $\alpha$ is not a completable basis of $\Lambda_R$, except when $l = 1$ (in which case $(\alpha, \tau\alpha)$ is clearly a canonical basis of $\Lambda_R \oplus \tau\Lambda_R$). We assume henceforth that $l \geq 2$, and let us consider a slightly different basis

$$\beta = (\beta_1, ..., \beta_l) := (\alpha_1, ..., \alpha_{l-1}, n\lambda_1),$$

where $\lambda_1$ is the fundamental weight dual to $\alpha_1$. Since $n\lambda_1 = l\alpha_1 + (l-1)\,\alpha_2 + ... + \alpha_l$, then $\beta = \alpha A$ for a unimodular matrix $A$, which means that $\beta$ is another basis of $\Lambda_R$. To prove that this $\beta$ is indeed completable, consider $\delta_1 = ... = \delta_{l-1} = 1$ and $\delta_l = n$, and write $\beta\Delta^{-1} = (\alpha_1, ..., \alpha_{l-1}, \lambda_1)$; this implies that the matrix relating $\lambda$ to $\beta\Delta^{-1}$ is an integer matrix; since this matrix is $CA\Delta^{-1}$ (from the lemma), it has determinant one, which proves that $\beta\Delta^{-1}$ is a basis of $\Lambda_W$. (Note that $\det C = \det \Delta = n$).

Simple calculations now give the following explicit expressions. Recall that the period matrix $\Omega$ is defined by $\tilde{\beta} = \beta\Omega\Delta$.

Proposition A.1. *For $l \geq 2$, a canonical basis of $\Lambda_R \oplus \tau\Lambda_R$ with respect to $E$ is $(\beta_1, ..., \beta_l, \tilde{\beta}_1, ...., \tilde{\beta}_l)$, where*

$$\begin{cases} \beta_i = \alpha_i, & i = 1, ..., l-1 \\ \beta_l = n\lambda_1 = l\alpha_1 + (l-1)\,\alpha_2 + ... + \alpha_l & \\ \tilde{\beta}_i = \tau\lambda_i - (n-i)\,\tau\lambda_l, & i = 1, ..., l-1 \\ \tilde{\beta}_l = n\tau\lambda_l & \end{cases}$$

*The period matrix is then given by*

$$\begin{cases} \Omega_{i,i+k} = \Omega_{i+k,i} = \tau(n-i)(l-i-k), & i+k \leq l-1 \\ \Omega_{i,l} = \tau(i-l), & i \leq l-1 \\ \Omega_{l,l} = \tau\frac{l}{n}. & \end{cases}$$

*Proof.* We have verified the formula for $\beta$, and the one for $\tilde{\beta} = (\tilde{\beta}_1, ...., \tilde{\beta}_l)$ is a direct calculation from $\tilde{\beta} = \tau\alpha\tilde{A} = \tau\lambda\,(A^t)^{-1}\,\Delta$ (using (*)). Since we also have $\tilde{\beta} = \tau\alpha\tilde{A} = \tau\beta A^{-1}\tilde{A}$, we get

$$\Omega = \tau A^{-1}\tilde{A}\Delta^{-1} = \tau A^{-1}C^{-1}(A^t)^{-1},$$

which readily gives the above expression for the period matrix (note that all its entries have to lie in $\frac{\tau}{n}\mathbb{Z}$). ∎

Let us now examine the other possible canonical bases, i.e, all the solutions to (*). If the pairs $(A_1, \tilde{A}_1)$ and $(A_2, \tilde{A}_2)$ are both solutions of (*), with $A_2 = A_1 B$, and $\tilde{A}_2 = \tilde{A}_1 \tilde{B}$, for some unimodular matrices $B$ and $\tilde{B}$, then necessarily

$$B^t \Delta \tilde{B} = \Delta. \tag{**}$$

It is not difficult to see that the set of matrices $B$ such that there exists $\tilde{B}$ satisfying (**) form a group, and that this is the subgroup $\Gamma_n \subset SL(l, \mathbb{Z})$ of modular matrices of the form

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a$ is an $(l-1) \times (l-1)$ matrix, $d$ is an integer, $b$ and $c^t$ are $(l-1)$-vectors and $b \in n\mathbb{Z}^{l-1}$ (all entries in $b$ are multiples of $n$). Finally, if $\Omega_i$ are period matrices given by $\Omega_i = \tau A_i^{-1} C^{-1} \left(A_i^t\right)^{-1}$, $i = 1, 2$, as in the proposition above, and $A_2 = A_1 B^{-1}$, for some $B \in \Gamma_n$, then we have

$$\Omega_2 = B \Omega_1 B^t.$$

This means that two period matrices which are related in this way for some $B \in \Gamma_n$ should be regarded as equivalent.

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