Coherent State Transforms and Abelian Varieties

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We extend the coherent state transform (CST) of Hall to the context of abelian varieties by considering them as quotients of the complexification of the abelian group $K = U(1)^g$. We show that this transform, applied to appropriate distributions on K, gives all classical theta functions, and that, by defining on this space of theta functions an inner product related to the K-averaged heat kernel, the unitarity of the CST transform is still preserved.

1. INTRODUCTION

The Segal-Bargmann or coherent state transform (CST) was introduced in the context of the quantum theory [20, 21, 3] as a transform from the Hilbert space of square integrable functions on the configuration space to the space of holomorphic functions on the phase space. In finite dimensions this transform maps $L^2(\mathbb{R}^n, d^n x)$ to the space $\mathcal{H}(\mathbb{C}^n)$ of holomorphic functions on $\mathbb{C}^n$. With an appropriate choice of measure $d\mu$ on $\mathbb{C}^n$ the transform is a unitary transform (on)to $L^2(\mathbb{C}^n, d\mu) \cap \mathcal{H}(\mathbb{C}^n)$.

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In [13] Hall proposed a generalization of the CST in which $\mathbb{R}^n$ is replaced by an arbitrary compact connected Lie group K and $\mathbb{C}^n$ by the complexification $K_{\mathbb{C}}$ of K (see [16]). When K is abelian this CST is an easier extension of the one defined by Segal and Bargmann; however, the normalization proposed by Hall is particularly convenient for our purposes. This Segal-Bargmann-Hall CST was further generalized to gauge theories with applications to gravity in the context of Ashtekar variables in [1] and to Yang-Mills theories in two space-time dimensions in [8]. For reviews and recent developments see [14] and [15, 24].

In the present paper we extend the CST in a different direction. By considering a general complexified invariant metric $-i\Omega$ on $K = U(1)^g$ and applying the appropriately extended CST to certain spaces of distributions, we obtain spaces of (level k) theta functions on a rank g abelian variety M with period matrix Ω. These theta functions are not square integrable on $K_{\mathbb{C}} = (\mathbb{C}^*)^g$ with respect to the averaged heat kernel measure introduced in [13]. However, since level k theta functions correspond to holomorphic sections of the k-th power L^k of the determinant bundle $L \to M$, their natural inner product is given instead by an integral over M with respect to the Liouville measure and an appropriate hermitian structure on L^k. We show that the averaged heat kernel does satisfy the necessary quasi-periodicity conditions to define the pull-back of an hermitian structure on L^k. With this inner product the unitarity of the CST is recovered by choosing a natural Ω-independent inner product on the initial spaces of distributions.

Analogous techniques can be applied to the study of non-abelian theta functions [2, 4, 6, 7, 23] on the moduli space of semi-stable holomorphic vector bundles [18] on a compact Riemann surface. The CST on Lie groups appears in this context through the use of the Schottky uniformization of the moduli space [5, 9] and it provides a purely finite dimensional framework for the explicit study of non-abelian theta functions. The application of these techniques to genus one non-abelian theta functions will appear in [10]. Applications to higher genus will appear in [11].

The paper is organized as follows. In section 2, for convenience of the reader and also to establish notation, we start by briefly recalling from [13] the coherent state transform for Lie groups. In section 3, we recall from [12] some aspects of theta functions on abelian varieties. Finally, in section 4 we define the CST for abelian varieties, study its relation to classical theta functions, and prove its unitarity.

2. COHERENT STATE TRANSFORM FOR LIE GROUPS

Let K be a compact connected Lie group, $K_{\mathbb{C}}$ its complexification [16] and let ρ_t be the heat kernel for a Laplacian Δ on K associated with a given bi-invariant metric. By definition, ρ_t for $t > 0$ is the fundamental solution at the identity of the heat equation on K

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u. \tag{1}$$

As proved in [13], ρ_t has a unique analytic continuation to $K_{\mathbb{C}}$, also denoted by ρ_t. The K-averaged coherent state transform is defined as the map

$$\begin{aligned} C_t \ &: \ L^2(K, dx) \mapsto \mathcal{H}(K_{\mathbb{C}}) \\ (C_t f)(z) \ &= \ \int_K \rho_t(zx^{-1}) f(x) dx, \qquad f \in L^2(K, dx), \ z \in K_{\mathbb{C}} \end{aligned}$$

where dx is the normalized Haar measure on K and $\mathcal{H}(K_{\mathbb{C}})$ is the space of holomorphic functions on $K_{\mathbb{C}}$. For each $f \in L^2(K, dx)$, the restriction of $C_t f$ to K is just $e^{t\frac{\Delta}{2}} f$, the solution of the heat equation (1), with initial condition given by $u(0, x) = f(x)$. Therefore, $C_t f$ can be also defined by

$$(C_t f)(z) = (\mathcal{C} \circ \rho_t \star f)(z) = \left(\mathcal{C} \circ e^{t\frac{\Delta}{2}} f\right)(z), \tag{2}$$

where $\star$ denotes the convolution in K and $\mathcal{C}$ denotes analytic continuation from K to $K_{\mathbb{C}}$. In [13], Hall defines the K-averaged (or K-invariant) heat kernel measure $d\nu_t$ on $K_{\mathbb{C}}$, and proves the following result

Theorem 2.1 (Hall). *For each $t > 0$, the K-averaged coherent state transform C_t, is an unitary isomorphism from $L^2(K, dx)$ onto the Hilbert space $L^2(K_{\mathbb{C}}, d\nu_t) \bigcap \mathcal{H}(K_{\mathbb{C}})$.*

A useful expression for $C_t f$ can be obtained from the expansion of $f \in L^2(K, dx)$ given by Peter-Weyl's theorem,

$$f(x) = \sum_{\pi} \mathrm{Tr}(\pi(x) A_{\pi,f}), \tag{3}$$

where the sum is taken over the set of (equivalence classes of) irreducible representations of K, and $A_{\pi,f} \in End\ V_\pi$ is given by

$$A_{\pi,f} = (\dim V_\pi) \int_K f(x)\pi(x^{-1}) dx, \tag{4}$$

V_π being the representation space for π. Then we obtain

$$(C_t f)(z) = \sum_{\pi} e^{-t\frac{\lambda_\pi}{2}} \mathrm{Tr}(\pi(z) A_{\pi,f}), \tag{5}$$

where λ_π is the eigenvalue of $-\Delta$ on functions of the type

$$\mathrm{Tr}(A\pi(x)), \quad A \in End(V_\pi).$$

As we will show, it turns out that the image of appropriately chosen distributions on K with respect to a natural extension of C_t, gives functions satisfying quasi-periodicity conditions in the imaginary directions of $K_{\mathbb{C}}$. These are theta functions corresponding to holomorphic sections of the pull-back of line bundles over abelian varieties.

3. ABELIAN VARIETIES AND THETA FUNCTIONS

Let V be a g-dimensional complex vector space and $\Lambda \cong \mathbb{Z}^{2g}$ a maximal lattice in V such that the quotient

$$M = V/\Lambda \tag{6}$$

with canonical projection $p : V \to M$, is an abelian variety, i.e. a complex torus which can be holomorphically embedded in projective space. We assume that M is endowed with a principal polarization. The case with a general polarization is treated analogously [12]. We will address the CST transform for the general case of abelian varieties in [10], in the context of genus 1 non-abelian theta functions. We can always find a basis $\lambda_1, \cdots, \lambda_{2g}$ for Λ, such that $\lambda_1, \cdots, \lambda_g$ is a basis of V and

$$\lambda_{g+\alpha} = \sum_{\beta=1}^{g} \Omega_{\beta\alpha}\lambda_\beta, \quad \alpha = 1, \cdots, g \quad , \tag{7}$$

where $\Omega = (\Omega_{\alpha\beta})_{\alpha,\beta=1}^{g}$ is a $g \times g$ matrix in the Siegel upper half space $\mathbb{H}_g$ of symmetric matrices with positive definite imaginary part. Conversely, principally polarized abelian varieties are parametrized by matrices in $\mathbb{H}_g$, so we will consider once and for all an arbitrary but fixed matrix $\Omega \in \mathbb{H}_g$ and refer to $M = V/\Lambda$ as the abelian variety determined by the (period) matrix Ω.

Let $L \to M$ be the holomorphic line bundle defined by the automorphy factors

$$\begin{aligned} e_{\lambda_\alpha}(z) &= 1 \\ e_{\lambda_{g+\alpha}}(z) &= e^{-2\pi i z_\alpha - \pi i \Omega_{\alpha\alpha}} \qquad , \quad z \in V \ , \alpha = 1, \cdots, g \ , \end{aligned} \tag{8}$$

where $z_1, \cdots, z_g$ are complex coordinates on V dual to the complex basis $\lambda_1, \cdots, \lambda_g$, and let k be a nonnegative integer. Level k (abelian) theta functions $\tilde{\theta}$ are holomorphic sections of L^k, $\tilde{\theta} \in H^0(M, L^k)$. Using the automorphy factors (8), we see that the space $H^0(M, L^k)$ is naturally isomorphic with the space of holomorphic functions θ on V such that

$$\theta(z + \lambda_\alpha) = \theta(z) \tag{9}$$

$$\theta(z + \lambda_{g+\alpha}) = e^{-2\pi i k z_\alpha - \pi i k \Omega_{\alpha\alpha}} \theta(z) \quad . \tag{10}$$

Let us denote the latter space by $\mathcal{H}_{\Omega,k}$,

$$\mathcal{H}_{\Omega,k} \subset \mathcal{H}((\mathbb{C}^*)^g) \quad .$$

From (9) it follows that there exists a uniformly convergent series such that

$$\theta(z) = \sum_{n \in \mathbb{Z}^g} a_n e^{2\pi i n \cdot z} \quad , \tag{11}$$

where $n \cdot z \equiv n_1 z_1 + \cdots + n_g z_g$. By substituting this function in (10), we conclude that only the first coefficients, a_n, $n = (n_1, \cdots, n_g)$ with $0 \leq n_\alpha < k$, can be chosen freely while the others are fixed by the quasi-periodicity conditions (10). In fact, the general level k theta function, $\theta \in \mathcal{H}_{\Omega,k}$, is of the form

$$\theta(z) = \sum_{l \in (\mathbb{Z}/k\mathbb{Z})^g} a_l \theta_l(z, \Omega) \quad , \tag{12}$$

where the functions

$$\theta_l(z, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i (l+kn) \cdot \frac{\Omega}{k} (l+kn)} e^{2\pi i (l+kn) \cdot z}, \qquad l \in (\mathbb{Z}/k\mathbb{Z})^g \tag{13}$$

form a basis of $\mathcal{H}_{\Omega,k}$, and are commonly called theta functions with characteristics $\begin{bmatrix} l/k \\ 0 \end{bmatrix}$ and usually denoted by $\vartheta \begin{bmatrix} l/k \\ 0 \end{bmatrix} (kz, k\Omega)$ (see [17]). In particular we see that

$$\dim H^0(M, L^k) = \dim \mathcal{H}_{\Omega,k} = k^g \quad .$$

4. COHERENT STATE TRANSFORM FOR ABELIAN VARIETIES

As is well known the theta functions (12,13) are, for $\Omega_t = t\,\Omega$, $t > 0$, analytic continuations of solutions of a heat equation on $U(1)^g$. This is a defining property of the CST in [13] (see section 2 above), the main difference coming from the fact that, in the present case the "initial" (i.e. for $t=0$) values of the theta functions (12) are distributional rather than L^2, indicating that the CST could be extended to distribuitions on $U(1)^g$. Here, we define this extension of the CST for a given abelian variety, study its main properties related to theta functions, and prove that it is unitary when restricted to the subspace of linear combinations of Dirac delta distributions supported on points of order dividing the level k.

4.1. CST for Complexified Laplacians on $U(1)^g$

From now on, we consider the abelian Lie group $K=U(1)^g=(S^1)^g=\mathbb{R}^g/\mathbb{Z}^g$, and its complexification $K_{\mathbb{C}}=(\mathbb{C}^*)^g$. Let $\Omega\in\mathbb{H}_g$ and consider on $K=U(1)^g$ the invariant Laplacian given by

$$\Delta^{(Y)}=\sum_{\alpha,\beta=1}^{g}\frac{Y_{\alpha\beta}}{2\pi}\frac{\partial^2}{\partial x_\alpha\partial x_\beta}\quad , \tag{14}$$

where $Y=\mathrm{Im}(\Omega)$ and $x\in[0,1]^g$ are periodic coordinates. Let $w=(e^{2\pi i z_1},\cdots,e^{2\pi i z_g})$ be the coordinates in $K_{\mathbb{C}}=(\mathbb{C}^*)^g$, with $z=x+iy$, $y\in\mathbb{R}^g$, and denote by $dw\equiv dxdy$ the Haar measure on $K_{\mathbb{C}}$. The K-averaged heat kernel measure $d\nu_t$ is defined in the following way. First, we consider the fundamental solution μ_t at the identity, of the heat equation on $K_{\mathbb{C}}$

$$\frac{\partial u}{\partial t}=\frac{1}{4}\Delta_{\mathbb{C}}^{(Y)}u, \tag{15}$$

where $\Delta_{\mathbb{C}}^{(Y)}$ is the K-invariant laplacian given by

$$\Delta_{\mathbb{C}}^{(Y)}=\sum_{\alpha,\beta=1}^{n}\frac{Y_{\alpha\beta}}{2\pi}\left(\frac{\partial^2}{\partial x_\alpha\partial x_\beta}+\frac{\partial^2}{\partial y_\alpha\partial y_\beta}\right)\quad , \tag{16}$$

Then, the K-invariant measure $d\nu_t=\nu_t dw$ is obtained by averaging $\mu_t dw$ with respect to the action of $(S^1)^g$,

$$\nu_t(z)=\int_K\mu_t(z+x)dx, \tag{17}$$

which in this case yelds the explicit formula

$$\nu_t(z)=\left(\frac{2}{t}\right)^{g/2}(\det W)^{1/2}e^{\frac{\pi}{2t}\sum_{\alpha,\beta}(z_\alpha-\bar z_\alpha)W_{\alpha\beta}(z_\beta-\bar z_\beta)} \tag{18}$$

with $W=(W_{\alpha\beta})=Y^{-1}$.

We are now in the situation described in Theorem 2.1, and we conclude that the transform

$$\begin{aligned} C_t^{(Y)} \quad &: \quad L^2((S^1)^g, dx) \;\to \mathcal{H}((\mathbb{C}^*)^g) \cap L^2((\mathbb{C}^*)^g, d\nu_t) \\ \left(C_t^{(Y)} f\right)(w) \;&=\; \left(\mathcal{C} \circ e^{\frac{t}{2}\Delta^{(Y)}} f\right)(w) \quad , \; t > 0, \end{aligned} \tag{19}$$

is unitary.

To obtain the theta functions (12,13) as solutions of a heat equation, we need to consider the non-self-adjoint Laplace operator on $(S^1)^g$

$$\Delta^{(-i\Omega)} = -\sum_{\alpha,\beta=1}^{g} \frac{i}{2\pi} \Omega_{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \quad . \tag{20}$$

It is easy to see that the imaginary part of this Laplacian does not affect the unitarity properties of Hall's CST. Therefore we have the following

PROPOSITION 4.1. *For any $\Omega \in \mathbb{H}_g$ and $t > 0$ the transform*

$$C_t^{(-i\Omega)} = \mathcal{C} \circ e^{\frac{t}{2}\Delta^{(-i\Omega)}} \;:\; L^2((S^1)^g, dx) \to \mathcal{H}((\mathbb{C}^*)^g) \cap L^2((\mathbb{C}^*)^g, d\nu_t),$$

is unitary.

Proof. We can decompose this transform as

$$C_t^{(-i\Omega)} = \mathcal{C} \circ e^{\frac{t}{2}\Delta^{(Y)}} \circ e^{\frac{t}{2}\Delta^{(-iX)}} = C_t^{(Y)} \circ e^{\frac{t}{2}\Delta^{(-iX)}} \quad ,$$

where $\Omega = X + iY$. The unitarity of $C_t^{(Y)}$ follows from Theorem 2.1, and that of the operator

$$e^{\frac{t}{2}\Delta^{(-iX)}} = e^{-i\frac{t}{2}\Delta^{(X)}} \;:\; L^2((S^1)^g, dx) \to L^2((S^1)^g, dx)$$

follows from the fact that $\Delta^{(X)}$ is a self-adjoint operator. ∎

Consider $f \in L^2((S^1)^g, dx)$. The Peter-Weyl decomposition (3) corresponds in this case to the usual Fourier decomposition, as the irreducible representations of $U(1)^g$ are given by $\pi_n(e^{2\pi i x_1}, \cdots, e^{2\pi i x_g}) = e^{2\pi i x\cdot n}$ with $n \in \mathbb{Z}^g$. We have then

$$f(x) = \sum_{n\in\mathbb{Z}^g} a_n e^{2\pi i x\cdot n} \quad , \tag{21}$$

and its image under $C_t^{(-i\Omega)}$ is given by

$$\left(C_t^{(-i\Omega)} f\right)(z) = \sum_{n\in\mathbb{Z}^g} a_n e^{t i\pi n\cdot\Omega n} e^{2\pi i z\cdot n} \quad . \tag{22}$$

This function is the analytic continuation to $(\mathbb{C}^*)^g$ of the solution of the complex heat equation on $(S^1)^g$

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta^{(-i\Omega)}u \tag{23}$$

with initial condition given by f in (21).

4.2. Extension to distributions on $U(1)^g$

We now extend the CST transform of Proposition 4.1, from $L^2((S^1)^g, dx)$ to the space of distributions $(C^\infty((S^1)^g))'$. This is the space of Fourier series of the form:

$$f(x) = \sum_{n\in\mathbb{Z}^g} a_n e^{2\pi i x\cdot n} \quad , \tag{24}$$

for which there exists an integer $N > 0$, such that [19]

$$\lim_{||n||\to\infty} \frac{|a_n|}{(1+||n||^2)^N} = 0, \tag{25}$$

where $||n|| = \sqrt{n\cdot n}$. The Laplace operator and its powers act as continuous linear operators on this space of distributions by duality from the corresponding action on $C^\infty((S^1)^g)$, and define (for $\Omega \in \mathbb{H}_g$ and $t > 0$) the action of the operator $e^{\frac{t}{2}\Delta^{(-i\Omega)}}$ on a distribution $f \in (C^\infty((S^1)^g))'$. If f has the form (24), we have

$$\left(e^{\frac{t}{2}\Delta^{(-i\Omega)}} f\right)(x) = \sum_{n\in\mathbb{Z}^g} a_n e^{ti\pi n\cdot\Omega n} e^{2\pi i x\cdot n} \quad . \tag{26}$$

Lemma 4.1. *If $f(x) = \sum_{n\in\mathbb{Z}^g} a_n e^{2\pi i x\cdot n} \in (C^\infty((S^1)^g))'$ then the series*

$$\sum_{n\in\mathbb{Z}^g} a_n e^{ti\pi n\cdot\Omega n} e^{2\pi i z\cdot n}, \quad t > 0, \tag{27}$$

defines a holomorphic function on $(\mathbb{C}^*)^g$.

Proof. The growing condition (25) implies that there exists an $A > 0$ such that for all $n \in \mathbb{Z}^g$ we have

$$|a_n| \leq A e^{||n||}.$$

Since the imaginary part of Ω is positive definite, there exists a $c > 0$ such that

$$|a_n e^{ti\pi n\cdot\Omega n}| \leq e^{-c||n||^2},$$

for n sufficiently large. Therefore the series (27) is uniformly convergent on compact subsets of $(\mathbb{C}^*)^g$ and its sum is holomorphic. ∎

Recall that $M = V/\Lambda$ is the abelian variety defined by the period matrix $\Omega \in \mathbb{H}_g$. We introduce now the definition

Definition 4.1. The coherent state transform for the abelian variety M and for $t > 0$ is the map:

$$\begin{aligned} C_t^M \ : \ (C^\infty((S^1)^g))' \ &\rightarrow \ \mathcal{H}((\mathbb{C}^*)^g)) \\ f \ &\mapsto \ \left(\mathcal{C} \circ e^{\frac{t}{2}\Delta^{(-i\Omega)}}\right) f \quad . \end{aligned} \tag{28}$$

This definition has the following justification. Clearly when we extend the CST of Proposition 4.1 to distributions we lose unitarity in the sense of this proposition. The holomorphic images of distributions in Definition 4.1 are in general not square integrable with respect to the averaged heat kernel measure on $(\mathbb{C}^*)^g$. However, as we will see below, by considering discrete values of $t, t = 1/k, k \in \mathbb{N}$ and by taking the image under $C_{1/k}^M$ of certain finite dimensional spaces $\mathcal{F}_k$ of Dirac delta distributions supported on points of order dividing k in $(S^1)^g$ we obtain the space $\mathcal{H}_{\Omega,k}$ of level k theta functions (12,13). As we mentioned these are the pullback of holomorphic sections of $L^k \rightarrow M$. The amazing fact is (Lemma 4.2 below) that the averaged heat kernel $\nu_{\frac{1}{k}}$ introduced by Hall in $(\mathbb{C}^*)^g$ descends to the abelian variety M and defines an hermitian structure there. The Hall expression for the inner product in $\mathcal{H}((\mathbb{C}^*)^g) \cap L^2((\mathbb{C}^*)^g, d\nu_t)$ remains then almost unchanged for $\mathcal{H}_{\Omega,k}$ with the only difference that one integrates in $(\mathbb{C}^*)^g$ only over one fundamental region of the projection $(\mathbb{C}^*)^g \rightarrow M = V/\Lambda$ (see eqs. (36,39)).

Consider first, for $l \in (\mathbb{Z}/k\mathbb{Z})^g$, the following distributions on $(S^1)^g$

$$\theta_l^0(x) := \theta_l(x, 0) = \sum_{n \in \mathbb{Z}^g} e^{2\pi i (l + kn)\cdot x}. \tag{29}$$

From the discussion above, it is now easy to verify some of the claimed properties of the transform C_t^M.

Proposition 4.2. *The restriction to $L^2((S^1)^g, dx)$ of the CST transform for the abelian variety M coincides with the CST transform of Proposition 4.1. If we write $f(x) = \sum_{n\in\mathbb{Z}^g} a_n e^{2\pi i x\cdot n} \in (C^\infty((S^1)^g))'$, then $C_t^M f$ is given explicitly by:*

$$\left(C_t^M f\right)(z) = \sum_{n \in \mathbb{Z}^g} a_n e^{t i \pi n\cdot\Omega n} e^{2\pi i z\cdot n} \quad . \tag{30}$$

In particular, we obtain all level k theta functions (12,13), since

$$\left(C^M_{1/k}\theta^0_l\right)(z) = \theta_l(z,\Omega). \tag{31}$$

Proof. The first and second statements follow immediately from the coincidence of the formulas (22) and (26), for $z = x$. The third follows from (13,29). ∎

Let us denote by M_t the abelian variety defined by the period matrix $\Omega_t = t\Omega$. Then from the Definition 4.1 we see that the CST for the abelian variety M_t at time $s > 0$ coincides with the CST for the abelian variety M_1 at time ts

$$C^{M_1}_{ts} \equiv C^{M_t}_s, \quad \forall s,t > 0. \tag{32}$$

With this notation, we can obtain all level k theta functions for the period matrix $t\,\Omega$ using the CST for M_1 but at time $s = \frac{t}{k}$,

$$\left(C^{M_t}_{1/k}\theta^0_l\right)(z) = \left(C^{M_1}_{t/k}\theta^0_l\right)(z) = \theta_l(z,t\Omega). \tag{33}$$

4.3. Distributions supported on points of finite order

The subgroup of $K = U(1)^g$ consisting of points whose order divides $k \in \mathbb{N}$ (the elements $g \in K$ such that g^k is the identity) is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^g$ and its elements are given, in the coordinates $x \in \mathbb{R}^g/\mathbb{Z}^g$, by the points $\{l/k,\ l \in (\mathbb{Z}/k\mathbb{Z})^g\}$. Let $\mathcal{F}_k$ be the k^g dimensional complex linear subspace of $(C^\infty((S^1)^g))'$ spanned by the Dirac delta distributions

$$\delta_l(x) := \delta(x - \frac{l}{k}), \quad l \in (\mathbb{Z}/k\mathbb{Z})^g \quad , \tag{34}$$

which are supported on the points of order dividing k. Distributions of this type appear naturally in the context of geometric quantization of M with L^k as the pre-quantum line bundle, in a real polarization, where the points of order dividing k correspond to the so-called Bohr-Sommerfeld conditions [22, 25, 26]. This aspect will be further explored in [11]. We will now see that the image of $\mathcal{F}_k$ under the CST for M_t, at time $s = \frac{1}{k}$ consists of theta functions of level k. More precisely, we have

PROPOSITION 4.3. *For any $t > 0$ and level k, $C^{M_t}_{1/k}$ restricts to an isomorphism of vector spaces between $\mathcal{F}_k$ and $\mathcal{H}_{t\Omega,k}$.*

Proof. This is a consequence of the well known relation between the distributions θ_l^0 in (29) and the Dirac delta distributions δ_l, supported at the points l/k,

$$
\begin{aligned}
\delta_l(x) = \delta(x - \frac{l}{k}) &= \sum_{n \in \mathbb{Z}^g} e^{2\pi i n \cdot (x - \frac{l}{k})} \\
&= \sum_{0 \le l'_\alpha < k} e^{-2\pi i \frac{l' \cdot l}{k}} \sum_{n \in \mathbb{Z}^g} e^{2\pi i (l' + kn) \cdot x} \\
&= \sum_{0 \le l'_\alpha < k} e^{-2\pi i \frac{l' \cdot l}{k}} \, \theta_{l'}^0(x) \quad .
\end{aligned}
$$

This implies that $C_{1/k}^{M_t}(\delta_l)$ is a linear combination of the $\theta_{l'}(z, t\Omega) = \left(C_{1/k}^{M_t} \theta_{l'}^0 \right)(z)$, by (33), and so it belongs to $\mathcal{H}_{t\Omega,k}$. The lemma then follows by checking that the $k^g \times k^g$ matrix A which appears above, and whose l, l' entry is

$$
A_{l',l} = \left(e^{-2\pi i \frac{l' \cdot l}{k}} \right) \quad , \tag{35}
$$

is invertible. ▌

4.4. Unitarity of the CST transform

Let us consider now the question of defining an appropriate inner product on the space $\mathcal{H}_{\Omega,k}$ of level k theta functions (12,13). These functions are not square integrable in the non compact space $K_\mathbb{C} = (\mathbb{C}^*)^g$ with respect to the heat kernel measure $d\nu_t$. However, by considering the sequence

$$
\begin{array}{ccccc}
V & \overset{p'}{\to} & (\mathbb{C}^*)^g & \overset{p''}{\to} & M \\
z = \sum_{\alpha=1}^g z_\alpha \lambda_\alpha & \mapsto & (e^{2\pi i z_1}, \cdots, e^{2\pi i z_g}) & \mapsto & z \,(\mathrm{mod}\Lambda),
\end{array} \tag{36}
$$

such that $p'' \circ p'$ is the canonical projection $p : V \to M$, we will find and work instead with an appropriate measure on the compact manifold M.

Let $\eta_1, \cdots, \eta_g, \xi_1, \cdots, \xi_g$ be the coordinates on V which are dual to the generators of the lattice Λ, $\lambda_1, \cdots, \lambda_{2g}$. The ξ_α, η_α can also be considered as periodic coordinates in M, and are related to the complex ones by

$$
z_\alpha = \eta_\alpha + \sum_{\beta=1}^g \Omega_{\alpha\beta} \xi_\beta, \quad \alpha = 1, \cdots, g \quad . \tag{37}
$$

Then the symplectic form ω on M (which coincides with the first Chern class of L) can be written as

$$\omega = \sum_{\alpha=1}^{g} d\eta_\alpha \wedge d\xi_\alpha \quad , \tag{38}$$

and the Liouville measure on M is $d\eta d\xi = \frac{\omega^g}{g!}$ (see [12]).

Let $\mathcal{L}$ be the pull-back of L to $(\mathbb{C}^*)^g$ with respect to p''. Since by holomorphic trivialization of $\mathcal{L}$ we have

$$\mathcal{H}_{\Omega,k} = (p'')^*(H^0(M, L^k)) \subset \mathcal{H}((\mathbb{C}^*)^g)$$

relating the space of level k theta functions on $(\mathbb{C}^*)^g$ and the holomorphic sections of L, it is natural to define an inner product on $\mathcal{H}_{\Omega,k}$ in terms of integration on the abelian variety M, after defining an appropriate hermitean structure on L. This can be accomplished by using the K-averaged heat kernel (18) defined above. In view of Proposition 4.3, we will consider $d\nu_t$ with $t = \frac{1}{k}$.

LEMMA 4.2. *The function on $(\mathbb{C}^*)^g$ given by $h(z) = \nu_{1/k}(z)$ defines an hermitean structure on $\mathcal{L}^k$ which is the pull-back of an hermitean structure on L^k.*

Proof. Using the formula (18) for $\nu_{1/k}(z)$, we have that

$$\begin{aligned} h(z + \lambda_\alpha) &= h(z) \\ h(z + \lambda_{g+\alpha}) &= |e^{2\pi i k z_\alpha}|^2 |e^{\pi i k \Omega_{\alpha\alpha}}|^2 h(z), \end{aligned}$$

for $\alpha = 1, ..., g$. Therefore, from the quasi-periodicity relations (9,10) we conclude that h is the pull-back of a section of $(L^k \otimes \bar{L}^k)^*$ and so, it defines an hermitean structure on $\mathcal{L}^k$. ▮

Using this hermitean structure we define the inner product on $\mathcal{H}_{\Omega,k}$ by

$$< \theta, \theta' >:= \int_{[0,1]^g \times [0,1]^g} \bar{\theta}\, \theta' \nu_{1/k}\, d\eta d\xi = \int_M \bar{\tilde{\theta}}\, \tilde{\theta}'\, \tilde{h}\, \frac{\omega^g}{g!} \tag{39}$$

where $\theta = (p'')^* \tilde{\theta}$, $h = \nu_{1/k} = (p'')^* \tilde{h}$. From this definition of the inner product, the normalization of which was fixed by the averaged heat kernel, we obtain

LEMMA 4.3. *The set of level k theta functions $\{\theta_l(z, \Omega)\}$, $l \in (\mathbb{Z}/k\mathbb{Z})^g$, with the inner product (39) forms an orthonormal basis of $\mathcal{H}_{\Omega,k}$.*

Proof. Applying the formulas for $\theta_l(z,\Omega)$ (13) and $\nu_{1/k}(z)$ (18), we have

$$
\begin{aligned}
< \theta_l(z,\Omega), \theta_{l'}(z,\Omega) > &= \int_{[0,1]^g\times[0,1]^g} \left(\sum_{n\in\mathbb{Z}^g} e^{-i\pi(l+kn)\cdot\frac{\bar\Omega}{k}(l+kn)} e^{-2\pi i (l+kn)\cdot\bar z} \right) \cdot \\
&\cdot \left(\sum_{n'\in\mathbb{Z}^g} e^{i\pi(l'+kn')\cdot\frac{\Omega}{k}(l'+kn')} e^{2\pi i (l'+kn')\cdot z} \right) (2k)^{g/2} (\det Y)^{1/2} e^{-2k\pi\xi\cdot Y\xi} d\eta d\xi = \\
&= \delta_{l,l'} (2k)^{g/2} (\det Y)^{1/2} \sum_{n\in\mathbb{Z}^g} \int_{[0,1]^g} e^{-2k\pi(\xi+\frac{l+kn}{k})\cdot Y(\xi+\frac{l+kn}{k})} d\xi \\
&= \delta_{l,l'} (2k)^{g/2} (\det Y)^{1/2} \int_{\mathbb{R}^g} e^{-2k\pi\xi\cdot Y\xi} d\xi = \\
&= \delta_{l,l'},
\end{aligned}
$$

where $\delta_{l,l'}$ denotes Kronecker's delta. ▮

Finally, consider on the space $\mathcal{F}_k$, the Ω independent inner product such that the distributions $\{\delta_l\}$ (34) form an orthogonal basis, with

$$
< \delta_l, \delta_l >= k^g. \tag{40}
$$

We can now prove unitarity of the CST.

THEOREM 4.1. *For any $\Omega \in \mathbb{H}_g$, level $k \in \mathbb{N}$, and $t > 0$, the CST transform of the abelian variety M_t, at time $\frac{1}{k}$ restricted to $\mathcal{F}_k$ is a isometric isomorphism between $\mathcal{F}_k$ and $\mathcal{H}_{t\Omega,k} \cong H^0(M_t, L^k)$.*

Proof. Observe that the $k^g \times k^g$ matrix A (35) in Lemma 4.3, relating the two basis $\{\theta_l^0\}$ and $\{\delta_l\}$ of the space $\mathcal{F}_k$, is unitary up to scaling, namely is such that $A\bar{A}^t = k^g I$, where I is the identity $k^g\times k^g$ matrix. This implies, by (40) and (34), that the set $\{\theta_l^0(x)\}$, $0 \le l_\alpha < k$, is an orthonormal basis of $\mathcal{F}_k$, and the theorem follows from

$$
< \left(C^{M_t}_{1/k}\theta^0_l\right)(z) \ , \ \left(C^{M_t}_{1/k}\theta^0_{l'}\right)(z) \ > = < \theta_l(z,t\Omega), \theta_{l'}(z,t\Omega) > = \ \delta_{l,l'} \quad ,
$$

by (33) and Lemma 4.3. ▮

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