Coherent State Transforms and Theta Functions

Carlos Florentino[†], José Mourão[†] and João P. Nunes[†]

December 4, 2003

[†]cfloren, jmourao and jpnunes@math.ist.utl.pt
Department of Mathematics, Instituto Superior Técnico,
Av. Rovisco Pais, 1049-001 Lisboa, Portugal

Abstract

We review aspects of the relations between analytic and geometric properties of theta functions, coherent state transforms for Lie groups, and geometric quantization. In this context, we study classic theta functions on abelian varieties and non-abelian theta functions for vector bundles on elliptic curves. Some applications of these ideas to rank 2 non-abelian theta functions for genus $g \geq 2$ are also discussed.

Contents

1 Introduction

It is an honour to contribute to this volume dedicated to the memory of our friend and collaborator Professor Andrei Tyurin. He dedicated some of his last works to the study of theta functions [Ty1, Ty2], a subject notoriously marked by the interdisciplinarity of different areas of mathematics as, it seems to us, Andrei Tyurin very much appreciated.

Although of algebraic geometric origin, theta functions can also be studied from the points of view of analysis, symplectic geometry and geometric quantization and even from quantum field theory. Of central interest is the Knizhnik-Zamolodchikov-Bernard-Hitchin (KZBH) connection [KZ, Ber, Hi, AdPW, Wi], which allows the comparison of theta functions for different complex structures. (For the conformal field theory approach, see also [EMSS, G].) In this contribution, we will give the structure of an analytic theory for theta functions, with a view towards present and future applications to the non-abelian case.

Below, we will review our work on classic theta functions on abelian varieties [FMN1] and on non-abelian theta functions for vector bundles on elliptic curves [FMN2]. During the development of this work, we had the privilege of having many discussions with Andrei Tyurin, especially during his extended stays at our institute in Lisbon, in the Fall of 2000 and Spring 2001. We were deeply touched by his constant enthusiasm, encouragement and friendship.

In section two, we review the classic theory of theta functions and the algebro-geometric setting of non-abelian theta functions. In the following section, we explore the relation between classic theta functions and a heat equation on the compact Lie group $U(1)^g$. Here, we emphasize the role of the distributional initial conditions and their relation to geometric quantization. Section 4 contains an introduction to coherent state transforms for Lie groups as well as some of its generalizations. In section 5, in which we state the main results, the development of an analytic theory for theta functions is proposed in the framework of the coherent state transform. We treat the classic case of theta functions on abelian varieties and the higher rank theta functions for vector bundles over elliptic curves. An important role is played by the Schottky map for $SL(n,\mathbb{C})$.

In the final section, we will present some generalizations of the previous ideas to the much more difficult problem of constructing an analytic theory of non-abelian theta functions for genus $g \geq 2$. Many of these ideas were explored in collaboration with Andrei Tyurin and some of the results stated will be presented in a work in preparation [FMNT].

2 Theta functions

Riemann's classical theta functions are holomorphic functions on the product $\mathbb{C}^g \times \mathbb{H}_g$, where $\mathbb{H}_g$ is the Siegel upper half space consisting of complex symmetric $g \times g$ matrices with positive definite imaginary part. They depend on a non-negative integer k, called the level, and can be defined as the functions $\theta(z, \Omega)$ which satisfy the following quasiperiodicity conditions

$$\begin{aligned} \theta(z+\lambda_j, \Omega) &= \theta(z, \Omega) \\ \theta(z+\tilde{\lambda}_j, \Omega) &= e^{-2\pi i k z_j - \pi i k \Omega_{jj}} \theta(z, \Omega), \end{aligned} \tag{1}$$

where $\{\lambda_j\}_{j=1,\ldots,g}$ denotes the standard basis in $\mathbb{C}^g$, and $\tilde{\lambda}_j = \Omega\lambda_j$. Due to the integer periods, the theta functions can be regarded as holomorphic functions in the variables $w_j = e^{2\pi i z_j} \in \mathbb{C}^*$, and it is easy to see that holomorphic functions on $(\mathbb{C}^*)^g$ satisfying (1) form a vector space $\mathcal{H}_{\Omega,k}$, of dimension k^g, for which an explicit basis is [LB, Ke]

$$\theta_l(z, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i (l+kn)\cdot \frac{\Omega}{k}(l+kn)} e^{2\pi i (l+kn)\cdot z}, \qquad l \in (\mathbb{Z}/k\mathbb{Z})^g . \tag{2}$$

These functions have a geometric interpretation as follows. The lattice $\Lambda_\Omega = \mathbb{Z}^g \oplus \Omega\mathbb{Z}^g$ acts naturally on the trivial line bundle over $V = \mathbb{C}^g$ as

$$\lambda \cdot (z, t) = (z + \lambda, e_\lambda(z) t), \qquad (z, t) \in V \times \mathbb{C}, \tag{3}$$

where $e_{\lambda_j}(z) = 1$ and $e_{\tilde{\lambda}_j}(z) = e^{-2\pi i k z_j - \pi i k \Omega_{jj}}$, and this action defines a holomorphic line bundle

$$L = (V \times \mathbb{C})/\Lambda_\Omega \tag{4}$$

over the principally polarized abelian variety

$$M_\Omega = V/\Lambda_\Omega = \mathbb{C}^g/(\mathbb{Z}^g \oplus \Omega\mathbb{Z}^g).$$

It is then clear that theta functions of level k in (1) are the holomorphic sections of L^k:

$$\mathcal{H}_{\Omega,k} = H^0(M_\Omega, L^k). \tag{5}$$

By the standard correspondence between line bundles and divisors, L can also be defined by the locus of points in M_Ω where a given section vanishes. In the important case when M_Ω is the Jacobian $J(X)$ of a compact Riemann surface X of genus g and period matrix Ω, this locus, called the *theta divisor*, can also be described as follows. In this case, $M_\Omega = J(X)$ is the moduli space of holomorphic line bundles of degree 0 on X, and for a fixed line bundle m of degree $g-1$ on X, the set

$$\Theta_{\mathsf{m}} = \{\mathsf{n} \in J(X) : H^0(X, \mathsf{m} \otimes \mathsf{n}) \neq 0\}$$

is the zero set of a section of $L \to J(X)$ [LB].

One generalization of this theory to the non-abelian setting can be obtained by considering a reductive Lie group $G_{\mathbb{C}}$ and by replacing the Jacobian (which is also the moduli space of $\mathbb{C}^* = GL(1,\mathbb{C})$ principal bundles over X) by a suitable moduli space of principal $G_{\mathbb{C}}$-bundles over X. In the case $G_{\mathbb{C}} = SL(n,\mathbb{C})$, which is our main example, one deals with the moduli space of semistable vector bundles of rank n and trivial determinant over X, $\mathcal{M}_n = \mathcal{M}_n(X)$. This is a projective variety with singularities, except in the cases of genus 1, where $\mathcal{M}_n \cong \mathbb{P}^{n-1}$, and the case $g = n = 2$, for which it is $\mathbb{P}^3$ [NS, NR]. Choosing a line bundle $\mathsf{m} \in J^{g-1}(X)$ as before, the set

$$\Theta_{\mathsf{m}} = \{E \in \mathcal{M}_n : H^0(X, E \otimes \mathsf{m}) \neq 0\} \tag{6}$$

can be proved to be a divisor on $\mathcal{M}_n$ [DN], and defines a line bundle $L \to \mathcal{M}_n$ which is independent of m. Moreover, this bundle is canonical and any other line bundle on $\mathcal{M}_n$ is a power of L. Therefore, one naturally defines a non-abelian theta function of level k as a section of L^k,

$$\theta \in H^0(\mathcal{M}_n, L^k). \tag{7}$$

There are however important differences and technical difficulties in the non-abelian case. First of all, since $\pi_1(\mathcal{M}_n)$ is trivial, we cannot expect to realize L as quotient of some action as in (3); this constitutes a problem when trying to obtain explicit expressions for non-abelian theta functions; on the other hand, if one considers embedding $\mathcal{M}_n$ into a projective space to obtain such expressions, it is clear that they will depend on the particular embedding, which in turn depends on the complex structure of X. Another problem concerns the Kähler quantization of $\mathcal{M}_n(X)$ for which it is still an open question whether one can define naturally a unitary projectively flat connection on the bundle of conformal blocks. For a survey of these questions, see [Bea1, Bea2].

The Verlinde formula [Ve], which computes the dimension of the space of non-abelian theta functions $H^0(\mathcal{M}_n, L^k)$, in terms of g, n and k, is by now well established from the mathematical point of view [So, Fa, BL]. Also, for some particular values of these integers, relations have been found between abelian and non-abelian theta functions.

3 Abelian theta functions and the heat equation

As mentioned above, abelian theta functions of level k on the abelian variety

$$M_\Omega = \mathbb{C}^g/(\mathbb{Z}^g \oplus \Omega\mathbb{Z}^g), \quad \Omega \in \mathbb{H}^g, \tag{8}$$

form a k^g dimensional vector space of holomorphic functions on $(\mathbb{C}^*)^g \cong \mathbb{C}^g/\mathbb{Z}^g$ with basis given by (2). These functions are the pull-back by

$$p : (\mathbb{C}^*)^g \to M_\Omega = (\mathbb{C}^*)^g / \, \Omega\mathbb{Z}^g \tag{9}$$

of holomorphic sections of L^k, where L is the holomorphic line bundle over M_Ω defined in (4).

It is a well known classical fact, which goes back to Jacobi, that these functions are the analytic continuation to $(\mathbb{C}^*)^g$ of solutions of the following heat equation on $U(1)^g$,

$$\frac{\partial u}{\partial t} = -\sum_{\alpha,\beta=1}^{g} \frac{i}{4\pi}\Omega_{\alpha\beta}\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta}, \tag{10}$$

evaluated at $t = \frac{1}{k}$ and with (special) distributional initial conditions for $t = 0$

$$\varphi_l(x) = \sum_{n\in\mathbb{Z}^g} e^{2\pi i(l+kn)\cdot x} = \sum_{l'\in(\mathbb{Z}_k)^g} \delta(x - \frac{l'}{k})e^{2\pi i\frac{l'\cdot l}{k}}. \tag{11}$$

These distributions define fixed k^g-dimensional subspaces of the space of distributions $C^\infty(U(1)^g)'$,

$$V_{k,g}^{U(1)} = \mathrm{span}_{\mathbb{C}}\left\{\delta(x - \frac{l'}{k}),\ l' \in (\mathbb{Z}_k)^g\right\} \subset C^\infty(U(1)^g)'. \tag{12}$$

We remark that the unitary matrix S given by

$$S_{l,l'} = k^{-\frac{g}{2}}e^{2\pi i\frac{l'\cdot l}{k}} \tag{13}$$

which appears in (11) characterizes the modular properties of the integrable representations of the affine Lie algebra $\widehat{U(1)}^g_k$. A similar remark will take place in the non-abelian case.

A related observation is that the distributions in $V_{k,g}^{U(1)}$ are supported precisely at the points of order dividing k

$$\left(\frac{1}{k}\mathbb{Z}_k\right)^g \subset U(1)^g = (\mathbb{R}/\mathbb{Z})^g, \tag{14}$$

on the "real part" of M_Ω, which are also the k-Bohr-Sommerfeld points of a real polarization in the geometric quantization of the symplectic manifold underlying this abelian variety.

Let us recall the general notion of a real polarization and of Bohr-Sommerfeld points. Consider a symplectic manifold M of dimension $2m$, with symplectic form ω, and let $L \to M$ be a Hermitean line bundle with connection ∇, with curvature $F_\nabla = 2\pi i\omega$. A real polarization of the data (M, ω, L, ∇) (usually called prequantization data) is a surjective map $\pi : M \to B$ onto a manifold B of dimension m such that $\omega_{|\pi^{-1}(b)} = 0$, for all $b \in B$. Then, for a generic b, the fiber $M_b = \pi^{-1}(b)$, will be a lagrangean submanifold and the restriction of L to M_b will have a connection with zero curvature. The holonomy of this connection gives rise to a homomorphism

$$H_b : \pi_1(M_b) \to U(1)$$

One defines $b \in B$ to be a k-Bohr-Sommerfeld (k-BS) point and $\pi^{-1}(b)$ to be a k-Bohr-Sommerfeld fiber if $(H_b)^k = 1$. The quantization of this system is then provided by the vector space $H^n(M, \mathcal{S}_\pi)$, with its natural Hilbert space structure [Sn]. Here, $\mathcal{S}_\pi$ denotes the sheaf of local sections of L^k which are covariantly constant along the fibers. Below, following [We, JW] we will also consider non-smooth real polarizations with singular Bohr-Sommerfeld fibers, for which the notion of covariantly constant section still makes sense.

When M is a Kähler manifold and L is a positive holomorphic line bundle on M, there is an alternative path for the quantization of M. In this case, one uses a complex polarization and defines the quantum Hilbert space as $H^0(M, L^k)$. It is expected that, in most cases, there should be a natural unitary isomorphism between Hilbert spaces corresponding to different complex structures on M. Moreover, it is natural to expect that the cardinality of the k-Bohr-Sommerfeld set $B_{k-\mathrm{BS}} \subset B$ coincides with the dimension of this space (see [We, JW, Ty2]).

When M_Ω is our abelian variety, and L^k are the bundles in (4), the quantizations in the real and complex polarizations can be compared in the following way. The symplectic form on M_Ω is given by

$$\omega = \sum_{j=1}^{g} d\eta_j \wedge d\xi_j,$$

where $\eta_1, ..., \eta_j, \xi_1, ..., \xi_j$ are the coordinates on V dual to the generators of the lattice Λ_Ω, which are periodic coordinates on M_Ω and related to the complex ones by

$$z = x + iy = \eta + \Omega\xi.$$

The canonical projection

$$\begin{array}{rcl} \pi : M_\Omega & \to & (\mathbb{R}/\mathbb{Z})^g \\ z & \mapsto & \eta \end{array}$$

defines a real polarization of M_Ω such that a connection 1-form of L^k restricted to the fibre M_η over η is $\alpha = k \sum_{j=1}^{g} \eta_j d\xi_j$. It is easy to see that the holonomy map H_η is trivial if and only if $k\eta \in \mathbb{Z}^g$, so that the k-Bohr-Sommerfeld points are the points of order dividing k on the real torus

$$U(1)^g_{k-\mathrm{BS}} = \left(\frac{1}{k}\mathbb{Z}_k\right)^g.$$

In the next section, we will describe an analytic tool that will allow us to study theta functions, the heat equation and geometric quantization in a unified way.

4 Coherent State Transforms

4.1 CST for Lie groups

In this section we describe the coherent state transforms (CST) for Lie groups introduced by Hall [Ha1]. For simplicity, we will restrict ourselves in this section, to the case when K is simple. The general compact case can be treated in a similar way. In particular $K = U(1)^g$ will be considered in sections 4.2 and 5.1. Let K be a compact connected simple Lie group, $K_{\mathbb{C}}$ its complexification (see [Ho]) and Δ_K the Laplacian on K associated to an Ad-invariant inner product on its Lie algebra $Lie(K)$.

For each $f \in L^2(K, dx)$, where dx is the normalized Haar measure on K, the image of f by the CST, $C_t f$, is the analytic continuation to $K_{\mathbb{C}}$ of the solution of the heat equation,

$$\frac{1}{\pi}\frac{\partial u}{\partial t} = \Delta_K u, \tag{15}$$

with initial condition given by $u(0, x) = f(x)$. The K-averaged coherent state transform is then defined by

$$\begin{array}{rcl} C_t : L^2(K, dx) & \rightarrow & \mathcal{H}(K_{\mathbb{C}}) \\ f & \mapsto & C_t(f) = \mathcal{C} \circ e^{\pi t \Delta_K}(f), \end{array} \tag{16}$$

where $\mathcal{H}(K_{\mathbb{C}})$ denotes the space of holomorphic functions on $K_{\mathbb{C}}$ and $\mathcal{C}$ denotes analytic continuation.

Let $d\nu_t$ be the K-averaged heat kernel measure on $K_{\mathbb{C}}$ defined in [Ha1, Ha2]. It coincides with the Haar measure the compact directions (corresponding to K) and is asymptotically Gaussian in the imaginary directions. Then the following result holds.

Theorem 1. [Hall] *For each $t > 0$, the mapping C_t defined in (16) is a unitary isomorphism from $L^2(K, dx)$ onto the Hilbert space $L^2(K_{\mathbb{C}}, d\nu_t) \bigcap \mathcal{H}(K_{\mathbb{C}})$.*

To obtain a more explicit description of this CST, consider the expansion of $f \in L^2(K, dx)$ given by the Peter-Weyl theorem,

$$f(x) = \sum_{\mathrm{R}} \mathrm{tr}(\mathrm{R}(x) A_{\mathrm{R}}), \tag{17}$$

where the sum is taken over the set of (equivalence classes of) irreducible representations of K, and $A_{\mathrm{R}} \in End\ V_{\mathrm{R}}$ is given by

$$A_{\mathrm{R}} = (\mathrm{dim} V_{\mathrm{R}}) \int_K f(x) \mathrm{R}(x^{-1}) dx, \tag{18}$$

V_{R} being the representation space for R. Let $\{X_i,\, i=1,...,\dim K\}$ be an orthonormal basis for the Ad-invariant inner product on $Lie(K)$ for which the longest root has squared norm 2. Viewing $Lie(K)$ as the space of left-invariant vector fields on K, we have

$$\Delta_K = \sum_{i=1}^{\dim K} X_i X_i$$

and one obtains

$$C_t f(g) = \sum_{\mathrm{R}} e^{-\pi t c_{\mathrm{R}}}\, \mathrm{tr}(\mathrm{R}(g) A_{\mathrm{R}}), \tag{19}$$

where $c_{\mathrm{R}} \geq 0$ is the eigenvalue of $-\Delta_K$ on functions of the type

$$\mathrm{tr}(A\mathrm{R}(x)), \quad A \in End(V_{\mathrm{R}}).$$

For the application to non-abelian theta functions on an elliptic curve of the form $X = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ with $\tau \in \mathbb{H}_1$, it is convenient to consider instead of Δ_K the operator $-i\tau\Delta_K$. The extension of theorem 1 to this case is straightforward. We denote the extended CST by the same symbol C_t, and write

$$C_t f(g) = \sum_{\mathrm{R}} e^{i\pi t\tau c_{\mathrm{R}}}\, \mathrm{tr}(\mathrm{R}(g) A_{\mathrm{R}}). \tag{20}$$

Similarly, for classic theta functions on M_Ω the relevant operator is the one in (10) [FMN1].

4.2 CST for distributions

Let first K be a simple group. The CST transform (16) can be extended to the space of distributions $C^\infty(K)'$ [FMN1, FMN2]. As we saw in the abelian case, see (11) and (12), this is a necessary step if we want to apply the CST to the study of theta functions. The distributions in $C^\infty(K)'$ can be written in the form

$$f = \sum_{\lambda \in \Lambda_W^+} \mathrm{tr}(\mathrm{R}_\lambda A_\lambda), \tag{21}$$

where we labelled irreducible representations of K by the highest weights λ in the set of dominant weights Λ_W^+. The Fourier series (21) defines a distribution if the coefficients A_λ are such that there exists an integer $N > 0$ satisfying

$$\lim_{\|\lambda\| \to \infty} \frac{\|A_\lambda\|}{(1 + \|\lambda\|^2)^N} = 0, \tag{22}$$

where $\|A_\lambda\|$ is the operator norm of the endomorphism $A_\lambda \in End(\mathrm{R}_\lambda)$.

The Laplace operator Δ_K and its powers act as continuous linear operators on the space $C^\infty(K)'$ and define the action of the operator $e^{-\pi it\tau\Delta_K}$ on it, where $t \geq 0$ and $\tau \in \mathbb{H}_1$. For f of the form (21) we have

$$e^{-\pi it\tau\Delta_K} f = \sum_{\lambda\in\Lambda_W^+} e^{i\pi t\tau c_\lambda}\mathrm{tr}(\mathrm{R}_\lambda A_\lambda). \tag{23}$$

Proposition 1. *[FMN2] If* $f = \sum_{\lambda\in\Lambda_W^+} \mathrm{tr}(\mathrm{R}_\lambda A_\lambda) \in C^\infty(K)'$ *then the series*

$$\sum_{\lambda\in\Lambda_W^+} e^{i\pi t\tau c_\lambda}\mathrm{tr}(\mathrm{R}_\lambda(g)A_\lambda) \tag{24}$$

where $g \in K_\mathbb{C}$*, defines a holomorphic function on* $K_\mathbb{C} \times \mathcal{T}_1$ *which we denote by* $C_t f$*. Here,* $\mathcal{T}_1 = \mathbb{H}_1$ *stands for the Teichmüller space of genus 1 Riemann surfaces.*

We therefore have the following commutative diagram

$$\begin{array}{ccc} L^2(K, dx) & \xrightarrow[\sim]{C_t} & \mathcal{H}(K_\mathbb{C}) \bigcap L^2(K_\mathbb{C}, d\nu_t) \\ \big\downarrow & & \big\downarrow \\ C^\infty(K)' & \xhookrightarrow{C_t} & \mathcal{H}(K_\mathbb{C}) \end{array} \tag{25}$$

where $\xrightarrow[\sim]{}$ stands for isomoprhism and $\hookrightarrow$ for an inclusion.

For the case $K = U(1)^g$, and for a complex Laplacian as in (10) the result is

Proposition 2. *[FMN1] If* $f(x) = \sum_{n\in\mathbb{Z}^g} A_n e^{2\pi inx} \in C^\infty(U(1)^g)'$ *then the series*

$$\sum_{n\in\mathbb{Z}^g} A_n e^{i\pi tn\cdot\Omega n} e^{2\pi inx} \tag{26}$$

where $g \in (\mathbb{C}^*)^g$*, defines a holomorphic function on* $(\mathbb{C}^*)^g \times \mathcal{T}_g$ *which we denote by* $C_t f$*, where* $\mathcal{T}_g$ *is the Teichmüller space of genus* g *Riemann surfaces.*

The analog of the diagram above is now the commutative diagram

$$\begin{array}{ccc} L^2(U(1)^g, dx) & \xrightarrow[\sim]{C_t} & \mathcal{H}((\mathbb{C}^*)^g) \bigcap L^2((\mathbb{C}^*)^g, d\nu_t) \\ \big\downarrow & & \big\downarrow \\ C^\infty(U(1)^g)' & \xhookrightarrow{C_t} & \mathcal{H}((\mathbb{C}^*)^g) \end{array} \tag{27}$$

The bottom rows of diagrams (27) and (25) preserve the injectivity of the CST of the upper rows. However, the surjectivity and unitarity are lost when taking the space of distributions. In the next section, we will see how to remedy these problems by looking at theta functions.

5 Theta functions and generalized CST

5.1 Abelian varieties

As mentioned above, the maps C_t in the bottom rows of diagrams (27) and (25) are not surjective nor unitary. Considering first the abelian case in (27), this problem gives rise to the following questions.

Q1. How can the unitarity of C_t be recovered in a natural way, taking into account the fact that on the right hand side of (27) we are interested not in $(\mathbb{C}^*)^g$ but in $M_\Omega \cong (\mathbb{C}^*)^g/\Omega\mathbb{Z}^g$?

Q2. How to select in a natural way the subspaces $V_{k,g}^{U(1)} \subset (\mathbb{C}^\infty(U(1)^g))'$ of dimensions $\mathrm{dim} V_{k,g}^{U(1)} = h^0(M_\Omega, L^k) = k^g$, such that their image by the CST corresponds to theta functions ?

We have already answered Q2 in section 3. On the other hand, we want to emphasize that, due to "hidden" properties of the averaged heat kernel measure $d\nu_t$ on $(\mathbb{C}^*)^g$, the consideration of question Q1 leads *independently* to the same answer to question Q2. Motivated by the map $p : (\mathbb{C}^*)^g \to M_\Omega$, we consider the modified inner product

$$\langle\!\langle F_1, F_2 \rangle\!\rangle := \int_D \bar{F}_1 F_2 d\nu_t, \tag{28}$$

where $D \subset (\mathbb{C}^*)^g$ is a fundamental domain for the quotient under p. This definition makes sense only if the holomorphic functions F_1, F_2 are such that the integrand in (28),

$$\bar{F}_1 F_2 \nu_t, \tag{29}$$

where ν_t is the Gaussian function in $d\nu_t = \nu_t dz$ and dz denotes the Lebesgue measure in $\mathbb{C}^g$, is invariant with respect to the action of $\Omega\mathbb{Z}^g$ so that (28) does not depend on the choice of fundamental domain D. The crucial property of ν_t is:

Proposition 3. *[FMN1] The function $\nu_{1/k}$ on $(\mathbb{C}^*)^g$ is the pull-back by $p : (\mathbb{C}^*)^g \to M_\Omega$ of an Hermitean structure on L^k.*

From this follows the main result.

Theorem 2. *[FMN1]*

(i) Nontrivial functions F such that the inner products (28) do not depend on the choice of D exist if and only if $t = \frac{1}{k}, k \in \mathbb{N}$. For $t = \frac{1}{k}$ the set of such functions is the union of vector spaces intersecting only at $F = 0$. The space of classical theta functions $H^0(M_\Omega, L^k)$ is one of these spaces, the others corresponding also to theta functions but translated with respect to (2).

(ii) The space $H^0(M_\Omega, L^k)$ is in the image of $(C^\infty(U(1)^g))'$ by $C_{1/k}$ in the second row of (27). Its inverse image coincides with the space $V_{k,g}^{U(1)}$ in (12)

It is now clear that, in the process of addressing $Q1$, we were led to $C_{1/k}$ and then to an alternative path to the answer to question 2, expressed in (ii) above. Moreover, we obtain in this way a natural inner product on $V_{k,g}^{U(1)}$. We see from (11) and (12) that there is a natural isomorphism from a subspace $U_{k,g}^{U(1)}$ of $L^2(U(1)^g, dx)$,

$$U_{k,g}^{U(1)} = \mathrm{span}_{\mathbb{C}} \left\{ e^{2\pi i l \cdot x},\ l :\ 0 \leq l_\alpha < k,\ 1 \leq \alpha \leq g \right\} \tag{30}$$

to $V_{k,g}^{U(1)}$ defined by

$$\begin{array}{rcl} \Phi_k\ :\ U_{k,g}^{U(1)} & \to & V_{k,g}^{U(1)} \\ f_l(x) = e^{2\pi i l \cdot x} & \mapsto & \varphi_l(x) = \displaystyle\sum_{\tilde{l} \in [l]_k} e^{2\pi i \tilde{l} \cdot x}, \end{array} \tag{31}$$

where $[l]_k = l + k\mathbb{Z}^g$. We define on $V_{k,g}^{U(1)}$ the inner product $(\cdot,\cdot)$ for which Φ_k is unitary. Therefore the basis $\{\varphi_l,\ l\ :\ 0 \leq l_\alpha < k\}$ is orthonormal. The answer to $Q1$ is then [FMN1]

Theorem 3. *[FMN1] The restriction of the CST $C_{1/k}$ to $V_{k,g}^{U(1)}$ is an isometric isomoprhism from $(V_{k,g}^{U(1)}, (\cdot,\cdot))$ onto $(H^0(M_\Omega, L^k), \langle\!\langle \cdot,\cdot \rangle\!\rangle)$.*

Therefore for principally polarized abelian varieties the diagram (27) can be completed to the commutative diagram in which the top and bottom horizontal arrows are unitary isomorphisms.

$$\begin{tikzcd} L^2(U(1)^g, dx) \arrow[r, "C_{1/k}", "\sim"'] \arrow[d, hook] & \mathcal{H}((\mathbb{C}^*)^g) \bigcap L^2((\mathbb{C}^*)^g, d\nu_t) \arrow[d, hook] \\ C^\infty(U(1)^g)' \arrow[r, hook, "C_{1/k}"] & \mathcal{H}((\mathbb{C}^*)^g) \\ V_{k,g}^{U(1)} \arrow[r, "C_{1/k}", "\sim"'] \arrow[u, hook] & H^0(M_\Omega, L^k) \arrow[u, hook] \end{tikzcd} \quad . \tag{32}$$

5.2 The Schottky map

As we saw above there were two crucial ingredients that allowed the use of a CST transform for $K = U(1)^g$ to study abelian theta functions. The surjective map (9) from the complex group $K_{\mathbb{C}} = (\mathbb{C}^*)^g = (U(1)^g)_{\mathbb{C}}$ to the abelian variety M_Ω and the quasiperiodicity properties of the averaged heat kernel measure with respect to the same map (9).

Our aim is to apply a CST for $K = SU(n)^g$ to the study of $SL(n, \mathbb{C})$ non-abelian theta functions. As we saw in section 2, given a genus g Riemann surface X, the $SL(n, \mathbb{C})$ theta functions are holomorphic sections of line bundles L^k over the moduli space, $\mathcal{M}_n(X)$, of rank n semistable holomorphic vector bundles with trivial determinant. We will consider the Schottky map

$$\widetilde{S} : SL(n, \mathbb{C})^g / SL(n, \mathbb{C}) \to \mathcal{M}_n(X) \tag{33}$$

which is, in many respects, in the Jacobian case, a higher rank analogue of the map p in (9), and study the pull-back of theta functions to

$$SL_g(n, \mathbb{C}) \equiv SL(n, \mathbb{C})^g / SL(n, \mathbb{C}).$$

In fact, this map is not defined everywhere in $SL_g(n, \mathbb{C})$ and it is not known whether it is surjective. However, it is known that the Schottky map is locally (in a neighborhood of $SU_g(n) \equiv SU(n)^g/SU(n) \subset SL_g(n, \mathbb{C})$) bi-holomorphic onto a neighbourhood in the moduli space of stable bundles [Fl]. It is also conjectured that its image is dense in $\mathcal{M}_n(X)$. We will try to identify properties that characterize holomorphic function on $SL_g(n, \mathbb{C})$ whose restriction to the domain of the Schottky map is the pullback of a non-abelian theta function.

Let us now define the Schottky map for a complex linear subgroup $G_\mathbb{C}$ of $GL(n, \mathbb{C})$ in the case of semistable bundles. There is a well know bijection between the space of flat $G_\mathbb{C}$-bundles over X, $H^1(X, G_\mathbb{C})$, and the space of $G_\mathbb{C}$-representations of the fundamental group of X, $\pi_1(X)$, modulo overall conjugation $Hom(\pi_1(X), G_\mathbb{C})/G_\mathbb{C}$; it is given explicitly by:

$$\begin{aligned} V : Hom(\pi_1(X), G_\mathbb{C})/G_\mathbb{C} &\to H^1(X, G_\mathbb{C}) \\ \rho &\mapsto V_\rho := \tilde{X} \times_\rho G_\mathbb{C}, \end{aligned} \tag{34}$$

where the notation means that $\pi_1(X)$ acts diagonally through ρ on the trivial $G_\mathbb{C}$-bundle over the universal cover $\tilde{X}$ of X and V_ρ is the quotient with respect to that action.

Let us fix a canonical basis of $\pi_1(X)$: elements $a_1, ..., a_g, b_1, ..., b_g$ that generate $\pi_1(X)$, subject to the single relation $\prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} = 1$. Then an element $B = (B_1, ..., B_g) \in G_\mathbb{C}^g$ induces a representation class

$$\rho_B \in Hom(\pi_1(X), G_\mathbb{C})/G_\mathbb{C} \tag{35}$$

defined by $\rho_B(a_i) = 1,\ \rho_B(b_i) = B_i$. The Schottky map is then

$$S : (G_\mathbb{C}^g)^o \to \mathcal{M}_n \tag{36}$$
$$B \mapsto [V_{\rho_B}]\ , \tag{37}$$

where $[V_{\rho_B}]$ denotes the (isomorphism class of) holomorphic vector bundle of rank n over X, defined by the flat $G_\mathbb{C}$-bundle V_{ρ_B}. Here, $(G_\mathbb{C}^g)^o$

denotes the open dense subset of $G_{\mathbb{C}}^g$ which corresponds to semistable bundles. Note that for $G_{\mathbb{C}} = SL(n,\mathbb{C})$, $[V_{\rho_B}]$ has always degree 0. In other words, this map sends a g-tuple of $n \times n$ invertible matrices $(B_1, ..., B_n) \in G_{\mathbb{C}}^g \subset GL(n,\mathbb{C})^g$ to the flat rank n holomorphic vector bundle determined by the holonomies $(1, ..., 1, B_1, ..., B_g)$ around the loops $(a_1, ..., a_g, b_1, ..., b_g)$, respectively.

In the case of line bundles the situation is simple, $GL(1,\mathbb{C}) = \mathbb{C}^*$ is abelian and the degree 0 line bundles in $H^1(X, \mathcal{O}^*)$ form an abelian variety, the Jacobian, $J(X)$, of X. The Schottky map reduces then to the map p in (9) (for more details see [FMN2]).

5.3 $SU(n)$ and $g = 1$

Let us now consider the Schottky map for the case of an elliptic curve $X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, and for semistable bundles of rank n with trivial determinant over X_τ. This will be a map

$$S : SL(n,\mathbb{C}) \to \mathcal{M}_n(X_\tau).$$

From geometric invariant theory (see [FKM, N]), we know that under the adjoint action, $SL(n,\mathbb{C})$ has a good quotient, which is a map $SL(n,\mathbb{C}) \to T_{\mathbb{C}}/W$, where $T_{\mathbb{C}}$ is the maximal torus of $SL(n,\mathbb{C})$ and W the Weyl group; this map sends a matrix to the unordered set of its eigenvalues. Therefore the Schottky map factors through $T_{\mathbb{C}}/W$, as in the following diagram.

$$\begin{array}{ccc} SL(n,\mathbb{C}) & \xrightarrow{q} & T_{\mathbb{C}}/W \\ & \searrow S & \downarrow \widetilde{S} \\ & & \mathcal{M}_n(X_\tau) \end{array}$$

We are therefore reduced to describing $\widetilde{S}$. Fixing the following Cartan subalgebra of $sl(n,\mathbb{C})$

$$\mathfrak{h} = \{A \in sl(n,\mathbb{C}) : A \text{ is diagonal}\},$$

and using the corresponding coroot lattice $\check{\Lambda}_R$, define M to be the abelian variety

$$M = \check{\Lambda}_R \otimes X_\tau = \check{\Lambda}_R \otimes (\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}) = \mathfrak{h}/\left(\check{\Lambda}_R \oplus \tau\check{\Lambda}_R\right).$$

The Weyl group W acts naturally on M, via the usual action on $\check{\Lambda}_R$, and, as shown in [FM],[FMW],[L] $\mathcal{M}_n(X_\tau)$ is isomorphic to the $l = (n-1)$-dimensional complex projective space $\mathbb{P}^l(\mathbb{C})$ obtained as the quotient under this action,

$$\mathcal{M}_n(X_\tau) \cong M/W \cong \mathbb{P}^{n-1}.$$

Let now

$$p : \mathbb{C}^* \to J(X_\tau) \cong X_\tau,$$

be the abelian Schottky map (9) for genus one, where we identify, as usual the elliptic curve with its Jacobian. It is easy to see that we have [FMN2]

Proposition 4. *The following diagram is commutative*

$$\begin{array}{cccccccc} SL(n,\mathbb{C}) & \xrightarrow{q} & T_{\mathbb{C}}/W & \xleftarrow{\tilde{\pi}} & T_{\mathbb{C}} & \hookrightarrow & (\mathbb{C}^*)^n & \\ & S\searrow & \downarrow \widetilde{S} & & \downarrow p^n|_{T_{\mathbb{C}}} & & \downarrow p^n & \\ & & \mathcal{M}_n(X_\tau) & \xleftarrow{\pi} & M & \hookrightarrow & J\left(X_\tau\right)^n & \end{array} \tag{38}$$

We can write explicitly the map $p^n|_{T_{\mathbb{C}}}$ as

$$\begin{array}{rcl} p^n|_{T_{\mathbb{C}}} : \mathfrak{h}/\check{\Lambda}_R \cong T_{\mathbb{C}} & \longrightarrow & M = \mathfrak{h}/(\check{\Lambda}_R \oplus \tau\check{\Lambda}_R) \\ v + \check{\Lambda}_R & \mapsto & v + \check{\Lambda}_R + \tau\check{\Lambda}_R. \end{array} \tag{39}$$

Proposition 5. *The space $H^0(\mathcal{M}_n(X_\tau), L^k)$ of level k non-abelian theta functions on the moduli space of semistable rank n holomorphic vector bundles with trivial determinant, $\mathcal{M}_n(X_\tau)$ is naturally identified with the space $\mathcal{H}^+_{k,\tau}$ of holomorphic Weyl invariant functions on $\mathfrak{h}$ satisfying the quasiperiodicity conditions*

$$\theta(v + \check{\alpha} + \tau\check{\beta}) = e^{-2\pi i k\beta(v) - \pi i k\tau <\beta,\beta>}\theta(v), \quad \alpha, \beta \in \Lambda_R \tag{40}$$

i.e,

$$H^0(\mathcal{M}_n(X_\tau), L^k) \xrightarrow{\sim} \mathcal{H}^+_{k,\tau} \equiv \{\theta \in \mathcal{H}(\mathfrak{h}),\ \theta \text{ satisfies } (40), w\theta = \theta, \forall w \in W\}.$$

Before we can formulate the analogue of questions Q1 and Q2 in section 5.1, we need an Ad-invariant version of the $SU(n)$ CST or, more precisely, an Ad-invariant version of the diagram (25). Ad-invariant distributions have a decomposition (21) with $A_\lambda = a_\lambda I_\lambda$, where I_λ denotes the identity operator in V_{R_λ} and the coefficients $a_\lambda \in \mathbb{C}$ satisfy growth conditions implied by (22). For such a distribution, we have

$$f = \sum_{\lambda \in \Lambda^+_W} a_\lambda \chi_\lambda, \tag{41}$$

where χ_λ denotes the character corresponding to λ.

By restricting the coherent state transform (16) to the closed subspace of Ad-invariant functions on $SU(n)$ we obtain easily,

Proposition 6. *[FMN2] The restriction C_t^{Ad} of the CST (16) to the Hilbert space $L^2(SU(n), dx)^{Ad_{SU(n)}}$ is an isometric isomorphism onto the Hilbert space $L^2(SL(n,\mathbb{C}), d\nu_t)^{Ad_{SL(n,\mathbb{C})}} \bigcap \mathcal{H}(SL(n,\mathbb{C})$.*

The extension to distributions is straightforward and can be summarized as the following *Ad*-invariant version of (25).

Theorem 4. *[FMN2] The following diagram is commutative*

$$\begin{tikzcd} L^2(SU(n),dx)^{Ad} \arrow[r, "C_t^{Ad}", "\sim"'] \arrow[d, hook] & \mathcal{H}(SL(n,\mathbb{C}))^{Ad} \bigcap L^2(SL(n,\mathbb{C}),d\nu_t) \arrow[d, hook] \\ C^\infty(SU(n))'^{Ad} \arrow[r, hook, "C_t^{Ad}"] & \mathcal{H}(SL(n,\mathbb{C}))^{Ad} \end{tikzcd} \quad . \tag{42}$$

We can now repeat the questions of section 5.1 in the form

Q1. How can the unitarity of C_t^{Ad} in the second row of (42) be recovered in a natural way taking into account the fact that on the right hand side we are interested not in $SL(n,\mathbb{C})/SL(n,\mathbb{C}) \cong \mathfrak{h}/W \rhd \check{\Lambda}_R$ but in $\mathcal{M}_n(X_\tau) \cong \left[\mathfrak{h}/(\check{\Lambda}_R + \tau\check{\Lambda}_R)\right]/W$?

Q2. How to select in a natural way subspaces

$$V_{k,1}^{SU(n)} \subset \left(C^\infty(SU(n))\right)'^{Ad}$$

of dimensions given by the Verlinde numbers, such that their image by the CST corresponds to $g = 1$ theta functions ?

Again, we will see that the answer to Q1 leads to the answer to Q2. As before, consider a modified inner product on $\mathcal{H}(SL(n,\mathbb{C}))^{Ad_{SL(n,\mathbb{C})}}$,

$$\langle\!\langle F_1, F_2 \rangle\!\rangle = \int_{q^{-1}([\mathfrak{h}_0])} \bar{F}_1 F_2 d\nu_t, \tag{43}$$

where $[\mathfrak{h}_0]$ is a fundamental domain for the action of $W \rhd (\check{\Lambda}_R + \tau\check{\Lambda}_R)$.

This definition makes sense only if we consider it for functions for which the inner product does not depend on the choice of $[\mathfrak{h}_0]$. The key technical result here is the following integration formula which is an analog for heat kernel measures of the Weyl integration formula. Let σ be the denominator in the Weyl character formula, $\sigma_{\mathbb{C}}$ its analytic continuation to $SL(n,\mathbb{C})$, and let $d\nu_t^{\mathrm{ab}}$ denote the averaged heat kernel measure on $T_{\mathbb{C}}$ corresponding to the Laplacian

$$\Delta^{\mathrm{ab}} = \sum_{i,j=1}^{n-1} \frac{1}{4\pi^2} C^{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

where C^{ij} is the inverse Cartan matrix and x_i are periodic coordinates on T.

Theorem 5. *[FMN2] If $f \in L^1(SL(n,\mathbb{C}), d\nu_t)$ is a class function, then, for all $t > 0$*

$$\int_{SL(n,\mathbb{C})} f(g)\nu_t(g)dg = \frac{e^{-2t\pi\|\rho\|^2}}{|W|} \int_{T_{\mathbb{C}}} f(h)|\sigma_{\mathbb{C}}(h)|^2 \nu_t^{\mathrm{ab}}(h)dh. \tag{44}$$

From (44), the independence of (43) with respect to the choice of $[\mathfrak{h}_0]$ corresponds to considering functions $F \in \mathcal{H}(SL(n,\mathbb{C}))^{Ad_{SL(n,\mathbb{C})}}$ such that

$$|F|^2 |\sigma_{\mathbb{C}}|^2 \nu_t^{\mathrm{ab}} \tag{45}$$

is $\tau\check{\Lambda}_R$-invariant as a function on $\mathfrak{h}$. Crucial here are, as in the abelian case of proposition 3, the quasiperiodicity properties satisfied by ν_t^{ab}.

Proposition 7. *[FMN2] The function ν_t^{ab} satisfies the following quasiperiodicity conditions*

$$\begin{aligned} \nu_t^{\mathrm{ab}}(v+\check{\alpha}) &= \nu_t^{\mathrm{ab}}(v), \\ \nu_t^{\mathrm{ab}}(v+\tau\check{\alpha}) &= |e^{2\pi i \frac{1}{t}\alpha(v)}|^2 |e^{\pi i \frac{1}{t}\tau<\alpha,\alpha>}|^2 \nu_t^{\mathrm{ab}}(v), \ \text{for all}\, \check{\alpha} \in \check{\Lambda}_R. \end{aligned} \tag{46}$$

which imply the following

Corollary 1. *The function $\nu_{1/k'}^{\mathrm{ab}}$, $k' \in \mathbb{N}$, considered as function on $T_{\mathbb{C}}/W$ is the pull-back by $\widetilde{S} : T_{\mathbb{C}}/W \to \mathcal{M}(X_\tau)$ of an Hermitean structure on L^k.*

Let us consider first the case $n > 2$. From theorem 5, proposition 7 and corollary 1 we obtain

Theorem 6. *[FMN2]*

(i) For $K = SU(n), n > 2$, nontrivial functions F such that the inner products (43) do not depend on the choice of $[\mathfrak{h}_0]$ exist if and only if $t = \frac{1}{k+n}, k \in \mathbb{N}\bigcup\{0\}$. For $t = \frac{1}{k+n}$ the set of such functions forms a vector space $\mathcal{H}_{k,\tau} \subset \mathcal{H}(SL(n,\mathbb{C}))^{Ad}$ naturally isomorphic to $H^0(M_n(X_\tau), L^k)$ with isomorphism given by

$$\begin{aligned} \varphi\ : \mathcal{H}_{k,\tau} &\stackrel{\sim}{\longrightarrow} H^0(M_n(X_\tau), L^k) \\ F &\longmapsto \frac{e^{i\pi\tau\frac{||\rho||^2}{k+n}}}{|W|}\sigma_{\mathbb{C}}\ F_{|T_{\mathbb{C}}}. \end{aligned} \tag{47}$$

(ii) The space $\mathcal{H}_{k,\tau}$ is in the image of $(C^\infty(SU(n,\mathbb{C}))'^{Ad}$ by $C_{1/(k+n)}$ in the second row of (42). Its inverse image defines the space $V_{k,1}^{SU(n)}$ as in Q2.

Proof. (of i)
If the function $|F|^2|\sigma_{\mathbb{C}}|^2\nu_t^{\mathrm{ab}}$ is $(\check{\Lambda}_R \oplus \tau\check{\Lambda}_R)$-invariant then from proposition 7 and the holomorphicity in $\mathfrak{h} \times \mathcal{T}_1$ it follows that F must also satisfy the following quasi-periodicity conditions

$$(F\sigma_{\mathbb{C}})(v+\tau\check{\alpha}) = e^{-2\pi i<d,\alpha>} e^{-2\pi i\frac{1}{t}\alpha(v)} e^{-\pi i \frac{1}{t}\tau<\alpha,\alpha>}(F\sigma_{\mathbb{C}})(v), \tag{48}$$

where $d \in \mathfrak{h}_{\mathbb{R}}^*$. Non-zero holomorphic functions on $\mathfrak{h}/\check{\Lambda}_R$ satisfying (48) do not exist if $1/t \notin \mathbb{N}$. This results from the fact that if $1/t \notin \mathbb{N}$

then the automorphy factors in (48) are not invariant under $v \mapsto v + \check{\beta}$, $\check{\beta} \in \check{\Lambda}_R$, which makes it impossible to solve (48) on $\mathfrak{h}/\check{\Lambda}_R$. We see from (40) that functions satisfying (48) with $\frac{1}{t} = k' \in \mathbb{N}$ are level k' theta functions. Since $\sigma_{\mathbb{C}}$ is W-anti-invariant and the functions F are W-invariant the functions $F\sigma_{\mathbb{C}}$ satisfying (48) are W-anti-invariant theta functions which exist only for level $k' \geq n$ [L]. $\square$

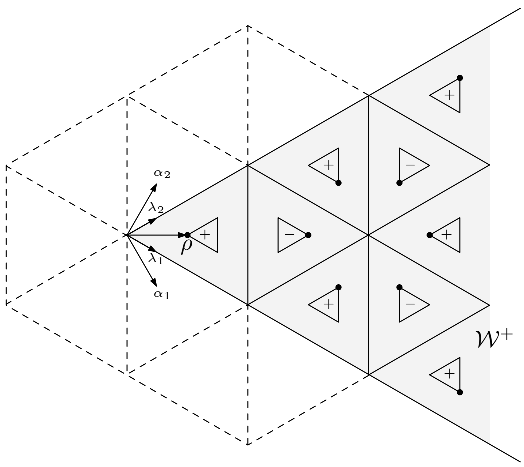
Let $D_k = \{\lambda \in \Lambda_W^+ | \langle \lambda, \hat{\alpha} \rangle \leq k\}$, where $\hat{\alpha}$ is the highest root of $sl(n,\mathbb{C})$. This set labels the integrable representations of the affine Kac-Moody algebra $\widehat{sl(n,\mathbb{C})}_k$. Consider also the affine Weyl group $W^{\text{aff}} := W \rhd \Lambda_R$, and let us define the level $k+n$ action of W^{aff} on $\mathfrak{h}^*_{\mathbb{R}}$ by

$$\tilde{w} \cdot \lambda = w \cdot (\lambda + (n+k)\alpha), \quad \tilde{w} = (w, \alpha) \in W^{\text{aff}}. \tag{49}$$

The vector space $V_{k,1}^{SU(n)}$ has a basis given by

$$\varphi_{\gamma,k} = \sum_{\lambda+\rho \in [\gamma+\rho]_k \cap \Lambda_W^+} \epsilon_\lambda \chi_\lambda, \qquad \gamma \in D_k, \tag{50}$$

where $[\gamma+\rho]_k$ denotes the orbit of $\gamma+\rho$ under the action (49) of W^{aff}, and $\epsilon_\lambda = \epsilon(w)$ where $\tilde{w}$ is the unique element of W^{aff} such that $\lambda + \rho = \tilde{w} \cdot (\gamma + \rho)$. Figure 1 below, shows an example of the orbit $[\gamma+\rho]_k$ inside the fundamental Weyl chamber $\mathcal{W}^+ = \mathbb{R}_{\geq 0} \cdot \Lambda_W^+$.



The points $\lambda + \rho \in [\gamma+\rho]_k \cap \Lambda_W^+$, in the case $SU(3)$, $k=1$, $\gamma=0$, together with the corresponding sign ϵ_λ

Figure 1

As in section 5.1, we see that there is a natural isomorphism from a subspace $U_{k,1}^{SU(n)}$ of $L^2(SU(n), dx)^{Ad}$,

$$U_{k,1}^{SU(n)} = \text{span}_{\mathbb{C}} \{\chi_\gamma, \gamma \in D_k\} \tag{51}$$

to $V_{k,1}^{SU(n)}$ defined by

$$\begin{array}{rcl} \Phi_k \ : \ U_{k,1}^{SU(n)} & \to & V_{k,1}^{SU(n)} \\ \chi_\gamma & \mapsto & \varphi_{\gamma,k}. \end{array} \tag{52}$$

We define on $V_{k,1}^{SU(n)}$ the inner product $(\cdot,\cdot)$ for which Φ_k is unitary. Therefore the basis $\{\varphi_\gamma,\ \gamma \in D_k\}$ is orthonormal. The answer to $Q1$ is then [FMN2]

Theorem 7. *[FMN2] The restriction of the CST $C^{Ad}_{1/(k+n)}$ to $V_{k,1}^{SU(n)}$ is a unitary transform from $(V_{k,1}^{SU(n)}, (\cdot,\cdot))$ onto $(\mathcal{H}_{k,\tau}, \langle\!\langle \cdot,\cdot \rangle\!\rangle)$.*

Therefore, for theta functions on the moduli space of holomorphic vector bundles on elliptic curves we have instead of diagram (32), the following commutative diagram

$$\begin{array}{ccc} L^2(SU(n),dx)^{Ad} & \xrightarrow[\sim]{C^{Ad}_{1/(k+n)}} & \mathcal{H}(SL(n,\mathbb{C}))^{Ad} \bigcap L^2(SL(n,\mathbb{C}),d\nu_t) \\ \downarrow & & \downarrow \\ C^\infty(SU(n))'^{Ad} & \xhookrightarrow{C^{Ad}_{1/(k+n)}} & \mathcal{H}(SL(n,\mathbb{C}))^{Ad} \\ \uparrow & & \uparrow \\ V_{k,1}^{SU(n)} & \xrightarrow[\sim]{C^{Ad}_{1/(k+n)}} & \mathcal{H}_{k,\tau} \\ & & \varphi \downarrow \wr \\ & & H^0(\mathcal{M}_n(X_\tau), L^k) \ , \end{array} \tag{53}$$

where all the isomorphisms are unitary.

Note that the extended CST in theorem 7 defines a unitary connection on the bundle of conformal blocks which coincides with the KZBH connection.

Consider now the geometric quantization in a real polarization of

$$\mathcal{M}_n(X_\tau) = \left[\mathfrak{h}/(\check\Lambda_R + \tau\check\Lambda_R)\right]/W$$

with the symplectic form given by the polarizing form of the abelian variety $M = \mathfrak{h}/(\check\Lambda_R + \tau\check\Lambda_R)$. We can view $\mathcal{M}_n(X_\tau)$ as the symplectic manifold of flat $SU(n)$ connections on X_τ with its canonical symplectic structure [AB]

$$Hom(\pi_1(X_\tau), SU(n))/SU(n).$$

Since $\pi_1(X_\tau)$ is abelian, this space is isomorphic to the product of the maximal torus of $SU(n)$ by itself, quotiented by W acting diagonally, $(T\times T)/W$ and then a (non-smooth) real polarization is just given by projection onto the first factor

$$(T\times T)/W \to T/W$$

(see [We]). Under the canonical identification of T/W with the Weyl alcove

$$\mathfrak{A} = \{\nu \in \mathcal{W}^+ | \langle \nu, \alpha \rangle \le 1\} \subset \mathfrak{h}^*_{\mathbb{R}},$$

the Bohr-Sommerfeld points of level k are the points of the weight lattice divided by $k+n$ inside the Weyl alcove

$$(T/W)_{k-\mathrm{BS}} \cong (\tfrac{1}{k+n}\Lambda_W) \cap \mathfrak{A}.$$

As in the abelian case, we can relate the distributions $\varphi_{\gamma,k}$ in (52) with Dirac delta distributions supported on these Bohr-Sommerfeld points. In fact, for $\gamma \in D_k \subset \Lambda_W^+$ and using additive notation for points on $T/W \cong \mathfrak{A}$, we can expand such a delta distribution as follows

$$\delta\left(v - \tfrac{\gamma+\rho}{k+n}\right) = \sum_{\lambda \in \Lambda_W^+} \overline{\chi_\lambda(\tfrac{\gamma+\rho}{k+n})} \chi_\lambda(v), \tag{54}$$

since the characters form an orthonormal basis for $L^2(SU(n), dx)^{Ad}$. From (50) we obtain

$$\delta\left(v - \tfrac{\gamma+\rho}{k+n}\right) = \sum_{\gamma' \in D_k} \chi_{\gamma'}(-\tfrac{\gamma+\rho}{k+n})\ \varphi_{\gamma'}(v). \tag{55}$$

Again, notice that the matrix

$$A_{\gamma,\gamma'} = \chi_{\gamma'}(-\tfrac{\gamma+\rho}{k+n}) \tag{56}$$

is, up to scale, the unitary S-matrix that characterizes the modular properties of the integrable representations of the affine Lie algebra $\widehat{SU(n)}_k$ [Ka]. Note also that the delta distributions in (55) are supported on regular elements of the level 1 action of W^{aff} on $\mathfrak{h}^*_{\mathbb{R}}$. This justifies the consideration of Bohr-Sommerfeld points of level $k+n$, in the level k quantization of $\mathcal{M}_n(X_\tau)$.

Finally, in the case when $K = SU(2)$, theorem (6) is replaced by:

Theorem 8. *[FMN2]*

(i) For $K = SU(2)$, nontrivial functions F such that the inner product (43) does not depend on the choice of $\mathfrak{h}_0$ exist if and only if $t = \frac{2}{2k+3}$ or $t = \frac{1}{k+2}$, $k \in \mathbb{N}\bigcup\{0\}$. For each of these values of t, such functions form the union of two vector spaces intersecting only at 0, with dimensions given by the Verlinde number $k+1$. One of these vector spaces is naturally isomorphic to $H^0(\mathcal{M}_2(X_\tau), L^k)$. The other one corresponds to sections of orbifold line bundles over $\mathcal{M}_2(X_\tau)$.

(ii) The space $H^0(\mathcal{M}_2(X_\tau), L^k)$ is in the image of $(C^\infty(SU(2,\mathbb{C}))'^{Ad}$ by $C_{1/(k+2)}$ in the second row of (4). Its inverse image defines the space $V_{k,1}^{SU(2)}$ as in Q2.

6 $SU(2)$ and $g \geq 2$

6.1 Unitary Schottky bundles and trinion decompositions

It is known from the theorem of Narasimhan and Seshadri [NS] that $\mathcal{M}_n$ is the moduli space of flat $SU(n)$ connections on X :

$$\mathcal{M}_n = Hom(\pi_1(X), SU(n))/SU(n).$$

In the higher genus case $g \geq 2$, we will restrict in this section to rank 2 bundles, since a suitable real polarization of $\mathcal{M}_2$ was explicitly defined in [JW]. To describe it, one uses a trinion decomposition of X, $\alpha = (\alpha_1, ..., \alpha_{3g-3})$, where the α_i form a maximal set of simple closed non-intersecting pairwise non-homotopic curves on X. A trinion decomposition defines a trivalent graph Γ_α, (where each trinion (a three-holed sphere) corresponds to a vertex and each α_i to an edge) and also a 3 dimensional handlebody, whose boundary is X and where the α_i are homotopically trivial.

To any fixed trinion decomposition α, we can associate the following map

$$\begin{array}{rcl} \pi_\alpha : \mathcal{M}_2 & \rightarrow & [0,1]^{3g-3} \subset \mathbb{R}^{3g-3} \\ A & \mapsto & \frac{1}{\pi}\arccos(\frac{1}{2}Tr(H_{\alpha_i}(A)))_{i=1}^{3g-3}, \end{array}$$

where $H_\gamma(A)$ denotes the holonomy of the flat connection A around the loop γ (π_α does not depend on a choice of basepoint since the trace is conjugation invariant). This map π_α defines a (non-smooth) real polarization of $\mathcal{M}_2$ whose image is the convex polytope Δ_α defined by the inequalities

$$\begin{array}{rcl} \theta_{i_1} + \theta_{i_2} - \theta_{i_3} & \geq & 0 \\ \theta_{i_1} + \theta_{i_2} + \theta_{i_3} & \leq & 2, \end{array} \tag{57}$$

whenever $\alpha_{i_1}, \alpha_{i_2}$ and α_{i_3} are the 3 curves bounding a trinion, and where θ_i denotes the ith coordinate in $\mathbb{R}^{3g-3}$. It is known that the fibers of π_α are generically lagrangean tori, although there are some singular fibers [JW]. There is, in particular, a maximally degenerate fiber $\pi_\alpha^{-1}(0)$ which consists of connections which are trivial on all of the α_i, and therefore define connections on the handlebody. Since the fundamental group of the handlebody is free on g generators, this fiber (which of course, as a subset of $\mathcal{M}_2$, depends on the trinion decomposition α), consists of unitary Schottky representations, so we define:

$$\mathcal{S}_\alpha := \pi_\alpha^{-1}(0) \subset \mathcal{M}_2, \qquad \mathcal{S}_\alpha \cong SU(2)^g/SU(2)$$

Then, we have the following,

Theorem 9. *Given a trinion decomposition α, there is another ("dual") trinion decomposition β such that the map*

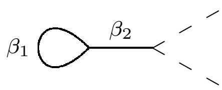
$$\phi := \pi_\beta|_{\mathcal{S}_\alpha} : \mathcal{S}_\alpha \to \Delta_\beta$$

is surjective. For $g = 2$, ϕ is a homeomorphism, and for $g > 2$, it is a finite cover of Δ_β.

We can make the following remarks

1. One cannot hope that ϕ is one-to-one and onto, since it is known that the space $SU(2)^g/SU(2)$ is not contractible in general. (For instance $SU(2)^3/SU(2) \cong S^6$, see [BC]).

2. Clearly, the statement only depends on the graphs of the decompositions. But the roles of α and β are not interchangeable, since the graph of β cannot contain a separating arc β_2 (equivalently a loop β_1), i.e, a configuration of the form

   (58)

   which in X has the shape of a "handle" X_1 (a torus with one hole). If this were the case, any other trinion decomposition α would have a curve α_1 in X_1 which would be either homotopic to β_1 or to a curve which intersects β_1 in one point and together with α_1 generates the fundamental group of the handle, and introducing the relation $\beta_2 = [\alpha_1, \beta_1]$ in $\pi_1(X)$. Then, since any connection in the space $\mathcal{S}_\alpha$ has holonomy 1 around α_1, this would imply either $B_1 = 1$ or $B_2 = 1$, (according to either $\alpha_1 = \beta_1$ or $\beta_2 = [\alpha_1, \beta_1]$), which means that ϕ cannot be onto.

3. It is not difficult to prove the theorem in the case $g = 2$. We have only two genus 2 trivalent graphs. Consider the trinion decomposition α giving the Θ-shaped graph (the other one can be treated in an analogous way). Assume the corresponding theta-shaped handlebody lays out flat on a table, so that the 3 α's are like meridians, and separate two ε -shaped trinions. This defines π_α and the unitary Schottky set $\mathcal{S}_\alpha \subset \mathcal{M}$. Let β be the decomposition with the 3 horizontal curves on the intersection of the surface with the horizontal plane cutting the handlebody in two pieces, the "equators", so that with respect to β, we have north and south trinions (Γ_β is also the theta graph, as it should by remark 2). Let $\theta = (\theta_1, \theta_2, \theta_3) \in \Delta_\beta$ be arbitrary numbers satisfying equations (57), and $A \in \pi_\beta^{-1}(\theta)$. Notice that, if $B_i = H_{\beta_i}(A)$, $i = 1, 2, 3$, we have $B_1 B_2 B_3 = 1$ (for an appropriate choice of orientations of the β's). Note also that with 2 of the α's and 2 of the β's (call them $\alpha_1, \alpha_2, \beta_1, \beta_2$) we can form

a symplectic basis of $\pi_1(X) = \{\alpha_1, \alpha_2, \beta_1, \beta_2 : [\alpha_1, \beta_1][\alpha_2, \beta_2] = 1\}$, so that

$$\mathcal{M} = \{(a_1, a_2, b_1, b_2) \in SU(2)^4 : [a_1, b_1][a_2, b_2] = 1\}/SU(2).$$

It is now clear that, under the above identification, the matrices $a_1 = a_2 = 1$, $b_1 = B_1, b_2 = B_2$, define another connection $\tilde{A}$, with the properties $\pi_\beta(\tilde{A}) = \theta$, and $\tilde{A} \in \mathcal{S}_\alpha$, so that ϕ is onto. From [JW], prop. 3.1 we see also that given $\theta = (\theta_1, \theta_2, \theta_3) \in \Delta_\beta$, the choice of matrices B_i so that $Tr(B_i) = 2\cos(\pi\theta_i)$ (and such that $B_1 B_2 B_3 = 1$) is unique up to simultaneous conjugation, so that ϕ is also injective.

The proof of theorem 9 and further discussions will appear in [FMNT]. The relevance of this result, from our point of view is related to the possibility of describing non-abelian theta functions as (complexification) of solutions of a certain heat equation on $SU(2)^g$, as happens for genus 1, and for abelian theta functions.

From [JW], we know that given an integer k, the polytopes Δ_β have special points labeling the Bohr-Sommerfeld fibers of the real polarization π_β. The number of these fibers is equal to the Verlinde number for rank 2 and level k. Hence it is desirable to compare the polytope with the unitary Schottky spaces, and in particular to see if "Bohr-Sommerfeld" points on $\mathcal{S}_\alpha$ have some special significance for the "initial conditions" of these theta functions.

6.2 Spaces of distributions with Verlinde dimensions

Let us recall three facts that were behind the success of the strategy of section 5, in the following two cases:

A. Abelian varieties, including moduli spaces of line bundles on curves of arbitrary genus, or the $G = U(1)$, arbitrary genus case.

B. $G = SU(n), g = 1$ case.

The facts, valid for cases **A** and **B**, are the following.

Fact 1. By considering the Schottky map (see definition in (36) and (9), (33))

$$K_{\mathbb{C}} \xrightarrow{S} \mathcal{M}(X), \tag{59}$$

where $K_{\mathbb{C}} = (\mathbb{C}^*)^g$ for case **A** and $K_{\mathbb{C}} = SL(n, \mathbb{C})$ for case **B**, one realizes the so called vector bundle of conformal blocks on the Teichmüller space $\mathcal{T}_g$ of genus g curves

$$\begin{array}{ccc} \widetilde{\mathcal{H}}_{k,g} & \longrightarrow & \mathcal{T}_g \\ \widetilde{\mathcal{H}}_{k_{|X}} & := & H^0(\mathcal{M}(X), L^k), \end{array} \tag{60}$$

as a subbundle $\mathcal{H}_k$ of the trivial bundle

$$\mathcal{H}_{k,g} \hookrightarrow \mathcal{T}_g \times \mathcal{H}(K_{\mathbb{C}}) \ . \tag{61}$$

The fibers $\mathcal{H}_{k,X}$ of $\mathcal{H}_{k,g}$ are finite dimensional subspaces of $\mathcal{H}(K_{\mathbb{C}})$ satisfying quasiperiodicity (see (1) and (40)) and other properties, *e.g.* Ad-invariance in the non-abelian case.

Fact 2. The fibers $\mathcal{H}_{k,X}$ for different curves X can be canonically identified via the CST, by taking the correspondence of a theta function for a given complex structure with a distribution corresponding to the degenerate complex structure $\Omega = 0$. In this context, the extended CST is unitary for a natural hermitean structure and defines on $\mathcal{H}_{k,g} \to \mathcal{T}_g$ a unitary flat connection, which coincides with the KZBH connection. Horizontal sections are solutions of

$$\left\{ \begin{array}{rcl} \frac{\partial \theta}{\partial s} & = & \Delta\theta \\ \theta(s=0) & \in & V^G_{k,g} \subset (C^\infty(K))'^{Ad} \end{array} \right. , \tag{62}$$

where $s = 0$ corresponds to $\Omega = 0$.

Fact 3. An orthonormal basis $\{f_\gamma\}_{\gamma\in\Gamma}$ of $L^2(K, dx)^{Ad}$, for the Haar measure on K, is labelled by the set Γ of n-tuples ($n = g$) of irreducible representations of G. So, $\Gamma = \widehat{U(1)^g}$ in case **A** and $\Gamma = \widehat{SU(n)}$ in case **B**. Let $\widetilde{\Gamma} = \Gamma$ in case **A** and $\widetilde{\Gamma} = \Lambda_W \supset \Gamma$ in case **B**. On $\widetilde{\Gamma}$ there is, for every level $k \in \mathbb{N}$, an action of an infinite discrete group P ($P = \mathbb{Z}^g$ in the case **A** and $P = W \rhd \Lambda_R = W^{\mathrm{aff}}$ for the case **B**). The quotient $\widetilde{\Gamma}/P_{k'}$ of $\widetilde{\Gamma}$ by the level k' action of P ($k' = k$ in case **A** and $k' = k + n$ in case **B**) has a number of classes m, that coincides with the Verlinde number in both cases. To every class $[\gamma + \rho]_{k'}$ ($\rho = 0$ in case **A**) there corresponds one distribution in $V^G_{k,g}$,

$$\varphi_{\gamma,k} = \sum_{\widetilde{\gamma}\in[\gamma]_k} f_{\widetilde{\gamma}}, \tag{63}$$

in case **A** and

$$\varphi_{\gamma,k} = \sum_{\widetilde{\gamma}+\rho\in[\gamma+\rho]_{k'}\cap\Lambda^+_W} \epsilon_{\widetilde{\gamma}} f_{\widetilde{\gamma}}, \tag{64}$$

in case **B**, where ϵ_γ can take the values ± 1. These distributions form a basis of $V^G_{k,g}$ and are therefore the initial conditions for θ functions in (62). Their CST image gives an orthonormal frame of the bundle $\mathcal{H}_{k,g} \to \mathcal{T}_g$.

For genus $g \geq 2$ and rank $n \geq 2$, we still have the Schottky map (see (36)), so that we can still, in principle, consider the bundle of conformal blocks as a sub-bundle of the trivial bundle $\mathcal{T}_g \times \mathcal{H}(SL(n, \mathbb{C})^g)^{Ad}$. The problem here, is that the Schottky map is rather complicated and we do not know the analog of the quasiperiodicity properties that the sections θ have to satisfy. Also, even though there exists a flat connection on $\mathcal{H}_{k,g} \to \mathcal{T}_g$, the form of the horizontal sections is not known, nor is their behaviour in the limit $\Omega \to 0$. However, fact 3 above remains valid (almost) precisely as stated for the cases **A** and **B**, except for the last part concerning orthonormality of the corresponding frame (still conjectural). Namely, for $n = 2$, there continue to exist natural spaces $V_{k,g}^{SU(2)} \subset (C^\infty(SU(2)^g))'^{Ad}$, with $\dim V_{k,g}^{SU(2)} =$ Verlinde number, obtained from the level $k + n$ action of the affine Weyl group on Λ_W. We conjecture that these distributions are the boundary values of horizontal holomorphic sections of the KZBH connection for $SL(2,\mathbb{C})$ and genus g, at particular singular boundary points of $\mathcal{T}_g$. Moreover, by choosing an orthonormal basis of these distribution spaces (see below), we should obtain non-abelian theta functions corresponding to orthonormal frames for $\mathcal{H}_{k,g}$, therefore fixing the Hermitean structure on this bundle.

These natural spaces $V_{k,g}^{SU(2)}$ can be obtained as follows. Let X be a given genus g Riemann surface with a fixed trinion decomposition α as in the previous section. As we will see, a basis for $V_{k,g}^{SU(2)}$ can be given which corresponds to labelling the $(3g-3)$ edges of the graph Γ_α with irreducible representations of $SU(2)$, subject to the condition that they are all level k integrable representations and that the tensor product of three representations meeting at a vertex must contain the trivial representation. As shown in [JW], the set of these labellings, Ξ_k, has cardinality equal to the Verlinde number for genus g, rank 2 and level k. To describe these distributions, we will first consider the associated orthonormal spin network basis on $L^2(SU(2)^g, dx)^{Ad}$, where dx denotes the Haar measure. Given the trivalent graph Γ_α, there is a natural isomorphism

$$\frac{SU(2)^{3g-3}}{SU(2)^{2g-2}} \xrightarrow{\sim} \frac{SU(2)^g}{SU(2)}, \tag{65}$$

where in the first quotient the denominator acts by gauge transformations at the vertices of the graph, while the second quotient is by the diagonal conjugation action of $SU(2)$. The advantage of the left-hand side in (65) is that it gives a natural orthonormal basis $\{f_\gamma\}_{\gamma\in\Xi}$ on $L^2(SU(2)^g, dx)^{Ad}$, where $\Xi = \cup_{k=1}^\infty \Xi_k$, namely the so-called spin network basis [Ba, Th]. Now Ξ is the set of labellings of the edges of Γ_α with irreducible representations of $SU(2)$ such that the tensor product of three representations meeting at a vertex contains the trivial representation. Note that for $SU(2)$ such tensor product contains only one copy of the trivial representation, *i.e.* there is only one invariant tensor. By assigning group elements to each of the $(3g-3)$ edges, taking their corresponding representation matrices and producing this invariant tensor, one defines a function on the left hand side of (65); this corresponds to an Ad-invariant function f_γ on $SU(2)^g$, which we can normalize so that $||f_\gamma|| = 1$ in $L^2(SU(2)^g, dx)^{Ad}$.

Theorem 10. *[Ba] The set $\{f_\gamma, \gamma \in \Xi\}$ forms an orthonormal basis for $L^2(SU(2)^g, dx)^{Ad}$.*

Consider the level $(k+2)$ orbits of the action of the affine Weyl group W^{aff} on Λ_W and the spaces

$$U_{k,g}^{SU(2)} = \mathrm{span}_{\mathbb{C}} \langle f_\gamma \in L^2(SU(2)^g, dx)^{Ad} : \gamma \in \Xi_k \rangle$$

as in (51). Consider the following map analogous to (52)

$$\begin{array}{rcl} \Phi_k : \; U_{k,g}^{SU(2)} & \to & \left(C^\infty(SU(2)^g)\right)'^{Ad} \\ f_\gamma & \mapsto & \varphi_{\gamma,k} = \displaystyle\sum_{\tilde\gamma + \hat\rho \in [\gamma+\hat\rho]_{k+2} \cap \Xi} \epsilon_{\tilde\gamma} f_{\tilde\gamma} \end{array} \tag{66}$$

where $\hat\rho = (\rho, \cdots, \rho) \in \Lambda_W^{3g-3}$, $[\gamma + \hat\rho]_{k+2}$ denotes the orbit of the $(3g-3)$-tuple of $SU(2)$ representations $\gamma + \hat\rho$ under $(W^{\mathrm{aff}})^{3g-3}$ and $\epsilon(\gamma) = \Pi_{j=1}^{3g-3} \epsilon_{\gamma_j}$ with ϵ_{γ_j} defined as in (50).

The map Φ_k is well defined and it is injective. By analogy with previous cases, we denote the image of $U_{k,g}^{SU(2)}$ by $V_{k,g}^{SU(2)}$. This reasoning, together with the results of the previous section and the situation in the cases **A** and **B** above, lead us to the following natural conjectures.

Conjecture 1. *The space $V_{k,g}^{SU(2)}$ is a linear span of Dirac delta functions supported on the points of $\mathcal{S}_\alpha$ which are in the Bohr-Sommerfeld fibers of the moment map associated to the dual trinion decomposition β, as in the previous section.*

Conjecture 2. *The distributions in $V_{k,g}^{SU(2)}$ are boundary values of KZBH-horizontal non-abelian theta functions of rank 2 and genus g when the curve X degenerates to the graph Γ_α.*

Further details and results concerning these conjectures will appear in [FMNT].

7 Acknowledgments

The authors were partially supported by the Center for Mathematics and Applications, IST, Lisbon (CF and JM), by the Center for Analysis, Geometry and Dynamical Systems, IST, Lisbon (JPN), and also by the FCT (Portugal) via the program POCTI and FEDER and by the projects POCTI/33943/MAT/2000 and CERN/FIS/43717/2002.

References

[AB] M.Atiyah, R. Bott, “The Yang-Mills equations over Riemann surfaces”, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), 523-615.

[AdPW] S.Axelrod, S.Della Pietra, E.Witten, “Geometric quantization of Chern-Simons gauge theory”, J. Diff. Geom. **33** (1991), 787-902.

[Ba] J.Baez, “Spin networks in gauge theory”, Adv. Math. **117** (1996) 253-272.

[Bea1] A.Beauville, “Vector Bundles on Curves and Generalized Theta Functions: recent results and open problems”, mathAG/9404001.

[Bea2] A.Beauville, “Conformal blocks, Fusion rings and the Verlinde formula” Proc. of the Hirzebruch 65 Conf. on Algebraic Geometry, Israel Math. Conf. Proc. **9** (1996) 75-96.

[BL] A.Beauville and Y.Laszlo, “Conformal blocks and generalized theta functions”, Comm. Math. Phys. **164** (1994) 385-419.

[Ber] D.Bernard, “On the Wess-Zumino-Witten model on the torus”, Nucl. Phys. **303** (1988) 77-93; “On the Wess-Zumino-Witten models on Riemann surfaces”, Nucl. Phys. **B309** (1988) 145-174.

[BC] S.Bratholdt and D. Cooper, “On the topology of the character variety of a free group”, preprint, available at www.math.ucsb.edu/∼cooper/freegroup_characters3text.pdf

[DN] J.Drezet, M.S.Narasimhan, “Groupe de Picard des varietés de modules de fibrés semi-stables sur les courbes algébriques”, Invent. Math. **97** (1989) 53-94.

[EMSS] S.Elitzur, G.Moore, A.Schwimmer, N.Seiberg, “Remarks on the canonical quantization of the Chern-Simons-Witten theory”, Nucl. Phys. **B326** (1989), 108-134.

[Fa] G.Faltings, “Stable G-bundles and projective connections”, Journ. Alg. Geom. **2** (1993), 507–568.

[Fl] C.Florentino, “Schottky uniformization and vector bundles over Riemann surfaces”, Manuscripta Mathematica **105** (2001) 68-83.

[FMN1] C.Florentino, J.Mourão, J.P.Nunes, “Coherent state transforms and abelian varieties”, Journ. Funct. Anal. **192** (2002) 410-424.

[FMN2] C.Florentino, J.Mourão, J.P.Nunes, “Coherent State Transforms and Vector Bundles on Elliptic Curves”, Journ. Funct. Anal. **204** (2003) 355-398.

[FMNT] C.Florentino, J.Mourão, J.P.Nunes, A.Tyurin, “Analytical aspects on non-abelian theta functions”, in preparation.

[FKM] J. Fogarty, F. Kirwan and D. Mumford “Geometric Invariant Theory”, Ergebnisse der Mathematik und ihrer Grenzgeibeite, 3rd edition, Springer Verlag, New York, 1994.

[FM] R.Friedman and J.Morgan, “Holomorphic principal bundles over elliptic curves”, math.AG/9811130.

[FMW] R.Friedman, J.Morgan and E.Witten, “Principal G-bundles over elliptic curves”, Math. Res. Lett. **5** (1998) 97-118.

[G] K.Gawędzki, “Lectures on Conformal Field Theory”, in: Quantum fields and strings: A course for mathematicians, Amer. Math. Soc. and Inst. Adv. St. (1999);
“Conformal Field Theory: a case study”, hep-th/9904145.

[Ha1] B.C.Hall, “The Segal-Bargmann coherent state transform for compact Lie groups”, J. Funct. Anal. **122** (1994), 103-151.

[Ha2] B.C.Hall, “Holomorphic methods in mathematical physics”, Contem. Math. **260** (2000) 1-59; “Quantum mechanics in phase space”, Contem. Math. **214** (1998) 47-62, “Geometric quantization and the generalized Segal-Bargmann transform for Lie groups of compact type”, quant-phys/0012105; “Harmonic analysis with respect to the heat kernel measure”, Bull. Amer. Math. Soc. **38** (2001) 43-78.

[Hi] N.Hitchin, “Flat connections and geometric quantization”, Comm, Math. Phys. **131** (1990) 347-380.

[Ho] G.Hochschild, “The structure of Lie groups”, Holden-Day, 1965.

[JW] L.Jeffrey and J.Weitsman, “Bohr-Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula”, Comm. Math. Phys. **150** (1992) 593-630.

[Ka] V. Kac, “Infinite dimensional Lie algebras,” (Third edition) Cambridge University Press, Cambridge, 1990.

[Ke] G.Kempf, “Complex abelian varieties and theta functions”, Springer-Verlag, 1991.

[KZ] V.Knizhnik, A.Zamolodchikov, “Current Algebras and the Wess-Zumino-Witten Model in Two Dimensions”, Nucl. Phys. **B247** (1984), 83-103.

[L] E.Looijenga, “Root systems and elliptic curves”, Invent. Math. **38** (1976) 17-32.

[LB] H.Lange, C.Birkenhake, “Complex Abelian Varieties”, Springer-Verlag, Berlin, 1992.

[N] M.S.Narasimhan, “Geometric Invariant Theory and Moduli Problems”, unpublished lecture notes.

[NS] M.S.Narasimhan, C.Seshadri, “Stable and unitary vector bundles on a compact Riemann surface”, Ann. Mathematics, **82** (1965) 540-567.

[NR] M.S.Narasimhan, Ramanan, “Stable and unitary vector bundles on a compact Riemann surface”, Ann. Mathematics, **82** (1965) 540-567.

[Sn] J.Sniaticky, “Geometric Quantization and Quantum Mechanics”, Springer, 1987.

[So] C.Sorger, “La formule de Verlinde”, Semin. Bourbaki, 1994-95, N^o 794.

[Th] T. Thiemann, “The inverse loop transform”, J. Math. Phys. **39** (1998) 1236-1248.

[Ty1] A.Tyurin, “Quantization, Classical and Quantum Field Theory and Theta Functions”, CRM Monograph Series, AMS, 2003. “Quantization and Theta-Functions”, math.AG /9904046;

[Ty2] A.Tyurin, “On Bohr-Sommerfeld Bases”, Izv. Math. **64** (2000) 1033-1064; “Complexification of Bohr-Sommerfeld Conditions”, math.AG/9909094.

[Ve] E.Verlinde, “Fusion rules and modular transformations in 2d conformal field theory”, Nucl. Phys. **B300** (1988) 360-376.

[We] J.Weitsman, "Quantization via real polarization of the moduli space of flat connections and Chern-Simons gauge theory in genus one", Comm. Math. Phys. **137** (1991) 175-190.

[Wi] E.Witten, "Quantum Field Theory and the Jones Polynomial", Comm. Math. Phys. **121** (1989), 351.