Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

DIFFERENTIAL TOPOLOGY - WEEKLY PLAN

Week 1: Quick review: topological spaces, Hausdorff, paracompact, countable basis, differentible manifolds, tangent map.
Immersions, submersions, embeddings, regular value theorem, transversality, intersection of manifolds.
Quick review: cuttoff functions, partitions of unity.

Week 2: Whitney embedding theorem.
Manifolds with boundary, neat submanifolds, regular value theorem for manifolds with boundary.
Strong and weak topologies on $C^r(M,N)$. Density of embeddings for $N=\mathbb{R}^k$ and $k>2\dim M$ and of immersions for $k\geq 2\dim M$.
The sets of immersions, submersions, embeddings, proper maps and diffeomorphisms are open in the strong topology.

Week 3: Convolutions, density of $C^s(M,N)$ in $C^r(M,N)$ in the strong topology for $s>r$. Density of C^s immersions, submersions, embeddings, proper maps and diffeomorphisms in the set of maps of the same type and class C^r.
Unicity of C^∞ manifold structure on a differentiable manifold.
Baire property of the strong and weak topologies in $C^r(M,N)$.
Morse-Sard theorem.

Week 4: Brouwer fixed point theorem. Compact M does not retract on its boundary.
Rich classes of maps. Globalization theorem. Transversality theorem. Parametric transversality. Vector bundles: review, exact sequences.

Week 5: Oriented vector bundles. Normal bundle of a submanifold.
Results on submanifolds of N that separate N.
Tubular neighbourhoods and collars.

Week 6: Degree, degree *mod* 2, invariance by homotopy.
Hairy ball theorem.
Hopf theorems for maps to S^n.

Week 7: Continuation of discussion and proof of the Hopf theorems.
Intersection number, invariance by homotopy and definition in homology.
Euler number of an oriented vector bundle and Euler characteristic of a manifold.

Index of a vector field at an isolated zero, degenerate or not. Gauss map and the Euler characteristic. Poincaré-Hopf theorem.

Week 8: Vector bundles with zero Euler number have sections without zeros.
Morse theory: introduction, example, Morse functions, index of a critical point, Morse lemma, abundance of Morse functions on a manifold.

Week 9: Morse theory: regular interval theorem, Reeb theorem, topology change across a critical level of index k, type of a Morse function, homotopy type of a manifold with a Morse function of a given type.
Morse inequalities, relation between the number of critical points of index k and the kth Betti number, Morse inequalities and the Euler characteristic.
A compact manifold has the homotopy type of a finite CW complex.

Week 10: Morse polynomial, Poincaré polynomial and Morse inequalities.
Perfect Morse functions, Morse lacunary principle, Morse theorem for Morse functions with completable critical poins. Examples.
Non-degenerate critical manifolds, Morse-Bott functions, Morse-Bott functions and pull-back, Morse-Bott polynomial and Morse inequalities. Examples.
Classification of vector bundles: Stiefel manifolds, extensions of monomorphisms of vector bundles to the trivial bundle of large enough rank.

Week 11: Classification of vector bundles: universal bundle over the Grassmannian $G_{k,s}$, classifying maps for vector bundles, classification theorem.
Cobordism: oriented and non-oriented cobordism groups, cobordism of pre-image of a manifold by homotopic maps.
Thom space of a vector bundle, oriented and non-oriented Thom homomorphism, the fundamental theorem of cobordism (Thom).

Week 12: Statement of Thom's theorem for oriented cobordism groups in dimensions equal to $0 \, mod \, 4$ and different from $0 \, mod \, 4$. Schematic description of some of the steps in the proof and of Pontrjagin and Stiefel-Whitney numbers.
Framed cobordism: Pontrjagin manifolds and submanifolds with trivial normal bundle, Pontrjagin theorem for cobordism classes of Pontrjagin manifolds. Rederivation of the theorems of Hopf.
Introduction to Morse homology: stable and unstable manifolds of a non-degenerate critical point, Morse-Smale flows, examples.

Week 13: Introduction to Morse homology: manifolds of trajectories, orientation of trajectories from orientation of unstable manifolds, examples.

Morse-Smale-Witten complex and boundary operator, schematic discussion of the nilpotency of the boundary operator, Morse homology. Examples and schematic discussion of the isomorphism between Morse homology and singular homology.
Supersymmetry and Morse theory: Witten's Laplacian Δ_t, the $t \to \infty$ limit, kernel of Δ_t in terms of k-forms localized on critical points of index k, cohomological reinterpretation of the Morse inequalities in terms of the Morse-Smale-Witten complex.
The $t \to \infty$ limit and the (co)boundary operator in Morse (co)homology, isomorphism between de Rham cohomology and Morse cohomology.