Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Geometria Riemanniana - 1º semestre de 2014/15

Teste de Recuperação 2 - 30/01/2015
Duration: 90 min.
Write down all calculations and relevant justifications

1) Consider the hyperboloid

$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$$

and the local parametrization $h(\theta, z) = (\sqrt{1+z^2}\cos\theta, \sqrt{1+z^2}\sin\theta, z), \theta \in]0, 2\pi[, z \in \mathbb{R}$. Let S be equipped with the metric g induced from the Euclidian metric in $\mathbb{R}^3$.

(2.5pt) a) Show that

$$g = (1+z^2)\, d\theta \otimes d\theta + \frac{1+2z^2}{1+z^2}\, dz \otimes dz.$$

(2.5pt) b) Consider the local orthonormal frame

$$X_1 = \frac{1}{\sqrt{1+z^2}} \frac{\partial}{\partial \theta}, \;\; X_2 = \sqrt{\frac{1+z^2}{1+2z^2}} \frac{\partial}{\partial z}.$$

Show that the corresponding connection forms for the Levi-Cicita connection are

$$\omega_1^2 = -\omega_2^1 = -\frac{z}{\sqrt{1+2z^2}}\, d\theta, \;\; \omega_1^1 = \omega_2^2 = 0.$$

(2.5pt) c) Determine the Gauss curvature of S.

(2.5pt) d) Is the equator $E = \{(x,y,z) \in S : z = 0\}$ a geodesic ?

(3pt) e) Determine the vector resulting from the parallel transport of $\frac{\partial}{\partial \theta}$ at $(\sqrt{2}, 0, 1)$ around the closed curve on S defined by $z = 1$.

(3pt) f) Write an explicit integral giving the distance on S between the points $(0,1,0)$ and $(0, \sqrt{5}, 2)$.

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2) Let M be a Riemannian manifold with Levi-Civita connection ∇. Suppose that there exists a nonzero vector field X on M such that

$$\nabla_Y X = 0, \quad \forall Y \in \mathcal{X}(M).$$

(2 pt) a) Show that if $\dim M = 2$ then the metric on M is flat.

(2 pt) b) For $\dim M > 2$, give an example of M and $X \in \mathcal{X}(M)$ with the above property such that the metric on M is not flat.

Formulae

- Cartan equations on an orthonormal local frame $\{X_i\}_{i=1,\ldots,n}$:

$$\begin{cases} d\omega^i = \sum_{j=1}^n \omega^j \wedge \omega_j^i \\ \omega_i^j = -\omega_j^i \\ d\omega_i^j = \Omega_i^j + \sum_{k=1}^n \omega_i^k \wedge \omega_k^j. \end{cases}$$

- $\nabla_X(X_i) = \sum_{k=1}^n \omega_i^k(X) X_k$.