Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Geometria Riemanniana - 1º semestre de 2015/16

Teste de Recuperação 1 - 29/01/2016
Duration: 90 min.

Write down all calculations and relevant justifications

1) Let $X \in \mathcal{X}(\mathbb{R}^2)$ be given by

$$X(x,y) = e^{-x}\frac{\partial}{\partial x} + e^{y}\frac{\partial}{\partial y},$$

where (x,y) are Cartesian coordinates.

(3pts) a) Determine the integral curve of X through $p = (x_0, y_0) \in \mathbb{R}^2$.

(3pts) b) Is X a complete vector field?

(3pts) c) Consider the diffeomorphism $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\psi(x,y) = (x+2, y+e^x).$$

Determine $(D\psi \cdot X)_{(2,3)}$.

(3pts) d) Let $Y \in \mathcal{X}(\mathbb{R}^2)$ be such that $[X,Y] = X$. Show that if $f : \mathbb{R}^2 \to \mathbb{R}$ is constant along the flow of X then so is $(Y \cdot f)$.

(2pts) 2) Let G be a Lie group and let $e \in G$ be the identity. Recall that the exponential map

$$\exp : T_eG \to G,$$

is defined by $\exp(A) = \phi_1(e)$ where $\phi_t, t \in \mathbb{R}$, is the flow of the left-invariant vector field on G whose value at e is A.

Show that exp defines a diffeomorphism from an open neighborhood of $0 \in T_eG$ to an open neighboorhood of e in G.

(Please turn the page)

3) Let M be a manifold with $\dim M = n$. Let ϕ_t be the flow of a complete vector field $X \in \mathcal{X}(M)$.

(2pts) a) Let $\omega \in \Omega^k(M)$. Recall the Lie derivative of ω along X

$$L_X \omega = \frac{d}{dt}(\phi_t^* \omega)_{|t=0}.$$

Show that

$$\frac{d}{dt}(\phi_t^* \omega)_{|t=s} = L_X(\phi_s^* \omega).$$

(2pts) b) Let N be a compact submanifold of M where $\dim N = k$ and $\partial N = \emptyset$. If $\omega \in \Omega^k(M)$ is closed show that

$$\frac{d}{dt}\int_N i_t^* \omega = 0,$$

where $i_t : \phi_t \circ i_0 : N \to M$ and $i_0 : N \to M$ is the inclusion.

(Note: Recall Cartan's formula: $L_X \omega = \iota_X d\omega + d\iota_X \omega$.)

(2pts) 4) Let $f : M \to N$ be a map between smooth manifolds and let $A \subset N$ be a submanifold. f is said to be *transverse* to A if $\forall p \in f^{-1}(A)$ one has

$$ImDf_p + T_q A = T_q N,$$

where $q = f(p)$.

Show that in this case $f^{-1}(A) \subset M$ is a submanifod and determine its dimension.

Hint: Recall how a submanifold $A \subset N$ is described locally.