Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Geometria Riemanniana - 1º semestre de 2014/15

Teste 2 - 14/01/15 - Duration: 90 min.

Write down all calculations and relevant justifications

1) Let $M = \mathbb{R}\times S^1$. Let (x,θ) be the usual coordinates on M and consider the metric
$$g = h(x)^{-1}dx^2 + h(x)d\theta^2,$$
where $h : \mathbb{R} \to \mathbb{R}^+$ is an even function.

(3 pt) a) Consider the local orthonormal frame
$$X_1 = h^{\frac{1}{2}}\frac{\partial}{\partial x},\ X_2 = h^{-\frac{1}{2}}\frac{\partial}{\partial \theta}.$$
Show that the corresponding connection forms for the Levi-Civita connection are
$$\omega_1^2 = -\omega_2^1 = \frac{h'}{2}d\theta,\ \omega_1^1 = \omega_2^2 = 0.$$

(3 pt) b) Determine the Gauss curvature of M.

(3 pt) c) Let $\theta \in \,]0, 2\pi[$. Show that the straight line $\mathbb{R} \times \{\theta\} \subset M$ is a geodesic.

(4 pt) d) *Using the geodesic equation*, determine for which $x \in \mathbb{R}$ we have that $\{x\} \times S^1 \subset M$ is a geodesic.

(3 pt) e) *Using the Gauss-Bonnet theorem*, and recalling that h is even, verify the result of d).

(4 pt) 2) Give examples of two non-isometric Riemannian metrics of constant curvature on the 2-torus with the same area.

Formulae

- Cartan equations on an orthonormal local frame $\{X_i\}_{i=1,\dots,n}$:

$$\begin{cases} d\omega^i = \sum_{j=1}^n \omega^j \wedge \omega_j^i \\ \omega_i^j = -\omega_j^i \\ d\omega_i^j = \Omega_i^j + \sum_{k=1}^n \omega_i^k \wedge \omega_k^j. \end{cases}$$

- $\nabla_X(X_i) = \sum_{k=1}^n \omega_i^k(X)X_k.$