Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Geometria Riemanniana - 1º semestre de 2015/16

Teste 1 - 11/11/2015 - Duration: 90 min.

Write down all calculations and relevant justifications

1) Consider the following vector fields in $\mathbb{R}^4$,

$$X(x) = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}, \ \ Y(x) = -x_3\frac{\partial}{\partial x_4} + x_4\frac{\partial}{\partial x_3}$$

and

$$Z(x) = \alpha(x_3, x_4)\frac{\partial}{\partial x_1} + \beta(x_3, x_4)\frac{\partial}{\partial x_2},$$

where $x = (x_1, x_2, x_3, x_4)$ are Cartesian coordinates and α, β are smooth functions.

(2.5pts) a) Determine the flow, ϕ_t, of X.

(2pts) b) Determine the Lie bracket $[Y, Z]$.

(2.5pts) c) For which functions α, β is Z a complete vector field ?

(3pts) d) Consider the diffeomorphism $f : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$f(x_1, x_2, x_3, x_4) = (x_1 + e^{x_2}, x_2 + 1, x_3 + x_4, x_4 - x_3 + e^{x_2}).$$

Determine

$$(Df \cdot X)_{(3,1,2,1)}\,.$$

(3pts) e) Let now α, β above be two fixed non-constant smooth functions. Suppose that $h : \mathbb{R}^4 \to \mathbb{R}^4$ is a diffeomorphism such that

$$Dh \cdot X = Y \ \text{ and } \ Dh \cdot Y = Z.$$

Construct such h or show that it does not exist.

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2) Let $\Omega_c^m(\mathbb{R}^l)$ be the space of m-forms in $\mathbb{R}^l$ with compact support. For $j = 0, \ldots, n$, consider a linear map

$$I : \Omega_c^{j+k}(\mathbb{R}^n \times \mathbb{R}^k) \to \Omega_c^j(\mathbb{R}^n)$$

defined as follows. Let x, y be Cartesian coordinates in $\mathbb{R}^n, \mathbb{R}^k$, respectively. Then, if $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ has compact support, we define

$$I(f(x,y)dx_{i_1} \wedge \cdots \wedge dx_{i_m} \wedge dy_{l_1} \wedge \cdots \wedge dy_{l_{j+k-m}}) = 0, \text{ if } m > j,$$

and

$$I(f(x,y)dx_{i_1} \wedge \cdots \wedge dx_{i_j} \wedge dy_1 \wedge \cdots \wedge dy_k) = \left(\int_{\mathbb{R}^k} f(x,y)dy \right) \; dx_{i_1} \wedge \cdots \wedge dx_{i_j}.$$

(2pts) a) Show that $I \circ d = d \circ I$.

(2pts) b) Let $\tau \in \Omega_c^{n-j}(\mathbb{R}^n)$ and $\omega \in \Omega_c^{j+k}(\mathbb{R}^n \times \mathbb{R}^k)$. Show that

$$\int_{\mathbb{R}^n \times \mathbb{R}^k} \pi^*(\tau) \wedge \omega = \int_{\mathbb{R}^n} \tau \wedge I(\omega),$$

where $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ is the usual projection in the first n coordinates.

(3pts) c) Let E be an $(n+k)$-manifold, B an n-manifold and let

$$\pi : E \to B$$

be a surjective submersion such that there exists a compact k-manifold F with the property that $\forall p \in B$, $\exists$ open neighborhood $U \ni p$, and $\exists$ diffeomorphism

$$\phi_U : U \times F \to \pi^{-1}(U).$$

Generalize I to obtain a map $\check{I} : H^{j+k}(E) \to H^j(B)$.

Hint: Suppose that F is a compact k-manifold. Then, one can define a linear map $\tilde{I} : \Omega^{j+k}(\mathbb{R}^n \times F) \to \Omega^j(\mathbb{R}^n)$ generalizing I. (So, if $U \subset F$ is a sufficiently small open set with U diffeomorphic to $\mathbb{R}^k$, and if ω has compact support contained in $\mathbb{R}^n \times U$, then $\tilde{I}(\omega) = I(\omega)$.)

Note: The map $\check{I}$ is called "integration along the fibers".